NON-DIFFERENTIABLE IRRATIONAL CURVES FOR C^1 TWIST MAP

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ABSTRACT. We construct a C^1 symplectic twist map g of the annulus that has an essential invariant curve Γ such that Γ is not differentiable and g restricted to Γ is minimal.

We denote $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the circle and $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ the annulus. An important class of area preserving maps of the annulus are the so called twist maps or maps that deviate the vertical, since this class of maps describes the behavior of area preserving surface diffeomorphisms in the neighborhood of a generic elliptic periodic point.

More precisely, a C^1 diffeomorphism g of the annulus that is isotopic to identity is a positive twist map (resp. negative twist map) if, for any given lift $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}^2$ of g, and for every $\theta \in \mathbb{R}$, the maps $r \mapsto \tilde{\pi} \circ \tilde{g}(\theta, r)$ is an increasing (resp. decreasing) diffeomorphisms. A twist map may be positive or negative. Here $\tilde{\pi}$ denotes the lift of the first projection map.

An *essential invariant curve* by a diffeomorphism g of the annulus is a homotopically non-trivial simple loop that is invariant by g.

When f is a C^1 twist map, it is known from Birkhoff theory that any invariant essential curve by g is the graph of a Lipschitz map over $\mathbb{T}$. Furthermore, it was proven by M.-C. Arnaud in [A09] that the latter map must be C^1 on a G_δ set of $\mathbb{T}$ of full measure.

Numerical experiment show that the invariant curves of a smooth twist map are actually more regular than just what the Birkhoff theory predicts. Moreover, in the perturbative setting of quasi-integrable twist maps, KAM theory provides a large measure set of smooth invariant curves. The question of the regularity of the invariant curves of twist maps is thus a natural question that is also related to the study of how the KAM curves disappear as the perturbation of integrable curves becomes large.

Another natural and related question is that of the regularity of the boundaries of Birkhoff instability zones. A Birkhoff instability zone of a twist map g is an open set of the annulus that is homeomorphic to the annulus, that does not contain any invariant essential curve, and that is maximal for these properties. By Birkhoff theory the boundary of an instability zone is an invariant curve that is a Lipschitz graph. We refer to the nice introduction of [A11] where many features and questions are discussed about the boundaries of Birkhoff instability zones.

In [EKMY98] the following question was asked.

Question 1. (J. Mather, [EKMY98, Problem 3.1.1]) Does there exist an example of a symplectic C^r twist map with an essential invariant curve that is not C^1 and that contains no periodic point?

In [H83][§III], Herman gave an example of a C^2 twist map of the annulus that has a C^1 invariant curve on which the dynamic is conjugated to the one of a Denjoy counterexample. By Denjoy theorem on topological conjugacy of C^2 circle diffeomorphisms with irrational rotation number, such a curve cannot be C^2.

In [A13], M.-C. Arnaud gave an example of a C^2 twist map g of the annulus that has an invariant curve Γ that is non-differentiable on which the dynamic is conjugated to the one of a Denjoy counterexample. In addition, she showed that in any C^1 neighborhood of g, there exist twist maps with Birkhoff instability zones having Γ for a boundary, and having the same dynamics as g on Γ.

In [A11] the following natural question is raised.

Question 2. (M.-C. Arnaud) Does there exist a regular symplectic twist map of the annulus that has an essential invariant curve that is non-differentiable on which the restricted dynamics is minimal?

In this note we give a positive answer to this question in low regularity.

Theorem 1. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exists a symplectic C^1 twist map of the annulus that has a non differentiable essential invariant curve Γ such that the restriction of g to Γ is C^0 conjugated to the circle rotation of angle α, hence minimal.*

Due to [A11, Theorem 2] we can deduce the following

Corollary 2. *There exists a symplectic C^1 twist map g of the annulus that has a non differentiable essential invariant curve Γ such that the restriction of g to Γ is minimal and such that Γ is at the boundary of an instability zone of g.*

The derivation of Corollary 2 from Theorem 1 follows from the general result of [A11], Theorem 2 asserting that any essential invariant curve of a C^1 twist map g that has an irrational rotation number, can be viewed as a boundary dynamics of Birkhoff instability zone of an arbitrarily nearby C^1 twist map. This result relies on a perturbation argument involving the Hayashi C^1-closing lemma.

