Lebesgue spectrum of countable multiplicity for conservative flows on the torus

Bassam Fayad, Giovanni Forni and Adam Kanigowski

7th April 2020

Abstract

We study the spectral measures of conservative mixing flows on the 2-torus having one degenerate singularity. We show that, for a sufficiently strong singularity, the spectrum of these flows is typically Lebesgue with infinite multiplicity.

For this, we use two main ingredients: 1) a proof of absolute continuity of the maximal spectral type for this class of non-uniformly stretching flows that have an irregular decay of correlations, 2) a geometric criterion that yields infinite Lebesgue multiplicity of the spectrum and that is well adapted to rapidly mixing flows.

1 Introduction

Smooth conservative, or area-preserving, flows on surfaces provide one of the fundamental examples in the theory of dynamical systems. These flows are often called *multi-valued, or locally, Hamiltonian flows*, following the terminology introduced by S. P. Novikov [44], who emphasized their relation with solid state physics [45]. In fact, smooth conservative surface flows preserve by definition a smooth area-form, hence they are generated by the symplectic dual of a closed 1-form, which is locally the exterior derivative of a multi-valued Hamiltonian function.

Multi-valued Hamiltonian flows can be viewed as special flows above circle rotations, or more generally above IETs (interval exchange transformations). One can thus also view them as time changes of translation flows on surfaces. When the flow has fixed points, the ceiling function has singularities, that often appear at the discontinuity points of the IET.

The study of conservative surface flows goes back to Poincaré, and it knew spectacular advances with the works of the Russian school starting from the beginning of the second half of last century till the early 90s. Recently, further substantial advances were made in their understanding and they attracted a lot of attention due to their connections with billiards on rational polygons and Teichmüller theory, as well as with parabolic dynamics such as the dynamics of horocycle flows and Ratner theory.

Questions on the ergodic and spectral theory of conservative surface flows have a long history. The simplest setting to be examined is that of smooth conservative flows on the 2-torus without periodic orbits. This setting is reduced to that of reparametrizations (time changes) of minimal translation flows (see for example the textbook [5] by I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai). A. N. Kolmogorov [35] showed that such reparametrized flows are typically conjugated to translation flows, since it suffices for this that the slope of the translation flow belongs to the full measure set of Diophantine numbers. He also observed that more exotic behaviors should be expected for the reparametrized flows in the case of Liouville slopes. M. D. Shklover indeed obtained in [53] examples of real analytic reparametrizations of linear flows on the 2-torus that were weak mixing (continuous spectrum). Not long after Shklover's result, A. B. Katok [24], and later A. V. Kochergin [29], showed the absence of mixing for non-singular conservative flows on the 2-torus, hence establishing that in a sense shear of nearby orbits near singularities is the only mixing mechanism available for smooth surface flows[1]. Note that analytic reparametrizations of Liouvillean irrational flows of the 2-torus can have a mixed singular continuous and discrete maximal spectral type [11, 19].

Kochergin mixing flows on surfaces.

The simplest mixing examples of conservative surface flows are those with one (degenerate) singularity on the 2-torus produced by Kochergin in the 1970s [30]. They are time changes of linear flows on the 2-torus with an irrational slope and with a single rest point (see Figure 1 and the last section of this introduction for a precise definition of *Kochergin flows*). Equivalently these flows can be viewed as special flows under a ceiling, or roof, function with at least one power singularity (see Figures 2 and 3 and the precise definition of special flows in Section 3).

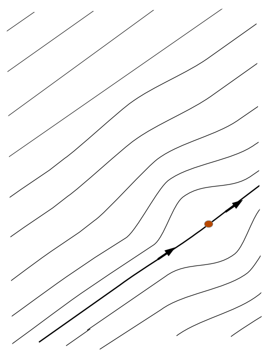

Figure 1: Torus flow with one degenerate saddle acting as a stopping point.

Multi-valued Hamiltonian flows on higher genus surfaces can also be mixing (or mixing on an open ergodic component) in the presence of non-degenerate saddle type singularities that have some asymmetry (see Figure 2). Such flows are called *Arnol'd flows* and their mixing property, conjectured by V. I. Arnol'd in [1], was obtained by K. Khanin and Ya. G. Sinai [27] and, later, in more generality by Kochergin [32, 34].

Kochergin also proved that for suspension flows under a roof function with *symmetric logarithmic* singularities over a circle rotation, mixing fails for almost every rotation number [31]. Many years later he proved in [33] that indeed mixing fails in this case for *all* rotation numbers.

Ulcigrai substantially extended Kochergin's results by proving in [61] that conservative flows with non-degenerate saddle singularities are generically not mixing (due to the cancellations coming from symmetry in the saddles). Recently, J. Chaika and A. Wright [4] gave mixing examples

[1]This confirmed Kolmogorov's intuition about the absence of mixing for analytic reparametrizations of translation flows, but only in this two-dimensional setting. Indeed, mixing analytic reparametrizations of translation flows on $\mathbb{T}^3$ were obtained in [8].

with finitely many non-degenerate saddles and no saddle connections on a closed surface of genus 5.

