ON MIXING DIFFEOMORPHISMS OF THE DISC

A. AVILA, B. FAYAD, P. LE CALVEZ, D. XU, Z. ZHANG

ABSTRACT. We prove that a C^k, $k \geq 2$ pseudo-rotation f of the disc with non-Brjuno rotation number is C^{k-1}-rigid.

The proof is based on two ingredients : 1) we derive from Franks' Lemma on free discs that a pseudo-rotation with small rotation number compared to its C^1 norm must be close to the identity map; 2) using Pesin theory, we obtain an effective finite information version of the Katok closing lemma for an area preserving surface diffeomorphism f, that provides a controlled gap in the possible growth of the derivatives of f between exponential and sub-exponential.

Our result on rigidity, together with a KAM theorem by Rüssmann, allow to conclude that analytic pseudo-rotations of the disc or the sphere are never topologically mixing. Due to a structure theorem by Franks and Handel of zero entropy surface diffeomorphisms, it follows that an analytic conservative diffeomorphism of the disc or the sphere that is topologically mixing must have positive topological entropy.

1. INTRODUCTION

1.1. **On rigidity of pseudo-rotations.** We denote the unit disc in $\mathbb{R}^2$ by $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. A homeomorphism of $\mathbb{D}$ that preserves Lebesgue measure, fixes the origin, and has no other periodic points is called a *pseudo-rotation*. Since every point of the disc apart from the origin must have the same irrational rotation number around the origin defined modulo 1 (see for example [6, Corollary 2.6] or [7, Theorem 3.3]), we call this number $\rho(f)$ and we designate the origin as an elliptic fixed point with rotation number $\rho(f)$. If the pseudo-rotation is differentiable, its Jacobian at the origin would be conjugated to the rotation matrix of angle $\rho(f)$.

In [3], Bramham showed that when the rotation number of a pseudo-rotation is very well approximated by rationals then the pseudo-rotation is C^0-rigid, in the sense that a subsequence of its iterates

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converges to the identity map in the uniform topology. In this paper, we generalize Bramham's result in two ways: first, we extend the result to non-Brjuno rotation numbers, and second we establish rigidity in higher regularity.

We recall that a number $\alpha \in \mathbb{R}/\mathbb{Z} - \mathbb{Q}$ is said to be of Brjuno type if

$$\sum_{n=0}^{\infty} \frac{\ln(q_{n+1})}{q_n} < +\infty$$

where q_n is the sequence of denominators of the best rational approximations to α.

A diffeomorphism of class C^k, $k \in \mathbb{N} \cup \{\infty\}$, is said to be C^k *rigid* if there exists a sequence q_n such that f^{q_n} converges to the identity map in the C^k topology. If we just know that the latter holds in a fixed neighborhood of some point p, we say that f is C^k *locally rigid* at p.

Our main rigidity result for pseudo-rotations is the following.

Theorem 1. *If f is a C^k, $k \geq 2$, pseudo-rotation of $\mathbb{D}$ with rotation number $\alpha = \rho(f)$ that is not of Brjuno type, then f is C^{k-1} rigid : There exists a subsequence q_{n_j} of the sequence $q_n(\alpha)$ such that $f^{q_{n_j}} \to \mathrm{Id}_{\mathbb{D}}$ in the C^{k-1} topology.*

When the rotation number is well approximated by rationals we do not need too much regularity to show rigidity of a pseudo-rotation. Indeed, a number $\alpha \in \mathbb{R}/\mathbb{Z} - \mathbb{Q}$ is said to be *super-Liouville* if

$$\limsup q_n^{-1} \ln q_{n+1} = +\infty.$$

We then have the following rigidity result in which the case Hölder with exponent $a = 1$ corresponds to Lipschitz maps.

Theorem 2. *If f is a pseudo-rotation of the disc $\mathbb{D}$ that is Hölder with exponent $a \in (0, 1]$ and if $\alpha = \rho(f)$ satisfies*

$$\limsup q_n^{-1} a^{q_n} \ln q_{n+1} = +\infty$$

then $f^{q_{n_j}} \to \mathrm{Id}_{\mathbb{D}}$ in the uniform topology for any sequence n_j such that $\lim {q_{n_j}}^{-1} a^{q_{n_j}} \ln q_{n_j+1} = +\infty$.

For any given modulus of continuity, we can by the same method obtain arithmetic criteria that imply rigidity of a pseudo-rotation with this modulus of continuity.

Theorems 1 and 2 generalize an earlier result by Bramham [3]. Our proof is based on Brouwer's theory of fixed point free planar homeomorphisms, and more precisely on Franks' free disc Lemma. As such, it is much more elementary than Bramham's proof that relies on the theory of pseudo-holomorphic curves. On the other hand Bramham's approach yields more information on irrational pseudo-rotations than rigidity since he obtains for them periodic approximations with estimates (see [2] and also [15] for an approach based on generating functions).

We will always denote by $\|g\|$ the supremum norm of a map g. The essential idea in our proof of Theorem 1 is that if a pseudo-rotation f has a rotation number that is small compared to $\|Df\|^{-2}$ then f must be close to the identity map, because otherwise f would have a periodic point outside the origin which is a contradiction. The proof of the latter fact is based on a simple observation that combines Franks' free disc lemma with Kac's lemma to give a lower bound on the measure of free topological discs of a pseudo-rotation with a small rotation number (see Lemma 2.1 and Corollary B in Section 2). The rigidity for super-Liouville pseudo-rotations then follows immediately since $\|Df^q\|$ is at most exponential in q. To obtain the rigidity of non-Brjuno numbers, we need a better bound than exponential on the growth of $\|Df^{q_n}\|$. Such a bound exists *a priori* for pseudo-rotations, and more generally for any surface area preserving C^2 diffeomorphism of a surface without hyperbolic periodic points. It is provided by an *effective finite information version of the Katok closing lemma for area preserving surface diffeomorphisms* with some hyperbolic behavior.

1.2. **A growth gap for area preserving surface diffeomorphisms.**

The following result that gives a quantitative gap in the growth of $\|Df^m\|$ between exponential and sub-exponential for an area preserving disc diffeomorphism of class C^2 is sufficient to prove rigidity for non-Brjuno rotation numbers.

Let S be a compact smooth surface with a Riemannian metric. Denote $\mathrm{Diff}^r_{\mathrm{vol}}(S)$ the group of C^r diffeomorphisms which preserve the volume form induced by the Riemannian metric.

Theorem A. *For any compact subset $K \subset \mathrm{Diff}^2_{\mathrm{vol}}(S)$, there exists $\theta \in (0,1)$ and $H \in \mathbb{N}^*$ satisfying the following property: Let $\{q_n\}$ be a sequence such that $q_0 \geq H$, $q_n \geq H^{q_{n-1}}$. If for some $f \in K$ there exists $n \geq 0$ such that*

$$\frac{1}{q_n} \ln \|Df^{q_n}\| > \theta^n, \tag{1.1}$$

then f has a hyperbolic periodic point.

Observe that our result implies that in case (1.1) holds for some n, then actually $\limsup_n \frac{1}{q_n} \ln \|Df^{q_n}\| > 0$, hence establishing a gap in the possible rate of growth of $\|Df^n\|$. In spirit, our approach gives an effective instance of what Katok described as Anosov and Bowen's observation "that assuming some hyperbolicity conditions, dynamical phenomena which are observed to almost occur for some diffeomorphisms usually do occur for that diffeomorphism" [14, Introduction].

Note that a finite information version of Katok's closing lemma was given by Climenhaga and Pesin in [4] that relies on *the notion of effective hyperbolicity*. A piece of orbit being called effectively hyperbolic if along this piece there are nicely behaving *stable* and *unstable* directions for the differential in the sense that expansion along the *unstable* direction dominates the defect from domination over the *contracted* direction and dominates also the decay of the angles between the directions under iteration.

Since we want a finitary version of the closing lemma that only involves the growth of derivatives we prefer to give an independent statement from [4], with a proof that takes some ideas from [4], but differs in that the condition of conservation of area is used to control *a posteriori* the angles as soon as contraction and expansion are controlled. In the general setting of [4], it is not assumed that the maps are area preserving surface diffeomorphisms. Also, our proof does not use graph transforms but rather follows the orbit of a small box along a selected piece of orbit (a kind of effectively hyperbolic piece of orbit) and shows that the return of the box to its neighborhood enjoys hyperbolic-like properties that force the existence of a hyperbolic periodic point (see Section 3).

We will give the proofs of Theorems 1 and 2 in Section 2. Theorem A, that is of independent interest and of independent flavor from the rest of the paper is discussed and proven in Section 3.

1.3. **KAM theory and local rigidity of real analytic pseudo-rotations with Brjuno rotation numbers.** In Theorems 1 and 2, to get C^0 rigidity of a pseudo-rotation, we only require it to be Lipschitz regular in the case of super-Liouville rotation number and C^2 in the non-Brjuno case. We will see in the sequel that our direct approach to rigidity of pseudo-rotations cannot go much beyond non-Brjuno type numbers (for example to include all Liouville numbers), irrespective of the regularity of the pseudo-rotation.

So, to cover the case of less Liouvillean numbers, we resort to KAM theory (named after Kolmogorov, Arnol'd, and Moser). Indeed, to bridge up from the non-Brjuno type to the ones that are of Brjuno type, we use the KAM result of Rüssmann that asserts that an elliptic fixed point, that is precisely of Brjuno type, of an analytic area preserving surface diffeomorphism is surrounded by invariant curves [20], and in the case of pseudo-rotations yields local rigidity. We thus get the following consequence of Theorem 1 and Rüssmann's result (see Theorem 3 below)

Theorem B. Real analytic pseudo-rotations of $\mathbb{D}$ are C^∞ locally rigid around the center.

It is worth mentioning here (see also Question 2) that no examples are available of real analytic pseudo-rotations that are not conjugate to a rotation. Of course, if Birkhoff's conjecture that such examples do not exist turns out to be true, then the content of Theorem B would obviously follow. If to the contrary such examples do exist, then Theorem B would be saying that analytic pseudo-rotations still share some constrictive properties of rigid rotations.

As mentioned above, to go from Theorem 1 to Theorem B we use a KAM result of Rüssmann, that does not require any twist condition, and allows to deal with the pseudo-rotations having Brjuno type rotation numbers.

Theorem 3 (Rüssmann [20])**.** *If f is an area preserving real analytic map defined in some open neighborhood of $0 \in \mathbb{R}^2$, and if $f(0) = 0$ and the rotation number α of f at 0 is of Brjuno type, then 0 is surrounded by a positive measure set of real analytic invariant closed simple curves. Moreover, If f has no periodic points accumulating 0 then f is analytically conjugated to the disc rotation R_α in some neighborhood of 0.*

Theorem B is then an immediate consequence of Theorem 3 (in the Brjuno type case) and our rigidity result of Theorem 1 (in the non-Brjuno type case).

Actually, if in Rüssmann's Theorem f is in addition supposed to be a pseudo-rotation close to the rotation R_α, α of Brjuno type, it holds that f is analytically conjugated to R_α and *a fortiori* C^ω rigid (see Corollary 1 in [11] and its proof). From this rigidity result, and from our Theorem 1, we get the following *local* result, in which the notation $|\cdot|_\delta$ stands for the sup norm inside the analyticity band of width δ.

Corollary A. *For any $\alpha \in \mathbb{R} - \mathbb{Q}$ and any $\delta > 0$, there exists $\varepsilon(\delta, \alpha) > 0$ such that if f is a real analytic pseudo-rotation with rotation number*

$\rho(f)=\alpha$ and if

$$|f-R_\alpha|_\delta \leq \varepsilon(\delta,\alpha)$$

then f is C^∞-rigid.

Remark 1.1. Observe that $\varepsilon(\delta,\alpha)$ essentially depends on δ and $\sum \ln(q_{n+1})/q_n$ where $q_n=q_n(\alpha)$ and that $\varepsilon(\delta,\infty)=\infty$ by Theorem 1.

1.4. **Some consequences on zero entropy disc dynamics.** Examples of smooth dynamical systems with a rich variation of ergodic properties were constructed on the disc and the sphere. Katok constructed a smooth conservative map of the disc that is isomorphic to a Bernoulli shift [14]. His example, that is based on slowing down fixed points of an Anosov automorphism of the two torus before projecting down the dynamics on the sphere, can be made real analytic if the slowing down is chosen adequately.

On the zero entropy side, the Anosov-Katok construction method [1] *via* successive conjugations of rational rotations provided a multitude of constructions with various ergodic properties, starting with the first examples of smooth ergodic conservative diffeomorphisms on the disc or the sphere.

All constructions of transitive smooth area preserving diffeomorphisms on $\mathbb{D}$ by the Anosov-Katok method are pseudo-rotations with a Liouville rotation number. That the rotation number must be Liouville follows from Herman's last geometric Theorem asserting that if this is not the case, then the center and the boundary are accumulated by invariant closed curves which clearly excludes transitivity [11]. Weak mixing pseudo-rotations with arbitrary Liouville rotation number are obtained by the Anosov-Katok method in [12].

While it is not known if a smooth pseudo-rotation with Liouville rotation number is rigid (see Section 1.5), it is a fact that all the smooth examples on the disc or the sphere obtained up to now *via* the Anosov-Katok method are C^∞ *rigid* by construction. Obviously, rigidity or local rigidity precludes mixing. Hence, the following natural question was raised in [9] in connection with the smooth realization problem and the Anosov-Katok construction method.

Question 1. Does there exist a smooth area preserving diffeomorphism of the disc that is mixing with zero metric entropy? with zero topological entropy?

In fact, when topological entropy is assumed to be zero, the question reduces to the pseudo-rotations setting. Indeed, a structure Theorem by Franks and Handel for area preserving diffeomorphisms of genus zero surfaces with zero topological entropy shows that transitivity in such a context implies that the diffeomorphism must be a pseudo-rotation [8]. More precisely, they show that if F is an area preserving diffeomorphism of the sphere with at least three fixed points and if $M = \mathbb{S}^2 - \mathrm{Fix}(F)$ then an open and dense set $\mathcal{W} \subset M$ (of points with a weak type of recurrence called *weakly free disc recurrent points*) decomposes into a countable union of disjoint invariant annuli on which the dynamics of F is similar to an integrable map albeit with possibly complicated sets (such as pseudo-circles) for the constant rotation number sets instead of nice simple closed curves. This decomposition clearly excludes transitivity of F, hence transitive area preserving diffeomorphisms on $\mathbb{D}$ must belong to the class of pseudo-rotations.