Due to a construction by Herman in [H83][§II.2], Theorem 1 can be derived from the following result on circle homeomorphisms.

Theorem 3. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exists a non-differentiable orientation preserving (minimal) homeomorphism of the circle f, topologically conjugate to the circle rotation of angle α, such that $f + f^{-1}$ is of class C^1.*

Proof of Theorem 1. To prove that Theorem 3 implies Theorem 1, we use Herman's beautiful trick that associates to some circle homeomorphisms f a twist map of the annulus which preserves a circle that is determined by f and on which the restricted dynamics is given by f. For completeness, we state and prove Herman's observation.

First of all, note that if $\tilde{f}$ is a lift of a circle homeomorphism f and if there exists a C^1 function $\phi : \mathbb{T} \to \mathbb{R}$, that we also view as a function from $\mathbb{R}$ to $\mathbb{R}$ of period 1, satisfying

$$\mathrm{Id}_{\mathbb{R}} + \frac{\phi(\cdot)}{2} = \frac{1}{2}(\tilde{f}(\cdot) + \tilde{f}^{-1}(\cdot)), \tag{1}$$

then the map

$$g : \mathbb{A} \to \mathbb{A} : (\theta, r) \mapsto (\theta + r, r + \phi(\theta + r)), \tag{2}$$

is a C^1 Hamiltonian twist map of the annulus (this follows from the fact that $\int_{\mathbb{T}} \phi(\theta) d\theta = 0$ as proved in [H83][§II.2.3]). Moreover, with $\psi := \tilde{f} - id_{\mathbb{R}}$, that we view as function from $\mathbb{T}$ to $\mathbb{R}$, or from $\mathbb{R}$ to $\mathbb{R}$ of period 1, the following holds:

Claim [H83][§II.2]. *The graph Γ of $\theta \mapsto \psi(\theta)$ is invariant by g and the dynamics of g restricted to Γ is conjugated (via the first projection) to f.*

Proof. From (1), we have that

$$2\tilde{f} + \phi \circ \tilde{f} = \tilde{f} \circ \tilde{f} + \mathrm{Id}_{\mathbb{R}} = \tilde{f} + \psi \circ \tilde{f} + \mathrm{Id}_{\mathbb{R}}$$

hence

$$\psi \circ f - \psi = \phi \circ f,$$

and finally (2) implies

$$\begin{aligned} g(\theta, \psi(\theta)) &= (f(\theta), \psi(\theta) + \phi(f(\theta))) \\ &= (f(\theta), \psi(f(\theta))). \end{aligned}$$

□

The proof that Theorem 3 implies Theorem 1 is now straightforward from the claim. □

To prove Theorem 3, we will work with a special class of circle diffeomorphisms that we call C-great.

Definition 1. *Let $f : \mathbb{T} \to \mathbb{T}$ be a C^1 minimal diffeomorphism. For $C > 1$, we say that x is C-good if $\sum_{n\geq 0} |Df^n(x)|^{-2} < C$, while $\sup_{n<0} |Df^n(x)| = \infty$ and $\sum_{n<0} |Df^n(x)|^{-2} = \infty$.*

We say that f is C-great if the set of C-good points is uncountable and $\sup_{x\in\mathbb{T}} |Df(x) + Df^{-1}(x)| < C$.

Moreover, we say that x is C-great if it is accumulated by an uncountable set of C-good points. We then say that the pair (f, x) is C-great.

It is not hard to see that if f is C-great then it has a C-great point x. Moreover, a C-great point is actually accumulated by an uncountable set of C-great points.

Let $0 < u < v$. For an injective map f defined from $[x - v, x + v]$ to $\mathbb{R}$, we define

$$\Delta(f, x, u, v) = \frac{v}{u}\frac{f(x + u) - f(x)}{f(x + v) - f(x)}.$$

The following is a straightforward criterion of non-differentiability of f.

