Super-exponential stability for generic real-analytic elliptic equilibrium points

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Abstract

We consider the dynamics in a neighborhood of an elliptic equilibrium point with a Diophantine frequency of a symplectic real analytic vector field and we prove the following result of effective stability. Generically, both in a topological and measure-theoretical sense, any solution starting sufficiently close to the equilibrium point remains close to it for an interval of time which is doubly exponentially large with respect to the inverse of the distance to the equilibrium point.

1 Introduction

The aim of this paper is to study the effective stability of elliptic equilibrium points of real analytic Hamiltonian systems in n degrees of freedom, for any $n \in \mathbb{N}^*$. Our main result can be stated informally as follows. The exact definitions and statements are gathered in Section 1.6 below.

THEOREM. *If the frequency vector of the elliptic equilibrium is Diophantine, and under an additional open and dense condition of full Lebesgue measure on the coefficients of the power expansion of H at the equilibrium up to order $[\frac{n^2+4}{2}]$, the equilibrium is doubly exponentially stable.*

This result will be derived from a more general effective stability result for non-resonant elliptic equilibrium points, which crucially uses a version of Nekhoroshev estimates for steep real-analytic elliptic equilibrium points which is proved in a companion paper [BFN20]. It extends previous results of super-exponential stability that were only proved for invariant tori, and where much stronger non-generic assumptions were required. Inspired by the techniques of the current paper, we proved in the subsequent paper ([BFN17]) that an invariant Lagrangian Diophantine torus is generically doubly exponentially stable. As we will discuss in detail in the sequel, the study of the stability of tori and that of points have some intrinsic differences. Our proofs build on the idea introduced by Giorgilli and Morbidelli in [MG95] of combining averaging estimates due to Birkhoff normal forms with the Nekhoroshev geometric stability theory. Before stating the exact results and giving more explanation about the proof, let us start by describing the general setting.

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1.1 Stability of elliptic equilibrium points

We consider a symplectic manifold (M, Ω) of dimension $2n$, $n \in \mathbb{N}$, where Ω is an everywhere non-degenerate closed 2-form, a smooth symplectic vector field X on M (meaning that the one-form $i_X\Omega$ is closed, or, equivalently, that the Lie derivative $\mathcal{L}_X\Omega$ vanishes identically) and an equilibrium point $p^* \in M$, that is $X(p^*) = 0$. We are interested in studying whether p^* is *stable* in the following sense (in the sense of Lyapounov): given any neighborhood U of p^*, there exists a smaller neighborhood V of p^* such that for any point $p_0 \in V$, the unique solution $p(t)$ of X starting at p_0 (that is, the unique curve $p(t)$ satisfying $\dot{p}(t) = X(p(t))$ and $p(0) = p_0$) is defined and contained in U for all time $t \in \mathbb{R}$.

The problem being local, there are some obvious simplifications. First, by the classical theorem of Darboux, we may assume without loss of generality that $(M, \Omega) = (\mathbb{R}^{2n}, \Omega_0)$ where Ω_0 is the canonical symplectic structure of $\mathbb{R}^{2n}$, and that $p^* = 0 \in \mathbb{R}^{2n}$. Then, we may also assume that the one-form $i_X\Omega_0$ is in fact exact, meaning that X is *Hamiltonian*: given a primitive H of $i_X\Omega_0$ and letting J_0 be the canonical complex structure of $\mathbb{R}^{2n}$, the vector field can be simply written $X = X_H = J_0\nabla H$, where the gradient is taken with respect to the canonical Euclidean structure of $\mathbb{R}^{2n}$. Therefore 0 is an equilibrium point of X_H if and only if it is a critical point of H, that is $\nabla H(0) = 0$. Moreover, the Hamiltonian function H being defined only modulo a constant, it is not a restriction to impose that $H(0) = 0$.

Let $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$ be symplectic coordinates defined in a neighborhood of the origin $0 \in \mathbb{R}^{2n}$ so that $(\dot{x}(t), \dot{y}(t)) = X_H(x(t), y(t))$ is equivalent to the system

$$\dot{x}(t) = \partial_y H(x(t), y(t)), \quad \dot{y}(t) = -\partial_x H(x(t), y(t)).$$

Since $H(0) = 0$ but also $\nabla H(0) = 0$, the Taylor expansion of H at the origin is of the form

$$H(x, y) = H_2(x, y) + O_3(x, y)$$

where H_2 is the quadratic part of H at the origin and where $O_3(x, y)$ contains terms of order at least 3 in (x, y). We can now define the *linearized* Hamiltonian vector field at the origin to be the Hamiltonian vector field associated to H_2:

$$X_{H_2} = J_0\nabla H_2 = J_0 A$$

where A is the symmetric $2n \times 2n$ matrix (corresponding to the Hessian of H at the origin) such that $H_2(x, y) = A(x, y) \cdot (x, y)$. In order to study the stability of the equilibrium point, it is useful to first study its *linear stability*, that is, the stability of the origin for the linearized vector field (the latter is obviously equivalent to the boundedness of all its solutions). The matrix J_0A possesses symmetries which imply, in particular, that if λ is an eigenvalue then so is $-\lambda$. It follows that if J_0A has an eigenvalue with a non zero real part, it also has an eigenvalue with positive real part and in this case one can find solutions of the linear system that converges to infinity at an exponential rate: this implies linear instability but also instability in the sense of Lyapounov. We will say that the equilibrium point is *elliptic* if the spectrum of the matrix J_0A is both purely imaginary and simple. This implies linear stability, while linear stability is equivalent to J_0A being semi-simple and its spectrum purely imaginary (but the assumption that the spectrum is simple, which is already a non-resonance assumption, will be important for us in the sequel). Note that if we only assumed the spectrum to be purely imaginary, then, if the matrix J_0A has a non-trivial Jordan block, one can find solutions for the linearized vector field converging to infinity at a polynomial rate, implying linear instability (but not necessarily instability in the sense of Lyapounov).

So from now on, $0 \in \mathbb{R}^{2n}$ is assumed to be an elliptic equilibrium point of the Hamiltonian system defined by a smooth function H. Since the spectrum of the matrix JA is invariant by complex conjugation, it has necessarily the form $\{\pm i\alpha_1, \dots, \pm i\alpha_n\}$ for some vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with distinct components: this is usually called the *frequency vector*. By a result of linear symplectic algebra (a simple case of a theorem due to Williamson, see [AKN06]) one can find a linear symplectic map which puts the quadratic part into diagonal form (this result requires the components of α to be distinct): hence we can assume that H is of the form

$$H(x,y) = \sum_{j=1}^{n} \alpha_j (x_j^2 + y_j^2)/2 + O_3(x,y), \tag{1.1}$$

where our standing assumption from now on is that the Hamiltonian H is *real-analytic*, hence it can be extended as a holomorphic function on some complex neighborhood of the origin. Also, we will always assume that the frequency vector α is *non-resonant*, that is for any non-zero $k \in \mathbb{Z}^n$, the Euclidean scalar product $k \cdot \alpha$ is non-zero.

Note that fixing such coordinates imposes a sign on the components of the vector $\alpha \in \mathbb{R}^n$. Given a point $(x,y) \in \mathbb{R}^{2n}$, let us define $I(x,y) \in \mathbb{R}^n_+$ by

$$I(x,y) = (I_1(x_1,y_1), \dots, I_n(x_n,y_n)), \quad I_j(x_j,y_j) = (x_j^2 + y_j^2)/2, \quad 1 \le j \le n$$

so that H can be written again as

$$H(x,y) = \alpha \cdot I(x,y) + O_3(x,y) := h_1(I(x,y)) + O_3(x,y)$$

The linearized vector field, associated to $h_1(I(x,y)) = \alpha \cdot I(x,y)$, is easily integrated: given an initial condition (x_0,y_0), the corresponding solution $(x(t),y(t))$ is quasi-periodic. More precisely, letting $I_0 = I(x_0,y_0) \in \mathbb{R}^n_+$, one obviously has $I(x(t),y(t)) = I_0$ for all time $t \in \mathbb{R}$ and so the set $T(I_0) = \{(x,y) \in \mathbb{R}^{2n} \mid I(x,y) = I_0\}$ is an invariant torus, the dimension of which equals the number of strictly positive components of I_0, and on which the flow is just a flow of translation. The same holds true in fact for an arbitrary Hamiltonian depending only on the quantity $I(x,y)$, and such Hamiltonians will be called here *integrable*.

A central question in Hamiltonian dynamics is then the following.

Problem 1. *For a Hamiltonian H as in* (1.1)*, is the origin stable or unstable?*

By stable we mean Lyapunov stable in the sense that points near the origin remain in a neighborhood of the origin. Other notions of stability may also be addressed as we will see below.

1.2 Perturbation of completely integrable systems.

If H is integrable, the origin is obviously stable. Now in general H is, in a small neighborhood of the origin, a small perturbation of the integrable Hamiltonian h_1 and thus classical techniques from perturbation theory (such as KAM theory, Aubry-Mather theory, Nekhoroshev estimates or Arnold diffusion) may be used to tackle the problem. However, this setting of singular perturbation theory is quite different from the usual context of a perturbation of an integrable Hamiltonian system in action-angle coordinates, that is, a Hamiltonian of the form $h(I) + \varepsilon f(\theta, I)$, where ε is the small parameter and $(\theta, I) \in \mathbb{T}^n \times \mathbb{R}^n$.

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A first obvious difference is that for a Hamiltonian H as in (1.1), one cannot introduce action-angle coordinates on a full neighborhood of the origin: indeed, if we let $I_j = I_j(x_j, y_j)$, then the symplectic polar coordinates

$$x_j = \sqrt{2I_j}\cos\theta_j, \quad y_j = \sqrt{2I_j}\sin\theta_j, \quad 1 \leq j \leq n$$

are analytically well-defined only away from the axes $I_j = 0$. This amounts to the fact that for a Hamiltonian integrable in a neighborhood of an elliptic equilibrium point, the foliation by invariant tori is singular in the sense that the dimension of each leaf is non-constant (it varies from 0 to n), whereas in action-angle coordinates this foliation is regular.