Although mixing was thoroughly studied for conservative surface flows, almost nothing was known about the spectral type and spectral multiplicity of the mixing examples (see for example the survey by Katok and J.-P. Thouvenot [26] and the discussion therein). By spectral type of a flow $\{T^t\}$ we mean the spectral type of the associated Koopman operator $U_t : L^2(M,\mu) : f \to f \circ T^t$.

The nature of the spectral type and multiplicity of mixing surface flows naturally arose as soon as such mixing examples were obtained, especially since, at that time, the spectral theory of dynamical systems was a matter of major interest for the Russian school in that second half of last century (see for example the textbook [5] or Kolmogorov's 1954 ICM address [36]). Since then, the question about the possibility of a Lebesgue maximal spectral type for mixing surface flows appeared in many monographs and surveys (see for example the discussions in [26], [40] or [6]). In the survey [26], by Katok and Thouvenot, it is remarked that "Some estimate of correlation decay have been obtained but they are too weak to conclude that the spectrum is absolutely continuous." Finally, Kochergin at the end of his paper [34] asks about rate of mixing and absolutely continuous spectrum (Problem 4) and multiple mixing (Problem 6) for flows on surfaces.

In this paper we treat the simplest mixing examples that are Kochergin flows on the torus with a single degenerate rest point. We give a general statement here that will be made more specific in the last section of this introduction.

Theorem 1. *There exists a real analytic conservative flow on $\mathbb{T}^2$ with exactly one singularity, with Lebesgue spectral type of countable multiplicity.*

Note that besides their own interest, mixing conservative flows attracted an additional attention since they stood as the main and almost only natural class of mixing transformations for which higher order mixing has not been established, nor disproved. The first and third author of this paper established multiple mixing only for a very special class of mixing Kochergin flows [12]. In our proof of Theorem 1 we will actually show that Kochergin flows with a sufficiently strong singularity have for almost every slope a countable Lebesgue spectrum. As a consequence, neither B. Host's celebrated theorem that establishes multiple mixing for mixing systems with purely singular spectral type [20], neither [12], can give a positive answer to the multiple mixing question for typical Kochergin flows.

Before we proceed to the precise statement of Theorem 1, we make some comments about the new phenomenon that is enclosed in the above result and about the mechanisms that yield it.

How chaotic can the lowest-dimensional, smooth, invertible dynamical systems be?

A circle diffeomorphism with an irrational rotation number that preserves a smooth measure is smoothly conjugate to a rotation. It is hence rigid in the sense that the iterates along a subsequence of the integers converge uniformly to identity. Rigidity implies the absence of mixing between any two measurable observables. This absence of mixing actually holds for all smooth circle diffeomorphisms with irrational rotation number since, by Denjoy theory, they are topologically conjugated to rotations. Circle diffeomorphisms with rational rotation number are even farther from mixing, since their non-wandering dynamics are supported on periodic points.

The *lowest dimensional setting* that can be investigated for dynamical complexity after circle diffeomorphisms is that of multi-valued Hamiltonian flows on surfaces. In the absence of periodic orbits, these flows can be viewed as reparametrizations of minimal translation flows on the torus. Combining Kolmogorov's result on the linearizability of Diophantine flows, and the theory of periodic approximations, A. Katok [24] proved that sufficiently smooth reparametrizations of linear flows on the torus are actually rigid. In particular, the maximal spectral type of smooth conservative flows of the torus without periodic orbits is always purely singular.

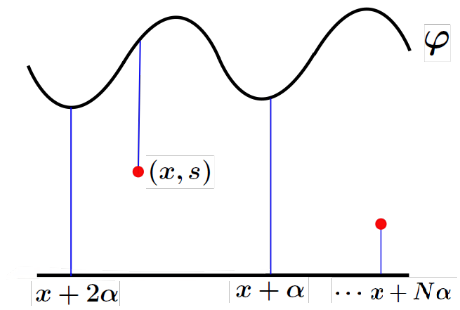

Figure 2: The orbit of a point by the special flow above a rotation of angle α and under a bounded ceiling function φ. Smooth reparametrizations of linear flows on $\mathbb{T}^2$ are equivalent to such flows.

As a consequence of Katok's result, in order to go beyond the purely singular maximal spectral type for smooth conservative flows on the 2-torus, one must allow the existence of singularities for the flow. When there is just one singularity, the phase portrait is actually similar to that of a minimal translation flow, apart from one orbit that contains the saddle point which acts as a stopping point (see Figure 1). Our result shows that in this situation the maximal spectral type can indeed be Lebesgue even in this 'almost one-dimensional' setting.