Lemma 4. *If f is a differentiable circle diffeomorphism then it must satisfy for every $x \in \mathbb{T}$ that the limit as u, v go to 0 of $\Delta(f, x, u, v)$ exists and equals 1.*

The proof of Theorem 3 is based on the following two lemmas.

Lemma 5. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exist $C > 1$ and a circle diffeomorphism f topologically conjugate to the circle rotation of angle α that is C-great.*

We will prove Lemma 5 at the end of the note.

We first show how starting from a C-great diffeomorphism we can prove Theorem 3. The proof is based on an inductive application of the following lemma.

Lemma 6. *For every $C > 1$ there exists $\epsilon_0 > 0$ with the following property. Let (f, x) be C-great. Then for every $\epsilon > 0$ there exists a C^1-diffeomorphism $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $g = h \circ f \circ h^{-1}$ satisfies :*

- *h is ϵ-close to $\mathrm{Id}_{\mathbb{T}}$ in the C^0 topology,*
- *$g + g^{-1}$ is ϵ-close to $f + f^{-1}$ in the C^1 topology,*

- *there exists $y \in \mathbb{T}$ such that $|y - x| < \epsilon$ and (g, y) is C-great, and there exists $u, v \in (0, \epsilon)$ such that $\Delta(g, y, u, v) > 1 + \epsilon_0$.*

Proof that Lemma 5 and Lemma 6 imply Theorem 3. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We want to construct a sequence of circle diffeomorphisms (f_n) that converges to a circle homeomorphism f_∞ that is topologically conjugated to R_α and such that f_∞ is not a diffeomorphism while $f_\infty + f_\infty^{-1}$ is C^1. We will use the criterion of Lemma 4 to gurantee the non-differentiability of f_∞.

In light of Lemma 5, we can start with a C^1 circle diffeomorphism f_0 and x_0 such that (f_0, x_0) is C-great and f_0 is topologically conjugate to the circle rotation of angle α. Using Lemma 6, we construct inductively, for $n \geq 1$, sequences (h_n) of circle diffeomorphisms, and (x_n) of points in $\mathbb{T}$, and (u_n) and (v_n) of positive numbers, such that for

$$f_n := h_n \circ f_{n-1} \circ h_n^{-1}, \quad F_n := f_n + f_n^{-1}, \quad H_n := h_n \circ \cdots \circ h_1,$$

the following holds

- (f_n, x_n) is C-great;
- $u_n + v_n \leq 2^{-n}$;
- $\|x_{n+1} - x_n\| \leq 2^{-n}$;
- $\Delta(f_n, x_n, u_n, v_n) > 1 + \epsilon_0$
- $\|H_n - H_{n-1}\|_{C^0} \leq 2^{-n}$;
- $\|F_{n+1} - F_n\|_{C^1} \leq 2^{-n}$.

We also ask that the convergence of (H_n) and (x_n) is sufficiently fast so that $\Delta(f_m, x_m, u_n, v_n) > 1 + \epsilon_0$ for all $m \geq n$.

From the convergence of H_n we get that f_n converges to a circle homeomorphism f_∞ that is topologically conjugated to R_α. In addition, the C^1 limit of F_n must be $f_\infty + f_\infty^{-1}$. Since $\Delta(f_m, x_m, u_n, v_n) > 1 + \epsilon_0$ for all $m \geq n$, we guarantee that the limit point x_∞ of (x_n) satisfies $\Delta(f_\infty, x_\infty, u_n, v_n) > 1 + \epsilon_0$ for every n. Hence f_∞ is not differentiable by Lemma 4.