A second difference lies in the fact that for Hamiltonians of the form $h(I) + \varepsilon f(\theta, I)$ the perturbation f is usually considered as arbitrary whereas in (1.1) the perturbation is more restricted as it is given by the higher order terms $O_3(x, y)$.

Finally, a third difference is that, under the assumption that α is non-resonant, a Hamiltonian H as in (1.1) possesses infinitely many integrable approximations h^m, for any integer $m \geq 2$ (given by the Birkhoff normal form, see below for more details) which are uniquely determined (once the vector α is fixed). This is in sharp contrast with a Hamiltonian of the form $h(I) + \varepsilon f(\theta, I)$ which does not have, in general, further integrable approximations.

As we will see below, these differences have the following general effect: in a neighborhood of an elliptic equilibrium point, as opposed to a perturbation of an integrable system in action-angle coordinates, stability properties are stronger and instability properties are harder to exhibit.

1.3 KAM stability

Due to the classical KAM (Kolmogorov-Arnold-Moser) theory, one can prove, for any number of degrees of freedom and assuming some non-degeneracy assumption (on the higher order terms $O_3(x, y)$), that the elliptic equilibrium point is *KAM stable*: in any sufficiently small neighborhood of the origin, there exist a positive measure set of Lagrangian invariant tori, on which the dynamics is conjugated to a linear flow, having the origin as a Lebesgue density point.

Related to the results that we will expose in the following sections, let us mention that it is sometimes possible to replace the non-degeneracy assumption in the study of stability by arithmetic conditions on the frequency vector α of the linear part of the flow at the equilibrium. Indeed, in the analytic setting, Herman conjectured the KAM stability (without the Lebesgue density requirement) of Diophantine equilibria without any non-degeneracy assumption. In ([Her98]) he made the following conjecture (in the slightly different context of symplectic maps).

Conjecture 1 (Herman). *Assuming that α is Diophantine, in any sufficiently small neighborhood of the origin there exists a set of positive Lebesgue measure of Lagrangian invariant tori.*

Recall that $\alpha \in \mathbb{R}^n$ is said to be Diophantine if for some constant $\gamma > 0$ and exponent $\tau \geq n-1$ it holds that $|k \cdot \alpha| \geq \gamma |k|_1^{-\tau}$ for all $k = (k_1, \dots, k_n) \in \mathbb{Z}^n \setminus \{0\}$, where $|k|_1 := |k_1| + \dots + |k_n|$. We then use the notation $\alpha \in \mathrm{DC}(\tau, \gamma)$.

Herman's conjecture is true for $n = 2$, even in the smooth category, as it was proved by Rüssmann (see for instance [Rüs02] and [FK09] in the discrete case, for respectively real-analytic and smooth maps, and [EFK13] or [EFK15, Section 7.1] in the continuous case) but unknown in general (see [EFK13, EFK15] for partial results).

Observe also that this KAM stability phenomenon without any non-degeneracy condition has no counterpart for perturbed integrable system in action-angle coordinates, since any integrable system that does not satisfy the so-called Rüssmann non-degeneracy condition can be simply perturbed so that no invariant torus survives (see [Sev03]).

1.4 Arnold's diffusion conjecture

In general, KAM stability does not have direct implications on Lyapounov stability. There are however two cases for which one knows that stability holds true for a Hamiltonian H as in (1.1).

The first case is when the quadratic part H_2 is sign-definite, or, equivalently, when the components of the vector $\alpha \in \mathbb{R}^n$ have the same sign (and this includes, as a trivial instance, the case $n = 1$). Indeed, the Hamiltonian function has then a strict minimum (or maximum) at the origin, and as this function is constant along the flow (it is in particular a Lyapounov function) one can construct, using standard arguments, a basis of neighborhoods of the origin which are invariant, and the latter property is obviously equivalent to stability.

The second case is when $n = 2$ and when the so called Arnold iso-energetic non-degeneracy condition is satisfied. Then, KAM stability occurs in every energy level passing sufficiently close to the origin, implying Lyapounov stability as the two-dimensional tori disconnect each three-dimensional energy level (see for instance [Arn61] and [Mos62]). It is easy to see that the Arnold iso-energetic non-degeneracy condition is generic in measure and topology as a function of the coefficients of the $O_4(x, y)$ part of the Taylor expansion of H around the origin. Arnold conjectured that apart from these two cases (the case of a sign-definite quadratic part, and generically for $n = 2$), an elliptic equilibrium point is generically unstable. More precisely, in [Arn94] one can find the following conjecture.

Conjecture 2 (Arnold). *An elliptic equilibrium point of a generic analytic Hamiltonian system is Lyapounov unstable, provided $n \geq 3$ and the quadratic part of the Hamiltonian function at the equilibrium point is not sign-definite.*

This conjecture is wide open, to such an extent that under our standing assumptions (real-analyticity of the Hamiltonian and a non-resonance condition on the frequency vector), it was only recently that some particular unstable examples were introduced in [Fay18]. For instance, the Birkhoff normal form in the examples of [Fay18] are not Kolmogorov non degenerate.

If the frequency vector is resonant, it is quite easy to construct an example of unstable elliptic equilibrium point (see [Mos60]). But even in this case, the genericity is still open (see [KMV04] for an announcement on some partial results).

If the Hamiltonian is smooth non-analytic, examples have been constructed by Douady-Le Calvez ([DLC83]) for $n = 3$ and by Douady ([Dou88]) for any $n \geq 3$, but there also, genericity was out of reach with their methods.

1.5 Effective stability

The aim of this paper is to investigate the so called effective stability of an elliptic equilibrium point. More precisely, given r sufficiently small and any initial condition (x_0, y_0) at a distance at most r from the origin, we are interested in the largest positive time $T(r)$ for which the solution $(x(t), y(t))$, starting at (x_0, y_0), stays at a distance at most $2r$ from the origin, for all $|t| \leq T(r)$. Arnold's conjecture states that for $n \geq 3$, it holds generically that $T(r) < \infty$.

At the moment there is no other conjectural upper bound on $T(r)$. In this paper, we will be interested in lower bound on $T(r)$. Let us first recall some previous results.

First, without any assumptions, it is easily seen from the equations of motion that $T(r)$ is at least of order r^{-1}. Then, given an integer $K \geq 4$, with the assumption that H is smooth and α is non-resonant up to order K, that is

$$k \in \mathbb{Z}^n, \quad 0 < |k|_1 \leq K \Longrightarrow k \cdot \alpha \neq 0$$

the following statement can be proved (see [Bir66] or [Dou88]): there exists a symplectic transformation Φ^K, well-defined in a neighborhood of the origin, such that

$$H \circ \Phi^K(x,y) = \alpha \cdot I(x,y) + h^m(I(x,y)) + f^K(x,y) \tag{BNF}$$

where h^m is a polynomial of degree at most $m = [K/2]$ (the integer part of $K/2$) in n variables, with vanishing constant and linear terms, and f_K is of higher order $O_{K+1}(x,y)$. The polynomial $\alpha \cdot I(x,y) + h^m(I(x,y))$ is usually called the Birkhoff normal form of H of order K. Since the term $\alpha \cdot I(x,y)$ will be fixed in the sequel we will denote $h^m(I(x,y))$ by $\mathrm{BNF}_K(H)$. The polynomial $\mathrm{BNF}_K(H)$ is uniquely defined, but, in general, this is not the case for the coordinate change function Φ^K (although there is a distinguished choice of a generating function for Φ^K). An obvious consequence of (BNF) is that, in this case, $T(r)$ is at least of order r^{-K+1} at the origin (naturally, the neighborhood in which the effective stability holds depends on K and may be very small depending in particular on the arithmetics of α). Thus if α is non-resonant and H is of class C^∞, $T(r)$ becomes larger near the origin than any power of r^{-1}. Observe that if α is non-resonant, one can find a formal symplectic transformation Φ^∞ and a unique formal series h^∞ in n variables such that $H \circ \Phi^\infty(x,y) = h^\infty(I(x,y))$. However, the formal transformation Φ^∞ is in general divergent (see [Sie41]), and the convergence problem for the formal series h^∞ is still an open problem (see [PM03] for some results).

Now with the assumption that the Hamiltonian H is real-analytic, exponentially large lower bounds for $T(r)$ have been obtained in two different contexts.

First, if α is Diophantine, $\alpha \in \mathrm{DC}(\tau,\gamma)$, one can prove that $T(r)$ is at least of order $\exp\left((\gamma r^{-1})^{\frac{1}{\tau+1}}\right)$. This is obtained by estimating the size of the remainder term f^K in the Birkhoff normal form of order K, and then choosing $K = K(r)$ as large as possible in terms of r (see [GDF+89] or [DG96] for slightly better estimates). One should point out here that actually for any non-resonant α one can associate a function $\Delta_\alpha(r)$ and prove that $T(r)$ is at least of order $\exp\left(\Delta_\alpha(r^{-1})\right)$ (see Section 1.6 below for the definition of this function Δ_α). In the Diophantine case one has $\Delta_\alpha(x) \geq (\gamma x)^{\frac{1}{\tau+1}}$ and the classical result is thus recovered.

Then, in a different direction, assuming only that α is non-resonant up to order K, for some $K \geq 4$, but requiring that the quadratic form h^2 is positive definite (which implies that $h_1 + h^2$, and then $h_1 + h^m$ for any $m \geq 2$, is convex in a neighborhood of the origin), it has been proved that $T(r)$ is at least of order $\exp\left(r^{-\frac{K-3}{2n}}\right)$: this was established independently by Niederman ([Nie98]) and Fasso-Guzzo-Benettin ([FGB98]) and later clarified by Pöschel ([Pös99]). The proof is based on the implementation of Nekhoroshev's estimates ([Nek77], [Nek79]): observe that in the absence of action-angle coordinates, this implementation is not straightforward and it was only conjectured by Nekhoroshev.