An immediate consequence of Franks and Handel's classification result and our Theorem B is the following

Theorem C. Area preserving real analytic diffeomorphisms of the disc or the sphere that are topologically mixing have positive topological entropy.

A homeomorphism f of a metric space X is said to be topologically mixing if for any non empty open sets A and B there exists $N \geq 0$ such that for any $n \geq N$, $f^n(A) \cap B \neq \emptyset$.

Observe that the existence of mixing homeomorphisms of the sphere (or the disc) with zero metric entropy follows from Katok's construction of an area preserving ergodic diffeomorphism f of the sphere with positive entropy [14], with an approach similar to that of [16]. Indeed, f as constructed in [14] is isomorphic to a Bernoulli shift T, and it is possible to construct using this isomorphism a mixing invariant measure for T with zero entropy that gives a positive mass to every cylinder. The almost product structure of stable and unstable leaves of f then imply that the constructed measure assigns a positive mass to every open set in the sphere. Applying Oxtoby-Ulam's lemma, we get a C^0 conjugacy to a mixing conservative homeomorphism of $\mathbb{S}^2$ with zero metric entropy.

We recall that on the other end of the regularity spectrum, the real analytic category, nothing is known, apart from Theorem C, about the possible limitations on the dynamical properties of disc and sphere area preserving zero entropy diffeomorphisms. The following natural question is wide open (see [9, 10] and discussion therein).

Question 2. Does there exist real analytic area preserving diffeomorphisms of the disc or the sphere with zero metric entropy that are transitive? ergodic?

Note that Theorem B has an interesting consequence on the C^∞ centralizer of an analytic pseudo-rotation f (smooth diffeomorphisms that commute with f), since it easily implies that the latter is always uncountable. Similarly, Theorem 1 shows that the C^{k-1} centralizer of a C^k pseudo-rotation f with rotation number that is not of Brjuno type is uncountable. This shows that real analytic conservative diffeomorphisms of the disc, or smooth non-Brjuno pseudo rotations are in this extent less flexible than smooth circle diffeomorphisms for which Yoccoz constructed smooth examples with a C^∞ centralizer that is reduced to the powers of the diffeomorphism [21]. We observe that it is not known if there exists real analytic examples of the latter construction of Yoccoz.

We conclude this introduction with a series of comments and questions.

1.5. **Some questions around the rigidity of pseudo-rotations.** The following question was raised by Bramham in [3].

Question 3. Is every C^k pseudo-rotation f C^0 rigid? The question can be asked for any $k \geq 1$, $k = \infty$ or $k = \omega$.

In the case $k = \omega$ or $\rho(f)$ Diophantine and $k = \infty$, the latter question becomes an intermediate question relative to the Birkhoff-Herman problem on the conjugability of f to the rigid disc rotation of angle $\rho(f)$ [13].

As discussed earlier a better *a priori* control on the growth of $\|Df^m\|$ for a pseudo-rotation is sufficient to deduce rigidity for larger classes of rotation numbers. In the case of a circle diffeomorphism f a gap in the growth is known to hold between exponential growth in the case f has a hyperbolic periodic point or a growth bounded by $O(m^2)$ if not [19]. Does a similar dichotomy hold for area preserving disc diffeomorphisms?

If for example a polynomial bound holds on the growth of $\|Df^m\|$ for a smooth pseudo-rotation, then C^∞ rigidity would follow for any Liouville rotation number by the same proof as that of Theorem 1.

Question 4. Is there any polynomial bound on the growth of the derivatives of a pseudo-rotation? Is every C^∞ pseudo-rotation with Liouville rotation number C^0 (or even C^∞) rigid?

With Herman's smooth version of Rüssmann's Theorem 3 for Diophantine rotation numbers (see [11]), a positive answer to the second part of Question 4 would imply that smooth pseudo-rotations, and therefore area preserving smooth diffeomorphisms of the disc with zero topological entropy are never topologically mixing.

The local result of Corollary A that holds for *every* analytic pseudo-rotation raises the following natural question on its *semi-global* potential validity.

Question 5. Does the local result of Corollary A hold for an ε that is independent on the rotation number of the pseudo-rotation f?

Non rigid C^0 pseudo-rotations. In the case of homeomorphisms, observe that Crovisier constructed on the sphere $\mathbb{S}^2$, C^0 pseudo-rotations of arbitrary rotation number that have positive topological entropy [5]. It is simple to see that positive topological entropy precludes C^0 rigidity.

Here is a direct construction of non rigid C^0 pseudo-rotations. Let f_t, $t \in [0,1]$, be a continuous family of circle homeomorphisms such that $f_0 = R_\alpha$, $\rho(f_t) = \alpha$ for every $t \in [0,1]$, and f_t is of class C^2 for every $t \in [0,1] - \{\frac{1}{2}\}$ while $f_{\frac{1}{2}}$ is a Denjoy counterexample. Hence f_t is topologically conjugate to R_α for each $t \neq \frac{1}{2}$ by Denjoy Theorem while $f_{\frac{1}{2}}$ has a wandering interval. Then, we consider a homeomorphism of the disc f that leaves each circle of radius $t \in [0,1]$ invariant and acts on this circle by the homeomorphism f_t. It is clearly not C^0 rigid since it is not rigid on the circle of radius $1/2$. On the other hand, f leaves invariant the measure μ given by the integration of the invariant measures of f_t on each circle of radius t, and it is clear that $\mu(O) > 0$ for any open set O, therefore f can be conjugated by Oxtoby Ulam lemma to an area preserving homeomorphism that will be a non C^0 rigid pseudo-rotation of angle α. These examples fail however to be mixing.

Question 6. Does there exist a mixing C^0 pseudo-rotation of the disc?

Pseudo-rotation of the annulus. Pseudo-rotations of the annulus $\mathbb{A} = \mathbb{T} \times [0,1]$ are area preserving maps of $\mathbb{A}$ that preserve the boundaries and have a unique irrational rotation number (in projection on the first variable).

Question 7. Are analytic pseudo-rotations of the annulus rigid?

Theorems 1 and 2 hold on the Annulus, with the same proof as for the Disc. But even in the real analytic category, we do not have a proof of Theorem C. Indeed, the proof of the KAM Theorem 3 can be adapted to the annulus only if the map is supposed to be analytically conjugated to R_α on the boundary. But to guarantee that this always holds one has to ask that α satisfies the so called condition $\mathcal{H}$ [22]. Hence, if α does not satisfy $\mathcal{H}$ but satisfies the Brjuno condition (see [22, Example 2.13]) then neither Theorem 2.1 nor Theorem 3 can be used, and we do not know if absence of mixing always holds for real analytic pseudo-rotation of the annulus with rotation number α.

The non-conservative case. Similar results as in Theorems 1 and 2 hold if instead of Lebesgue measure we suppose that f preserves any measure that assigns a lower bounded mass as a function of r to discs of radius r. Thus a natural question is the following.

Question 8. Is it true that a diffeomorphism f of the disc with zero topological entropy that has a unique rotation number is never topologically mixing?

Since Franks and Handel classification result holds for diffeomorphisms that preserve a measure that is strictly positive on every open set, a positive answer to Question 8 would imply a positive answer to the first part of the following question.

Question 9. Is it true that a diffeomorphism of the disc with zero topological entropy that preserves a measure that is strictly positive on every open set is never topologically mixing? Is it true that a diffeomorphism of the disc with zero topological entropy is never topologically mixing?

2. Rotation number, Recurrence, and Rigidity

2.1. **Rotation number and recurrence.** We give here the main ingredient of our approach to rigidity of pseudo-rotations. It is a simple observation that combines Franks' free disc lemma with Kac's lemma to give a lower bound on the measure of free topological discs of a pseudo-rotation with a small rotation number.

Lemma 2.1. *Let f be a continuous pseudo-rotation of $\mathbb{D}$. If $\rho(f) = \varepsilon + \mathbb{Z}$, then any topological disc D such that $\lambda(D) > \varepsilon$ satisfies $f(D) \cap D \neq \emptyset$. We will denote by λ the Lebesgue measure.*

Proof. Let D be a disc such that $f(D) \cap D = \emptyset$ and $\tilde{D}$ a fixed connected component of the lift of D to $\widetilde{A} = \mathbb{R} \times [0,1]$, the universal covering space of $\mathbb{D}^2 \setminus \{0\}$. Since f does not have periodic points on

$\mathbb{D}^2 \setminus \{0\}$, neither does the lift $\tilde{f}$ to $\tilde{\mathbb{A}}$, whose (real) rotation number $\rho(\widetilde{f})$ is ε. In particular $\tilde{f}$ is fixed point free. Note that $\tilde{f}(\tilde{D}) \cap \tilde{D} = \emptyset$. Hence, by Franks' Theorem on positively and negatively returning discs [7, Theorem 2.1] we have that either $\tilde{f}^n(\tilde{D}) \cap (\tilde{D} + l) = \emptyset$ for every $n \geq 1, l \geq 0$ or $\tilde{f}^n(\tilde{D}) \cap (\tilde{D} + l) = \emptyset$ for every $n \geq 1, l \leq 0$. For definiteness we assume the second condition holds. By Poincaré recurrence, λ-a.e. $x \in D$ returns infinitely many times to D under iteration by f, call $n_D(x)$ the first strictly positive return time of such a point and f_D the first return map $f_D(x) = f^{n_D(x)}(x)$. Write $l_D(x)$ for the integer such that $\tilde{f}^{n_D(x)}(\tilde{x}) \in \tilde{D} + l_D(x)$, where $\tilde{x}$ is the lift of x belonging to $\tilde{D}$. By our assumption we know that $l_D(x) \geq 1$. Next, since f is a pseudo-rotation and $\rho(\tilde{f}) = \varepsilon$, we necessarily have *for every x*

$$\lim_{N\to\infty} \frac{l_D(x) + l_D(f_D(x)) + \ldots + l_D(f_D^{N-1}(x))}{n_D(x) + n_D(f_D(x)) + \ldots + n_D(f_D^{N-1}(x))} = \varepsilon, \tag{2.1}$$

which implies

$$\liminf_{N\to\infty} \frac{n_D(x) + n_D(f_D(x)) + \ldots + n_D(f_D^{N-1}(x))}{N} \geq \frac{1}{\varepsilon}. \tag{2.2}$$

Now, Kac's Lemma asserts that

$$\int_D n_D(x) d\lambda = \lambda\left(\bigcup_{n\geq 0} f^n(D)\right) \leq 1. \tag{2.3}$$

Finally, by invariance of the restriction of λ to D by f_D and Fatou's Lemma we get from (2.2) and (2.3)

$$1 \geq \int_D \liminf_{N\to\infty} \frac{n_D(x) + n_D(f_D(x)) + \ldots + n_D(f_D^{N-1}(x))}{N} d\lambda$$

$$\geq \int_D \frac{1}{\varepsilon} d\lambda = \frac{\lambda(D)}{\varepsilon}.$$

□

An immediate consequence of Lemma 2.1 is the following estimate on the maximal displacement of a pseudo-rotation with respect to its rotation number.

Corollary B. *Let f be a continuous pseudo-rotation of $\mathbb{D}$ with $\rho(f) = \varepsilon + \mathbb{Z}$. Then*

$$\|f - \mathrm{Id}\| \leq \varepsilon^{\frac{1}{2}} + \max_{x\in\mathbb{D}} \mathrm{diam}\left(f(B(x, \varepsilon^{\frac{1}{2}}))\right)$$

Hence, in case f is differentiable we get

$$\|f - \mathrm{Id}\| \leq (1 + \|Df\|)\varepsilon^{\frac{1}{2}}$$

2.2. **Rigidity of super-Liouville pseudo-rotations.**

Proof of Theorem 2. Just observe that if f is Hölder with exponent a then there exists C such that for any m

$$|f^m(x) - f^m(y)| \leq C^m |x-y|^{a^m} \tag{2.4}$$

and apply Corollary B to the pseudo-rotation $f^{q_{n_j}}$ that satisfies $\rho(f^{q_{n_j}}) = \|q_{n_j}\alpha\| \leq \frac{1}{q_{n_j+1}}$. Hence

$$\|f^{q_{n_j}} - \mathrm{Id}\| \leq \frac{1}{\sqrt{q_{n_j+1}}} + C^{q_{n_j}} \left(\frac{2}{\sqrt{q_{n_j+1}}}\right)^{a^{q_{n_j}}}$$

and rigidity follows from the arithmetic condition

$$\lim {q_{n_j}}^{-1} a^{q_{n_j}} \ln q_{n_j+1} = +\infty.$$

□

2.3. **Rigidity of non-Brjuno type pseudo-rotations.** The goal of this section is to prove Theorem 1 using Corollary B and Theorem A.

We will need the following result derived from the growth of the denominators of a non Brjuno type number. For $x \in \mathbb{R}$ we denote the closest distance of x to the integers by $\|x\| := \min_{p\in\mathbb{Z}} |x-p|$.

Lemma 2.2. *Suppose $\alpha \in \mathbb{R} - \mathbb{Q}$ is not of Brjuno type. For any $H > 1$, there exists a sequence Q_j of integers such that*

$$Q_{j+1} \geq H^{Q_j} \tag{2.5}$$

and for infinitely many $j \in \mathbb{N}$.

$$\|Q_j\alpha\| < e^{-\frac{Q_j}{j^2}}. \tag{2.6}$$

Proof. Fix $H > 1$. Let q_n be the sequence of denominators of the best rational approximations of α. Define an increasing sequence of integers $\{m_j\}$ as follows. Let $m_1 = 1$. For $j \geq 2$, define m_j to be the first integer such that $q_{m_j} \geq H^{q_{m_{j-1}}}$.

Let

$$\sum_{m=m_{j_p}}^{m_{j_{p+1}-1}} \frac{\ln q_{m+1}}{q_m} := a_j. \tag{2.7}$$

The non Brjuno condition writes as

$$\sum_{j=1}^{\infty} a_j = \infty \tag{2.8}$$

CLAIM. For every integer j sufficiently large, such that $a_j \geq 1/j^{\frac{3}{2}}$, we have that there exists $l_j \in [m_j, m_{j+1}-1]$ such that $q_{l_j+1} \geq e^{4q_{l_j}/j^2}$.

