On the dynamics of the renormalization operator

A. Avila, M. Martens, and W. de Melo [1]

Abstract

An important part of the bifurcation diagram of unimodal maps corresponds to infinite renormalizable maps. The dynamics of the renormalization operator describes this part of the bifurcation pattern precisely. Here we analyze the dynamics of the renormalization operator acting on the space of C^k infinitely renormalizable maps of bounded type. We prove that two maps of the same type are exponentially asymptotic. We suppose $k \geq 3$ and quadratic critical point.

1 Introduction

We will consider the renormalization operator R acting on an open set in the space of C^k, $k \geq 3$, of unimodal interval maps of bounded combinatorial type. Each map f in the domain D of the operator has a periodic interval around the critical point whose period $q \geq 2$ is at most a given integer $N \geq 3$ and $R(f)$ is affinely conjugate to the restriction of f^q to this interval. A map f is infinitely renormalizable of combinatorial type bounded by N if all iterates $R^n(f)$ belong to D. Sullivan proved in [5], see also [3], the existence of a compact invariant subset $\Gamma \subset D$ such that the restriction of R to Γ is a homeomorphism which is topologically conjugate to a full shift on a finite number of symbols. Furthermore, each map $g \in \Gamma$ is real analytic with a holomorphic extension which is quadratic-like in the sense of Douady-Hubbard. He also proved that if f is infinitely renormalizable with combinatorial type bounded by N then there exist $g \in \Gamma$ such that the C^0 distance between $R^n(f)$ and $R^n(g)$ converges to zero. Using the fundamental result of Lyubich, [1], on the hyperbolicity of the renormalization operator in the space of germs of quadratic-like maps, it was proved in [4] that in fact the C^0 distance between $R^n(f)$ and $R^n(g)$ converges to zero exponentially fast. Here we complete the description of the dynamics of the renormalization operator by proving the exponential convergence of the C^k distances of these iterates.

[1]This work has been partially supported by the Pronex Project on Dynamical System, Faperj Grant E-26/151.896/2000 and NSF Grant DMS-0073069

To give a more precise formulation of our result we include for completeness the usual notions of unimodal renormalization . Fix $\alpha > 1$, the *critical exponent*, and $k \geq 3$, indicating the smoothness class. The *standard folding family* $q_t : [-1,1] \to [-1,1]$, $t \in [0,1]$ is

$$q_t(x) = -2t|x|^\alpha + 2t - 1.$$

Let $\mathrm{Diff}^k([-1,1])$ be the set of C^k orientation preserving diffeomorphisms of the interval $[-1,1]$. It is an open subset of a codimension 2 affine subspace of the Banach space $C^k([-1,1],\mathbb{R})$ of the C^k mappings endowed with the C^k norm $|f|_k = \sup_x\{|f(x)|, |Df(x)| \ldots |D^k f(x)|\}$. The class of unimodal maps we consider here is

$$U = \mathrm{Diff}^k([-1,1]) \times [0,1],$$

where an element $(\phi, t) \in U$ should be interpretated as the C^k unimodal map

$$f = \phi \circ q_t : [-1,1] \to [-1,1],$$

with critical exponent $\alpha > 1$. The diffeomorphism ϕ is called the *diffeomorphic part* of the unimodal map f. The metric on U is the product of the metric of the interval with the C^k distance on $\mathrm{Diff}^k([-1,1])$.

An unimodal map $f \in U$ is called *renormalizable* iff there exists an expanding periodic point $p \in (-1,1)$ such that the first return map to the *central* interval $C = [-p,p]$ is a of the form $f^q : C \to C$ with $f^q(p) = p$ and $q \geq 2$. The first return map to C will be, up to rescaling, a unimodal map. This unimodal map is a *renormalization* of f. Observe that a renormalization is completely determined by the periodic point p.

The combinatorial aspects of a renormalization are described by unimodal permutations. A permutation on a finite ordered set is a *unimodal* permutation if the following holds. Embed the set monotonically into the real line. Draw the graph of the permutation. If this graph can be extended to the graph of a unimodal map then the permutation is called unimodal.