It is a remarkable fact that both exponential stability results under one of the two hypothesis : 1) α is Diophantine or 2) h^2 is positive definite, can be combined into a double exponential stability result if both 1) and 2) hold. This was first done by Giorgilli and Morbidelli in [MG95] in the context of a quasi-periodic invariant Lagrangian torus. In our context of an elliptic equilibrium, the result of [MG95] would amount to double exponential stability of a Diophantine equilibrium provided h^2 is positive definite, or more precisely that $T(r)$ is at least of order $\exp\left((\exp((\gamma r^{-1})^{\frac{1}{1+\tau}}))^{\frac{1}{2n}}\right)$. Even though the condition that h^2 is positive definite is open, it is far from being generic in any sense and recently some efforts have been made to improve this result, especially in [Bou11] and [Nie13]. In [Bou11], using results from [Nie07] and [BN12], it was proved that under a certain condition on the formal Birkhoff series h_∞, the double exponential stability holds true. This condition, which includes the condition that h^2 is positive definite as a particular case, was proved to be prevalent (a possible generalization of "full measure" in infinite dimensional spaces) in the space of all formal series. This result has at least two drawbacks. First, although this condition can be termed generic in a measure-theoretical sense, it is far from being generic in a topological sense. Secondly, this condition was only formulated in the space of formal series, and it was unclear whether prevalent Hamiltonians have formal Birkhoff series satisfying this condition. This second issue was partially solved in [Nie13]: it is proved there that a prevalent Hamiltonian has a formal Birkhoff series satisfying a condition close to the one introduced in [Bou11], yielding a result which is only intermediate between exponential and double exponential stability.

The aim of this paper is to improve those results by establishing that generically, and in a strong sense, the double exponential stability holds true.

1.6 Main results

We start by some reminders and notations that will be useful in our statements. Let H be a real analytic Hamiltonian on $\mathbb{R}^{2n}$ having an elliptic equilibrium point at the origin with a non-resonant frequency vector α, that is H is as in (1.1).

- For vectors in $\mathbb{C}^{2n}$, $\|\,.\,\|$ denotes the norm defined as

$$\|z\| := \max_{1\le j\le n} \sqrt{|z_j|^2 + |z_{n+j}|^2}, \quad z = (z_1, \dots, z_n, z_{n+1}, \dots, z_{2n}) \tag{1.2}$$

  and for vectors in $\mathbb{C}^n$, $\|\,.\,\|$ denotes the usual Euclidean norm

$$\|I\| := \sqrt{|I_1|^2 + \cdots + |I_n|^2}, \quad I = (I_1, \dots, I_n,). \tag{1.3}$$

  It will be more convenient to use these different norms for vectors in $\mathbb{C}^{2n}$ or in $\mathbb{C}^n$, and we hope that this abuse of notations will not confuse the reader.

- We suppose that the radius of convergence of H is strictly larger than some $R > 0$ and let $\|H\|_R$ be the sup norm of H in the open complex ball in $\mathbb{C}^{2n}$ centered at the origin of radius R that we denote by

$$\mathcal{B}_R := \{z \in \mathbb{C}^{2n} \mid \|z\| < R\}. \tag{1.4}$$

  We also define the real ball $B_R := \mathcal{B}_R \cap \mathbb{R}^{2n}$.

- We denote by $P(n, m)$ the set of polynomials of degree at most m in n variables. We let $P_2(n, m) \subset P(n, m)$ be the subspace of polynomials with a vanishing affine part, and $P_3(n, m) \subset P(n, m)$ the subset of polynomials that have a vanishing affine and quadratic part.

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- We denote by $\tilde{H}_m \in P_3(2n, m)$ the part of the power expansion of H that contains the terms of degree between 3 and m included.
- Having fixed the number of degrees of freedom n, in all the sequel, we let

$$K_0 = K_0(n) := n^2 + 4, \quad m_0 = m_0(n) := [K_0(n)/2].$$

- The vector α is supposed to be non-resonant: this means that for any integer $K \geq 1$,

$$\Psi_\alpha(K) = \max\{|k \cdot \alpha|^{-1} \mid k \in \mathbb{Z}^n,\ 0 < |k|_1 = |k_1| + \cdots |k_n| \leq K\} < +\infty. \tag{1.5}$$

We define, as in [Bou12], the function

$$\Delta_\alpha(x) = \sup\{K \geq 1 \mid K\Psi_\alpha(K) \leq x\}.$$

Observe that if $\alpha \in \mathrm{DC}(\tau, \gamma)$, then $\Psi_\alpha(K) \leq \gamma^{-1}K^\tau$ and hence

$$\Delta_\alpha(x) \geq (\gamma x)^{\frac{1}{1+\tau}} \tag{1.6}$$

- Recall that for H as in (1.1), there exists for every integer $K \geq 4$ a real analytic symplectic transformation Φ^K defined in the neighborhood of the origin such that

$$H \circ \Phi^K(x, y) = \alpha \cdot I(x, y) + h^m(I(x, y)) + f^K(x, y)$$

where h^m is a polynomial of degree $m = [K/2]$ (the integer part of $K/2$) in n variables, with vanishing constant and linear terms, and f_K is of higher order $O_{K+1}(x, y)$. We denoted h^m by $\mathrm{BNF}_K(H)$. By uniqueness of the Birkhoff normal form we have a well defined map

$$\begin{array}{rll} \mathrm{BNF}_K : & P_3(2n, K) & \longrightarrow P_2(n, m) \\ & \tilde{H}_K & \longmapsto h^m = \mathrm{BNF}_K(\tilde{H}_K) = \mathrm{BNF}_K(H). \end{array}$$

Our main result is the following.

Theorem A. *Let H be a real analytic Hamiltonian on $\mathbb{R}^{2n}$ having an elliptic equilibrium point at the origin with a non-resonant frequency vector α. There exists an open and dense set of full Lebesgue measure $\mathcal{N}_n(\alpha) \in P_3(2n, K_0)$ such that if $\tilde{H}_{K_0} \in \mathcal{N}_n(\alpha)$, then there exists $r^*, c, c', c'' > 0$ that depend only on $n, R, \|H\|_R, \alpha$ and $\tilde{H}_{K_0}$ such that if $r \leq r^*$, then*

$$T(r) \geq \exp\left(cr^{-2}\exp\left(c'\Delta_\alpha\left(c''r^{-1}\right)\right)\right).$$

If $\alpha \in \mathrm{DC}(\tau, \gamma)$, there exists an open and dense set of full Lebesgue measure $\mathcal{N}_n(\alpha) \in P_3(2n, K_0)$ such that if $\tilde{H}_{K_0} \in \mathcal{N}_n(\alpha)$, then there exists r^* and C that depend only on $n, R, \|H\|_R, \alpha$, and $\tilde{H}_{K_0}$ such that if $r \leq r^*$, then

$$T(r) \geq \exp\left(\exp\left(Cr^{-\frac{1}{\tau+1}}\right)\right).$$

We will see that the second part of the statement is a direct consequence of the first part. Since c' and c'' will not depend on α (see (3.8)), it follows from (1.6) that the constant C that appears under the double exponential in the Diophantine case is actually of the form $C = \gamma^{\frac{1}{\tau+1}}C'$ where C' does not depend on α. Theorem A improves all previous results contained in [MG95], [Bou11] and [Nie13]. In the course of its proof, we will also have to extend the results on exponential stability contained in [Nie98], [FGB98] and [Pös99].

Remark 1.1. Observe that even though $\Delta_\alpha(r^{-1})$ goes to infinity as r goes to zero, the speed of convergence can be arbitrarily slow but the statement implies that $T(r)$ is always at least of order $\exp(cr^{-2})$. From the proof of the theorem, one can easily obtain the following statement: fixing $k \in \mathbb{N}^*$, $k \geq 2$, and allowing the constants r_k^* and c_k to depend also on k, one has

$$T(r) \geq \exp\left(c_k r^{-k} \exp\left(c' \Delta_\alpha\left(c'' r^{-1}\right)\right)\right)$$

which is always at least of order $\exp(c_k r^{-k})$. As a matter of fact, the weaker estimate

$$T(r) \geq \exp\left(c_k r^{-k}\right)$$

can be obtained if one only assumes α to be non-resonant up to a sufficiently high order depending on k and n.

Remark 1.2. The Diophantine condition $\alpha \in \mathrm{DC}(\tau, \gamma)$ is sometimes called an asymptotic Diophantine condition. A strictly weaker condition, called uniform Diophantine condition, requires the existence of an increasing sequence $K_j \in \mathbb{N}$, $K_j \to \infty$, such that $|k \cdot \alpha| \geq \gamma K_j^{-\tau}$ for every $k \in \mathbb{Z}^n \setminus \{0\}$ with $|k|_1 \leq K_j$. Under that weaker condition, we have $\Psi_\alpha(K_j) \leq \gamma K_j^\tau$ and Theorem A then implies that there exists a sequence $r_j \to 0$ such that

$$T(r_j) \geq \exp\left(\exp\left(C r_j^{-\frac{1}{\tau+1}}\right)\right).$$

The notion of *stably steep* polynomials, which can be implicitly found in the work of Nekhoroshev ([Nek73]), will be important in the proof of Theorem A.

Definition 1 (Stably steep polynomials). *A polynomial $P_0 \in P_2(n,m)$ is called stably steep if there exist a neighborhood V of P_0 in $P_2(n,m)$ and positive constants C, δ such that for any integer $l \in [1, n-1]$, any $P \in V$ and any vector subspace $\Lambda \subseteq \mathbb{R}^n$ of dimension l, letting P_Λ be the restriction of P to Λ, the inequality*

$$\max_{0 \leq \eta \leq \xi} \; \min_{||x|| = \eta,\, x \in \Lambda} ||\nabla P_\Lambda(x)|| > C \xi^{m-1}$$

holds true for all $0 < \xi \leq \delta$, where $||\,.\,||$ is the usual Euclidean norm defined in (1.3).

The set of stably steep polynomials in $P_2(n,m)$ will be denoted by $SS(n,m)$.

Theorem A will clearly follow from the combination of the following two statements, Theorems B and C, with the set $\mathcal{N}_n(\alpha)$ being defined as $\mathcal{N}_n(\alpha) := \mathrm{BNF}_{K_0}^{-1}(SS(n, m_0))$.