Before we prove this claim, let us observe how it leads to the conclusions of the lemma.

WLOG, we suppose that $\sum_{j\in\mathbb{N}} a_{2j} = \infty$. We define $Q_j = q_{l_{2j}}$.

Since $q_{m_{j+1}} \geq H^{q_{m_j}}$ for any $j \in \mathbb{N}$, we have for any $j \in \mathbb{N}$

$$q_{l_{j+2}} \geq q_{m_{j+2}} \geq H^{q_{m_{j+1}}} \geq H^{q_{l_j}},$$

hence (2.5) for the sequence Q_j.

On the other hand it follows from the claim that if $a_{2j} \geq \frac{1}{(2j)^{\frac{3}{2}}}$ then

$$\|Q_j\alpha\| \leq \frac{1}{q_{l_{2j}+1}} \leq e^{-Q_j/j^2}$$

Since $\sum a_{2j} = \infty$, it follows that for infinitely many j we have $a_{2j} \geq \frac{1}{(2j)^{\frac{3}{2}}}$, hence (2.6) for the sequence Q_p.

To prove the claim, observe first that since $q_{m_{j+1}-1} \leq H^{q_{m_j}}$, then we have that, for j sufficiently large, there exists at most one index $l \in [m_j, m_{j+1}-2]$ such that $\frac{\ln q_{h+1}}{q_h} \geq \frac{1}{\sqrt{q_h}}$, for $h = l$. Indeed, if this was not true, there should exist two such $l, l' \in [m_j, m_{j+1}-2]$, and we would have that $q_{m_{j+1}-1} \geq \exp\circ\exp(\frac{1}{2}\sqrt{q_{m_j}}) \gg H^{q_{m_j}}$.

For j sufficiently large, since

$$\sum_{m\in[m_j,m_{j+1}-2],m\neq l} \frac{\ln q_{m+1}}{q_m} \leq \sum_{m=m_j}^{m_{j+1}-2} \frac{1}{\sqrt{q_m}} \ll \frac{1}{j^2}$$

(because $q_{m+2} \geq 2q_m$ for any m and $m_j \geq j$). Hence, if $a_j \geq \frac{1}{j^{\frac{3}{2}}}$ in (2.7), it follows that either $h = l$ or $h = m_{j+1}-1$ must satisfy $\frac{\ln q_{h+1}}{q_h} \geq \frac{4}{j^2}$ as claimed. $\square$

Proof of Theorem 1. Fix $\alpha \in \mathbb{R} - \mathbb{Q}$ that is not of Brjuno type.

We want to obtain C^{k-1}-rigidity for any C^k pseudo-rotation f with rotation number α. Let H be given by Theorem A applied to the compact set of diffeomorphisms $K := \{f\}$.

Let $\{Q_j\}$ be as in Lemma 2.2. Since f has no hyperbolic periodic point, Theorem A implies that

$$\|Df^{Q_j}\| < e^{\theta^j Q_j} \leq e^{\frac{Q_j}{j^4}} \tag{2.9}$$

if p is sufficiently large. Now, for p such that (2.6) holds, we have $\rho(f^{Q_j}) = \|Q_j \alpha\| \leq e^{-\frac{Q_j}{j^2}}$, hence by Corollary B and (2.9) we get for p sufficiently large that

$$\|f^{Q_j} - \mathrm{Id}\| \leq e^{-\frac{Q_j}{2j^2}}.$$

By the Faà di Bruno formula, (2.9) also implies that for any $k \in \mathbb{N}$, for large j

$$\|D^k f^{Q_j}\| < e^{\frac{Q_j}{j^3}}. \tag{2.10}$$

Hence, by the convexity estimates that interpolate between C^s norms, we get

$$\begin{aligned}\|f^{Q_j} - \mathrm{Id}\|_{k-1} &\leq C_k \|f^{Q_j} - \mathrm{Id}\|^{\frac{1}{k}} \|f^{Q_j} - \mathrm{Id}\|_k^{\frac{k-1}{k}} \\ &= o(e^{-\frac{Q_j}{j^3}}),\end{aligned}$$

where $\|g\|_k$ denotes the C^k-norm of a map g, that is the maximum of the supremum norms of the first k derivatives of g. □

3. A growth gap for area preserving surface diffeomorphisms

In all this section, S will denote a compact smooth surface with a Riemannian metric. Denote $\mathrm{Diff}^r_{\mathrm{vol}}(S)$ the group of C^r diffeomorphisms which preserve the volume form induced by the Riemannian metric.

3.1. **A finite information closing lemma and the gap in the derivatives growth.**

Katok's closing lemma asserts that if a $C^{1+\epsilon}$ diffeomorphism f of a compact manifold has a hyperbolic invariant probability measure, that is a measure with all Lyapunov exponents nonzero, then f has (many) hyperbolic periodic points. When we restrict to an area preserving surface diffeomorphism $f \in \mathrm{Diff}^r_{\mathrm{vol}}(S), r > 1$, any invariant probability measure for f that has one non zero Lyapunov exponent will be hyperbolic. Therefore, if the sequence of sup norms of the differentials $\|Df^n\|$ grows exponentially, one can easily construct using the sub-additive ergodic theorem a hyperbolic invariant probability measure for f. With Katok's closing lemma, this implies that if f has no hyperbolic periodic points, then the differentials $\|Df^n\|$ grows sub-exponentially.

Hence, if one considers a compact set K inside the set of $\mathrm{Diff}^r_{\mathrm{vol}}(S)$ for say $r = 2$, then there must be an *a priori* gap between exponential and sub-exponential growth of $\|Df^n\|$. Our goal here is to prove Theorem A that gives an explicit estimate on this gap.

We first give a finite information version of Katok's closing lemma for a map $g \in \mathrm{Diff}^2_{\mathrm{vol}}(S)$ that only requires a growth of $\|Dg^q\|$ comparable to $\|Dg\|^{\theta q}$ where θ is close to 1 and q is sufficiently large compared to powers of the C^2 norm of g. We recall that the C^r norm of a diffeomorphism of a smooth compact manifold M can be defined by taking the supremum over all (finitely many) charts of the C^r norms of the corresponding maps from $\mathbb{R}^d$ to $\mathbb{R}^d$ where d is the dimension of M.

Theorem 4. *There exist constants $A_0 > 0, \theta_0 \in (0,1), H_0 > 0$ such that for all (g, q, A, D, θ) with $g \in \mathrm{Diff}^2_{vol}(S), q \in \mathbb{N}, A \geq A_0, \theta \in [\theta_0, 1), D \geq A, q \geq D^{H_0}$, if*

$$\|Dg\| \leq A \tag{3.1}$$

$$\|D^2 g\| \leq D \tag{3.2}$$

$$\|Dg^q\| \geq A^{\theta q} \tag{3.3}$$

Then g has a hyperbolic periodic point.

Let us first see how the gap on the growth of $\|Df^n\|$ announced in Theorem A follows from Theorem 4.

Proof of Theorem A. Consider $K' \subset K$ defined as

$$K' = \{f \in K : f \text{ has no hyperbolic periodic point }\}$$

By stability of hyperbolic periodic points, ${K'}^C$ is open, so K' is compact. We claim that

$$\lim_{n\to\infty} \sup_{x\in S, f\in K'} \frac{1}{n} \ln \|Df^n(x)\| = 0. \tag{3.4}$$

Proof of (3.4). Let $\alpha_n := \sup_{x\in S, f\in K'} \ln \|Df^n(x)\|$, we have for all $m, n \in \mathbb{N}$,

$$\alpha_n + \alpha_m \geq \alpha_{m+n}, \quad \alpha_n \geq 0.$$

Let A_0, θ_0, H_0 be as in Theorem 4. From the sub-additive lemma, we have

$$c := \lim_{n\to\infty} \frac{\alpha_n}{n} = \inf \frac{\alpha_n}{n} \geq 0.$$

We have to show that $c = 0$. Suppose the contrary is true, then there exists $\bar{n} \in \mathbb{N}$ such that

$$e^{c\bar{n}\sqrt{\theta_0}} \geq A_0$$

and for all $n \geq \bar{n}$

$$\alpha_n \in [cn\sqrt{\theta_0}, cn/\sqrt{\theta_0}] \tag{3.5}$$

From the compactness of K, we know that there exists a constant $C \geq 1$ such that for all $n \in \mathbb{N}, f \in K$,

$$\|D^2 f^n\| \leq e^{Cn}. \tag{3.6}$$

Let

$$A := e^{c\bar{n}/\sqrt{\theta_0}}, D := \max(e^{C\bar{n}}, A), \theta := \theta_0, q := D^{H_0}. \tag{3.7}$$

Since $\frac{\alpha_{q\bar{n}}}{q\bar{n}} \geq c\sqrt{\theta_0}$, we can find an $f_0 \in K'$ such that

$$\|Df_0^{q\bar{n}}\| \geq e^{cq\bar{n}\sqrt{\theta_0}} = A^{\theta_0 q}. \tag{3.8}$$

Let $g := f_0^{\bar{n}}$. It follows from (3.5),(3.6),(3.7) and (3.8) that (g, q, A, D, θ) satisfies the conditions of Theorem 4, therefore we have that f_0 has a hyperbolic periodic point, which contradicts the definition of K'. Hence (3.4) is proved. □

Fix A_0, θ_0, H_0 as in the hypothesis of Theorem 4. As a consequence of (3.4), there exists H' such that for all $H > H'$

$$\sup_{x\in S, f\in K'} \frac{1}{H} \ln \|Df^H(x)\| \leq 1. \tag{3.9}$$

On the other hand, let H'' be sufficiently large such that for any sequence $\{q_n, n \geq 0\}$ satisfying $q_0 \geq H'', q_n \geq H''^{q_{n-1}}$, then for each $n > 0$, we have

$$e^{\theta_0^{n-1} q_{n-1}} > A_0. \tag{3.10}$$

Indeed for large H'', q_n increases much faster than θ_0^{-n}. Let C be as in (3.6). Choose $H \geq \max(H', H'', e^{2CH_0}), \theta \geq \theta_0$. We claim that such (H, θ) satisfies the requirements of Theorem A.

Indeed, assume there exists $f \in K, n \geq 0$ satisfying (1.1), with $q_0 \geq H, q_n \geq H^{q_{n-1}}$. Take n to be the smallest integer satisfying (1.1).

If $n = 0$, there thus exists an $x \in S$ such that $\frac{1}{q_0} \ln \|Df^{q_0}(x)\| > 1$. Since $q_0 \geq H$, by (3.9) we know $f \notin K'$. Therefore f has a hyperbolic periodic point.

If $n > 0$, then let $g := f^{q_{n-1}}, A := e^{\theta^{n-1} q_{n-1}}, q := \frac{q_n}{q_{n-1}}, D := \max(A, e^{Cq_{n-1}})$, where we assumed WLOG that q_n / q_{n-1} is an integer, and where C is given by (3.6). We finish if we show that (g, A, θ, q, D) satisfy the conditions in Theorem 4. Indeed, if g has a hyperbolic periodic point, thus so does f.

Since $\theta \geq \theta_0$, (3.10) implies that $A > A_0$. Since $\ln H \geq 2CH_0$ and $q_n \geq H^{q_{n-1}}$ we get that $q \geq D^{H_0}$. By (3.6), we have that $\|D^2 g\| \leq D$. Finally, since n is the smallest integer satisfying (1.1), we get that $\|Dg\| \leq A$ as well as $\|Dg^q\| \geq A^{\theta q}$. □

Strategy of the proof of Theorem 4. Let us briefly recall how Katok derives his closing lemma from Pesin theory. For almost every point with respect to the hyperbolic measure, Lyapunov neighborhoods are constructed with stable and unstable manifolds attached to the point, which have definite size under the Lyapunov metric. Moreover, the transformation between two neighborhoods looks like a uniform hyperbolic map under these new metrics. Even though the Lyapunov neighborhoods of some points are very small, Pesin theory provides sets of positive measure, the so called Pesin basic sets, consisting of points having Lyapunov neighborhoods (and thus stable and unstable manifolds) of uniformly lower bounded sizes and having as well a lower bounded angle between the stable and unstable directions. Then, the Poincaré recurrence theorem is used to obtain an integer L and a point x from a given Pesin basic set that comes back under L iterations inside the same basic set close to itself. This then guarantees the existence of a hyperbolic periodic point z of period L that shadows the piece of orbit of x of length L, in the same way as the Anosov closing lemma gives a hyperbolic periodic point that shadows a recurrent piece of orbit in uniform hyperbolic dynamics. The proof can for example be done through graph transforms involving the first return map to the Lyapunov neighborhood of x. Of course, once the good Pesin point x is detected, the rest of the proof does not depend on informations on f beyond the ones we have for the first L iterates of a small neighborhood of x. The problem with this procedure when one is interested in finite information versions of the closing lemma, lies actually in the fact that the Lyapunov charts construction requires an infinite amount of information involving the full orbit of points.

In the spirit of the finite information versions of the closing lemma based on the concept of effective hyperbolicity of [4], our proof of Theorem 4 uses properties (3.1)–(3.3) to select a good point for which nice hyperbolic properties hold most of the time along a piece of its orbit until it comes back very close to itself, and then deduce from a hyperbolic-like property of the return map the existence of a hyperbolic periodic point next to the selected good point.

More precisely, our proof of Theorem 4 goes as follows : In Section 3.2, we use (3.3) to find "good points" $x \in S$ for which there exists $L \in \{1, \dots q\}$ and $v_s, v_u \in T_x S$, such that the forward orbit from x to $g^L(x)$ contracts and expands consistently at rate almost A the directions v_s and v_u respectively (this is possible because θ is close to 1), and the backward orbit by g^{-1} from $g^L(x)$ to x expands and contracts consistently at rate almost A in the directions $Dg^L(v_s)$ and $Dg^L(v_u)$ respectively. To neutralize the nonlinearities coming from the second derivatives, we also ask the distances $d_{T_1 S}(v_s, \frac{Dg^L(v_s)}{\|Dg^L(v_s)\|})$ and $d_{T_1 S}(v_u, \frac{Dg^L(v_s)}{\|Dg^L(v_s)\|})$ to be small compared to a power of D^{-1}. Here $d_{T^1 S}$ is a metric on the unit tangent bundle of S defined by some embedding into a Euclidean space (in particular, the point $g^L(x)$ is very close to x). All the former is possible to achieve due to Pliss' Lemma and the pigeonhole principle (the hypothesis $q \geq D^{H_0}$ is crucial here). From area conservation, we also get *for free* that the angles $\angle(v_s, v_u)$ and $\angle(Dg^L(v_s), Dg^L(v_u))$ are not too small. We do not assume any *a priori* control on the angles of the expanded and contracted direction within the L-orbit.