In conclusion f_∞ satisfies all the properties of Theorem 3. □

Proof of Lemma 6. To motivate what follows, note that if $g = h \circ f \circ h^{-1}$ with $Dh(x) = 1 + e(x)$ then in order to have $g + g^{-1} = f + f^{-1}$ one must have

$$h \circ f + h \circ f^{-1} = f \circ h + f^{-1} \circ h.$$

The latter implies that

$$e(f(x))Df(x) + \frac{e(f^{-1}(x))}{Df(f^{-1}(x))} = e(x)(Df(h(x)) + \frac{1}{Df(f^{-1}(h(x))}) + \Delta,$$

with

$$\Delta = Df \circ h - Df + Df^{-1} \circ h - Df^{-1}.$$

Assuming h is C^0 close to id, we see that for $g + g^{-1}$ to be C^1-close to $f + f^{-1}$ we should choose e such that $(e(f(x)) - e(x))Df(x)Df(f^{-1}(x))$ is close to $e(x) - e(f^{-1}(x))$.

To construct the conjugacy h of Lemma 6, we start by defining a sequence $(e_j)_{j\in\mathbb{Z}}$ that will serve to define h along the orbit of a point that is C-good for f.

Let x be a C-good point for f. Let $b_j = Df(f^j(x))$, $j \in \mathbb{Z}$. Define c_j, $j \in \mathbb{Z}$ by $c_0 = 1$ and $c_j b_j b_{j-1} = c_{j-1}$.

Sublemma 7. *For any $\nu > 0$, we can define a sequence $(e_j)_{j\in\mathbb{Z}}$ such that $e_j = 0$ for $j \notin [-N', N]$ for some $N, N' \in \mathbb{N}^*$ and*

- $e_j \geq 0, \quad \forall j \in \mathbb{Z}.$
- $e_1 - e_0 = -1$
- $e_0 \leq C^3$
- $\eta_j := |(e_{j+1} - e_j)b_j b_{j-1} + (e_{j-1} - e_j)| \leq \nu, \quad \forall j \in \mathbb{Z}$

Proof. We will need the following properties on the orbit of x that will be a consequence from the fact that x is C-good for f.

Claim. *We have that $\sum_{j\geq 0} c_j \leq C^3$, and $\sum_{j\leq 0} c_j = \infty$, $\liminf_{j\leq 0} c_j = 0$.*

Proof. We have that for $j \geq 0$ $c_j = \frac{Df(x)Df(f^j(x))}{(Df^{j+1}(x))^2}$. Hence it follows from Definition 1 that $\sum_{j\geq 0} c_j \leq C^3$.

For $j \leq -1$, we have that $c_j = \frac{Df(x)}{Df(f^j(x))}\frac{1}{(Df^j(x))^2}$. Hence it follows from Definition 1 that $\sum_{j<0} c_j = \infty$ as well as $\liminf_{j\leq 0} c_j = 0$. □

Let us now define e_j, $j \in \mathbb{Z}$ as follows. Fix N large such that c_{-N} is small. Note that by taking N sufficiently large we will have that c_N is also small. We let $e_j = 0$ for $j > N$. For $|j| \leq N$, we define e_j so that $e_{j+1} - e_j = -c_j$. We now let N' be much larger than N such that $c_{-N'+1}$ is small and such that $\alpha = \frac{\sum_{-N'\leq j<-N} c_j}{e_{-N}}$ is large. We set then $e_{j+1} - e_j = \frac{c_j}{\alpha}$ for $-N' \leq j \leq -N-1$. We now define $e_j = 0$ for $j < -N'$. By construction, $e_1 - e_0 = -c_0 = -1$.

Note that

- For $j \in [-N, N]$, we have that $e_j = \sum_{k=j}^{N} c_k$.
- For $j \in [-N', -N)$, we have that $e_j = e_{-N} - \frac{1}{\alpha}\sum_{k=j}^{-N-1} c_k$.

From the definition of α we deduce that $e_{-N'} = 0$, and that $e_j \geq 0$ for all $j \in \mathbb{Z}$. Also, $-e_0 = e_{N+1} - e_0 = -\sum_{0\leq j\leq N} c_j$, so $e_0 = \sum_{0\leq j\leq N} c_j \leq C^3$.