Quasi-minimal flows that are spectrally isomorphic to Bernoulli flows

The two extremes in describing the stochasticity of a dynamical system from a spectral point of view are simple pure point spectrum on one end and countable Lebesgue spectrum on the other. Translation flows on the torus have a simple pure point spectrum, while Bernoulli flows have countable Lebesgue spectrum.

The proof of the countable Lebesgue spectrum property for geodesic flows on negatively curved surfaces by I. M. Gelfand and S. V. Fomin [18], and later for general (open sets of) K-flows by A. N. Kolmogorov, Ya. G. Sinai and others, was considered as a major breakthrough by the Russian school of dynamical systems in the second half of the twentieth century, because of what it implied on the similarities between some deterministic systems and stochastic flows [55, 5, 56, 47, 38].

Parallel to this discovery was another major discovery, made by Kolmogorov, that quasi-periodic Diophantine motion is robust in many systems of mechanical origin (such as quasi-integrable Hamiltonian flows). This insight later developed into what is called today KAM theory (after Kolmogorov, Arnol'd and Moser). With these two phenomena in mind, motivated by the theory of K-systems and by KAM theory, Kolmogorov ventured in his ICM 1954 paper into the following interesting speculation : *It is not impossible that only these cases (a discrete spectrum with a finite number of independent frequencies and a countably-multiple Lebesgue spectrum) are admissible for analytic transitive measures or that, in a sense, only they alone are general typical cases.* This of course, is in stark contrast with the Halmos-Rokhlin general description of invariant ergodic measures as being continuous and purely singular for the generic system in the weak topology, and Kolmogorov insisted in his speculation on restricting to the analytic setting, avoiding even the smooth category, for the above dichotomy to have some chances to hold. We know today, that even in the analytic category, and even in low dimensional systems such as reparametrized irrational flows of the 2-torus, there are many other possibilities for the spectrum, including singular continuous, mixed, etc., but the validity of the dichotomy in some typical sense is still a possibility.

For systems with zero entropy, for many decades progress on spectral questions was restricted to the case of homogeneous flows, starting with O. S. Parasyuk's result [46] on countable Lebesque spectrum for horocycle flows (see for example [28] for a systematic exposition of ergodic theory of homogeneous flows and many references). The only instance of smooth systems for which Lebesgue spectrum was established beyond hyperbolic and algebraic case is in [16] where the second author and Ulcigrai show that the maximal spectral type of smooth time-changes of the horocycle flow of a compact hyperbolic surface is Lebesgue. Their work was motivated by a conjecture of A. Katok and J.-P. Thouvenot that time-changes of horocycle flows should also have countable Lebesgue spectrum (see [26], Conjecture 6.8). Independently, R. Tiedra [57], [58] (following a different approach) obtained the absolute continuity of the spectrum for the same flows.

Note that conservative flows on surfaces always have topological entropy zero[2]. Note also that conservative non-singular time changes of translation flows on the 2-torus were presented by Kolmogorov [36] as the basic context of real analytic systems in which discrete spectrum prevails. In the particular case that we are considering of flows with just one singularity, the phase portrait is, as mentioned above, very similar to that of a minimal translation flow, except for the existence of one rest point. It is a striking fact that some of these quasi-minimal flows, as in Theorem 1, turn out to have a countable Lebesque spectrum and thus are spectrally equivalent to ergodic Bernoulli flows.

In the next two sections we describe some aspects of these two steps as well as their relations to the existing literature. In a third section, we cast our results in a more general picture on conservative surface flows and describe their relations to the main recent advances in the field. In the last section of this Introduction, before we give the plan of the paper, we explain the shear mechanism that underlies mixing for conservative surface flows with singularities, and we precisely state our results .

Non uniform shear and irregular decay of correlations

To prove the absolute continuity of the spectrum of a dynamical system, it is natural to look for a control on the decay of correlations by the flow. The only result in the direction of getting power-like estimates for the decay of correlations of surface flows was obtained in [9], where the first author proved a polynomial bound $t^{-\eta}$ on the decay of correlations (as functions of time $t > 0$) for Kochergin flows with one power singularity and for the characteristic functions of rectangles. However, in that paper, the power of the decay η is bound to be less than $1/4$, so it is not possible to deduce from the decay anything about the spectral type of the corresponding flow.

[2] The situation is completely different for surface diffeomorphisms. Anosov automorphisms of the torus and their relatives constructed by A. Katok [25] on the sphere and the disc are classical examples of conservative Bernoulli surface diffeomorphisms. Later, Bernoulli diffeomorphisms and flows were shown to exist on any compact manifold of dimension larger than 2 and 3 respectively [7, 21].