Next we study the dynamics along the L-orbit of the good point x and show that it is possible to find a box B (think of a square) in the neighborhood of x that contains a vertical strip that is mapped by the return map g^L into a horizontal strip with in addition a strict cone contraction condition for g^L and g^{-L} inside these strips. This evidently implies the existence of a hyperbolic fixed point for g^L.

3.2. **Finding a suitable finite orbit.** We first make precise the notion of good points mentioned at the end of Section 3.1.

Definition 3.1. For all triples (x, v_s, v_u) such that $x \in S, v_s, v_u \in T_xS$, for all $i \in \mathbb{Z}$ we denote

$$v_i^s := \frac{Dg^i(v_s)}{\|Dg^i(v_s)\|}, v_i^u := \frac{Dg^i(v_u)}{\|Dg^i(v_u)\|},$$
$$\lambda_i^s := \ln \|Dg(v_i^s)\|, \lambda_i^u := \ln \|Dg(v_i^u)\|,$$
$$\overline{\lambda}_i^e := \min\{\lambda_i^u, -\lambda_i^s\}$$

Definition 3.2. (Forward and backward $(L,a)-$good triple)

For all $L \in \mathbb{N}, a > 0$, we say a triple (x, v_s, v_u) is forward $(L,a)-$good if the exponents $\lambda_i^{s,u}, \overline{\lambda}_i^e$ associated to (x, v_s, v_u) satisfy the following inequalities:

$$|\lambda_j^{s,u}| \leq a, \quad \forall 0 \leq j \leq L-1 \tag{3.11}$$
$$\frac{1}{k}\sum_{j=0}^{k-1} \overline{\lambda}_j^e > (1 - \frac{1}{1000})a, \quad \forall 1 \leq k \leq L. \tag{3.12}$$

We say a triple (x, v_s, v_u) is backward $(L,a)-$good if the exponents $\lambda_i^{s,u}, \overline{\lambda}_i^e$ associated to (x, v_s, v_u) satisfy the following inequalities:

$$|\lambda_j^{s,u}| \leq a, \quad \forall -L \leq j \leq -1 \tag{3.13}$$
$$\frac{1}{k}\sum_{j=-k}^{-1} \overline{\lambda}_j^e > (1 - \frac{1}{1000})a, \quad \forall 1 \leq k \leq L. \tag{3.14}$$

For later use, we denote

$$\lambda_i^e := \min\{\lambda_i^u, \lambda_i^u - \lambda_i^s, -2\lambda_i^s\} \tag{3.15}$$

We have the following inequalities for $\lambda_i^{e,s}$ associated to forward and backward $(L,a)-$good triples.

Lemma 3.1. *For a forward $(L,a)-$good triple (x, v_s, v_u), the exponents $\lambda_i^{e,s}$ associated to (x, v_s, v_u) satisfy the following inequalities:*

$$\frac{1}{k}\sum_{j=0}^{k-1} \lambda_j^s < -\frac{1}{2}a, \quad \forall 1 \leq k \leq L, \tag{3.16}$$
$$\frac{1}{k}\sum_{j=0}^{k-1} \lambda_j^e > (1 - \frac{1}{100})a, \quad \forall 1 \leq k \leq L, \tag{3.17}$$
$$\frac{1}{k}\sum_{j=0}^{k-1} \min(3\lambda_j^e, 0) > -\frac{1}{10}a, \quad \forall 1 \leq k \leq L. \tag{3.18}$$

For a backward $(L, a)-$good triple (x, v_s, v_u), the exponents $\lambda_i^{s,e}$ associated to (x, v_s, v_u) satisfy the following inequalities:

$$\frac{1}{k}\sum_{j=-k}^{-1}\lambda_j^s < -\frac{1}{2}a, \quad \forall 1 \le k \le L, \tag{3.19}$$

$$\frac{1}{k}\sum_{j=-k}^{-1}\lambda_j^e > (1-\frac{1}{100})a, \quad \forall 1 \le k \le L. \tag{3.20}$$

Proof. Notice that $\lambda_i^s \le -\overline{\lambda}_i^e$, the inequalities (3.16), (3.19) are immediate consequences of (3.12) and (3.14).

Consider $\lambda_i^e - \overline{\lambda}_i^e$, it is nonnegative when $\lambda_i^s \le 0$, otherwise it larger or equal to $-3a$. But by the definition of forward $(L, a)-$good point, for all $1 \le k \le L$, there are at most $\frac{1}{1000}k$ many j such that $0 \le j \le k-1$ and $\overline{\lambda}_j^e < 0$. Since $\lambda_i^s \le -\overline{\lambda}_i^e$ for all $0 \le i \le L-1$, we have there are most $\frac{1}{1000}k$ many j such that $0 \le j \le k-1$ and $\lambda_j^s > 0$. Then we have for all $1 \le k \le L$,

$$\frac{1}{k}\sum_{j=0}^{k-1}\lambda_j^e - \overline{\lambda}_j^e \ge \frac{1}{1000}\cdot(-3a) > -\frac{1}{200}a$$

thus by (3.12), we get (3.17). The proof of (3.20) is similar.

Now we prove (3.18). Since $\lambda_j^e \le \lambda_j^u \le a$ and we have already proved (3.17), there are at most $\frac{1}{100}k$ many j such that $\lambda_j^e \le 0$, for which we have $\lambda_j^e \ge -2a$. Then we get

$$\frac{1}{k}\sum_{j=0}^{k-1}\min(3\lambda_j^e, 0) \ge \frac{1}{100}(-6a) > -\frac{1}{10}a, \quad \forall 1 \le k \le L.$$

This proves (3.18) and thus completes the proof. □

Remark 3.1. Without loss of generality, we fix an isometric embedding $\Psi : S \to \mathbb{R}^W$. This allows us to define the angle between two arbitrary tangent vectors by translating them back to 0. We always consider the angle as a number in $[0, \pi]$. As a result, we can define the distance between any two unit tangent vectors of the surface $v_1 \in T_{x_1}^1 S, v_2 \in T_{x_2}^1 S$ as the following:

$$d_{T^1S}(v_1, v_2) = \angle(v_1, v_2) + d(x_1, x_2)$$

In particular, for $v_1 \in T_{x_1}^1 S, v_2 \in T_{x_2}^1 S, d_{T^1S}(v_1, v_2) \ge d(x_1, x_2)$.

Definition 3.3. ((q,a)-good point)

For all $q \in \mathbb{N}, a > 0$, we say a point $x \in S$ is $(q,a)-$good if there exist $v_s, v_u \in T_xS, L \in \mathbb{N}$ such that the following is true. Let $v_i^s, v_i^u, i \in \mathbb{Z}$ be defined as in Definition 3.1, associated to the triple (x, v_s, v_u). Then

(1) The triple (x, v_s, v_u) is forward $(L,a)-$good.

(2) The triple $(g^L(x), v_L^s, v_L^u))$ is backward $(L,a)-$good.

(3) There holds
$$\ln\cot\angle(v_s, v_u) \leq 3a$$
$$\ln\cot\angle(v_L^s, v_L^u) \leq 3a$$

(4) There holds
$$d_{T^1S}(v_s, v_L^s) < q^{-\frac{1}{100}}$$
$$d_{T^1S}(v_u, v_L^u) < q^{-\frac{1}{100}}$$

The goal of this section is to prove the following proposition which shows the existence of a $(q,a)-$good point under certain conditions. We will then show in the Section 3.3 that the existence of a $(q,a)-$good point for sufficiently large q (the largeness only depends on a and S) implies the existence of a hyperbolic periodic point.

Proposition 3.1. *There exists $\theta_0 \in (0,1), A_0 > 0, H_0 > 0$, such that the following holds. If (g, A, θ, q) satisfies $A > A_0, \theta \in [\theta_0, 1)$, $q \geq A^{H_0}$ and conditions* (3.1),(3.3) *in Theorem 4, then there exists a $(q,a)-$good point $x \in S$, where $a = \ln A$.*

Proof. Since (g, A, θ, q) satisfies (3.1) and (3.3) in Theorem 4, we know there exists $x \in S$ such that for $a = \ln A$

$$\frac{1}{q}\ln\|Dg^q(x)\| \geq \theta a \tag{3.21}$$

To prove the proposition, it suffices to show that if A_0, H_0 are large enough, and θ_0 is sufficiently close to 1, then there exists a $(q,a)-$good point in the orbit of any x satisfying (3.21).

In all this section v_0^s will designate the most contracting direction of $Dg^q(x)$ in T_xS (that is unique since A_0 is assumed large and θ_0 close to 1), and $v_0^u = {v_0^s}^{\perp}$ denotes the direction orthogonal to v_0^s.

From now on to the end of this subsection, we consider $v_i^s, v_i^u, \lambda_i^{u,s}, \overline{\lambda}_i^e$, for all $i \in \mathbb{Z}$, as in Definition 3.1 for the triple (x, v_0^s, v_0^u), where x satisfies (3.21). Then we have the following estimates for $\lambda_i^{u,s}, \overline{\lambda}_i^e$ which will be used later.

Lemma 3.2. *For any $0 < \eta < 1$, there exists $\theta_0 \in (0,1)$ depending only on η such that the following is true. Suppose $g \in \mathrm{Diff}^1_{vol}(S)$ satisfies $\|Dg\| \leq A$. If $\theta \in [\theta_0, 1)$ and (q, a, θ, x) satisfies* (3.21)*, where $a = \ln A$, then we have for all $0 \leq i \leq q-1$,*

$$|\lambda_i^s| \leq a, |\lambda_i^u| \leq a$$

Moreover we have the following inequality:

$$\frac{1}{q}\sum_{i=0}^{q-1} \overline{\lambda}_i^e \geq (1-\eta)\theta a$$

Proof. Since $\|Dg\| \leq A$, we have $|\lambda_i^u|, |\lambda_i^s| \leq a$.

Since v_0^s is the most contracting vector for $Dg^q(x)$, by (3.21), we have

$$\frac{1}{q}\sum_{i=0}^{q-1} \lambda_i^s = -\frac{1}{q}\ln ||Dg^q(x)|| \leq -\theta a$$
$$\frac{1}{q}\sum_{i=0}^{q-1} \lambda_i^u = -\frac{1}{q}\sum_{i=0}^{q-1} \lambda_i^s \geq \theta a$$

The second line follows from the fact that g preserves area.

As a result, we know that there are at most $\frac{4(1-\theta)}{\eta}q$ many i such that $\lambda_i^u < (1-\frac{\eta}{2})a$ or $\lambda_i^s > -(1-\frac{\eta}{2})a$, for which i we have $\overline{\lambda}_i^e \geq -a$. For the rest of i, $\overline{\lambda}_i^e \geq (1-\frac{\eta}{2})a$. Hence

$$\frac{1}{q}\sum_{i=0}^{q-1} \overline{\lambda}_i^e \geq (1-\frac{\eta}{2})a - \frac{4(1-\theta)}{\eta} \times a \geq a(1-\frac{\eta}{2} - \frac{8(1-\theta)}{\eta}).$$

If θ_0 is sufficiently close to 1 depending only on η, we get that

$$\frac{1}{q}\sum_{i=0}^{q-1} \overline{\lambda}_i^e \geq (1-\eta)\theta a$$

□

From now on to the end of the proof of Proposition 3.1, we will also use the following definition.

Definition 3.4. The point $g^n(x)$ is called *good* in the orbit of x if $n \in [1, q-1]$ satisfies the following conditions:

$$\frac{1}{k}\sum_{j=n}^{n+k-1}\overline{\lambda}_j^e > (1-\frac{1}{1000})a, \quad \forall 1 \le k \le q-n, \tag{3.22}$$

$$\frac{1}{k}\sum_{j=n-k}^{n-1}\overline{\lambda}_j^e > (1-\frac{1}{1000})a, \quad \forall 1 \le k \le n. \tag{3.23}$$

We will use the following lemma to estimate the number of the good points:

Lemma 3.3. *There exists $\theta_0 \in (0,1)$ such that if $\theta \in [\theta_0, 1)$, $g \in \mathrm{Diff}^1_{vol}(S)$ satisfies $\|Dg\| \le A$, and (q, a, θ, x) satisfies* (3.21), *where $a = \ln A$, then there are more than $\frac{q}{2}$ points in $\{g^k(x), 0 \le k \le q-1\}$ that are good in the orbit of x.*

Proof. We will need Pliss' lemma that we now state.

Lemma 3.4. *(Pliss' lemma* [18]*) Consider a sequence of n real numbers $a_1, ..., a_n$. Assume $l' < l$ are numbers such that $a_i \le l$ for all $1 \le i \le n$, and $\sum_{i=1}^n a_i > nl'$.Then, for any $l'' < l'$, there exist at least $\frac{l'-l''}{l-l''}n$ many indices $i \in [1, n]$ such that $\sum_{j=i}^{i+k-1} a_j > kl''$, for all k satisfying $i+k-1 \le n$.*

By Lemma 3.2, for any $\eta > 0$ for we have

$$\frac{1}{q}\sum_{i=0}^{q-1}\overline{\lambda}_i^e \ge (1-\eta)\theta a, \quad \overline{\lambda}_i^e \le a, \quad \forall 0 \le i \le L-1$$

if we let θ to be close to 1. Using Pliss' lemma, we know that there are at least $\frac{(1-\eta)\theta - \frac{999}{1000}}{1-\frac{999}{1000}}q$ points $g^n(x)$ in $\{g^k(x), 0 \le k \le q-1\}$ such that,

$$\frac{1}{k}\sum_{j=n}^{n+k-1}\overline{\lambda}_j^e > (1-\frac{1}{1000})a, \quad \forall 1 \le k \le q-n.$$

Similarly, we know that there are at least $\frac{(1-\eta)\theta - \frac{999}{1000}}{1-\frac{999}{1000}}q$ points $g^n(x)$ in $\{g^k(x), 0 \le k \le q-1\}$ such that,

$$\frac{1}{k}\sum_{j=n-k}^{n-1}\overline{\lambda}_j^e > (1-\frac{1}{1000})a, \quad \forall 1 \le k \le n.$$

As a result, if η is sufficiently small, θ_0 is sufficiently closed to 1, and $\frac{(1-\eta)\theta-\frac{999}{1000}}{1-\frac{999}{1000}} > \frac{3}{4}$, then at least $\frac{q}{2}$ points in $\{g^k(x), 0 \le k \le q-1\}$ are good in the orbit of x as in Definition 3.4. □

A key estimate due to area preservation is the following lower bound on the angle $\angle(v_n^s, v_n^u)$ at good points.