Note that by construction, since $c_j b_j b_{j-1} = c_{j-1}$, we have that

$$\eta_j = (e_{j+1} - e_j)b_j b_{j-1} + (e_{j-1} - e_j) = 0,$$

for all $j \in \mathbb{Z}$, except for $j = -N'$, $j = -N$ and $j = N+1$. We also have, since $e_{-N'-1} = e_{-N'} = 0$, that $\eta_{-N'} = c_{-N'}/\alpha$ is small since $c_{-N'}$ is small and α is large. Next, we compute $\eta_{-N} = -c_{-N-1} - \frac{c_{-N-1}}{\alpha}$, which is also small since c_N is small and α is large. Finally $\eta_{N+1} = c_N$ is also small by our choice of N. □

We want now to modify f by conjugation along a neighborhood of the orbit of x between the times $-N'$ and N to obtain the diffeomorphism g with the required properties of Lemma 6. We will look for the conjugacy under the form $Dh(x) = 1 + e(x)$. The choice of the sequence (e_j) in Sublemma 7, would essentially allow to get a good control on g and $g + g^{-1}$ along the orbit of the point x if the function e takes the values e_j along the orbit of x. We need however to define h in the neighborhood of the orbit of x without losing the required properties on g and $g + g^{-1}$ and this will will require some additional technicalities that we now address.

To guarantee a good control of $g + g^{-1}$ everywhere, we start by slightly modifying f to make it affine along the $-N'$ to N orbit of a small interval around x. For this, choose a small interval I_0 centered around x, and let $I_j = f^j(I_0)$, $j \in \mathbb{Z}$. We may assume that $I_j \cap I_{j'} = \emptyset$ if $0 < |j - j'| \leq 2N'$. Then for every $\delta > 0$ we can define a diffeomorphism $h' : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, which is the identity outside $\cup_{-N'+1\leq j\leq N+1} I_j$, such that $\sup_y |Dh'(y) - 1| < \delta$, and such that letting $f' = h' \circ f \circ h'^{-1}$ we have $f'(y) = f^{j+1}(x) + b_j(y - f^j(x))$ whenever y is near $f^j(x)$ and $-N' \leq j \leq N$.

Let us now select an interval $I'_0 \subset I_0$ centered around x such that letting $I'_j = f'^j(I'_0)$, $j \in \mathbb{Z}$, we have that $f'|I'_j$ is affine for $-N' \leq j \leq N$.

Note that by choosing $\delta > 0$ small we get that $F' = f' + f'^{-1}$ is C^1 close to $F = f + f^{-1}$.

We now define another diffeomorphism $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which is the identity outside $\cup_{-N'+1\leq j\leq N} I'_j$ as follows. In order to specify h it is enough to define Dh on I'_j for $-N'+1 \leq j \leq N$. We let $\phi : \mathbb{R} \to [-\delta, 1]$ be a smooth function supported on $(-1/2, 1/2)$, symmetric around 0 and such that $\phi(0) = 1$ and $\int \phi(x)dx = 0$. We then let $Dh = 1 + e_j \phi \circ A_j$ where $A_j : I'_j \to [-1/2, 1/2]$ is an affine homeomorphism. If $\delta > 0$ is chosen sufficiently small h is indeed a diffeomorphism since all the e_j are positive.

Finally, let

$$g := h \circ f' \circ h^{-1} = h \circ h' \circ f \circ h'^{-1} \circ h^{-1}.$$

Claim. The diffeomorphism g satisfies the requirements of Lemma 6.