However, and as it is often the case, characteristic functions of nice sets do not give the best rate of decay of correlations between observables. Our work takes inspiration from that of [16], especially in the use of coboundaries to estimate the decay of correlations, as well as in the proof of the equivalence to Lebesgue of the maximal spectral type. Our approach considerably refines the approach of [16] in two directions : 1) it handles non-uniformly parabolic flows, for which the correlation decay, even for coboundaries, is very irregular (not even bounded by $t^{-1/2}$), and 2) it gives a criterion for countable multiplicity, which applies to Kochergin flows, but also to a much wider class of mixing systems with square summable decay of correlations for a sufficiently rich class of functions. To prove the square summable decay of correlations for Kochergin flows, we also take inspiration from [9] where a quantitative approach to the mixing shear mechanism exhibited by Kochergin in [30] is adopted to obtain a speed of mixing for these flows.

Indeed, there is a fundamental difference between the decay of correlations for time-changes of horocycle flows and for mixing surface flows, that we will now explain.

For time-changes of horocycle flows, the decay of correlations for coboundaries exploited in [16] is based on the *uniform shear* of geodesic arcs, linear with respect to time, as in B. Marcus' proof in [43] of mixing for these flows. Such a shear can be readily derived from the commutation relations for the horocycle and the geodesic flows, and the unique ergodicity of the horocycle flow (and hence of all of its time-changes), first established by H. Furstenberg [17] (see also [42]). The amount of shear is asymptotically linear with respect to time, since it is given by the ergodic integral of a function of non-zero mean.

In the case of suspension flows above rotations, the shear of horizontal arcs is provided by the stretching of the Birkhoff sums of a ceiling function with a singularity (see Figure 4, and the last section of this introduction for a precise description of the shear mechanism). In this case, the amount of shear is non-uniform since it is given by the ergodic integrals of a integrable function of zero mean, hence it depends on deviation of ergodic integrals from the mean. This *non-uniform shear* has a strength that depends on the asymptotics of the roof function at the singular point. It is crucial for our argument that the singularity be chosen strong enough so that, over most of the phase space, the reciprocal of the stretching is a square integrable function of time. This means that our power singularity must be chosen with exponent in the interval $(1/2, 1)$. For *asymmetric* power singularities, the set where the reciprocal of the stretching of Birkhoff sums is not sufficiently small, that is, not square-integrable, has very small measure and can be neglected in the argument. However, such suspension flows cannot be realized as smooth flows on a surface. For *symmetric* power singularities of exponent close to 1, which indeed can be realized as smooth flows (see Proposition 2.2 below), the set of insufficient stretching is not negligible anymore, and we have to deal with it in the argument. This is a significant difficulty, both conceptual and technical, and in fact the summability of the correlations even when their decay is not uniform, is a new phenomenon that, to the authors' best knowledge, does not arise in any of the proofs of absolutely continuous spectrum of dynamical systems available in the literature (see [16] , [57], [58], [54]).

Indeed, we emphasize that in our situation, and in contrast with all the above-mentioned cases, in particular that of time-changes of horocycle flows investigated in [16] , we have that for any smooth functions, the correlation coefficients will not always be of order less than $t^{-1/2-\varepsilon}$ as t goes to infinity. To the contrary, along the subsequence t_n given by the denominators of the irrational rotation, the correlation coefficients will in fact be as large as $t_n^{-1/2+\varepsilon}$, for some $\varepsilon > 0$, because there is a set of measure of order $t_n^{-1/2+\varepsilon}$ on which the flow at time t_n is almost equal to the identity.

This *bad set* appears due to the cancellations in the stretching of the Birkhoff sums of the ceiling function that are caused by the symmetry at the singularity (a remnant of the Denjoy-Koksma property). The bad set is essentially a union of thin towers that follow in projection the orbit of the rotation on the base. Outside the bad set, the correlations are well controlled due to sufficiently strong uniform stretching. A crucial part of our argument, completely absent in the earlier works mentioned above, deals precisely with the bad set. Indeed, we use a bootstrap argument and the regular structure of the bad set, to show that for most of the times that are in a medium scale neighborhood of the time t_n, there is some *small power* decay of correlations on the bad set. This property, plus the smallness in measure of the bad set, plus the fast decay outside of this set, finally yield square summability of the total correlations (see Figures 3, 6 and 7).

We think our method will be useful in treating other parabolic flows where mixing is due to shear and where the shear is often sufficiently strong but not uniformly in time and space.

A criterion for countable Lebesgue multiplicity for parabolic flows

Once we know that the maximal spectral type of Kochergin flows is absolutely continuous, two natural questions arise, one about the equivalence of the spectral type to Lebesgue measure on $\mathbb{R}$, and one about the spectral multiplicity.

The type and multiplicity of mixing surface flows was often raised in connection with the question whether there exist flows with *simple* Lebesgue spectrum. This is the flow version of the famous Banach's problem on the existence of a measure preserving transformation having simple Lebesgue spectrum. However, no tools were available to understand the multiplicity question for these flows. A criterion that gives an *upper bound* on the multiplicity of the spectrum of a flow, introduced by Katok and Thouvenot ([26, Theorem 1.21]), does not apply to our Kochergin flows due to the strong shear near the singularity.