A collection $\mathcal{I} = \{I_1, I_2, \ldots, I_{q-1}, I_q\}$ of oriented closed intervals in $[-1,1]$ is called a cycle for the unimodal map f if it has following properties

- there is an expanding periodic point $p \in (-1,1)$ with $I_q = [-|p|, |p|]$,
- $f : I_i \to I_{i+1}$, $i = 1, 2, \ldots, q-1$, is monotone onto,
- $f(I_q) \subset I_1$ with $f(p) \in \partial I_1$, the boundary of I_1,

- the interiors of $I_1, I_2, \dots, I_q$ are pairwise disjoint,
- $\mathcal{I}$ inherits an order from $[-1,1]$,
- the map
$$\sigma(\mathcal{I}) : I_i \to I_{i+1}$$
on $\mathcal{I}$ is a unimodal permutation,
- the orientation
$$o_{\mathcal{I}} : \mathcal{I} \to \{-1,1\}$$
is such that $o_{\mathcal{I}}(I_i) = 1$ when $f^i(p)$ is the left boundary point of I_i and $o_{\mathcal{I}}(I_i) = -1$ otherwise.

Observe the following

- a unimodal map is renormalizable iff it has a cycle,
- the last three properties defining a cycle follow automatically once a unimodal map has a periodic point with the first three properties.
- the orientation $o_{\mathcal{I}}$ depends only on σ. We will use the notation o_σ.

Let σ be a unimodal permutation and

$$U_\sigma = \{f \in U | f \text{ has a cycle } \mathcal{I} \text{ with } \sigma(\mathcal{I}) = \sigma\}.$$

The unimodal maps in U_σ are sometimes called $\sigma-$renormalizable to emphasize the type of renormalization under consideration. The renormalization operator
$$R_\sigma : U_\sigma \to U$$
is defined to assign to each unimodal map in U_σ the affinely rescaled first return map to the smallest central interval giving rise to a cycle $\mathcal{I}$ with $\sigma(\mathcal{I}) = \sigma$. These sets of renormalizable maps U_σ are not empty, every family $t \to \phi \circ q_t \in U$ contains points in each U_σ. Often a unimodal has different cycles, the sets U_σ are not disjoint. However they are nested. For each σ there exists a unique maximal factorization $\sigma =< \sigma_n, \dots, \sigma_2, \sigma_1 >$ such that

$$R_\sigma = R_{\sigma_n} \circ \dots \circ R_{\sigma_2} \circ R_{\sigma_1}.$$

An unimodal permutation σ is called *prime* iff $\sigma =< \sigma >$. Clearly each permutation in the maximal factorization is prime. Using the prime unimodal permutations we obtain a partition of the set of renormalizable unimodal maps and the *renormalization* operator becomes

$$R : \{\text{renormalizable maps}\} = \bigcup_{\text{prime } \sigma} U_\sigma \to U,$$

with $R|U_\sigma = R_\sigma$.

An unimodal map $f \in U$ is infinitely renormalizable iff $R^n f$ is defined for all $n \geq 1$. In particular, for each infinitely renormalizable maps f there is a unique sequence $\sigma_1, \sigma_2, \sigma_3, \cdots$ of prime unimodal permutation such that

$$R^n f = \cdots \circ R_{\sigma_3} \circ R_{\sigma_2} \circ R_{\sigma_1} f.$$

Fix a finite set of prime unimodal permutations. Two infinitely renormalizable unimodal maps are said to be of the same bounded type if they have the same sequence of prime unimodal permutation and all these permutations are in the given finite set of permutations.

Theorem 1 *Let $\alpha = 2$. If $f, g \in U$ are infinitely renormalizable maps of the same bounded type then there exists $C > 0$ and $\lambda < 1$ such that*

$$|R^n f - R^n g|_k \leq C\lambda^n.$$

The proof of the Main Theorem is formulated in the following three section. The main ingredient, the notion of decompositions, is introduced in section 2. The technical lemmas are collected in the Appendix and the actual proof is given in section 3.

Acknowledgement The second author acknowledges the hospitality of IMPA where parts of this work were done.

2 Decompositions

Applying the renormalization operator repeatedly will produce a sequence of unimodal maps. Each of the unimodal maps of this sequence has a diffeomorphic part and a standard folding part. The renormalization process causes these diffeomorphic parts not to be arbitrarily. The main idea presented in [2] is the method of decomposing the diffeomorphic parts in a systematic manner. The decomposition of renormalizations shows naturally two parts:

an analytic part and a C^k part. This separation underlies the proof of the main theorem.