Proof. Let us show that $G = g + g^{-1}$ is C^1 close to $F' = f' + f'^{-1}$. Note that $G = F'$ in the complement of $\bigcup_{-N' \leq j \leq N} I'_j$, so it is enough to show that $DG - DF'$ is small in each I'_j, $-N' \leq j \leq N$. Indeed for $y \in I'_j$, letting $\kappa = \phi \circ A_j(y)$, we have

$$DG(h(y)) - DF'(h(y)) = \frac{\kappa}{(1+e_j\kappa)b_{j-1}}((e_{j+1}-e_j)b_jb_{j-1} + e_{j-1} - e_j),$$

which is small since the term $\frac{\kappa}{(1+e_j\kappa)b_{j-1}}$ is bounded : Indeed, $-\delta \leq \kappa \leq 1$, and $1 + e_j\kappa \geq 1 - e_j\delta \geq 1/2$ provided δ is chosen sufficiently small, and finally $b_{j-1} > C^{-1}$ since $\sup_{x\in\mathbb{T}} |Df(x) + Df^{-1}(x)| < C$.

Moreover, we have $Dg(x) - Df(x) = \frac{(1+e_1)b_0}{1+e_0} - b_0 = \frac{-b_0}{1+e_0} < -\frac{1}{C(1+C^3)}$. Since $\frac{g(x+u)-g(x)}{u} \sim Dg(x)$ for $u \ll |I_0|$, while $\frac{g(x+v)-g(x)}{v} \sim Df(x)$ for $1 \gg u \gg |I_0|$, this allows to exhibit u and v such that $\Delta(g, x, u, v) \geq 1 + \epsilon_0$ with $\epsilon_0 = \frac{1}{2C^2(1+C^3)}$.

To conclude, we must show that there exists arbitrary close to x a point y such that (g, y) is C-great. It suffices for this to show that in any interval J around x there is an uncountable set of C-good points for g. By definition of x, we know that the latter is true for f. Note that if y is C-good for f then $y' = h(h'(y))$ is C-good for g if $\sum_{n\geq 0} |Dg^n(y')|^{-2} < C$.

Fix some $\Lambda > 0$ much larger than $\sup e_j$. Notice that for small $\lambda > 0$, we can choose $m > 0$ such that there is an uncountable set $K' \subset J$ of y such that

$$\sum_{n\geq 0} |Df^n(y)|^{-2} < C - \lambda, \quad \text{and} \quad \sum_{n\geq m} |Df^n(y)|^{-2} < \lambda/\Lambda.$$

Now, notice that $h \circ h' = \mathrm{id}$ except in $\cup_{-N'+1\leq j\leq N+1} I_j$, and the derivative of $h \circ h'$ is bounded by $(1+\delta)(1+\sup e_j)$. In particular, if $y \in K'$ then $h(h'(y))$ will be C-good for g provided $g^n(y) \notin \cup_{-N'+1\leq j\leq N+1} I_j$ for $0 \leq n \leq m$. If we choose the size of I_0 sufficiently small, we will have the latter property for uncountably many $y \in K' \subset J$. □

The proof fo Lemma 6 is thus accomplished. □

To finish we still need to show the existence of C-great diffeomorphisms.

Proof of Lemma 5. Given an interval $I = [a, b]$, let $l(I) = a + \frac{3}{8}(b - a)$ and $r(I) = a + \frac{5}{8}(b - a)$.

Our construction will depend on a sequence of integers $m_n \in \mathbb{N}$, $n \geq 0$, such that m_0 is large and m_{n+1} is much larger than m_n.

Let Ω be the set of all finite sequences $\omega = (\omega_1, ..., \omega_n)$ of l's and r's and length $|\omega| \geq 0$. For $\omega \in \Omega$, we define intervals I_ω inductively as follows. Let $n = |\omega|$. If $n = 0$ we let I_ω be the interval of length 2^{-m_0} centered on 0. If $n \geq 1$, let ω' consist of ω stripped of its last digit, and let I_ω be the interval of length 2^{-m_n} centered on $t(I_{\omega'})$ where $t \in \{l, r\}$ is the last digit of ω.

Let $\phi : \mathbb{R} \to [0, 1]$ be a smooth function symmetric around 0, supported on $[-1/2, 1/2]$ and such that $\phi|[-1/4, 1/4] = 1$.