Our first statement is that the set of Hamiltonians with stably steep BNF of order K_0 have doubly exponentially stable equilibria.

Theorem B. *Let H be a real analytic Hamiltonian on $\mathbb{R}^{2n}$ having an elliptic equilibrium point at the origin with a non-resonant frequency vector α. If*

$$\mathrm{BNF}_{K_0}(H) = h^{m_0} \in SS(n, m_0)$$

then the conclusions of Theorem A hold.

The second statement shows that the condition $\mathrm{BNF}_{K_0}(H) = \mathrm{BNF}_{K_0}(\tilde{H}_{K_0}) \in SS(n, m_0)$ is generic in a strong sense.

Theorem C. *For any non-resonant $\alpha \in \mathbb{R}^n$, the complement of* $\mathrm{BNF}_{K_0}^{-1}(SS(n,m_0))$ *in* $P_3(2n,K_0)$ *is contained in a semi-algebraic subset of positive codimension. In particular,* $\mathrm{BNF}_{K_0}^{-1}(SS(n,m_0))$ *is a dense open subset of* $P_3(2n,K_0)$ *of full Lebesgue measure.*

Proof of Theorem A. Putting together Theorem B and C immediately yields Theorem A if we take $\mathcal{N}_n(\alpha) = \mathrm{BNF}_{K_0}^{-1}(SS(n,m_0))$. □

To prove Theorem C, we will show that the complement of $SS(n,m_0)$ in $P_2(n,m_0)$ is contained in a semi-algebraic subset of codimension at least one. This will be done in Sections 3.1, 3.2 and Appendix A.

Theorem B will follow (see Section 3.4) from a version of the Nekhoroshev exponential stability result adapted to our singular perturbation setting, which will be stated in Section 2 and proved in [BFN20].

1.7 Comments, open questions and prospects

It is natural to ask whether our main result, Theorem A, can be improved, and so we can ask the following two questions.

Question 1. *What can one say about the effective stability of an equilibrium as in* (1.1) *if no assumption is made on the Birkhoff normal form?*

As mentioned earlier, the sole fact that $\alpha \in \mathrm{DC}(\tau,\gamma)$, implies that $T(r)$ is at least of order $\exp\left((\gamma r^{-1})^{\frac{1}{\tau+1}}\right)$ (see [GDF+89] or [DG96] for slightly better estimates). In [FF19], for $d \geq 3$ and any $\tau > d$ and any $\varepsilon > 0$, a real analytic Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ was constructed that has an invariant quasi-periodic Lagrangian torus with frequency $\alpha \in \mathrm{DC}(\tau)$, for which the diffusion time satisfies $T(r) \leq \frac{1}{r}\exp\left(Cr^{-\frac{1}{\tau+1-\varepsilon}}\right)$. No such example is known to exist for Diophantine equilibria.

Question 2. *Is the estimate on the time $T(r)$ in Theorem A essentially optimal?*

A main difficulty in these questions is related to the fact that the construction of an unstable elliptic equilibrium point in the analytic category is a wide open problem as we emphasized in the Introduction. Concerning the second question, let us just mention that it may be possible to give an answer in the Gevrey category (a regularity which is intermediate between smooth and analytic). Indeed, on the one hand, one should expect that the statement of Theorem A holds true for Gevrey Hamiltonians, with only different constants. On the other hand, elaborating on the methods of [Her83] and [MS02], an example showing the optimality of double exponential stability for invariant tori in Gevrey class and in the quasi-convex case, was announced in [FD18].

Moreover, it is also natural to ask whether our result holds true for a quasi-periodic invariant Lagrangian torus, or more generally, for a quasi-periodic normally elliptic and reducible invariant torus (which includes both elliptic equilibrium points and quasi-periodic invariant Lagrangian tori as particular cases). This general case is described by a Hamiltonian of the form

$$H(\theta, J, x, y) = \beta \cdot J + \alpha \cdot I(x,y) + F(\theta, I, x, y)$$

where $(\theta, J) \in \mathbb{T}^m \times \mathbb{R}^m$ are action-angle coordinates, (x,y) symplectic coordinates around the origin in $\mathbb{R}^{2n}$ and F is at least of order 2 in I and 3 in (x,y). The set $\{(J,x,y) \mid J =$

$0, I(x,y) = 0\}$ is a normally elliptic torus of dimension n in a $n+m$ degrees of freedom Hamiltonian, and the question is as follows.

Question 3. *Assuming that the vector $(\beta, \alpha) \in \mathbb{R}^{m+n}$ is Diophantine, does the conclusion of Theorem A hold for a generic real-analytic function F: if $(J(0), I(x(0), y(0)))$ is at a distance r of zero in $\mathbb{R}^{n+m}$, with r sufficiently small, is it true that $(J(t), I(x(t), y(t)))$ stays at a distance $2r$ from 0 for a time $T(r)$ which is doubly exponentially large with respect to $r^{-1/(\tau+1)}$ (where τ is the exponent of the Diophantine condition on the vector (β, α))?*

In a subsequent paper ([BFN17]), we answered positively the above question in the case of an invariant Lagrangian Diophantine torus, which is of particular interest in the study of perturbed integrable systems. Indeed, by KAM theory, it is well-known that invariant Lagrangian Diophantine tori appear for arbitrary small perturbations of generic integrable Hamiltonian systems in action-angle coordinates. Furthermore, these tori are not isolated and appear as a family parametrized by some Cantor set of positive Lebesgue measure (tending to full measure as the size of the perturbation goes to zero). The goal of [BFN17] is to prove that under an additional generic assumption on the integrable Hamiltonian, most of the KAM tori are doubly exponentially stable.

2 Nekhoroshev exponential stability for a steep elliptic equilibrium

To state precisely the result, let us introduce further notations.

- For vectors in $\mathbb{C}^n$, it will be convenient to also use the sup norm $|\,.\,|$ defined as

$$|I| := \max\{|I_1|, \ldots, |I_n|\}, \quad I = (I_1, \ldots, I_n). \tag{2.1}$$

  This norm allows an easier comparison between $I(z) \in \mathbb{C}^n$ and $z \in \mathbb{C}^{2n}$: indeed, we have $|I(z)| \leq \|z\|^2/2$ and the equality holds true if $z \in \mathbb{R}^{2n}$.

- Given $r > 0$, we define the domain $\mathcal{D}_r$ to be the open ball centered at the origin in $\mathbb{C}^n$ of radius $r^2/2$ with respect to the norm $|\,.\,|$:

$$\mathcal{D}_r := \{I \in \mathbb{C}^n \mid |I| < r^2/2\}$$

  and we let $D_r := \mathcal{D}_r \cap \mathbb{R}^n$. This choice is motivated by the fact that if $I : z \in \mathbb{C}^{2n} \mapsto I(z) \in \mathbb{C}^n$, then $I(\mathcal{B}_r) \subseteq \mathcal{D}_r$ and $I(B_r) = D_r \cap \mathbb{R}^n_+$, where $\mathcal{B}_r$ and B_r have been defined in (1.4).

- We define $\|\,.\,\|_r$ to be the sup norm for functions defined on $\mathcal{B}_r$ or on $\mathcal{D}_r$. Extending the norm $\|\,.\,\|$ initially defined for vectors in $\mathbb{C}^n$ and $\mathbb{C}^{2n}$ (respectively in (1.2) and in (1.3)) to tensors in $\mathbb{C}^n$ and $\mathbb{C}^{2n}$, we extend the sup norm $\|\,.\,\|_r$ for tensor-valued functions defined on $\mathcal{B}_r$ or on $\mathcal{D}_r$. As an example, given $h : \mathcal{D}_r \to \mathbb{C}$, we have

$$\nabla h : \mathcal{D}_r \to \mathbb{C}^n, \quad \nabla^2 h : \mathcal{D}_r \to L(\mathbb{C}^n, \mathbb{C}^n)$$

  where $L(\mathbb{C}^n, \mathbb{C}^n)$ is the space of $\mathbb{C}$-linear maps from $\mathbb{C}^n$ to itself, and thus

$$||\nabla h||_r = \sup_{z \in \mathcal{D}_r} ||\nabla h(z)||, \quad ||\nabla^2 h||_r = \sup_{z \in \mathcal{D}_r} \left(\sup_{v \in \mathbb{C}^n,\, ||v||=1} ||\nabla^2 h(z).v|| \right).$$

The same notation $\|\,.\,\|_r$ will be used also for the real domains B_r and D_r: this will not cause confusion as it will be clear from the context if it is the complex or the real domains that are considered.

- We consider a Hamiltonian H of the form

$$H(z) = h(I(z)) + f(z), \quad h : \mathcal{D}_r \to \mathbb{C}, \quad f : \mathcal{B}_r \to \mathbb{C} \tag{$*$}$$

which is real analytic and such that

$$\|\nabla h\|_r \leq E, \quad \|\nabla^2 h\|_r \leq F, \quad \|X_f\|_r \leq \varepsilon \tag{2.2}$$

where X_f is the Hamiltonian vector field associated to f.

- The integrable Hamiltonian h is supposed to be *steep* on the domain D_r, as defined below.

Definition 2. *A differentiable function $h : D_r \to \mathbb{R}$ is steep if there exist positive constants C, δ, p_l, for any integer $l \in [1, n-1]$, and κ such that for all $I \in D_r$, we have $||\nabla h(I)|| \geq \kappa$ and, for all integer $l \in [1, n-1]$, for all vector space $\Lambda \in \mathbb{R}^n$ of dimension l, letting $\lambda = I + \Lambda$ the associated affine subspace passing through I and h_λ the restriction of h to λ, the inequality*

$$\max_{0 \leq \eta \leq \xi} \min_{||I'-I||=\eta,\, I' \in \lambda \cap D_r} ||\nabla h_\lambda(I') - \nabla h_\lambda(I)|| > C\xi^{p_l}$$

holds true for all $0 < \xi \leq \delta$. We say that h is $(r, \kappa, C, \delta, (p_l)_{l=1,\ldots,n-1})$-steep and, if all the $p_i = p$, we say that h is $(r, \kappa, C, \delta, p)$-steep.