Fix an infinitely renormalizable map $f = (\phi, q_t) \in U$ and let $\mathcal{I}^n = \{I_1^n, I_2^n, \cdots, I_{q_n}^n\}$ be the cycle corresponding to the n^{th} renormalization, $n \geq 1$. Each cycle will be partitioned in sets

$$\mathcal{I}^n = \bigcup_{k \geq 0} L_k^n.$$

The definition of these *level sets* is by induction. Let

$$\mathcal{I}^0 = L_0^0 = \{[-1,1]\}.$$

If $\mathcal{I}^{n+1} \ni I_i^{n+1} \subset I_j^n \in L_k^n$ and $0 \notin I_i^{n+1}$ then $I_i^{n+1} \in L_{k+1}^{n+1}$. If $0 \in I_i^{n+1}$ then $I_i^{n+1} \in L_0^{n+1}$.

Observe,

$$\mathcal{I}^n = \bigcup_{k \geq 0}^{n} L_k^n.$$

For each $I_i^n \in \mathcal{I}^n$, $i \neq 0$, define the orientation preserving diffeomorphisms $q_i^n : [-1,1] \to [-1,1]$ and $\phi_i^n : [-1,1] \to [-1,1]$ where

$$q_i^n = Z_{I_i^n}(q_t)$$

and

$$\phi_i^n = Z_{q_t(I_i^n)}(\phi),$$

where $q_t(I_i^n)$ and I_{i+1}^n are oriented in the same direction, namely the orientation $o(I_{i+1}^n)$ defined by the cycle $\mathcal{I}^n$. Furthermore, let

$$t_n = \frac{|q_t(I_0^n)|}{|\phi^{-1}(I_1^n)|}.$$

These definition describe the decomposition of $R^n f$. Namely,

$$R^n f = ((\phi_{q_n-1}^n \circ q_{q_n-1}^n) \circ \cdots \circ (\phi_2^n \circ q_2^n) \circ (\phi_1^n \circ q_1^n), t_n).$$

Let $I \subset [-1,1]$ be an oriented interval and consider the zoom operator

$$Z_I : \mathrm{Diff}^k([-1,1]) \to \mathrm{Diff}^k([-1,1]),$$

which assigns to each diffeomorphism ϕ the affinely rescaled version of the orientation preserving restriction $\phi : I \to \phi(I)$. The intervals I and $\phi(I)$ are oriented the same. A computation shows.

Lemma 1 *For every diffeomorphism ϕ there exists a constant C, which depends only on the C^k norm of ϕ and on the C^0 norm of $D\ln D\phi$, such that for every interval $I \subset [-1,1]$ we have*

$$|Z_I(\phi) - id|_k \leq C \cdot |I|.$$

Proof. For each diffeomorphism ψ we define its non-linearity by

$$\eta_\psi = D \log D\psi.$$

The zoom operators act as follows on the non-linearities

$$|\eta_{Z_I(\psi)}|_{k-2} \leq \frac{|I|}{2} |\eta_\psi|_{k-2}. \tag{2.1}$$

This is a straight forward chain rule computation. Moreover, the C^{k-2} norm of the non-linearity bounds the C^k distance to the identity in the following way.

For every $B > 0$ there exists a constant K such that

$$|\psi - \mathrm{id}|_k \leq K \cdot |\eta_\psi|_{k-2}, \tag{2.2}$$

for every diffeomorphism ψ with $|\eta_\psi|_{k-2} \leq B$.

The definition of non-linearity η of a diffeomorphism ψ implies

$$D\psi(x) = A \cdot \exp \int_{-1}^{x} \eta,$$

where

$$A = \frac{2}{\int_{-1}^{1} \exp \int_{-1}^{x} \eta}.$$

These expressions and the bound on the non-linearity of ψ implies

$$|D\psi - 1|_0 \leq K \cdot |\eta|_0.$$

The expression for higher derivatives of ψ is a sum of terms which are products of derivatives of η and a factor $\exp \int_{-1}^{x} \eta$. Using the bound on the non-linearity η we get an estimate

$$|D^j \psi|_0 \leq K \cdot |\eta|_{k-2}, j \leq k.$$

This proves 2.2. Observe, that the above constants do only depend on the C^{k-2} norm of η_ψ.