Lemma 3.5. *There exist $A_0 > 0, \theta_0 \in (0,1)$ such that if $A > A_0, \theta \in [\theta_0, 1)$, $g \in \mathrm{Diff}^1_{vol}(S)$ satisfies $\|Dg\| \le A$, and (q, a, θ, x) satisfies* (3.21), *where $a = \ln A$, then for all good points $g^n(x)$ in the orbit of x,*

$$\cot \angle(v_n^s, v_n^u) \le A^3$$

Proof. We first need a straightforward geometric lemma which only uses the fact that g is area-preserving.

Sublemma 3.1. *For all $i \in \mathbb{Z}$,*

$$\cot \angle(v_{i+1}^u, v_{i+1}^s) \le e^{2\lambda_i^s} \cot \angle(v_i^u, v_i^s) + A^2.$$

Proof. Consider the matrix of $Dg(g^i(x))$ under the basis $(v_i^s, {v_i^s}^\perp)$, $(v_{i+1}^s, {v_{i+1}^s}^\perp)$. Since g preserves area, for some $d_i \in \mathbb{R}$ we have:

$$Dg(v_i^s) = e^{\lambda_i^s} v_{i+1}^s, \quad Dg({v_i^s}^\perp) = \pm e^{-\lambda_i^s} ({v_{i+1}^s}^\perp) + d_i v_{i+1}^s.$$

The first bound implies that $|d_i| \le \|Dg\| \le A$ and the second that $e^{\lambda_i^s} \le \|Dg\| \le A$. Then we have

$$\begin{aligned} \cot \angle(v_{i+1}^u, v_{i+1}^s) &\le e^{2\lambda_i^s} \cot \angle(v_i^u, v_i^s) + |e^{\lambda_i^s} d_i| \\ &\le e^{2\lambda_i^s} \cot \angle(v_i^u, v_i^s) + A^2. \end{aligned}$$

□

As a consequence, we have

$$\cot \angle(v_n^u, v_n^s) \le e^{2\sum_{j=1}^n \lambda_{n-j}^s} \cot \angle(v_0^u, v_0^s) + A^2 (1 + \sum_{j=1}^n e^{2\sum_{k=1}^j \lambda_{n-k}^s})$$

Since $g^n(x)$ is a good point in the orbit of x, by (3.23) the triple $(g^n(x), v_n^s, v_n^u)$ is backward $(n, a)-$good. By (3.19) we have

$$\frac{1}{j} \sum_{k=1}^j \lambda_{n-k}^s < -\frac{1}{2} a$$

for each $1 \le j \le n$. Note that $\cot \angle(v_0^u, v_0^s) = 0$, we have

$$\cot \angle(v_n^u, v_n^s) \le \frac{A^2}{1 - e^{-a}} = \frac{A^2}{1 - A^{-1}} \le A^3$$

The last inequality holds when A_0 is large enough. □

Now we can conclude the proof of Proposition 3.1. By Lemma 3.3, there exist A_0, θ_0 such that for (g, A, θ, q, x) satisfying (3.21) and the conditions of Proposition 3.1, there are more than $\frac{q}{2}$ points in $\{g^k(x), 0 \le k \le q-1\}$ are good in the orbit of x.

We consider the space $S \times T^1S \times T^1S$. By the pigeonhole principle, if A_0, H_0 are large enough (recall that $q \ge A_0^{H_0}$),then q is large, and we can find two good points, $g^i(x), g^{i+L}(x)$ such that the distances $d_{T^1S}(v_i^s, v_{i+L}^s), d_{T^1S}(v_i^u, v_{i+L}^u)$ satisfy

$$d_{T^1S}(v_i^s, v_{i+L}^s) < q^{-\frac{1}{100}}$$

$$d_{T^1S}(v_i^u, v_{i+L}^u) < q^{-\frac{1}{100}}$$

We claim that the point $g^i(x)$ is a $(q,a)-$good point. In fact, we consider the triple $(g^i(x), v_i^s, v_i^u)$. From our choice of $g^i(x), g^{i+L}(x)$ and $\frac{Dg^L(v_i^{s,u})}{\|Dg^L(v_i^{s,u})\|} = v_{i+L}^{s,u}$, we know the triple $(g^i(x), v_i^s, v_i^u)$ satisfies condition (4) in Definition 3.3. By (3.22) of Definition 3.4, since $g^i(x)$ is a good point in the orbit of x, we have that the triple $(g^i(x), v_i^s, v_i^u)$ is forward $(L,a)-$good. Similarly by (3.23) in the Definition 3.4, since $g^{i+L}(x)$ is a good point in the orbit of x, we have the triple $(g^{i+L}(x), v_{i+L}^s, v_{i+L}^u)$ is backward $(L,a)-$good. Moreover, by Lemma 3.5 on the estimates of the angles at points that are good in the orbit of x, we have

$$\ln \cot \angle(v_i^s, v_i^u) \le 3a$$

$$\ln \cot \angle(v_{i+L}^s, v_{i+L}^u) \le 3a$$

As a result the triple $(g^i(x), v_i^s, v_i^u)$ satisfies conditions (1)-(4) in Definition 3.3, so $g^i(x)$ is a $(q,a)-$good point. The proof of Proposition 3.1 is thus complete. □

3.3. **From a good point to a hyperbolic periodic point.** We now claim that the existence of a $(q,a)-$*good* point implies the existence of a hyperbolic periodic point when q is large compared to the C^2 norm.

Proposition 3.2. *There exist $A_0 > 0$, $H_0 > 0$ such that the following is true. If (g, A, D, q) satisfies: $A \ge A_0, D \ge A, q \ge D^{H_0}$ and* (3.1),(3.2) *of Theorem 4 and if there exists a $(q,a)-$good point x, where $a = \ln A$, then g has a hyperbolic periodic point.*

Using Proposition 3.2 and Proposition 3.1, we can easily deduce Theorem 4.

Proof of Theorem 4. We take θ_0 as in Proposition 3.1, A_1,H_1 as in Proposition 3.1 and A_2,H_2 as in Proposition 3.2. Let $A_0 = \max(A_1, A_2)$,$H_0 = \max(H_1, H_2)$. Then for (g, q, A, D, θ) satisfying the hypothesis in Theorem 4, we know from Proposition 3.1 that there exists a $(q, a)-$good point $x_0 \in S$. From Proposition 3.2, we know that there exists a hyperbolic periodic point. □

The rest of the paper is devoted to the proof of Proposition 3.2. Hereafter, we will assume that the conditions of Proposition 3.2 are satisfied.

3.3.1. *Boxes, strips and hyperbolic-like maps.* Let x_0 be the $(q, a)-$ good point in Proposition 3.2, let $L \in \mathbb{N}$ be associated to x_0 as in Definition 3.3. To prove Proposition 3.2 we will show that g^L has a hyperbolic fixed point in the neighborhood of x_0. We will do so by applying to a conjugate of g^L(by a coordinate map), denoted by G, a standard result on hyperbolic-like maps, namely Proposition 3.3 below which essentially states that if a homeomorphism G of $\mathbb{R}^2$ maps a vertical "strip" $\mathcal{R}_1$ into a horizontal "strip" $\mathcal{R}_2$ that crosses $\mathcal{R}_1$ transversally then G has a fixed point inside the intersection of the strips. Moreover, if G is a diffeomorphism and if G and G^{-1} have strict cone conditions in $\mathcal{R}_1$ and $\mathcal{R}_2$ (see (3.25), (3.26)) then the fixed point is hyperbolic.

In this section, we first need to introduce some notations and then define the properties of hyperbolic-like maps that we will be interested in since they guarantee the existence of a hyperbolic fixed point. We then state Proposition 3.4 that reduces the proof of Proposition 3.2 to checking the hyperbolic-like properties for the map G. The rest of the paper will then be dedicated to the proof of Proposition 3.4.

• Given $r > 0, \tau > 0, \kappa > 0$, we define a $(r, \tau, \kappa)-$*Box* , that we denote by $U(r, \tau, \kappa)$, to be

$$U(r, \tau, \kappa) = \{(v, w) \in \mathbb{R}^2; |v| \leq r, |w| \leq \tau + \kappa|v|\}$$

We denote $U(\infty, \tau, \kappa) = \{(v, w) \in \mathbb{R}^2; |w| \leq \tau + \kappa|v|\}$

• A curve contained in $\mathbb{R}^2 = \mathbb{R}_x \oplus \mathbb{R}_y$ is called a $\kappa-$*horizontal graph* if it is the graph of a Lipschitz function from a closed interval $I \subset \mathbb{R}_x$ to $\mathbb{R}_y$ with Lipschitz constant less than κ. Similarly, we can define the $\kappa-$ *vertical graphs*.

• The boundary of an $(r, \tau, \kappa)-$Box U is the union of two $0-$ vertical graphs and two $\kappa-$ horizontal graphs. We call these graphs respectively, the *left (resp. right) vertical boundary of U* and the *upper (resp. lower) horizontal boundary of U*. We call the union of the left and right vertical boundary of U the *vertical boundary of U*. Similarly we call the union of the upper and lower horizontal boundary of U the *horizontal boundary of U*.

• Horizontal and vertical graphs that connect the boundaries of U will be called full horizontal and full vertical graphs as in the following definition. Given $r, \tau, \kappa, \eta > 0$, for each $(r, \tau, \kappa)-$Box U, an *$\eta-$ full horizontal graph of U* is an $\eta-$ horizontal graph L such that $L \subset U$ and the endpoints of L are contained in the vertical boundary of U. Similarly, we define the *$\eta-$full vertical graphs of U*.

• We define an *$\eta-$horizontal strip of U* to be a subset of U bounded by the vertical boundary of U and two disjoint $\eta-$ full horizontal graphs of U that are both disjoint from the horizontal boundary of U. Similarly we can define *$\eta-$vertical strips of U*.

• A homeomorphism that maps a strip $\mathcal{R}'$ to $\mathcal{R}$ is said to be *regular* if it maps the horizontal (resp. vertical) boundary of $\mathcal{R}'$ homeomorphically to the horizontal (resp. vertical) boundary of $\mathcal{R}$.

• For $\kappa > 0$, we denote

$$\begin{aligned} C(\kappa) &= \{(v, w) \in \mathbb{R}^2 : |w| < \kappa|v|\} \\ \tilde{C}(\kappa) &= \{(v, w) \in \mathbb{R}^2 : |v| < \kappa|w|\} \end{aligned}$$

we will refer to these sets as cones.

Using the above definitions and notations, we now define a class of maps on the plane that display some sort of local hyperbolicity that we will call *hyperbolic-like maps*.

Definition 3.5. Given $r, \tau > 0, 0 < \kappa, \kappa', \kappa'' < 1$. Denote $U = U(r, \tau, \kappa)$, and let $\mathcal{R}_1$ be a $\kappa-$vertical strip of U, $\mathcal{R}_2$ be a $\kappa-$horizontal strip of U. A diffeomorphism $G : \mathcal{R}_1 \to G(\mathcal{R}_1) \subset \mathbb{R}^2$ is called a hyperbolic-like map if it satisfies the following conditions:

$$G \text{ is a regular map from } \mathcal{R}_1 \text{ to } \mathcal{R}_2, \tag{3.24}$$

$$\forall x \in \mathcal{R}_1, DG_x(C(\kappa')) \subset C(\frac{1}{2}\kappa'), \tag{3.25}$$

$$\forall x \in \mathcal{R}_2, DG_x^{-1}(\tilde{C}(\kappa'')) \subset \tilde{C}(\frac{1}{2}\kappa'') \tag{3.26}$$

Our interest in hyperbolic-like maps comes from the following classical result.

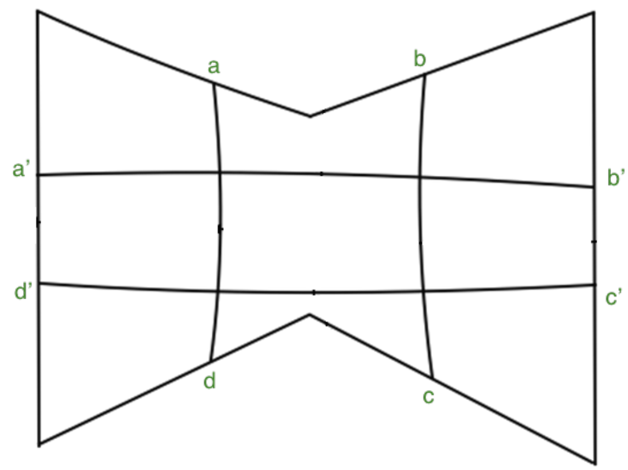

FIGURE 1. $\mathcal{R}_1$ is the topological rectangle $abcd$, $\mathcal{R}_2$ is the topological rectangle $a'b'c'd'$. Under a hyperbolic-like map G, the boundaries lines ab and dc are mapped to the boundary lines $a'b'$ and $d'c'$, and the vertical boundaries ad and bc are mapped to $a'd'$ and $b'c'$.

Proposition 3.3. *A hyperbolic-like map F has a hyperbolic fixed point in $\mathcal{R}_1 \bigcap \mathcal{R}_2$.*

Proof. By (3.24) the Jordan curve $\mathcal{C} = \partial\mathcal{R}_1$ contains no fixed point of G. Therefore the Lefschetz index of G relative to $\mathcal{C}$ is well defined. From (3.24), we see that it is equal to 1 therefore there is a fixed point inside $\mathcal{R}_1$ (then necessarily in $\mathcal{R}_2$, for $G(\mathcal{R}_1) = \mathcal{R}_2$, see Figure 1). The fixed point is hyperbolic by (3.25) and (3.26). $\square$

By Proposition 3.3, the proof of Proposition 3.2 is thus reduced to the following proposition.