We introduce here a geometric criterion based on rapid mixing that implies the pure Lebesgue and infinite multiplicity for flows that have an absolutely continuous maximal spectral type. It applies in particular to Kochergin flows with sufficiently degenerate power singularity and allows to complete the proof of Theorem 1, building on the absolute continuity of the maximal spectral type, and on the estimates that implied it.

The criterion, that we call CILS (Criterion for Infinite Lebesgue Spectrum), will be presented in detail in Section 6. Heuristically we see that if the flow admits a given number $n+1$ of functions, $n \geqslant 0$, such that each function is almost orthogonal to the cyclic space of any other one, and such that the spectral measures of the functions can be chosen to be not too small on any fixed bounded measurable set of $\mathbb{R}$, then the pure Lebesgue multiplicity of the flow is larger than $n+1$. In fact, in our formulation it is enough to construct $n+1$ functions such that the $(n+1)\times(n+1)$ matrix of Fourier transforms of their square-integrable mutual correlations has maximal rank equal to $n+1$ on any given positive measure subset of the real line.

The idea of constructing an arbitrarily large number of such independent functions for a rapidly mixing system is the following: one can choose the functions to be supported on one or several Rokhlin towers for the flow (or flow-boxes with an arbitrarily short base) and specify their values on these towers so that the conditions of the criterion are satisfied for a finite, arbitrarily large time. Heuristically, such finite systems of functions are constructed to have orthogonal cyclic subspaces on a large subinterval of the real line. Once more, it is in fact enough to control the Fourier transforms of all correlations of the functions in each finite system over a large time interval. The conditions of the criterion in the infinite complementary intervals are then derived from the mixing estimates, that is, from the square-integrability of the correlations (and their Fourier transforms).

Note that for $n = 0$, only the condition on the spectral measure is required and yields the equivalence of the maximal spectral type to Lebesgue. Our criterion in that case reduces to the one used by the second author and Ulcigrai in [16] in the prof that the maximal spectral type of smooth time-changes of horocycle flows is Lebesgue. Also, the construction of the function satisfying the criterion in that case $n = 0$ is very similar to the construction in [16] but has to be adapted to our context of non-uniformly stretching flows.

The main novelty in our CILS is the lower bound on the multiplicity. Indeed, our CILS gives an alternative to the much stronger K-property introduced by Kolmogorov [37], Sinai [55] and others to establish countable Lebesgue spectrum for uniformly hyperbolic systems. It is presented in a clear cut form that makes it applicable to a wide range of smooth mixing systems with a sufficiently fast rate of mixing for observables in some rich class of functions.

Besides K-flows, infinite Lebesgue spectrum was so far established only for homogeneous flows and other systems of algebraic origin. Even in one of the simplest non-algebraic cases, that of smooth time changes of horocycle flows, the *countable* Lebesgue spectrum property, conjectured, as we have recalled, by Katok and Thouvenot (see [26], Conjecture 6.8), was still open.

Our criterion allows to extend the work of the second author and Ulcigrai [16] and thereby complete the proof of the Katok-Thouvenot conjecture. However, the domain of applicability of our criterion is definitely wider than the class of *uniformly parabolic* flows (that is, flows with uniformly strong shear) such as horocycle flows and their time changes. Indeed we have applied it in this paper to the borderline case of mixing Kochergin flows, which are *non-uniformly parabolic*, with irregular decay of correlations, even for smooth coboundaries.

We believe that a systematic application of our CILS will allow to show that countable Lebesgue spectrum is a robust property in many non linear contexts, where many metric invariants (not just one metric invariant, like entropy in the case of K-systems) preclude the possibility of isomorphism classification.

Other recent advances in the study of ergodic properties of surface flows

Further advances in the ergodic theory of flows on higher genus surfaces came only in last couple of decades as a consequence of a deeper understanding of the behavior of ergodic sums (integrals) of Interval Exchange Transformations (translation flows), and several spectacular developments in that direction also brought renewed interest in multi-valued Hamiltonian flows on surfaces.

The second author studied *deviations of ergodic averages* for such flows [14] and proved a substantial part of the conjectures formulated by M. Kontsevich [39] and A. Zorich [62], [63], [64] on their deviation spectrum. From these results, A. Avila and the second author [2] derived the *weak mixing* property of non-toral translation flows and of Interval Exchange Transformations which are not rotations. The proof of the Kontsevich–Zorich conjectures was later completed by A. Avila and M. Viana [3].

Ergodic properties of multi-valued Hamiltonian flows on higher genus surfaces with non-degenerate saddle singularities were then studied by C. Ulcigrai, who established that such flows are generically *weak mixing* [60], but *not mixing* [61] (see also D. Scheglov's paper [52]).

For suspensions flows under ceiling functions with *asymmetric logarithmic* singularities, Ulcigrai generalized in her thesis [59] the result of Khanin and Sinai [27] to suspensions with one singularity above generic IET's. Only recently, D. Ravotti [51] has carried out the argument for any number of singularities, thereby establishing mixing, with (at least) logarithmic decay of correlations, for smooth flows of Arnol'd type on surfaces of higher genus.