To prove the Lemma observe that 2.1 implies that $|\eta_{Z_I(\phi)}|_{k-2} \leq |\eta_\phi|_{k-2}$. Hence, we can apply 2.2 with a constant independent of I.

$$|Z_I(\phi) - \mathrm{id}|_k \leq K \cdot |\eta_{Z_I(\phi)}|_{k-2}.$$

Now apply 2.1 again and the Lemma follows with a constant which depends only on $|\eta_\phi|_{k-2}$. Note, that this norm can be bounded by the C^0 norm of η_ϕ and the C^k norm of ϕ. ■

Lemma 2 *For every infinitely renormalizable map of bounded type $f \in U$ there exist $C > 0$ and $\lambda < 1$ such that*

- $\sum_{i=0}^{q_n-1} |\phi_i^n - id|_k \leq C\lambda^n$.
- $\sum_{I_i^n \in L_m^n} |q_i^n - id|_{k+1} \leq C\lambda^m$,

Proof. This Lemma relies on the a priori bounds formulated in the Appendix as Lemma 7. Observe, that the a priori bounds imply that for some constants $C > 0$ and $\lambda < 1$

$$\sum_{i=0}^{q_n-1} |I_i^n| \leq C\lambda^n.$$

Because the diffeomorphic part of f has derivative bounded away from zero these two constants can be adjusted such that also

$$\sum_{i=0}^{q_n-1} |\phi^{-1}(I_i^n)| \leq C\lambda^n.$$

Now Lemma 1, describing the restriction operator, assures the first estimate of Lemma 2.

The proof of the second estimate is by induction in $m \geq 1$. Observe that the a priori bounds imply the estimate for $m = 1$. Every diffeomorphism q_i^n with $I_i^n \in L_{m+1}^n$ is obtained from a diffeomorphism q_j^{n-1} with $I_j^{n-1} \in L_m^{n-1}$ by applying the restriction operator. In particular, the sum of the C^{k+1} norms of all the diffeomorphisms obtained in such a way out of a specific q_j^{n-1} is bounded, using the a priori bound, by a definite factor $1-b$ times the C^{k+1} norm of q_j^{n-1}.

Note that the obtained bounds only depend on the contraction constant on the restriction operator, which in turn only depends on the bounded geometry of the cycles. In particular, they do not depend on n but only on the number of times the restriction operator is applied, namely $m+1$ times. This finishes the induction step.

$\blacksquare$

For each infinitely renormalizable map $f \in U$ we consider the unimodal maps

$$T_n f = (q^n_{q_n-1} \circ \cdots \circ q^n_2 \circ q^n_1, t_n) \in U.$$

Observe, that the maps $T_n f$ are unimodal and that the diffeomorphic part is analytic. In particular, $R^n f$ consists of the analytic terms q_i^n and the restrictions ϕ_i^n of the diffeomorphic part ϕ of f. Lemma 2 states that the terms ϕ_i^n are all very close to the identity. The proof of the main Theorem will show C^k convergence of $T_n f$ and compare $T_n f$ with $R^n f$.

Lemma 3 *For each infinitely renormalizable map of bounded type, $f \in U$, there are constants C and $\lambda < 1$ such that*

$$|R^n f - T_n f|_k \le C\lambda^n.$$

Proof. Fix $n \ge 1$. Observe that Lemma 2 and 6 (see Appendix) imply that any composition of diffeomorphisms q_i^n is uniformly bounded in the C^{k+1} topology. In particular, the bound are independent of $n \ge 1$. Similarly, any composition of diffeomorphisms q_i^n and/or ϕ_i^n is uniformly bounded in the C^k topology.

These bounds allow us to apply Lemma 5 to the following diffeomorphisms. Define for each $i = 1, \cdots, q_n - 1$ the diffeomorphisms

$$\Psi_i^n = q^n_{q_n-1} \circ \cdots \circ q^n_{i+2} \circ q^n_{i+1},$$

and

$$\Phi_i^n = (\phi_i^n \circ q_i^n) \circ \cdots \circ (\phi_2^n \circ q_2^n) \circ (\phi_1^n \circ q_1^n).$$

Observe, the diffeomorphisms Ψ_i^n are uniformly bounded in the C^{k+1} topology and the diffeomorphisms Φ_i^n are uniformly bounded in the C^k topology. Moreover,

$$T_n f = (\Psi_0^n, t_n),$$

and

$$R^n f = (\Phi^n_{q_n-1}, t_n).$$

For $i = 0, \cdots, q_n - 1$ define also

$$H^n_i = \Psi^n_i \circ \Phi^n_i.$$

The sequence H^n_i starts at $T_n f = H^n_0$ and ends in $R^n f = H^n_{q_n-1}$. Hence

$$|R^n f - T_n f|_k \leq \sum_{i=1}^{q_n-1} |H^n_i - H^n_{i-1}|_k.$$

Because

$$H^n_i = \Psi^n_i \circ \phi^n_i \circ (q^n_i \circ \Phi^n_{i-1})$$

and

$$H^n_{i-1} = \Psi^n_i \circ (q^n_i \circ \Phi^n_{i-1})$$

Lemma 5 gives the following estimate

$$|H^n_i - H^n_{i-1}|_k \leq E \cdot |\phi^n_i - \mathrm{id}|_k.$$

The final step is to use Lemma 2 to get an exponential small distance:

$$|R^n f - T_n f|_k \leq E \cdot \sum_{i=1}^{q_n-1} |\phi^n_i - \mathrm{id}|_k \leq C\lambda^n.$$