Proposition 3.4. *Under the conditions of Proposition 3.2, there exists an integer $L > 0$, a Box $U \subset \mathbb{R}^2$, a vertical strip of U denoted by $\mathcal{R}_1$, a C^1 diffeomorphism $H : U \to S$, such that $g^L H(\mathcal{R}_1) \subset H(U)$ and $H^{-1}g^L H : \mathcal{R}_1 \to \mathbb{R}^2$ is a hyperbolic-like map.*

Now the rest of the paper is devoted to the proof of Proposition 3.4.

3.3.2. *Map from the n-th point to the (n+1)-st point.* Let x_0 be the $(q,a)-$ good point in Proposition 3.2, let $v_s, v_u \in T_{x_0}S, L \in \mathbb{N}$ be as in Definition 3.3. From now on to the end of this paper, we denote

$$x_n = g^n(x_0), \quad \forall 1 \leq n \leq L$$

To show Proposition 3.4, we will study the iterates of g in the neighborhoods of the piece of orbit of x_0 of length L. We want to show that the "return" map from x_0 to the neighborhood of x_0 (since x_L is very close to x_0) is a hyperbolic-like map. But for this, we need to specify the Box on which the "return" map will be considered as well as the neighborhoods of the piece of orbit of x_0 on which the dynamics of g will be studied.

For each $0 \leq n \leq L$, let $x_n, v_n^u, v_n^s, \lambda_n^s, \lambda_n^u$ be defined as in Definition 3.1 and λ_n^e be defined as in (3.15) associated to the triple (x_0, v_s, v_u). In the following we fix a splitting $T_{x_n}S = E_n^u \oplus E_n^s$, where $E_n^u = \mathbb{R}v_n^u, E_n^s = \mathbb{R}v_n^s$. We define $i_n : \mathbb{R}^2 \to T_{x_n}S$ as

$$i_n(a,b) = av_n^u + bv_n^s$$

By (3.1) and $D \geq A$, it is direct to see that there exists a constant $R > 0$ depending only on S such that $\exp_{x_{n+1}}^{-1}$ is a diffeomorphism over $g \exp_{x_n} i_n(B(0, D^{-1}R))$.

Denote g_n the C^2 diffeomorphism defined by

$$\begin{aligned} g_n : B(0, D^{-1}R) &\to \mathbb{R}^2 \\ g_n(v,w) &= i_{n+1}^{-1} \exp_{x_{n+1}}^{-1} g \exp_{x_n} i_n(v,w) \end{aligned} \tag{3.27}$$

In Section 3.3.4, we will construct the box U and the vertical strip $\mathcal{R}_1$ in Proposition 3.4, and we will let H be $exp_{x_0} i_0$ restricted to $\mathcal{R}_1$.

We let $\delta := \frac{1}{100}a$, where $a = \ln A$. The following parameters r_n, τ_n, κ_n will determine the domains and the cones that we will consider in the study of the dynamics of g_n. For $0 \leq n \leq L-1$ define

$$c_0 := 1, c_{n+1} := \min(e^{\lambda_n^e - \delta} c_n, 100), \tag{3.28}$$

and

$$r_n = \bar{r}c_n^3 \tag{3.29}$$

$$\tau_n = e^{\sum_{i=0}^{n-1}(\lambda_i^s+\delta)}\bar{r} \tag{3.30}$$

$$\kappa_n = \bar{\kappa}c_n^{-1} \tag{3.31}$$

$$\tilde{\kappa}_n = \bar{\kappa}c_n \tag{3.32}$$

$$\beta_n = \bar{\beta}c_n^{-1} \tag{3.33}$$

where we set

$$\bar{r} = D^{-3M}, \quad \bar{\kappa} = D^{-M}, \quad \bar{\beta} = D^M, \quad M := 1000. \tag{3.34}$$

Define for each $0 \le n \le L$ the Box and cones

$$U_n = U(r_n, \tau_n, \kappa_n), \quad C_n = C(\kappa_n), \quad \tilde{C}_n = \tilde{C}(\kappa_n).$$

In order to exploit the properties of the good point x_0 we chose to study the dynamics of the iterates of g in the *moving* frames $E_n^u \oplus E_n^s$, that is g_n as defined in (3.27). This comes at the cost of an extra factor in the C^2 norm of g_n coming from the angle between v_n^s and v_n^u. The parameter β_n gives a bound on how small can these angles become as we shall now see.

Lemma 3.6. *If we choose A_0 in Proposition 3.2 to be sufficiently large, we have*

$$\beta_i \ge \max(\cot\angle(E_i^u, E_i^s), 1), \quad \forall 0 \le i \le L-1 \tag{3.35}$$

Proof. Since $\delta = a/100$, if $A_0 > e^{100}$, then $\delta > 1$. By Lemma 3.1, we have

$$\cot\angle(E_{i+1}^u, E_{i+1}^s) \le e^{2\lambda_i^s}\cot\angle(E_i^u, E_i^s) + A^2$$

Then

$$\cot\angle(E_{i+1}^u, E_{i+1}^s) \le \max(\cot\angle(E_i^u, E_i^s)e^{2\lambda_i^s+\delta}, \frac{1}{100}\bar{\beta}) \tag{3.36}$$

since $\frac{1}{100}\bar{\beta} = \frac{1}{100}D^M \ge 10A^2$ when A_0 is sufficiently large.

Moreover, we have

$$\begin{aligned} \beta_{i+1} &= \frac{1}{\min(e^{\lambda_i^e-\delta}c_i, 100)}\bar{\beta} \\ &\ge \max(e^{2\lambda_i^s+\delta}\beta_i, \frac{1}{100}\bar{\beta}) \end{aligned} \tag{3.37}$$

The last inequality follows from $\lambda_i^e \le -2\lambda_i^s$. Since x_0 is a $(q,a)-$good point, by (3) in Definition 3.3 we have $\cot\angle(E_0^u, E_0^s) < A^3 < D^M = \beta_0$.

If for some $0 \leq i \leq L-1$, we have $\beta_i \geq \cot\angle(E_i^u, E_i^s)$. Then by (3.37) and (3.36) we have

$$\begin{aligned}\beta_{i+1} &\geq \max(e^{2\lambda_i^s+\delta}\beta_i, \frac{1}{100}\bar{\beta}) \\ &\geq \max(e^{2\lambda_i^s+\delta}\cot\angle(E_i^u, E_i^s), \frac{1}{100}\bar{\beta}) \\ &\geq \cot\angle(E_{i+1}^u, E_{i+1}^s)\end{aligned}$$

Thus we can show inductively that $\beta_i \geq \cot\angle(E_i^u, E_i^s)$ for all $0 \leq i \leq L$.

Since $c_n \leq 100$, $\beta_n \geq \frac{1}{100}\bar{\beta} > 1$ for all $0 \leq n \leq L$. Hence $\beta_i \geq \max(\cot\angle(E_i^u, E_i^s), 1)$ for $0 \leq i \leq L-1$. This completes the proof. □

The following lemmata exploit the fact that x_0 is good to control the shape and size of the Boxes at each time $n \leq L$. Most importantly, U_L is shown to be long in the horizontal direction and thin in the vertical direction compared to U_0.

Lemma 3.7. *If we choose A_0 in Proposition 3.2 to be sufficiently large, then we have that $c_L = 100$. As a consequence, we have*

$$r_L = 10^6\bar{r}, \quad \kappa_L = \frac{1}{100}\bar{\kappa}, \quad \tilde{\kappa}_L = 100\bar{\kappa}. \tag{3.38}$$

Proof. Using the fact that x_0 is a $(q,a)-$ good point, it is easy to show that $c_L = 100$ when A is large. Indeed, let i be the biggest index such that $0 \leq i \leq L-1$ and $c_i \geq 1$ ($i \geq 0$ because $c_0 = 1$). Then $c_L = \min(e^{\lambda_{L-1}^e-\delta}c_{L-1}, 100) = \min(e^{\sum_{k=i}^{L-1}(\lambda_k^e-\delta)}c_i, 100)$. Since the triple (x_L, v_L^u, v_L^s) is backward $(L,a)-$good, therefore by (3.20) $\sum_{k=i}^{L-1}(\lambda_k^e - \delta) > \frac{a}{2}$. Then $c_L = 100$ when $A \geq 10^4$. □

Lemma 3.8. *If we choose A_0 in Proposition 3.2 to be sufficiently large, we have*

$$\tau_L \leq \frac{1}{10}\bar{r} \tag{3.39}$$

$$r_n \geq \tau_n, \quad \forall 0 \leq n \leq L \tag{3.40}$$

Proof. By (3.16) and (3.30), when A_0 is sufficiently large

$$\tau_L \leq A^{-\frac{1}{5}}\tau_0 \leq \frac{1}{10}\tau_0 = \frac{1}{10}\bar{r}$$

Moreover, since $r_{n+1} = \bar{r}c_{n+1}^3 = \min(e^{3(\lambda_n^e-\delta)}r_n, 10^6\bar{r})$, then when $\lambda_n^e - \delta < 0$ we have

$$r_{n+1} \geq e^{3(\lambda_n^e-\delta)}r_n$$

and otherwise we have

$$r_{n+1} \geq r_n$$

In any case, for any $0 \leq n \leq L-1$ we have $r_{n+1} \geq e^{\min(3(\lambda_n^e-\delta),0)}r_n$. Since x_0 is a $(q,a)-$good point, Lemma 3.1 yields

$$\sum_{k=0}^{n-1} \min(3(\lambda_n^e - \delta), 0) \geq \sum_{k=0}^{n-1} \min(3\lambda_n^e, 0) - 3\delta \geq \sum_{k=0}^{n-1} (\lambda_n^s + \delta)$$

Thus we have that for any $0 \leq n \leq L$, $r_n \geq \tau_n$. □

Remark 3.2. Since $U_n \subset B(0, r_n + \tau_n + \kappa_n r_n)$, and

$$r_n + \tau_n + \kappa_n r_n \leq 2 \times 10^6 \bar{r} + 100\bar{\kappa}\bar{r} \leq D^{-M}$$

when D is sufficiently large. Thus $U_n \subset B(0, D^{-1}R)$ when D is sufficiently large depending only on S. This shows that g_n is defined over U_n.

The following proposition is purely analytic and relies on the relations between the parameters $r_n, \tau_n, \beta_n, \kappa_n, \tilde{\kappa}_n$. It will be used repeatedly in the sequel.

Proposition 3.5. *Recall that* $\{(r_n, \tau_n, \beta_n, \kappa_n, \tilde{\kappa}_n)\}_{0 \leq n \leq L}$ *are defined by* (3.29) *to* (3.33)*. Assume that we also have* (3.35) *and* (3.40)*. Then for all D sufficiently large depending only on S, for each* $0 \leq n \leq L-1$ *we have the following:*

(1). For any $(v,w) \in U_n$*, we have* $(Dg_n)_{(v,w)}(C_n) \subset C_{n+1}$*; For any* $(v,w) \in g_n(U_n) \bigcap U_{n+1}$*, we have* $(Dg_n^{-1})_{(v,w)}(\tilde{C}_{n+1}) \subset \tilde{C}_n$*. As a consequence, the image of any* κ_n- *horizontal graph contained in* U_n *under* g_n *is a* $\kappa_{n+1}-$ *horizontal graph; the image of any* $\tilde{\kappa}_{n+1}-$ *vertical graph contained in* $g_n(U_n) \bigcap U_{n+1}$ *under* g_n^{-1} *is a* $\tilde{\kappa}_n-$ *vertical graph.*

(2). If Γ *is a* κ_n-*full horizontal graph of* U_n*, then* $g_n(\Gamma) \bigcap U_{n+1}$ *is a* κ_{n+1}*-full horizontal graph of* U_{n+1}*. Moreover, the image of the horizontal boundary of* U_n *under* g_n *is disjoint from the horizontal boundary of* U_{n+1}*; the image of the vertical boundary of* U_n *under* g_n *is disjoint from the vertical boundary of* U_{n+1}*.*

The proof of Proposition 3.5 will be given in the Appendix.

Corollary C. *For any* $0 \leq k \leq L-1$*, we can find a* $\tilde{\kappa}_k-$ *vertical strip of* U_k*, denoted by* $\mathcal{R}'$*, such that* $\mathcal{R} = g_k(\mathcal{R}')$ *is a* $\kappa_{k+1}-$ *horizontal strip of* U_{k+1}*. Moreover,* g_k *is a regular map from* $\mathcal{R}'$ *to* $\mathcal{R}$*.*

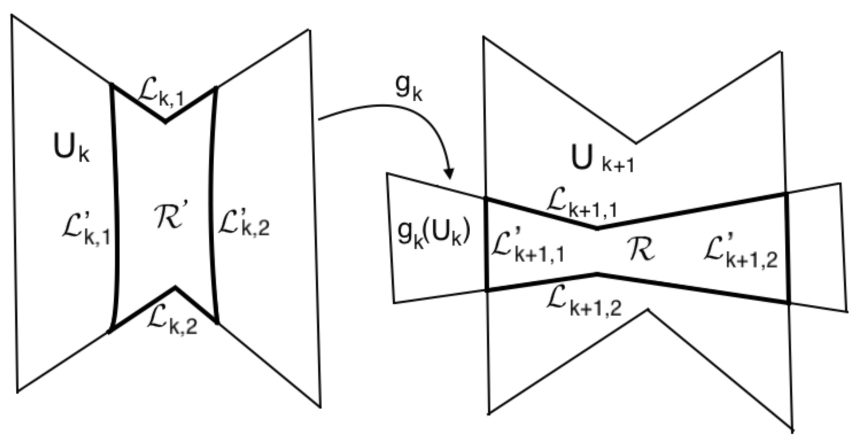

FIGURE 2. g_k maps $\mathcal{R}'$ to $\mathcal{R}$. Moreover $\mathcal{L}_{k,1}, \mathcal{L}_{k,2}$ are mapped to $\mathcal{L}_{k+1,1}, \mathcal{L}_{k+1,2}$ respectively and $\mathcal{L}'_{k,1}, \mathcal{L}'_{k,2}$ are mapped to $\mathcal{L}'_{k+1,1}, \mathcal{L}'_{k+1,2}$ respectively.