Let us point out that the definition of steepness that we use is not exactly the one given by Nekhoroshev but it is obviously equivalent to it (see [Nek73] or [Nek77]). Indeed, Nekhoroshev only requires steepness for subspaces Λ which are orthogonal to $\nabla h(I)$, in which case $\nabla h_\lambda(I) = 0$; for subspaces Λ such that $\nabla h_\lambda(I) \neq 0$, the inequality in Definition 2 is clearly satisfied (and one may even set $p_l = 0$ in this case). Here's the main result of [BFN20].

Theorem 2.1 (Bounemoura-Fayad-Niederman)**.** *Let $H(z) = h(I(z)) + f(z)$ be as in $(*)$ satisfying (2.2), such that h is $(r, \kappa, C, \delta, (p_l)_{l=1,\ldots,n-1})$-steep. Then there exist $\tilde{r}^*, \tilde{c}, \tilde{c}' > 0$, which depend only on n, E, F, κ, C and p_l for $1 \leq l \leq n-1$ such that if*

$$r \leq \tilde{r}^*, \quad r\varepsilon \leq \tilde{c} \min\left\{\delta^{2na}, r^{4na}\right\} \tag{2.3}$$

where

$$a := 1 + p_1 + p_1p_2 + \cdots + p_1p_2 \ldots p_{n-1},$$

then for any solution $z(t)$ of the Hamiltonian flow $(*)$ with $z(0) = z_0 \in B_{r/2}$ we have

$$|I(z(t)) - I(z_0)| \leq \tilde{c}'(r\varepsilon)^{\frac{1}{2na}}, \quad |t| \leq \exp\left((r\varepsilon)^{-\frac{1}{2na}}\right).$$

3 Genericity of steepness and Birkhoff normal forms

The aim of this section is to give a proof of Theorem B and Theorem C; we recall that this two statements together imply our main result Theorem A. The proof of Theorem C will be given in Section 3.2, while the proof of Theorem B will be given in Section 3.4 and will use Theorem 2.1 (which was just stated above).

3.1 Genericity of steepness

In Appendix A we will prove a general result on genericity of stably steep polynomials.

Theorem 3.1. *The complement of $SS(n, m_0)$ in $P_2(n, m_0)$ is contained in a semi-algebraic subset $\Upsilon(n, m_0)$ of codimension at least one.*

Theorem 3.1 has an immediate consequence on the genericity of steep functions as will be shown in the following Theorem 3.2.

Given $p \in \mathbb{N}$, $p \geq 3$ and $\rho > 0$, let $\mathcal{C}^p(D_\rho)$ be the set of functions p times continuously differentiable on D_ρ, and assume

$$||\nabla^2 h||_{p,\rho} = \max_{2 \leq j \leq p} \|\nabla^j h\|_\rho < \infty$$

where $\| \, . \, \|_\rho$ is the sup norm on D_ρ of the tensor-valued function $\nabla^j h$, and where we recall that by definition, D_ρ is the (real) open ball of radius $\rho^2/2$ with respect to the sup norm $| \, . \, |$. Given $h \in \mathcal{C}^p(D_\rho)$, we denote by $T_{p-1}h(I) \in P(n, p-1)$ the Taylor expansion of h of order $p-1$ at $I \in D_\rho$ (or the $p-1$-jet at I). We have the following statement (that will be used later with the value $p = m_0 + 1$).

Theorem 3.2. *Let $h \in \mathcal{C}^p(D_\rho)$ be such that $\|\nabla h(0)\| := \varpi$ and $P_{p-1} := T_{p-1}h(0) - T_1 h(0) - T_0 h(0) \in SS(n, p-1)$. Then, there exists positive numbers μ^*, δ^* and C that depend only on ϖ, P_{p-1}, $||\nabla^2 h||_{p,\rho}$ and n such that h is $(\mu, \kappa, C, \delta, p-2)$-steep, with*

$$\mu := \min\{\rho/2, \mu^*\}, \quad \kappa := \varpi/2, \quad \delta := \min\{\rho^2/4, \delta^*\}.$$

Proof of Theorem 3.2. Let $M := ||\nabla^2 h||_{p,\rho}$. Observe first that if $\mu^2 \leq \varpi/M$ then the condition $\|\nabla h(I)\| \geq \kappa = \varpi/2$ is satisfied for any $I \in D_\mu$.

Fix an arbitrary $I \in D_\mu$, and define $\mathcal{H}_I = T_{p-1}h(I) - T_1 h(I) - T_0 h(I) \in P_2(n, p-1)$. Since $\mathcal{H}_0 = T_{p-1}h(0) - T_1 h(0) - T_0 h(I) = P_{p-1}$ is stably steep, we have the existence of $\tilde{\mu}$ that depends on M, P_{p-1}, and n such that if $\mu \leq \tilde{\mu}$, $\mathcal{H}_I$ is sufficiently close to P_{p-1} so that for all integer $l \in [1, n-1]$, for all vector subspace $\Lambda \subseteq \mathbb{R}^n$ of dimension l, letting $\mathcal{H}_{I,\Lambda}$ be the restriction of $\mathcal{H}_I$ to Λ, the inequality

$$\max_{0 \leq \eta \leq \xi} \min_{||x|| = \eta, \, x \in \Lambda} ||\nabla \mathcal{H}_{I,\Lambda}(x)|| > C_0 \xi^{p-2}$$

holds true for all $0 < \xi \leq \delta_0$, where δ_0 and C_0 are the steepness constant related to P_{p-1}.

Now, we get by the Taylor formula (applied to ∇h at the order $p-1$) that

$$\|\nabla h(I + x) - \nabla h(I) - \nabla \mathcal{H}_I(x)\| \leq M(p-1)! \|x\|^{p-1}$$

provided $I + x \in D_\rho$, which is satisfied if $\mu \leq \rho/2$ and $|x| \leq \|x\| \leq \rho^2/4$. So for $\|x\| \leq \xi \leq \delta$, with $\delta := \min\{C_0(2M(p-1)!)^{-1}, \rho^2/4\}$, we have

$$\|\nabla h(I + x) - \nabla h(I) - \nabla \mathcal{H}_I(x)\| \leq (C_0/2)\xi^{p-2}$$

and then, letting $\lambda = I + \Lambda$,

$$\|\nabla h_\lambda(I + x) - \nabla h_\lambda(I) - \nabla \mathcal{H}_{I,\Lambda}(x)\| \leq (C_0/2)\xi^{p-2}.$$

From this we eventually obtain

$$\max_{0 \leq \eta \leq \xi} \min_{||x|| = \eta, \, x \in \Lambda} ||\nabla h_\lambda(I + x) - \nabla h_\lambda(I)|| > (C_0/2)\xi^{p-2}$$

and letting $I' = I + x$, $C := C_0/2$, $\delta^* := C_0(2M(p-1)!)^{-1}$ and $\mu^* := \min\{\tilde{\mu}, \sqrt{\varpi/M}\}$, the steepness of f is thus established with the constants given in the statement. □

3.2 Generic steepness of the BNF.

The proof of Theorem C will be an easy consequence of Theorem 3.1 and the following two lemmas on the map BNF_K.

Lemma 3.3. *The map* BNF_K *is algebraic.*

Proof. This follows by construction of the Birkhoff normal form, and we refer to [PM03] for more details. □

Now given a polynomial $Q = Q_2 + \cdots + Q_m \in P_2(n,m)$, where each Q_j is homogeneous of degree j, we can define a polynomial $\tilde{Q} \in P_3(2n,K)$ by setting $\tilde{Q}(\xi) := Q(I(\xi))$. For $K \geq 4$, we can define a map by

$$\begin{aligned} F^K : \quad & P_2(n,m) && \longrightarrow P_2(n,m) \\ & Q && \longmapsto \mathrm{BNF}_K(\tilde{H}_K + \tilde{Q}). \end{aligned}$$

Lemma 3.4. *The map* F^K *preserves Lebesgue measure.*

Proof. This also follows by construction of the Birkhoff normal form. More precisely, it can be shown that decomposing the map F^K as $F^K = (F_2^K, \dots, F_m^K)$, where F_j^K is the component with respect to homogeneous polynomials of degree j, then we have $F_2^K(Q) = Q_2 + \mathrm{BNF}^4(\tilde{H}_4) = Q_2 + h^2$, and for $3 \leq j \leq K$, we have $F_j^K(Q) = Q_j + \mathcal{F}_j^K(\tilde{H}_{2j}, Q_2, \dots, Q_{j-1})$ where $\mathcal{F}_j^K$ is an algebraic map (see [Nie13], where this property has already been used). This expression clearly implies that F^K is smooth with Jacobian one, therefore it preserves Lebesgue measure. □

Proof of Theorem C. Our aim is to show that the complement of $\mathrm{BNF}_{K_0}^{-1}(SS(n,m_0))$ in $P_3(2n,K_0)$ is contained in a semi-algebraic subset of positive codimension. Since the inverse image of a semi-algebraic subset by an algebraic map is semi-algebraic, from Theorem 3.1 and Lemma 3.3, it follows that the complement of $\mathrm{BNF}_{K_0}^{-1}(SS(n,m_0))$ in $P_3(2n,K_0)$ is contained in a semi-algebraic subset. It remains to prove that this set has positive codimension, or equivalently, zero Lebesgue measure in $P_3(2n,K_0)$. By Lemma 3.3, for any $\tilde{H}_{K_0} \in P_3(2n,K_0)$, the Lebesgue measure in $P_2(n,m_0)$ of the set

$$\{Q \in P_2(n,m_0) \mid \tilde{H}_{K_0} + \tilde{Q} \notin \mathrm{BNF}_{K_0}^{-1}(SS(n,m_0))\}$$

is zero. By Fubini-Tonelli theorem, this implies that the complement of $\mathrm{BNF}_{K_0}^{-1}(SS(n,m_0))$ in $P_3(2n,K_0)$ has zero Lebesgue measure, and this concludes the proof. □

3.3 Birkhoff normal forms with estimates

For a real analytic Hamiltonian with an elliptic equilibrium point, as in (1.1), it is known that the estimates on the Birkhoff normal form are given by the arithmetic properties of α and the analytic norm of H. We summarize in the following Proposition 3.5 the estimates on the BNF that will be useful for us in the sequel. The proof of Proposition 3.5 is relatively standard, we include it in Appendix B following [DG96].