After Ulcigrai's work on *mixing properties*, a few major questions remained open in the ergodic theory of flows on surfaces: whether mixing is at all possible for conservative smooth flows with non-degenerate saddles in higher genus, whether smooth flows on surfaces can have Lebesgue spectrum, and finally whether mixing implies multiple mixing (Rokhlin's question).

As explained above, a better understanding of the various possible behaviors of IET's allowed J. Chaika and A. Wright [4] to answer the first question in the affirmative. They proved the *existence of mixing special flows* over non-generic, uniquely ergodic, interval exchange transformations, in the case of a smooth ceiling function with symmetric logarithmic singularities at the interval endpoints. Their result implies in particular the existence of an (exceptional) mixing smooth flow with only Morse saddle singularities on a surface of genus 5.

Other important advances came from a better understanding of the similarities between the dynamics of uniformly parabolic flows, such as the horocycle flow, and locally Hamiltonian flows on surfaces. The first and third author proved *multiple mixing* for a class of such flows on the torus [12] (a restricted atypical class in the case of Kochergin flows, but a typical class for Arnol'd asymmetric flows). For this, they showed that these flows display a generalization of the so called Ratner property on slow divergence of nearby orbits (introduced by M. Ratner [48], [49], [50] in her study of ergodic properties of horocycle flows), that implies strong restrictions on their joinings, which in turn yield higher order mixing. Multiple mixing was later generalized to many mixing flows on higher genus surfaces in [22]. This was the first application of the Ratner property to prove multiple mixing outside its original context of horocycle flows. Finally, a *disjointess criterion*, based on the Ratner property, has very recently been introduced in [23], and systematically applied to disjointness results for time-changes of horocycles and Arnol'd flows (see also [15] for an application of the criterion to Heisenberg nilflows and [13] for a refinement of Ratner disjointness result for time-changes of horocycle flows). As for the question on the spectral type and multiplicity of mixing surface flows, no results were known up to now.

In the next section, we describe the mixing mechanism that comes from shear for surface flows with singularities, or for special flows above rotations. We will also give the precise definition of the Kochergin flows for which we will establish the countable Lebesgue multiplicty.

Shear of Birkhoff sums and mixing

Consider a section of a Kochergin's flow with one singularity on the torus, that is transversal to all orbits and does not contain the singularity. The dynamics can then be viewed as that of a special flow above an irrational rotation of the circle with a return time function (called a *ceiling* or *roof* function) having a power-like singularity (see Figure 3).

The singularity is precisely the last point where the section intersects the incoming separatrix of the fixed point. The strength of the singularity depends on how abruptly the linear flow is slowed down in the neighborhood of the fixed point (see Proposition 2.2). In the case of other surfaces and several singularities, the flows obtained by Kochergin are equivalent to special flows above interval exchange transformations (IET's) with ceiling functions having power-like singularities at the discontinuity points of the IET.

The mechanism of mixing in Kochergin examples is, in part, the same as in the weak mixing examples of Shklover, namely the stretching of the Birkhoff sums of the ceiling function above the

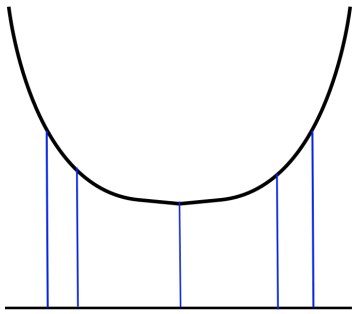

Figure 3: Representation of a 2-torus flow with one degenerate saddle as a special flow under a ceiling function (symmetric) power-like singularity.

iterates of the ergodic base dynamics. Whenever these sums are uniformly stretched above small intervals, the image of small rectangles by the special flow for large times decomposes into long and thin strips (see Figure 4). These strips are well distributed in the fibers due to uniform stretch, and well distributed in projection on the base because of ergodicity of the base dynamics.

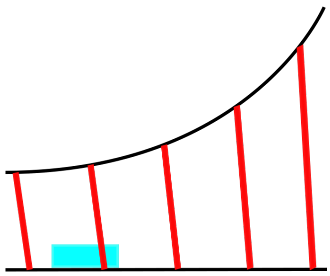

Figure 4: Mixing mechanism for special flows: the image of a rectangle is a union of long narrow strips which fill densely the phase space.

However, the reason behind the uniform stretching is different for Shklover's flows and Kochergin's ones. For the first ones, uniform stretching of the Birkhoff sums of the ceiling function is due to a Liouville phenomenon of accumulation, along a subsequence of time, of the oscillations of the ceiling function due to periodic approximations. In the case of Kochergin's flows, it is the shear between orbits as they get near the fixed points that is responsible for mixing. As a consequence, for the latter uniform stretching holds for *all* large times, while for the former, the existence of Denjoy-Koksma (DK for short) times impedes mixing. Denjoy-Koksma times are integers for which the Birkhoff sums have an *a priori* bounded oscillation around the mean value on all or on a positive measure proportion of the base (see for example the discussion around property DK in [6]). Hence, a key fact behind Kochergin's result is that the Denjoy-Koksma property does not necessarily hold for ceiling functions having infinite asymptotic values at some singularities.