Proof. For any $0 \leq k \leq L-1$, denote the upper and lower horizontal boundary of U_k by $\mathcal{L}_1, \mathcal{L}_2$ respectively. Then by Proposition 3.5, for $i = 1, 2$, there exists a κ_k- horizontal graph contained in $\mathcal{L}_i$, denoted by $\mathcal{L}_{k,i}$, such that $g_k(\mathcal{L}_{k,i})$ is a $\kappa_{k+1}-$ full horizontal graph of U_{k+1}, denoted by $\mathcal{L}_{k+1,i}$. By Proposition 3.5, $\mathcal{L}_{k+1,1}, \mathcal{L}_{k+1,2}$ are disjoint from the horizontal boundary of U_{k+1}. Then there exist a horizontal strip of U_{k+1}, denoted by $\mathcal{R}$, bounded by $\mathcal{L}_{k+1,1}$ and $\mathcal{L}_{k+1,2}$ (see Figure 2).

Again by Proposition 3.5, denote the $\tilde{\kappa}_{k+1}-$vertical graph contained in the left vertical boundary of U_{k+1}, connecting the endpoints of $\mathcal{L}_{k+1,1}$ and $\mathcal{L}_{k+1,2}$ by $\mathcal{L}'_{k+1,1}$, then $g_k^{-1}(\mathcal{L}'_{k+1,1})$ is a $\tilde{\kappa}_k-$vertical graph, denoted by $\mathcal{L}'_{k,1}$. From Proposition 3.5, we see that the images of the vertical boundary of U_k under g_k is disjoint from the vertical boundary of U_{k+1}. Then $\mathcal{L}'_{k,1}$ is disjoint from the vertical boundary of U_k. Since by construction $\mathcal{L}'_{k,1}$ connect the upper and lower horizontal boundary of U_k, we have $\mathcal{L}'_{k,1} \subset U_k$. Hence $\mathcal{L}'_{k,1}$ is a $\tilde{\kappa}_k-$full vertical graph of U_k. In a similar fashion, we construct $\mathcal{L}'_{k+1,2}$ and $\mathcal{L}'_{k,2}$. Then there exists a vertical strip of U_k, denoted by $\mathcal{R}'$, bounded by $\mathcal{L}'_{k,1}$ and $\mathcal{L}'_{k,2}$. It is straightforward to check that the statement is true for $\mathcal{R}'$ and $\mathcal{R}$. □

3.3.3. *Concatenation.* In this section, we will use Corollary C inductively to construct a vertical strip of U_0 that is mapped by $g_{L-1}\cdots g_0$ to a horizontal strip of U_L. We will also prove the cone preservation by the derivative map using Proposition 3.5. The result of the induction is summarized in Corollary D.

First we need the following lemma.

Lemma 3.9. *If $D > 1$, for any $0 \le n \le L$, any $\tilde{\kappa}_n-$vertical full graph $\mathcal{L}'$, any κ_n-horizontal full graph $\mathcal{L}$, $\mathcal{L}'$ intersects L at a unique point.*

Proof. A simple topological argument shows that there is at least one intersection point.

If there are two intersection points, then by the Rolle's theorem, there exists $p \in \mathcal{L}'$ and $q \in \mathcal{L}$, such that the Lipschitz constant of $\mathcal{L}$ at q, denoted by t, equals to the inverse of the Lipschitz constant of $\mathcal{L}'$at p, denoted by t'^{-1}. We have $\bar{\kappa} = D^{-M} < 1$. Thus, we have $\tilde{\kappa}_n \ge t' = t^{-1} \ge \kappa_n^{-1} = c_n\bar{\kappa}^{-1} > c_n\bar{\kappa} = \tilde{\kappa}_n$. This is a contradiction. □

Corollary D. *When $A_0 > 0$ is sufficiently large, there exists a $\tilde{\kappa}_0-$ vertical strip of U_0, denoted by $\mathcal{R}'$, such that $\mathcal{R} = g_{L-1}\cdots g_0(\mathcal{R}')$ is a κ_L- horizontal strip of U_L and $g_{L-1}\cdots g_0$ is a regular map from $\mathcal{R}'$ to $\mathcal{R}$. Moreover, for any $x \in \mathcal{R}'$, $D(g_{L-1}\cdots g_0)_x$ maps $C(\bar{\kappa})$ into $C(\frac{1}{100}\bar{\kappa})$; For any $x \in \mathcal{R}$, $D(g_0^{-1}\cdots g_{L-1}^{-1})_x$ maps $\tilde{C}(100\bar{\kappa})$ into $\tilde{C}(\bar{\kappa})$.*

Proof. We will inductively prove a stronger statement : for any $1 \le n \le L$, there exists a $\tilde{\kappa}_0-$ vertical strip R'_0 of U_0 and a κ_n- horizontal strip R_n of U_n such that $g_{n-1}\cdots g_0$ is a regular map from R'_0 to R_n.

First, we see that the above statement for $n = 1$ follows from Corollary C.

Now we assume that the statement is true for $n-1$. By the induction hypothesis, we get a $\tilde{\kappa}_0-$ vertical strip of U_0, denoted by Q'_0, such that $g_{n-2}\cdots g_0(Q'_0)$ is a $\kappa_{n-1}-$ horizontal strip of U_{n-1}, denoted by Q_{n-1}. From Corollary C, we get a $\tilde{\kappa}_{n-1}-$ vertical strip of U_{n-1}, denoted by Q'_{n-1}, such that $g_{n-1}(Q'_{n-1})$ is a κ_n- horizontal strip of U_n, denoted by Q_n. Denote two horizontal boundaries of Q_{n-1} by M_1, M_2, two vertical boundaries of Q'_{n-1} by M'_1, M'_2 (see Figure 3).

By Lemma 3.9, we easily see that $\tilde{Q} = Q_{n-1} \bigcap Q'_{n-1}$ is bounded by: two $\tilde{\kappa}_{n-1}-$ vertical graphs contained in M'_1 and M'_2 respectively, and

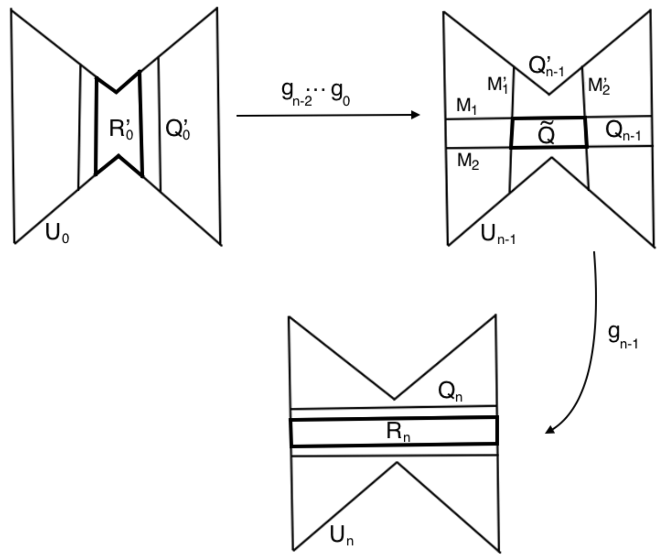

FIGURE 3. $g_{n-2}\cdots g_0$ maps R_0', Q_0' to $\tilde{Q}, Q_{n-1}$ respectively; g_{n-1} maps $\tilde{Q}, Q_{n-1}'$ to R_n, Q_n respectively.

two $\kappa_{n-1}-$ horizontal graphs contained in M_1 and M_2 respectively. Again by Proposition 3.5, we have that $g_0^{-1}\cdots g_{n-2}^{-1}(\tilde{Q})$ is a $\tilde{\kappa}_0-$ vertical strip of U_0,denoted by R_0', and $g_{n-1}(\tilde{Q})$ is a κ_n- horizontal strip of U_n, denoted by R_n. It is clear from the definitions that $g_{n-1}\cdots g_0$ is a regular maps from R_0' to R_n. This completes the induction.

The first part of Corollary D is thus proved if we take $n = L$, and $\mathcal{R}' = R_0'$ and $\mathcal{R} = R_L$. The construction also shows that for any $x \in \mathcal{R}'$, any $0 \leq n \leq L-1$, $g_n \cdots g_0(x) \in U_{n+1}$. This allows us to apply (1) in Proposition 3.5 and get from (3.38) the second part of the statement of Corollary D. $\square$

3.3.4. *Return map and the end of the proof of Proposition 3.4.* Since we have to consider a map from a vertical strip of a box to a horizontal strip of the *same* box. We have to compare the exponential coordinate charts at x_0 and x_L. Now we define the return map $I : U_L \to \Omega_0$ as $I := i_0^{-1} \exp_{x_0}^{-1} \exp_{x_L} i_L$.

Lemma 3.10. *If A_0, H_0 are sufficiently large, the map I satisfies the following inequalities:*

$$\forall x \in U_L, DI_x(C(\frac{1}{100}\bar{\kappa})) \subset C(\frac{1}{2}\bar{\kappa}) \tag{3.41}$$

$$\forall x \in I(U_L), DI_x^{-1}(\tilde{C}(2\bar{\kappa})) \subset \tilde{C}(100\bar{\kappa}) \tag{3.42}$$

Moreover, the image of any $\frac{1}{100}\bar{\kappa}-$ full horizontal graph of U_L under I contains a $\frac{1}{2}\bar{\kappa}-$ full horizontal graph of U_0. The image of the vertical boundary of U_L under I is disjoint from U_0.

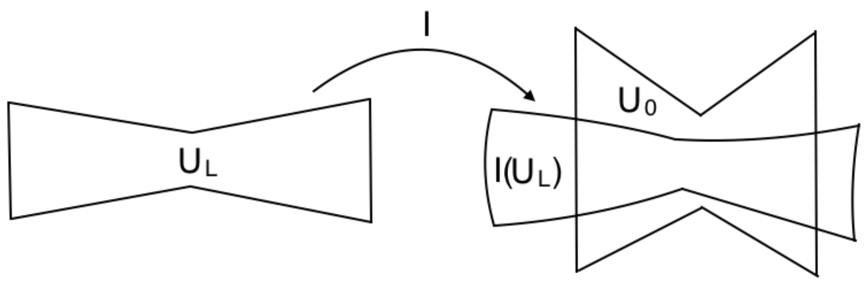

FIGURE 4. By Lemma 3.7,3.8, $U_L \subset U(10^6\bar{r}, \frac{1}{10}\bar{r}, \frac{1}{100}\bar{\kappa})$, $U_0 = U(\bar{r}, \bar{r}, \bar{\kappa})$. The image of U_L under I intersects U_0 transversely.

Proof. By (3) and (4) in the definition of a $(q, a)-$ good point, we have

$$d(x_0, x_L), d_{T^1S}(v_0^s, v_L^s), d_{T^1S}(v_0^u, v_L^u) < q^{-\frac{1}{100}} \leq D^{-\frac{H_0}{100}}$$
$$\cot\angle(v_0^s, v_0^u), \cot\angle(v_L^s, v_L^u) \leq A^3 \leq D^3$$

which shows that the map $i_0^{-1}exp_{x_0}^{-1}exp_{x_L}i_L$ is $D^{-O(H_0)}$ close to the identity map in C^1 over a D^{-1} neighborhood of the origin when D is sufficiently large (Note that the C^2 norm of the exponential map x_0,x_L depends only on S. Restricted to a D^{-1} neighborhood of the origin, $exp_{x_0}^{-1}exp_{x_L}$ tends to the identity map in C^1 as $d(x_0, x_L)$ tends to zero and D tends to infinity). By letting H_0 to be larger than some constant times M we obtain

$$\|I^{\pm 1} - id\|_{C^1} \leq D^{-10M} \tag{3.43}$$

We get (3.41),(3.42) as simple consequences of (3.43).

Denote the left and right vertical boundary of U_L by L_1, L_2 respectively. By (3.43) and (3.38), we see that

$$\sup_{(v,w)\in L_1} \pi_1(I(v,w)) \leq \sup_{(v,w)\in L_1} v + D^{-10M} = -10^6\bar{r} + D^{-10M} \leq -10\bar{r} \tag{3.44}$$

and similarly

$$\inf_{(v,w)\in L_2} \pi_1(I(v,w)) \geq 10\bar{r} \tag{3.45}$$

here for any $(a,b) \in \mathbb{R}^2$, we denote $\pi_1(a,b) := a$. Since $U_0 = U(\bar{r},\bar{r},\bar{\kappa})$, this proves the last statement in the lemma.

Take any $\frac{1}{100}\bar{\kappa}-$ full horizontal graph of U_L, denoted by $\mathcal{L}$. By (3.39), $\mathcal{L}$ is contained in $U(10^6\bar{r}, \frac{1}{10}\bar{r}, \frac{1}{100}\bar{\kappa})$. By (3.43) and (3.41), $I(\mathcal{L})$ is contained in $U(2\times 10^6\bar{r}, \frac{1}{2}\bar{r}, \frac{1}{2}\bar{\kappa})$. By (3.41), (3.42), $I(\mathcal{L})$ is a $\frac{1}{2}\bar{\kappa}-$horizontal graph. Using (3.44) and (3.45), we see that $I(\mathcal{L})$ contains a $\frac{1}{2}\bar{\kappa}-$full horizontal graph of U_0. This completes the proof. □

We are finally ready to prove Proposition 3.4.

Proof of Proposition 3.4. Consider the map $G := IG'$ with $G' := g_{L-1} \circ \cdots \circ g_0$. We have

$$G = i_0^{-1} exp_{x_0}^{-1} g^L exp_{x_0} i_0$$

When $D > 10$, we have $2\bar{\kappa} < 1$. If we show that there exists a vertical strip $\mathcal{R}_1$ of U_0 and a horizontal strip $\mathcal{R}_2$ of U_0 such that

(1) G is a regular map from $\mathcal{R}_1$ to $\mathcal{R}_2$,
(2) for any $(v,w) \in \mathcal{R}_1, DG_{(v,w)}(C(\bar{\kappa})) \subset C(\frac{1}{2}\bar{\kappa})$
(3) for any $(v,w) \in \mathcal{R}_2, DG^{-1}_{(v,w)}(\tilde{C}(2\bar{\kappa})) \subset \tilde{C}(\bar{\kappa})$

Then we define $U = U_0$ and $H : \mathcal{R}_1 \to S$ by $H = exp_{x_0} i_0$, we have $(H^{-1}g^L H)|_{\mathcal{R}_1} = G$. This finishes the proof.