■

Lemma 4 *For each infinitely renormalizable map of bounded type $f \in U$ the sequence $T_n f$, $n \geq 1$ forms a precompact family of analytic maps.*

Proof. The diffeomorphic part of each $T_n f$ is an analytic diffeomorphism which maps the interval $[-1, 1]$ onto itself. Namely, they are composition of pieces of standard folding maps. To prove the Lemma it suffices to show that these diffeomorphisms have univalent extensions on a fixed simply connected domain containing the interval $[-1, 1]$.

The construction of such a disk is based on Lemma 8 of the Appendix. We will use the notion of Lemma 3. In particular, we will apply Lemma 8 to the sequence of maps Ψ^n_i

$$\Psi^n_{i-1} = \Psi^n_i \circ q^n_i.$$

The diffeomorphisms q_i^n are pieces of standard folding maps and because of the a priori bounds they are all univalent on a fixed disk D_0 containing $[-1,1]$. Moreover, the second estimate of Lemma 2 gives a constant $K>0$ which is independent of $n \geq 1$ such that

$$\sum_i |(q_i^n - \mathrm{id})|D_0|_0 < K.$$

This estimate and Lemma 8 assure that there will be a definite simply connected domain containing $[-1,1]$ on which the maps Ψ_i^n are all univalent. In particular the diffeomorphic parts of $T_n f$ extend univalently to this domain.

$\blacksquare$

3 Proof of the Theorem

In this section we will prove that C^k exponential convergence follows from C^0 exponential convergence in any class of unimodal maps (e.g. $\alpha > 1$ and $k \geq 3$). In particular, let $f, g \in U$ be infinitely renormalizable C^k maps of the same bounded type and assume that

$$|R^n f - R^n g|_0 \leq C\lambda^n,$$

where $\lambda < 1$. Observe, that this exponential convergence actually has been shown in the class of C^3 unimodal maps with quadratic critical point, e.g. $\alpha = 2$.

To prove the main Theorem it suffices to show that there are constants, say also denoted by C and $\lambda < 1$, such that

$$|R^n f - R^n g|_k \leq C\lambda^n.$$

First, observe that the above C^0 exponential convergence together with Lemma 3 gives for some constants $C > 0$ and $\lambda < 1$

$$|T_n f - T_n g|_0 \leq C\lambda^n.$$

Then Lemma 4 gives that the sequences $T_n f$ and $T_n g$ form precompact families of analytic maps. That implies that the C^k norm is Hölder equivalent to the C^0 norm for these families. In particular, the constants $C > 0$ and $\lambda < 1$ can be adjusted such that

$$|T_n f - T_n g|_k \leq C\lambda^n.$$

Now the proof is finished by a second adjustment of the constants and the inequality

$$|R^n f - R^n g|_k \leq |R^n f - T_n f|_k + |T_n f - T_n g|_k + |T_n g - R^n g|_k \leq C\lambda^n.$$

4 Appendix

The following two Lemmas are estimates on the behavior of the composition operator. The proof of the next Lemma is a straightforward chain rule computation.

Lemma 5 *There is a constant $E = E(k, B)$ with the following property. Let $\psi_2 \in \mathit{Diff}^{k+1}([-1,1])$ and $\psi_1, \phi \in \mathit{Diff}^{k}([-1,1])$ such that*

$$|\psi|_{k+1}, |\phi|_k, |\psi_1|_k \leq B,$$

then

$$|\psi_2 \circ \phi \circ \psi_1 - \psi_2 \circ \psi_1|_k \leq E \cdot |\phi - id|_k.$$