A threshold is given by smooth ceiling functions having *logarithmic* singularities. When such a singularity is *symmetric*, it is known that for a typical irrational rotation a Denjoy-Koksma like property holds that prevents mixing of the special flow (see [41] and [6, Section 8]). In higher genus, Ulcigrai [61] proved that, despite the presence of polynomial deviations of Birkhoff sums from the mean [64], [14], for almost all IET's there are still sufficient cancellations to prevent mixing. A different, special, cancellation mechanism was found slightly earlier by Scheglov [52] in genus 2. However, as proven by Chaika and Wright [4], these cancellations do not happen for all IET's, as the speed of convergence of Birkhoff sums to the mean can be very slow, and this is why mixing is possible in some special cases.

The case of *asymmetric logarithmic* singularities is different. In [1], Arnol'd showed that multi-valued Hamiltonian flows with non-degenerate saddle points have a phase portrait that decomposes into elliptic islands (topological disks bounded by saddle connections and filled up by periodic orbits) and one open uniquely ergodic component. On this component, the flow can be represented as the special flow over an interval exchange map of the circle and under a ceiling function that is smooth except for some logarithmic singularities. The singularities are typically asymmetric since the coefficient in front of the logarithm is twice as big on one side of the singularity as the one on the other side, due to the existence of homoclinic loops (see Figure 5). As we mentioned above, Khanin and Sinai [27] proved that, as conjectured by Arnol'd, this asymmetry produces mixing.

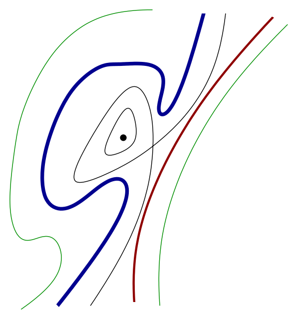

Figure 5: Multivalued Hamiltonian flow. Note that the orbits passing to the left of the saddle spend approximately twice longer time comparing to the orbits passing to the right of the saddle and starting at the same distance from the separatrix since they pass near the saddle twice.

2 Statements and plan of the work

A class of Kochergin flows

In this paper, we will show that Kochergin flows with a single sufficiently strong degenerate singularity typically have a Lebesgue spectral type with countable multiplicity. We now formulate our results more precisely. The flows which we will consider are special flows given by a base dynamics that is an irrational rotation of the circle (we refer to Section 3.1 for an exact definition of special flows).

The base dynamics.

The irrational rotation in the base dynamics is supposed to have a rotation number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfying a Diophantine condition of the following form. For $\xi > 0$, we will say that $\alpha \in D_{\log,\xi}$ if and only if there exists a constant $C(\alpha) > 0$ such that for any $p \in \mathbb{Z}, q \in \mathbb{Z}^*$,

$$|\alpha - \frac{p}{q}| \geqslant \frac{C(\alpha)}{q^2 \log^{1+\xi} q}. \qquad (H_1)$$

It is a classical and easy to prove fact that for any $\xi > 0$, $D_{\log,\xi}$ has full Haar measure in $\mathbb{T}$. We remark that an immediate, but relevant consequence of the condition that $\alpha \in D_{\log,\xi}$ is that, for all the convergents p_n/q_n of its continued fraction expansion we have

$$q_{n+1} \leqslant C(\alpha)^{-1} q_n \log^{1+\xi} q_n .$$

The ceiling function.

We prove our results on Kochergin flows under the following assumptions on the ceiling function.

Definition 2.1. *For all $\eta > 0$, we let $\mathscr{C}_\eta$ denote the class of functions $\varphi \in C^2(\mathbb{T} \setminus \{0\})$, with $\varphi > 0$, which for some constant $\varphi_0 > 0$ satisfy:*

$$\lim_{\theta\to 0^+} \frac{\varphi(\theta)}{\theta^{-(1-\eta)}} = \frac{\varphi_0}{1-\eta} \quad \textit{and} \quad \lim_{\theta\to 0^-} \frac{\varphi(\theta)}{|\theta|^{-(1-\eta)}} = \frac{\varphi_0}{1-\eta}, \tag{$H_{2.1}$}$$

$$\lim_{\theta\to 0^+} \frac{\varphi'(\theta)}{\theta^{-(2-\eta)}} = -\varphi_0 \quad \textit{and} \quad \lim_{\theta\to 0^-} \frac{\varphi'(\theta)}{|\theta|^{-(2-\eta)}} = \varphi_0, \tag{$H_{2.2}$}$$

$$\lim_{\theta\to 0^+} \frac{\varphi''(\theta)}{\theta^{-(3-\eta)}} = (2-\eta)\varphi_0 \quad \textit{and} \quad \lim_{\theta\to 0^-} \frac{\varphi''(\theta)}{|\theta|^{-(3-\eta)}} = (2-\eta)\varphi_0, \tag{$H_{2.3}$}$$

where $\eta \in (0, 1/1000)$.