Let $A^\omega : I_\omega \to [-1/2, 1/2]$ be an affine homeomorphism. Let us now define functions smooth functions $\phi^\omega : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ supported inside the intervals $I_\omega + k\alpha$, $1 \leq k \leq 2n - 1$, such that

$$\phi^\omega(x + k\alpha) = (n - |n - k|)\phi(A^\omega(x)).$$

Note that by selecting m_n sufficiently large, we may assume the following property

(*Low recurrence*) - For any $n \in \mathbb{N}$, for all $|\omega| = |\omega'| = n$ for any $0 \leq |k| \leq 2^{2^n}$, $I_\omega + k\alpha$ does not intersect $I_{\omega'}$. In particular the supports of ϕ^ω and $\phi^{\omega'}$ do not intersect.

We let $\phi_n = \sum_{|\omega|=n} \phi_\omega$. Note that low recurrence implies for any $x \in \mathbb{R}/\mathbb{Z}$:

(L1) $|\phi_n(x + \alpha) - \phi_n(x)| \leq 1$.
(L2) $\phi_n(x - m\alpha) = 0$ if $m \geq 0$ and $2n - 1 + m \leq 2^{2^n}$.
(L3) $\phi_n(x) \leq n$.

Finally, define a non-decreasing sequence of smooth function $\Phi_N : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ by

$$\Phi_N = \sum_{0 \leq n \leq N} \frac{1}{(n+1)^{4/3}} \phi_n.$$

Define the uncountable set $K := \bigcap_n \bigcup_{|\omega|=n} I_\omega$. Note that for $x \in K$ we have for $N \geq 1$

(K1) $\Phi_N(x) = 0$.
(K2) $\Phi_N(x + N\alpha) \geq \frac{1}{10} N^{2/3}$.
(K3) For $m \in [0, 2^{2^n} - 2n + 1]$, $\Phi_N(x - m\alpha) \leq \frac{n(n+1)}{2}$.

To see the last item, just observe that if $\Phi_N(x - m\alpha) \geq \frac{n(n+1)}{2}$, then $\phi_{n'}(x - m\alpha) > 0$ for some $n \leq n' \leq N$ (since $\phi_n \leq n$ by (*L3*)), and by (*L2*) we must have $m > 2^{2^n} - 2n + 1$.

Next, observe that $(L1)$ implies that $\Phi_N(\cdot+\alpha)-\Phi_N(\cdot)$ converges in the C^0 topology to some continuous function $\Theta:\mathbb{R}/\mathbb{Z}\to\mathbb{R}$.

Introduce $\xi_N := \int e^{\Phi_N(x)}dx$ and $\Psi_N := e^{\Phi_N}/\xi_N$. Again by letting the integers m_n grow fast we obtain that ξ_N is close to 1 for all $N \geq 1$, and that if we consider the circle homeomorphism $h_n : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $Dh_n = \Psi_n$ and $h_n(0) = 0$, we get that h_n converges in the C^0 topology to some homeomorphism $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

Since $\Phi_N(x+\alpha) - \Phi_N(x)$ converges in C^0 to a continuous function Θ, we get that $f_n(x) = h_n(h_n^{-1}(x)+\alpha)$ is converging in the C^1 topology to some f satisfying $f(x) = h(h^{-1}(x)+\alpha)$ and $\ln Df = \Theta \circ h^{-1}$. In particular, f is minimal.

Note that all $x \in h(K)$ are C-good for some absolute C, since $Df^n(x) \geq e^{\frac{n^{2/3}}{10}}$ for $x \in h(K)$ by $(K2)$.

On the other hand, $(K2)$ also implies that for each x, from time to time $h^{-1}(x) - n\alpha$, for $n \geq 0$, will visit regions where $\sup_N \Phi_N$ is large, so for $x \in h(K)$, $Df^{-n}(x)$ will be large, since $\Phi_N(h^{-1}(x)) = 0$ by $(K1)$. Moreover, if $x \in h(K)$, $(K3)$ implies that $\Phi_N(h^{-1}(x) - n\alpha)$ is at most of order $(\ln\ln n)^2$, so $\sum_{n\geq 1} |Df^{-n}(x)|^{-2} = \infty$. □

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