Lemma 6 *For every $B > 0$ and $k \geq 1$ there exists a constant K such that the following holds. If*

$$\sum_{i=1}^{n} |\phi_i - id|_k \leq B$$

then

$$|\phi_n \circ \cdots \circ \phi_2 \circ \phi_1|_k \leq K.$$

Proof. Let S_k^n be the maximum C^k norm of diffeomorphism of the form

$$\phi = \phi_n \circ \cdots \circ \phi_2 \circ \phi_1,$$

with

$$\sum_{i=1}^{n} |\phi_i - \mathrm{id}|_k \leq B.$$

The product rule implies easily that

$$S_1 = \sup_{n \geq 1} S_1^n \leq \exp(B).$$

An inductive argument will show that each

$$S_k = \sup_{n \geq 1} S_k^n$$

is finite. Assume $S_k < \infty$. Given are the diffeomorphisms ϕ_i, $i = 1, \cdots, n+1$, whose sum of C^{k+1} norms is bounded by B. Let Ψ be the composition of the first n diffeomorphisms ϕ_i, $i = 1, \cdots, n$ and $\Phi = \phi_{n+1} \circ \Psi$. The induction hypothesis imply that Φ is bounded in the C^k topology, the bound given by S_k.

It suffices to estimate the $(k+1)^{th}$ derivative of Φ. Observe, that this derivative is a polynomial expression whose terms which are products of derivatives of Ψ and ϕ_{n+1}. There is one term of the form $(D\phi_{n+1} \circ \Psi) \cdot D^{k+1}\Psi$. The other terms involve only lower derivatives of Ψ, that is lower than $k+1$. However, these terms have exactly one factor which is a specific higher derivative of ϕ_{n+1}, that is not a first derivative. This implies that there is a specific polynomial P_{k+1} such that the following estimate holds

$$S_{k+1}^{n+1} \leq (1 + |\phi_{n+1} - \mathrm{id}|_{k+1}) \cdot S_{k+1}^n + P_{k+1}(S_k) \cdot |\phi_{n+1} - \mathrm{id}|_{k+1}.$$

The term $P_{k+1}(S_k)$ is independent of n. Hence we get

$$S_{k+1} \leq \exp(B) \cdot P_{k+1}(S_k) \cdot B.$$

This finishes the induction step.

■

The following Lemma is known as the a priori bounds on the the geometry of cycles of infinitely renormalizable maps of bounded type.

Lemma 7 *Let $f \in U$ be an infinitely renormalizable map of bounded type. There exists constants $0 < b < 1$ with the following property. Let $I_{i_l}^{n+1} \subset I_j^n$, $l = 1, \cdots, m_n$ be the intervals of the $(n+1)^{th}$ cycle which are contained in the interval I_j^n of the n^{th} cycle. Then*

- $\frac{\sum_l |I_{i_l}^{n+1}|}{|I_j^n|} < 1 - b$,
- $\frac{|I_{i_l}^{n+1}|}{|I_j^n|} > b$, $l = 1, \cdots, m_n$.

A proof can be found in [3], see Theorem 2.1 Chapter VI. The proof of the following Lemma is left to the reader.

Lemma 8 *Let $D_0 \supset D_1 \supset [-1,1]$ be strictly nested disks in the complex plane. There exists a constant $K > 0$ such that the following holds. Let $\phi : D_0 \to \mathbb{C}$ and be a univalent map that fix the interval $[-1,1]$ and $[-1,1] \subset \tilde{D} \subset D_1$. There exists a simply connected domain $D_\phi \subset \tilde{D}$ containing $[-1,1]$ such that*

- $\phi(D_\phi) \subset \tilde{D}$,
- $\rho_\phi \geq (1 - K \cdot |(\phi - id)|D_0|_0)\tilde{\rho}$, *where ρ_ϕ and $\tilde{\rho}$ are the distances between the boundary of resp. D_ϕ and $\tilde{D}$ to the interval $[-1,1]$.*

References

[1] M. Lyubich, *Feigenbaum-Coullet-Tresser Universality and Milnor's Hairiness Conjecture*, Ann. of Math. v. 149 (1999) 319-420.

[2] M.Martens, *The Periodic Points of Renormalization*, Ann. of Math. **147** (1998), 543-584.

[3] W.de Melo and S.van Strien, *One-dimensional Dynamics*, Springer-Verlag, 1993.

[4] W. de Melo and A. Pinto, *Rigidity of C^2 Infinitely Renormalizable Unimodal Maps*, Commun. Math. Phys., 208, (1999), 91-105

[5] D. Sullivan *Bounds, quadratic differentials and renormalization conjectures*, AMS Centenial Publications. **2**: Mathematics into Twenty-first Century, (1992).