Here it will be more convenient to perform a linear change of complex canonical coordinates $z = S(\xi)$, where $S : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is defined by

$$z_j = \frac{1}{\sqrt{2}}(\xi_j + i\xi_{n+j}), \quad z_{n+j} = \frac{i}{\sqrt{2}}(\xi_j - i\xi_{n+j}).$$

It is easy to check that this linear transformation S and its inverse S^{-1} have unit norm (with respect to the norm $||\,.\,||$ defined in (1.2)), hence H and $H \circ S$ have the same radius of convergence around the origin and $||H \circ S||_R = ||H||_R$. Abusing notations, we will still write H instead of $H \circ S$ to denote the Hamiltonian in these new coordinates. Observe that

$$H_2(\xi) = h_1(I(\xi)) = \alpha \cdot I(\xi) = i \sum_{j=1}^{n} \alpha_j \xi_j \xi_{n+j}$$

where

$$I(\xi) = (I_1(\xi), \dots, I_n(\xi)), \quad I_j(\xi) = i\xi_j \xi_{n+j}, \quad 1 \le j \le n.$$

Recall the definition of Ψ_α given in Section 1.6 and define also for any integer $j \ge 3$,

$$\psi_\alpha^j := \prod_{i=3}^{j} \Psi_\alpha(i).$$

For $K \ge 1$, define

$$\rho_K := (548ncdK\Psi(K))^{-1}, \tag{3.1}$$

where the positive constants c and d depend only on n, R and $||H||_R$ and are defined in (B.3).

Proposition 3.5. *Let H be as in (1.1) with α as in (1.5), and fix integers $p \ge 2$, $K \ge 2p$ and $0 \le q \le K - 4$. There exist constants $b(p)$ and $\tilde{b}(q)$ that depend respectively on $p, n, R, \|H\|_R, \psi_\alpha^{2p-1}$ and on $q, n, R, \|H\|_R, \psi_\alpha^{q+2}$ such that if we assume*

$$0 < \rho \le \rho_K/e, \tag{3.2}$$

then there exists a real-analytic symplectic transformation $\Phi^K = \mathrm{Id} + O(\xi^2)$ defined on $\mathcal{B}_{\rho_K}$ such that

$$H \circ \Phi^K(\xi) = \alpha \cdot I(\xi) + h^m(I(\xi)) + f^K(\xi),$$

with $f^K = O(\xi^{K+1})$ and the following estimates hold

$$||\nabla^2 h^m||_{p,\rho} = \max_{2 \le j \le p} ||\nabla^j h^m||_\rho \le b(p) \tag{3.3}$$

$$||\nabla f^K||_\rho \le \tilde{b}(q) \rho^q e^{-K}. \tag{3.4}$$

3.4 From Nekhoroshev stability to double exponential stability

In this section we prove that Theorem 2.1 implies Theorem B. As a corollary of Proposition 3.5 and Theorem 3.2 we get the following

Proposition 3.6. *Let H be as in (1.1) with α as in (1.5), and such that*

$$\mathrm{BNF}_{K_0}(\tilde{H}_{K_0}) = h^{m_0} \in SS(n, m_0).$$

There exists $C > 0$ and $K^* \ge 4$ that depend only on n, R, $\|H\|_R$, h_{m_0}, $||\alpha||$ and $\psi_\alpha^{2m_0+1}$ such that if

$$K \ge K^*, \quad 0 < \rho \le \rho_K/e,$$

then there exists a real-analytic symplectic transformation $\Phi^K = \mathrm{Id} + O(\xi^2)$ *defined on* $\mathcal{B}_{\rho_K}$ *such that*

$$H \circ \Phi^K(\xi) = \alpha \cdot I(\xi) + h^m(I(\xi)) + f^K(\xi) := h(I(\xi)) + f^K(\xi) \tag{3.5}$$

with $f^K = O(\xi^{K+1})$ *and*

$$||\nabla^2 h^m||_{m_0+1,\rho} = \max_{2\leq j\leq m_0+1} ||\nabla^j h^m||_\rho \leq b(m_0+1) \tag{3.6}$$

$$||\nabla f^K||_\rho \leq \tilde{b}(q)\rho^q e^{-K}, \quad 0 \leq q \leq K-4, \tag{3.7}$$

and such that h *is* $(\rho/2, ||\alpha||/2, C, \rho^2/4, m_0-1)$*-steep.*

Proof of Proposition 3.6. For $K \geq K^* \geq 2(m_0+1)$ apply Proposition 3.5 with $p = m_0+1$ and $q \leq K-4$ and get (3.5) with estimates (3.6) and (3.7). We want to apply Theorem 3.2 with $p = m_0+1$ and $\varpi = ||\alpha||$. Observe first that $T_{m_0}h(0) - T_1 h(0) - T_0 h(0) = h^{m_0} \in SS(n, m_0)$. Then observe also that $\nabla^2 h = \nabla^2 h^m$ and that by (3.6), we have the bound

$$||\nabla^2 h||_{m_0+1,\rho} = ||\nabla^2 h^m||_{m_0+1,\rho} \leq b(m_0+1)$$

which is independent of ρ, hence the constants C, μ^* and δ^* in the statement of Theorem 3.2 do not depend on ρ, and choosing K^* sufficiently large, ρ_K and then ρ become sufficiently small so that $\rho/2 \leq \mu^*$ and $\rho^2/4 \leq \delta^*$ therefore h is $(\rho/2, ||\alpha||/2, C, \rho^2/4, m_0-1)$-steep. □

We now use Proposition 3.6 and Theorem 2.1 to give the

Proof of Theorem B. Let H be as in (1.1) with α as in (1.5) and

$$\mathrm{BNF}_{K_0}(\tilde{H}_{K_0}) = h^{m_0} \in SS(n, m_0).$$

For $r > 0$ we define

$$K = \Delta_\alpha((1644encdr)^{-1})$$

so that $\rho_K/e \geq 3r$, and observe that $K \geq K^*$ is satisfied (with K^* given by Proposition 3.6) provided $r \leq r^*$ for some sufficiently small $r^* > 0$. Hence we can apply the latter proposition with our choice of K and with $\rho = 3r$.

Next we want to apply Theorem 2.1 to (3.5). First observe that Theorem 2.1 is stated and proved in the z variables whereas the estimate of Proposition 3.6 are given in the ξ variables: however since $z = S(\xi)$ with S and S^{-1} of unit norm, Theorem 2.1 also holds true, with the same estimates, if one uses the ξ variables.

From Proposition 3.6 and our choice of ρ, the function h is $(3r/2, ||\alpha||/2, C, 9r^2/4, m_0-1)$-steep and (2.2) is satisfied with

$$E := 3/2||\alpha||, \quad F := b(2), \quad \varepsilon := \tilde{b}(q)\rho^q e^{-K}$$

for some $0 \leq q \leq K-4$ yet to be chosen. Up to taking r^* smaller one easily checks that (2.3) (with r replaced by $\rho = 3r$) is satisfied provided we choose $q = 4na-1$. Thus Theorem 2.1 can be applied and we obtain the following statement: given an arbitrary solution $\tilde{\xi}(t)$ of the system associated to $H \circ \Phi^K$ in (3.5), if $||\tilde{\xi}(0)|| = ||\tilde{z}(0)|| \leq \rho/2 = 3r/2$, then

$$|I(\tilde{z}(t)) - I(\tilde{z}(0))| \leq \tilde{c}'(\rho\varepsilon)^{\frac{1}{2na}}, \quad |t| \leq \exp\left((\rho\varepsilon)^{-\frac{1}{2na}}\right).$$

For r sufficiently small, this implies in particular that $||\tilde{\xi}(t)|| = ||\tilde{z}(t)|| < 7r/4$ for times

$$|t| \leq \exp\left((3r\varepsilon)^{-\frac{1}{2na}}\right).$$

Recalling the definition of ε and with our choices of q and K, the previous estimate implies that $||\tilde{\xi}(t)|| = ||\tilde{z}(t)|| < 7r/4$ for times

$$|t| \leq \exp(cr^{-2}\exp(c'\Delta_\alpha(c''r^{-1})))$$

with

$$c := 3^{-2}\tilde{b}(4na-1)^{-\frac{1}{2na}}, \quad c' := (2na)^{-1}, \quad c'' := 1644encd. \tag{3.8}$$

To conclude, observe that H is related to (3.5) by a symplectic transformation $\Phi^K = \mathrm{Id} + O(\xi^2)$, which can be made close enough to the identity (as well as its inverse) by taking r small enough. Hence, given any solution $\xi(t)$ of the system associated to H with $||\xi(t)|| = ||z(t)|| \leq r$, the corresponding solution $\tilde{\xi}(t)$ of (3.5) satisfy $||\tilde{\xi}(0)|| = ||\tilde{z}(0)|| \leq 3r/2$ for r small enough, and therefore $||\tilde{\xi}(t)|| = ||\tilde{z}(t)|| < 7r/4$, and also $||\xi(t)|| = ||z(t)|| < 2r$, for times

$$|t| \leq \exp(cr^{-2}\exp(c'\Delta_\alpha(c''r^{-1}))).$$

We eventually arrives at the estimate

$$T(r) \geq \exp(cr^{-2}\exp(c'\Delta_\alpha(c''r^{-1})))$$

and this concludes the proof of the lower bound on $T(r)$ in the general case. The estimate in the Diophantine case follows from the general case and from (1.6). □

A Proof of generic steepness

The aim of this section is to give the proof of Theorem 3.1. The latter will be an immediate consequence of Propositions A.2 and A.3 below. We shall use in the proof of these propositions basic results concerning semi-algebraic subsets; for proofs and more information we refer to [BCR98]. Our main ingredient to prove Theorem 3.1 is a result of Nekhoroshev on stably expanding polynomials that we will now state.