Proof of (1). By Corollary D, we know that there is a $\tilde{\kappa}_0-$vertical strip $\mathcal{R}'$ of U_0 and $\mathcal{R} := G'(\mathcal{R}')$ is a horizontal strip of U_L. Denote $\mathcal{R}_2 := I(\mathcal{R}) \cap U_0$.

By Corollary D, $\mathcal{R}$ is a κ_L-horizontal strip of U_L. By Lemma 3.10 (3.41), the image of the upper horizontal boundary of $\mathcal{R}$ under I is a $2\kappa_L-$horizontal graph, and by Lemma 3.10, it intersects the vertical boundary of U_0 at two points, one in each component. We have the same for the lower horizontal boundary of $\mathcal{R}$. Then $\mathcal{R}_2$

is a $2\kappa_L-$horizontal strip of U_0 and the vertical boundary of R_2 is contained in the vertical boundary of U_0.

By Corollary D and Lemma 3.10, $(DG^{-1})_{(v,w)}(\tilde{C}(2\bar{\kappa})) \subset \tilde{C}(\bar{\kappa})$ for any $(v,w) \in \mathcal{R}_2$, this implies that the image of the vertical boundary of R_2 under G^{-1} is the union of two $\bar{\kappa}-$ vertical graphs connecting two components of the horizontal boundary of $\mathcal{R}'$. By Lemma 3.10, these two vertical graphs are contained in $\mathcal{R}'$, disjoint from each other and disjoint from the vertical boundary of $\mathcal{R}'$, hence are $\bar{\kappa}-$ full vertical graphs of $\mathcal{R}'$. Hence $G^{-1}(\mathcal{R}_2)$ is contained in $\mathcal{R}'$, and is bounded by two $\bar{\kappa}-$ full vertical graphs of $\mathcal{R}'$. Then $G^{-1}(\mathcal{R}_2)$ is a vertical strip of U_0, denoted by $\mathcal{R}_1$. From the construction, we see that the horizontal (resp. vertical) boundary of $G^{-1}(\mathcal{R}_2)$ is mapped homeomorphically to the horizontal (resp. vertical) boundary of $\mathcal{R}$. Hence G is a regular map between these two strips. The proof of (1) is thus finished. □

Combining Lemma 3.10 and Corollary D, we immediately have (2) and (3) , which ends the proof of Proposition 3.4. □

4. Appendix

Proof of Proposition 3.5. We denote the first and second coordinates of g_n as $\bar{v}, \bar{w}$. We expand g_n into linear terms and higher order terms at the origin. For each $(v,w) \in U_n$, we have

$$\bar{v}(v,w) = A_n v + f_n(v,w) \tag{4.1}$$

$$\bar{w}(v,w) = B_n w + h_n(v,w) \tag{4.2}$$

where $\partial_v f_n(0,0) = \partial_w f_n(0,0) = \partial_v h_n(0,0) = \partial_w h_n(0,0) = 0$. ($f_n, h_n$ are uniquely determined by these conditions). Following the notations in Section 3.2, we have

$$A_n = e^{\lambda_n^u}, B_n = e^{\lambda_n^s}$$

Recall that we have assumed $\|Dg\| \leq A$ and $\|D^2 g\| \leq D$. By Lemma 3.6, we have the following lemmata which are essentially proved in [4].

Lemma 4.1. *For all* $0 \leq n \leq L-1$ *we have*

$$\|D^2 f_n\|, \|D^2 h_n\| \leq C_0 D \beta_{n+1}$$

As a consequence, for all $0 \leq n \leq L-1$ *we have*

$$|\partial_v f_n(v,w)|, |\partial_w f_n(v,w)|, |\partial_v h_n(v,w)|, |\partial_w h_n(v,w)| \leq C_0 D \beta_{n+1}(|v| + |w|)$$

Here C_0 *is a constant depending only on* S *and the norm is taken restricted to* U_n.

Denote $\epsilon_n = 2C_0 D\beta_{n+1} r_n(1+\kappa_n)$, then we have the following estimate:

Lemma 4.2. *For each* $0 \leq n \leq L-1$, *we have*

$$\|\partial_v f_n\|_{C^0} \leq \epsilon_n, \|\partial_w f_n\|_{C^0} \leq \epsilon_n$$
$$\|\partial_v h_n\|_{C^0} \leq \epsilon_n, \|\partial_w h_n\|_{C^0} \leq \epsilon_n$$

Here C^0 *norm is taken restricted to* U_n.

It is easy to see that

$$\begin{aligned}\epsilon_n = 2C_0 D\beta_{n+1} r_n(1+\kappa_n) &\leq 2C_0 D \max(A^2 c_n^{-1}, \frac{1}{100})\bar{\beta}\bar{r} c_n^3 (1 + c_n^{-1}\bar{\kappa}) \\ &\leq 400 C_0 c_n D A^2 \bar{\beta}\bar{r}\end{aligned}$$

Then by (3.31) and (3.32), we have

$$\epsilon_n \leq D^{-\frac{M}{2}}\bar{\kappa} \tag{4.3}$$
$$\epsilon_n\kappa_n \leq D^{-M}\bar{\kappa} \tag{4.4}$$
$$\epsilon_n\tilde{\kappa}_n \leq D^{-M}\bar{\kappa} \tag{4.5}$$

when D is sufficiently large depending only on S.

Then by Lemma 4.1 and (4.1), for any $(v,w) \in U_n$ we have,

$$\begin{aligned}|\bar{v}| &\geq A_n|v| - |f_n(v,w)| \\ &\geq A_n|v| - \epsilon_n(|v|+|w|) \\ &\geq (A_n - \epsilon_n - \epsilon_n\kappa_n)|v| - \epsilon_n\tau_n\end{aligned} \tag{4.6}$$

and by (4.2)

$$\begin{aligned}|\bar{w}| &\leq B_n|w| + |h_n(v,w)| \\ &\leq B_n|w| + \epsilon_n(|v|+|w|) \\ &\leq B_n(\tau_n + \kappa_n|v|) + \epsilon_n((1+\kappa_n)|v| + \tau_n) \\ &\leq (B_n\kappa_n + \epsilon_n + \epsilon_n\kappa_n)|v| + B_n\tau_n + \epsilon_n\tau_n\end{aligned} \tag{4.7}$$

We will first show that :

(4.8) $g_n(U_n)$ is contained in the interior of $U(\infty, \tau_{n+1}, \kappa_{n+1})$

Combining (4.6) and (4.7), (4.8) follows from :

$$\kappa_{n+1} > \frac{e^{\lambda_n^s}\kappa_n + \epsilon_n + \epsilon_n\kappa_n}{e^{\lambda_n^u} - \epsilon_n - \epsilon_n\kappa_n} \tag{4.9}$$

$$\tau_{n+1} > (e^{\lambda_n^s} + \epsilon_n + \epsilon_n\frac{e^{\lambda_n^s}\kappa_n + \epsilon_n + \epsilon_n\kappa_n}{e^{\lambda_n^u} - \epsilon_n - \epsilon_n\kappa_n})\tau_n \tag{4.10}$$

By (4.3), (4.4), then when D is sufficiently large we get (4.9) from

$$\begin{aligned}\frac{e^{\lambda_n^s}\kappa_n+\epsilon_n+\epsilon_n\kappa_n}{e^{\lambda_n^u}-\epsilon_n-\epsilon_n\kappa_n} &< e^{-\lambda_n^u+\frac{1}{2}\delta}(e^{\lambda_n^s}\kappa_n+D^{-\frac{1}{4}M}\bar{\kappa})\\ &\leq \max(e^{-\lambda_n^u+\lambda_n^s+\delta}\kappa_n,\frac{1}{100}\bar{\kappa})\\ &\leq \kappa_{n+1}\end{aligned}$$

Similarly, by (4.3),(4.4), (3.30), we have (4.10) when D is sufficiently large.

Now we get to the proof of (1) in Proposition 3.5. Differentiate (4.1) and (4.2), we get

$$\frac{d\bar{v}}{dv} = A_n+\partial_v f_n(v,w) \tag{4.11}$$

$$\frac{d\bar{v}}{dw} = \partial_w f_n(v,w) \tag{4.12}$$

$$\frac{d\bar{w}}{dv} = \partial_v h_n(v,w) \tag{4.13}$$

$$\frac{d\bar{w}}{dw} = B_n+\partial_w h_n(v,w) \tag{4.14}$$

Take any $(a,b)\in C_n$, $(v,w)\in U_n$, then

$$(Dg_n)_{(v,w)}(a,b) = (\bar{a},\bar{b})$$

with

$$\begin{aligned}\bar{a} &= (A_n+\partial_v f_n(v,w))a+\partial_w f_n(v,w)b\\ \bar{b} &= \partial_v h_n(v,w)a+(B_n+\partial_w h_n(v,w))b\end{aligned}$$

Since $|b|\leq\kappa_n|a|$, by Lemma 4.2 we have

$$\begin{aligned}|\bar{a}| &\geq (e^{\lambda_n^u}-\epsilon_n)|a|-\epsilon_n|b|\\ &\geq (e^{\lambda_n^u}-\epsilon_n-\epsilon_n\kappa_n)|a|\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}|\bar{b}| &\leq \epsilon_n a+(e^{\lambda_n^s}+\epsilon_n)b\\ &\leq (\epsilon_n\kappa_n+e^{\lambda_n^s}\kappa_n+\epsilon_n)|a|\end{aligned} \tag{4.16}$$

Then we have

$$|\bar{b}|\leq|\bar{a}|\frac{\epsilon_n\kappa_n+e^{\lambda_n^s}\kappa_n+\epsilon_n}{e^{\lambda_n^u}-\epsilon_n-\epsilon_n\kappa_n}$$

Then by (4.9), we have that $(\bar{a},\bar{b})\in C_{n+1}$.

On the other hand, take any $(\bar{a},\bar{b}) \in \tilde{C}_{n+1}, (v,w) \in g_n(U_n) \bigcap U_{n+1}$, denote $(a,b) = (Dg_n^{-1})_{(v,w)}(\bar{a},\bar{b})$, by (4.15) and (4.16) we have

$$\tilde{\kappa}_{n+1}(\epsilon_n|a| + (e^{\lambda_n^s} + \epsilon_n)|b|) \geq \tilde{\kappa}_{n+1}|\bar{b}| \geq |\bar{a}| \geq (e^{\lambda_n^u} - \epsilon_n)|a| - \epsilon_n|b|$$

hence

$$|a| \leq |b|\frac{\tilde{\kappa}_{n+1}(e^{\lambda_n^s} + \epsilon_n) + \epsilon_n}{e^{\lambda_n^u} - \epsilon_n - \tilde{\kappa}_{n+1}\epsilon_n}$$

In order to prove that $(a,b) \in \tilde{C}_n$, it is enough to verify that

$$\frac{\tilde{\kappa}_{n+1}(e^{\lambda_n^s} + \epsilon_n) + \epsilon_n}{e^{\lambda_n^u} - \epsilon_n - \tilde{\kappa}_{n+1}\epsilon_n} \leq \tilde{\kappa}_n$$

that is

$$\tilde{\kappa}_{n+1} \leq \frac{\tilde{\kappa}_n e^{\lambda_n^u} - \epsilon_n\tilde{\kappa}_n - \epsilon_n}{e^{\lambda_n^s} + \epsilon_n + \epsilon_n\tilde{\kappa}_n}$$

when D is sufficiently large. This is proved in a similar way as (4.9) using (4.3), (4.5). This completes the proof of (1).

Now we prove (2) in Proposition 3.5. If Γ is κ_n-full horizontal graph of U_n, we can denote Γ as the graph of $\phi : [-r_n, r_n] \to \mathbb{R}$, such that:

(1) The Lipschitz constant of ϕ is smaller than κ_n everywhere;
(2) For any $v \in [-r_n, r_n]$, we have $(v, \phi(v)) \in U_n$.

By (4.11), we have for any $(v,w) \in U_n$

$$\begin{aligned}\frac{d\bar{v}}{dv}(v,w) &> e^{\lambda_n^u} - \epsilon_n(|v| + |w|)\\ &\geq e^{\lambda_n^u} - \epsilon_n(|v| + \kappa_n|v| + \tau_n)\\ &\geq e^{\lambda_n^u} - \epsilon_n(r_n + \kappa_n r_n + \tau_n)\\ &> 0\end{aligned}$$

when D is sufficiently large. By (4.6), for any w such that $(r_n, w) \in U_n$, if D is sufficiently large then

$$\begin{aligned}\bar{v}(r_n,w) &\geq (A_n - \epsilon_n - \epsilon_n\kappa_n)r_n - \epsilon_n\tau_n\\ &> e^{\lambda_n^u - \frac{1}{2}\delta}r_n > r_{n+1}\end{aligned}$$

Here the second inequality follows from (4.3), (4.4) and (3.40). Similar calculation shows that $\bar{v}(-r_n,w) < -r_{n+1}$ for all w such that $(-r_n,w) \in U_n$. Then $\bar{v}^{-1}$ is defined over $[-r_{n+1}, r_{n+1}]$, and we have that the image of the vertical boundary of U_n under g_n is disjoint from the vertical boundary of U_{n+1}.

Then by (4.8), we have

$$g_n(\Gamma)\bigcap U_{n+1} \text{ is contained in } U_n$$
$$\text{and disjoint from the horizontal boundary of } U_n$$

Moreover, $g_n(\Gamma)\bigcap U_{n+1}$ is the graph of function $\phi\bar{v}^{-1}$ restricted to $[-r_{n+1}, r_{n+1}]$. Since we already showed (1), we know that $g_n(\Gamma)\bigcap U_{n+1}$ is a $\kappa_{n+1}-$full horizontal graph, and the image of the horizontal boundary of U_n under g_n is disjoint from the horizontal boundary of U_{n+1}.

This completes the proof of (2). □

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