Let us first recall that $P(n,m)$ denotes the space of polynomials of degree m in n variables with real coefficients, and $P_2(n,m)$ the subspace of $P(n,m)$ consisting of polynomials with vanishing homogeneous parts of order zero and one. The following definition, which is related to the definition of stably steep polynomials, is due to Nekhoroshev ([Nek73]).

Definition 3. *Let $1 \leq l \leq n-1$. A polynomial $Q_0 \in P_2(l,m)$ is called stably expanding if there exist a neighborhood U_l of Q_0 in $P_2(l,m)$ and positive constants C_l', δ_l' such that for any $Q \in U_l$, the inequality*

$$\max_{0\leq\eta\leq\xi} \min_{||y||=\eta} ||\nabla Q(y)|| > C_l'\xi^{m-1}$$

holds true for all $0 < \xi \leq \delta_l'$.

The set of stably expanding polynomials in $P_2(l,m)$ will be denoted by $SE(l,m)$.

Theorem A.1 (Nekhoroshev)**.** *Let $1 \leq l \leq n-1$. The complement of $SE(l,m)$ in $P_2(l,m)$ is contained in a closed semi-algebraic subset $\Sigma(l,m)$ of codimension $[m/2]$.*

Let us denote by $L(n,l)$ the space of rectangular matrices with n rows and l columns, with real coefficients, and by $L_1(n,l)$ the open subset of $L(n,l)$ consisting of matrices of maximal rank. Any $A \in L(n,l)$ induces a linear map $A : \mathbb{R}^l \to \mathbb{R}^n$, hence given $P \in P(n,m)$, we can define $P_A \in P(l,m)$ by setting $P_A(x) = P(Ax)$, $x \in \mathbb{R}^l$. Moreover, if $P \in P_2(n,m)$, then $P_A \in P_2(l,m)$. Let us define the set

$$\Theta(l,n,m_0) = \{(P,A,Q) \in P_2(n,m_0) \times L_1(n,l) \times \Sigma(l,m_0) \mid P_A = Q\}.$$

Then we define $\Upsilon(l,n,m_0)$ to be the projection of $\Theta(l,n,m_0)$ on the first factor $P_2(n,m_0)$, and finally

$$\Upsilon(n,m_0) = \bigcup_{l=1}^{n-1} \Upsilon(l,n,m_0).$$

Theorem 3.1 is a straightforward consequence of the following two properties of the set $\Upsilon(n,m_0)$.

Proposition A.2. *The set $\Upsilon(n,m_0)$ is a semi-algebraic subset of $P_2(n,m_0)$ of codimension at least one.*

Proposition A.3. *The complement of $SS(n,m_0)$ in $P_2(n,m_0)$ is contained in $\Upsilon(n,m_0)$.*

The second proposition is true for any $m \geq 2$ and not just for $m = m_0$, but this will not be needed.

Let us now give the proof of Proposition A.2 and Proposition A.3, following the arguments in [Nek73].

Proof of Proposition A.2. The set $P_2(n,m_0)$ is a real vector space hence it is algebraic, $L_1(n,l)$ is obviously an algebraic subset of $L(n,l)$ whereas, by Theorem A.1, $\Sigma(l,m_0)$ is a semi-algebraic subset of $P_2(l,m_0)$. Moreover, for $(P,A,Q) \in P_2(n,m_0) \times L_1(n,l) \times \Sigma(l,m_0)$, the equality $P_A = Q$ corresponds to a system of algebraic equations in the coefficients of P, A and Q. This implies that $\Theta(l,n,m_0)$ is a semi-algebraic subset of $P_2(n,m_0) \times L(n,l) \times P_2(l,m_0)$. Now since the projection of a semi-algebraic subset is a semi-algebraic subset, $\Upsilon(l,n,m_0)$ is a semi-algebraic subset of $P_2(n,m_0)$. Then, as a finite union of semi-algebraic subsets is semi-algebraic, $\Upsilon(n,m_0)$ is a semi-algebraic subset of $P_2(n,m_0)$. We need to prove that the codimension of $\Upsilon(n,m_0)$ in $P_2(n,m_0)$ is at least one; to do this it is sufficient to prove that the codimension of $\Upsilon(l,n,m_0)$ in $P_2(n,m_0)$ is at least one for any $1 \leq l \leq n-1$. So let us fix $1 \leq l \leq n-1$. Given $(A,Q) \in L(n,l) \times P_2(l,m_0)$, we define $\Theta_{A,Q}(l,n,m_0)$ to be the intersection of $\Theta(l,n,m_0)$ with the set

$$\{(P',A',Q') \in P_2(n,m_0) \times L(n,l) \times P_2(l,m_0) \mid A' = A,\ Q' = Q\}.$$

If $(A,Q) \in L_1(n,l) \times \Sigma(l,m_0)$, it is easy to see that

$$\dim \Theta_{A,Q}(l,n,m_0) = \dim P_2(n,m_0) - \dim P_2(l,m_0)$$

and therefore

$$\begin{aligned} \dim \Theta(l,n,m_0) &= \dim \Theta_{A,Q}(l,n,m_0) + \dim L_1(n,l) + \dim \Sigma(l,m_0) \\ &= \dim P_2(n,m_0) - \dim P_2(l,m_0) + \dim L_1(n,l) + \dim \Sigma(l,m_0) \\ &= \dim P_2(n,m_0) + \dim L_1(n,l) - \operatorname{codim} \Sigma(l,m_0) \\ &= \dim P_2(n,m_0) + nl - [m_0/2] \end{aligned}$$

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where in the last equality we used the fact that $\dim L_1(n,l) = \dim L(n,l) = nl$ and Theorem A.1. Now given $P \in P_2(n,m_0)$, we define $\Theta_P(l,n,m_0)$ to be the intersection of $\Theta(l,n,m_0)$ with the set

$$\{(P',A',Q') \in P_2(n,m_0) \times L(n,l) \times P_2(l,m_0) \mid P' = P\}.$$

Recall that if $GL(l)$ denotes the group of square invertible matrix of size l, with real coefficients, then $GL(l)$ acts freely on $L_1(n,l)$ (the quotient space is nothing but the Grassmannian $G(l,n)$, that is, the space of all l-dimensional subspaces of $\mathbb{R}^n$). It is then easy to see that $GL(l)$ acts freely on $\Theta_P(l,n,m_0)$, therefore the dimension of an orbit of this action equals the dimension of $GL(l)$, which is l^2, and hence,

$$\dim\Theta_P(l,n,m_0) = l^2.$$

Since $\Upsilon(l,n,m_0)$ is the projection of $\Theta(l,n,m_0)$ on the first factor $P_2(n,m_0)$, we have

$$\begin{aligned}
\dim\Upsilon(l,n,m_0) &\leq \dim\Theta(l,n,m_0) - l^2 \\
&\leq \dim P_2(n,m_0) + nl - [m_0/2] - l^2 \\
&\leq \dim P_2(n,m_0) - [m_0/2] + l(n-l) \\
&\leq \dim P_2(n,m_0) - [m_0/2] + [n^2/4] \\
&\leq \dim P_2(n,m_0) - 1
\end{aligned}$$

where the last inequality follows from the definition of m_0. This proves that $\Upsilon(l,n,m_0)$ has codimension at least one in $P(n,m_0)$ for any $1 \leq l \leq n-1$, therefore $\Upsilon(n,m_0)$ has codimension at least one in $P(n,m_0)$ and this concludes the proof. □

Proof of Proposition A.3. To prove that the complement of $SS(n,m_0)$ in $P_2(n,m_0)$ is contained in $\Upsilon(n,m_0)$, we will prove that the complement of $\Upsilon(n,m_0)$ in $P_2(n,m_0)$ is contained in $SS(n,m_0)$. So we fix $P_0 \in P_2(n,m_0) \setminus \Upsilon(n,m_0)$ and $1 \leq l \leq n-1$. We denote by $O(n,l)$ the subset of $L_1(n,l)$ consisting of matrices whose columns are orthonormal vectors for the Euclidean scalar product. Recalling that the Grassmannian $G(l,n)$ is the quotient of $L_1(n,l)$ by $GL(l)$, it is also the quotient of $O(n,l)$ by the group $O(l)$ of orthogonal matrices of $\mathbb{R}^l$. Therefore given any $\Lambda_0 \in G(l,n)$, there exist an open neighborhood B_{Λ_0} of Λ_0 in $G(l,n)$ and a continuous map $\Psi : B_{\Lambda_0} \to O(n,l)$ such that, if $\pi : O(n,l) \to G(l,n)$ denotes the canonical projection, then $\pi \circ \Psi$ is the identity. Let us now consider the continuous map

$$F : P_2(n,m_0) \times B_{\Lambda_0} \to P_2(l,m_0), \quad F(P,\Lambda) = P_{\Psi(\Lambda)}.$$

Since P_0 does not belong to $\Upsilon(n,m_0)$, by definition of the latter set it comes that $F(P_0,\Lambda)$ does not belong to $\Sigma(l,m_0)$ and therefore, by Theorem A.1, $F(P_0,\Lambda) \in SE(l,m_0)$ for any $\Lambda \in B_{\Lambda_0}$. Hence, by definition of $SE(l,m_0)$, there exist a neighborhood U_l of $F(P_0,\Lambda)$ in $P_2(l,m)$ and positive constants C'_l, δ'_l such that for any $Q \in U_l$, the inequality

$$\max_{0 \leq \eta \leq \xi} \min_{||y||=\eta} ||\nabla Q(y)|| > C'_l \xi^{m_0-1}$$

holds true for all $0 < \xi \leq \delta'_l$. Now by continuity of F, we can find a neighborhood V_l of P_0 in $P_2(n,m_0)$ and an open neighborhood $B'_{\Lambda_0} \subseteq B_{\Lambda_0}$ of Λ_0 in $G(l,n)$ such that $F(V_l \times B'_{\Lambda_0})$ is contained in U_l. So for any $P \in V_l$ and any $\Lambda \in B'_{\Lambda_0}$, we have

$$\max_{0 \leq \eta \leq \xi} \min_{||y||=\eta} ||\nabla F(P,\Lambda)(y)|| > C'_l \xi^{m_0-1}$$

for all $0 < \xi \leq \delta_l'$. Now since the columns of the matrix $\Psi(\Lambda)$ form an orthonormal basis of Λ, setting $x = \Psi(\Lambda)y$, $x \in \Lambda$, $||x|| = ||y||$ and hence

$$\min_{||y||=\eta} ||\nabla F(P,\Lambda)(y)|| = \min_{||x||=\eta,\, x\in\Lambda} ||\Pi_\Lambda \nabla P(x)|| = \min_{||x||=\eta,\, x\in\Lambda} ||\nabla P_\Lambda(x)||$$

where Π_Λ is the orthogonal projection onto Λ, and P_Λ is the restriction of P to Λ. Therefore, for any $P \in V_l$ and any $\Lambda \in B'_{\Lambda_0}$, we have

$$\max_{0\leq\eta\leq\xi} \min_{||x||=\eta,\, x\in\Lambda} ||\nabla P_\Lambda(x)|| > C_l'\xi^{m_0-1}$$

for all $0 < \xi \leq \delta_l'$. To conclude, since the Grassmannian $G(l,n)$ is compact, it can be covered by a finite number of neighborhoods of the form B'_{Λ_0}, $\Lambda_0 \in G(l,n)$, and hence one can certainly find positive constants C_l, δ_l such that for any $P \in V_l$ and any $\Lambda \in G(l,n)$, the inequality

$$\max_{0\leq\eta\leq\xi} \min_{||x||=\eta,\, x\in\Lambda} ||\nabla P_\Lambda(x)|| > C_l\xi^{m_0-1}$$

holds true for all $0 < \xi \leq \delta_l$. This means that $P_0 \in SS(n, m_0)$, and this finishes the proof. □

B Birkhoff normal forms with estimates

The goal of this section is to give the proof of Proposition 3.5 using the work of Delshams and Gutiérrez ([DG96]).

Given $l \in \mathbb{N}$ and P a homogeneous polynomial in ξ of degree l, if $P(\xi) = \sum_{|\nu|=l} P_\nu \xi^\nu$, we define the norm

$$||P|| := \sum_{|\nu|=l} |P_\nu|. \tag{B.1}$$

By our analyticity assumption on the Hamiltonian H in (1.1), we have the following expansion at the origin

$$H(\xi) = \sum_{l\geq 2} H_l(\xi) = i\sum_{j=1}^{n} \alpha_j \xi_j \xi_{n+j} + \sum_{l\geq 3} H_l(\xi)$$

and there exist positive constants c and d, which depends only on n, R and $||H||_R$ such that for any integer $l \geq 2$,

$$||H_l|| \leq c^{l-2}d. \tag{B.2}$$

Using Cauchy formula one easily proves that

$$||H_l|| \leq (2R)^{-l}(e(2n+1))^l ||H||_R$$

and therefore one can choose

$$c := (2R)^{-1}e(2n+1), \quad d := (2R)^{-2}(e(2n+1))^2||H||_R. \tag{B.3}$$

Given any function f that can be written as $f = \sum_k P_k$, with each P_k homogeneous of degree k in ξ, one easily check that

$$\sup_{\xi\in\mathcal{B}_\rho} |f(\xi)| \leq \sum_k ||P_k||\rho^k, \tag{B.4}$$

and, if $g = \sum_{k,\, k \text{ even}} Q_k$, with each Q_k homogeneous of degree $k/2$ in $I(\xi)$, then

$$\sup_{I \in \mathcal{D}_\rho} |g(I)| \leq \sum_{k,\, k \text{ even}} ||Q_k||(\rho^2/2)^{k/2}. \tag{B.5}$$

Moreover, the above estimates hold true if f is replaced by a tensor-valued function. Recall the definition of Ψ_α given in (1.5). Recall that we also defined for any integer $j \geq 3$, $\psi_\alpha^j = \prod_{i=3}^j \Psi_\alpha(i)$ and for convenience, we set $\psi_\alpha^2 := 1$. We can finally state the main technical proposition of [DG96].

Proposition B.1 (Delshams-Gutiérrez)**.** *Let H be as in* (1.1) *with α as in* (1.5) *and consider an integer $K \geq 4$. If we define*

$$\rho_K := (548ncdK\Psi(K))^{-1},$$

then there exists a real-analytic symplectic transformation $\Phi^K = Id + O(\xi^2)$ defined on $\mathcal{B}_{\rho_K}$ such that $H \circ \Phi^K$ is in Birkhoff normal form up to order K, that is

$$H \circ \Phi^K(\xi) = \alpha \cdot I(\xi) + \sum_{k \text{ even},\, 4 \leq k \leq K} h_k(I(\xi)) + \sum_{k \geq K+1} f_k(\xi)$$

where h_k is a homogeneous polynomial of degree $k/2$ in $I(\xi)$, f_k a homogeneous polynomial of degree k in ξ, with the following estimates:

$$||h_k|| \leq 6^{-1}(6cd)^{k-2}(k-2)!\psi_\alpha^{k-1}, \quad k \text{ even},\ 4 \leq k \leq K;$$

$$||f_k|| \leq 20d^2(20cd)^{k-2}(K-3)!(K-2)^{k-K+2}\psi_\alpha^{K-1}\Psi_\alpha(K)^{k-K+2}, \quad k \geq K+1.$$

This is exactly the statement of Proposition 1 in [DG96], to which we refer for the proof. We will now arrange these estimates in a way that will be more convenient for us.

Proposition B.2. *Let H be as in* (1.1) *with α as in* (1.5)*. Given an integer $p \geq 2$ and $K \geq 2p$, we have the following estimates on the homogeneous polynomials of Proposition B.1:*

$$||h_k|| \leq \beta(p)\rho_K^{-(k-2p)}, \quad 2p \leq k \leq K; \tag{B.6}$$

where

$$\beta(p) := 6^{-1}(6cd)^{2p-2}(2p-2)!\psi_\alpha^{2p-1}, \tag{B.7}$$

and, given an integer $0 \leq q \leq K-4$, we have

$$||f_k|| \leq \tilde{\beta}(q)\rho_K^{-(k-q-1)}, \quad k \geq K+1. \tag{B.8}$$

where

$$\tilde{\beta}(q) := c^{-1}d(20cd)^q(q+2)!\psi_\alpha^{q+2}.$$

The proof of Proposition B.2 is straightforward from Proposition B.1. Now from these estimates on the homogeneous parts of h^m and f^K, we will deduce the estimates of Proposition 3.5.

Proof of Proposition 3.5. Recall that

$$h^m(I(\xi)) = \sum_{k \text{ even}, \, 4 \le k \le K} h_k(I(\xi))$$

where each h_k is homogeneous of degree $k/2$. For $p \geq 2$ and $k \geq 2p$, $\nabla^p h_k$ is a tensor-valued homogeneous polynomial of degree $(k-2p)/2$ and one can easily check (see [DG96], estimates (24)), that

$$||\nabla^p h_k|| \leq (k/2)^p ||h_k||.$$

Using this inequality, inequality (B.5) and the estimate (B.6) we get

$$\begin{aligned}
||\nabla^p h^m||_\rho &\leq \sum_{k \text{ even}, \, 2p \le k \le K} ||\nabla^p h_k|| (\rho^2/2)^{(k-2p)/2} \\
&\leq \sum_{k \text{ even}, \, 2p \le k \le K} (k/2)^p ||h_k|| (\rho^2/2)^{(k-2p)/2} \\
&\leq \beta(p) \sum_{k \text{ even}, \, 2p \le k \le K} (k/2)^p \rho_K^{-(k-2p)} (\rho^2/2)^{(k-2p)/2} \\
&\leq \beta(p) \sum_{k \text{ even}, \, 2p \le k \le K} (k/2)^p (1/2)^{(k-2p)/2} (\rho/\rho_K)^{k-2p} \\
&\leq b(p)
\end{aligned}$$

since the sum can be bounded by the corresponding series which is convergent. The same bound applies to $||\nabla^j h^m||_\rho$ for any j such that $2 \leq j \leq p$, hence

$$||\nabla^2 h^m||_{p,\rho} = \max_{2 \le j \le p} ||\nabla^j h^m||_\rho \leq b(p).$$

Concerning

$$f^K(\xi) = \sum_{k \geq K+1} f_k(\xi),$$

since ∇f_k is a vector-valued homogeneous polynomial of degree $k-1$, we have

$$||\nabla f_k|| \leq k ||f_k||$$

and so, using this inequality together with inequality (B.4) and the estimate (B.8) we obtain

$$\begin{aligned}
||\nabla f^K||_\rho &\leq \sum_{k \geq K} ||\nabla f_k|| \rho^{k-1} \leq \sum_{k \geq K} k ||f_k|| \rho^{k-1} \\
&\leq \tilde{\beta}(q) \sum_{k \geq K} \rho_K^{-(k-q-1)} \rho^{k-1} \leq \tilde{\beta}(q) \rho^q \sum_{k \geq K} (\rho/\rho_K)^{k-q-1} \\
&\leq \tilde{b}(q) \rho^q e^{-K}.
\end{aligned}$$

This concludes the proof. □

Comment. The preprint "Double exponential stability for generic real-analytic elliptic equilibrium points" was first submitted to the server HAL (with automatic submission on Arxiv) in September 2015; in order to make it more accessible, we decided to withdraw this preprint in March 2019 and split it into two parts. This article corresponds to the second part, the first part being [BFN20]. Unfortunately, the HAL server did not allow us to mention that the old preprint has been transformed into two new preprints. Even worse, because of technical problems between the servers HAL and Arxiv, the situation could not be updated on the Arxiv: while the old preprint has been removed from HAL and the two new preprints appeared, on Arxiv the old preprint is still present but the two new preprints do not appear. We apologize for this situation and we hope that this will resolved in the future. Note that the results of [BFN17], which were published in 2017, are applications of the ideas developed in this paper.

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