Note that ($H_{2.3}$) implies ($H_{2.2}$) and ($H_{2.1}$) by L'Hôpital's rule, but we prefer to state these properties separately for future reference.

Note also that, since for any $c > 0$ the special flows under the ceiling functions $\varphi \in \mathscr{C}_\eta$ and $c\varphi \in \mathscr{C}_\eta$ are, up to an isomorphism, a linear time-change of each other, it is not restrictive to normalize either the mean $\int_{\mathbb{T}} \varphi d\lambda_{\mathbb{T}}$ or the singularity constant φ_0 of the ceiling function (but a priori not both at the same time). In fact, the spectral properties of a flow are invariant under isomorphisms and linear scaling of the time.

We will therefore assume below the the integral of ceiling function is normalized, that is, we will explicitly consider only ceiling functions such that $\int_{\mathbb{T}} \varphi d\lambda_{\mathbb{T}} = 1$.

We denote by $T^t_{\alpha,\varphi}$ the special flow defined above the rotation of angle α and under the ceiling function φ (we refer to Section 3.1 for an exact definition of special flows).

We let $M = \{(\theta, s) \in (\mathbb{T} \setminus \{0\}) \times \mathbb{R} \,:\, s \leqslant \varphi(\theta)\}$ and denote by μ the measure equal to the restriction to M of the product of the Haar measures $\lambda_{\mathbb{T}}$ on the circle $\mathbb{T}$ and $\lambda_{\mathbb{R}}$ on the real line $\mathbb{R}$. The measure μ is the only invariant probability measure for the special flow $T^t_{\alpha,\varphi}$.

Our main result is the following.

Theorem 2. *For all positive $\xi < 1/10$ and $\eta < \min(1/1000, \xi)$, the special flow $(T^t_{\alpha,\varphi}, M, \mu)$ over the rotation of angle $\alpha \in D_{\log,\xi}$ (see equation (H_1)) and under a ceiling function $\varphi \in \mathscr{C}_\eta$ (see Definition 2.1) has Lebesgue spectral type with countable multiplicity.*

Theorem 1 follows directly from Theorem 2 and the following well-known result.

Proposition 2.2 (Analytic Kochergin flows on the two torus, [30]). For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and any $\eta_0 > 0$, there exists a real analytic conservative flow on $\mathbb{T}^2$ with exactly one singularity which can be represented as a special flow $(T^t_{\alpha,\varphi}, M, \mu)$ over the rotation of angle α and under a ceiling function $\varphi \in \mathscr{C}_\eta$ for some $\eta < \eta_0$.

Proof. In [30], the following method is adopted to obtain conservative flows on the torus with a degenerate saddle-node fixed point as in ($H_{2.1}$)–($H_{2.3}$). Consider first some Hamiltonian flow on $\mathbb{R}^2$ with the $x-$axis invariant and with a unique singularity at the origin. In the neighborhood of the origin, the orbits of such a flow are as described in Figure 1. It is then possible to cut a small neighborhood of the origin and paste it smoothly inside the phase portrait of a linear flow of $\mathbb{T}^2$ with any given slope. As a result, one gets a multi-valued Hamiltonian flow that has a unique singularity of saddle-node type. An easy calculation shows that if we consider the Hamiltonian given by $H_l(x,y) = y(x^2+y^2)^l$ then the corresponding special flow has a unique symmetric power-like singularity as in ($H_{2.1}$)–($H_{2.3}$) with $\eta = \frac{2}{2l+1}$, hence arbitrarily close to 0 as $l \to \infty$.

One can also obtain analytic examples with one fixed point as in ($H_{2.1}$)–($H_{2.3}$). To do so, one starts with the smooth construction of a multi-valued Hamiltonian described above. Then, for an arbitrary $k > 2l+4$, one considers a real analytic approximation of the smooth multi-valued Hamiltonian that continues to have the same slope and a unique singularity at $(0,0)$, with the same jets of order k at $(0,0)$ (that is, those of H_l). From there it follows that the corresponding flow has a special flow representation with a ceiling function having a unique symmetric power-like singularity as in ($H_{2.1}$)–($H_{2.3}$). □

Other mixing surface flows.

Some questions naturally arise from our result.

Question 1. *Do Kochergin flows always have Lebesgue spectral type (with countable multiplicity)?*

To answer this question, one has to treat several singularities and with smaller powers as well as general IET's on the base.

Question 2. *What is the spectral type in the case of non degenerate saddles?*

Arnol'd conjectured a power-like decay of correlation in the asymmetric case, but the decay is more likely to be logarithmic, at least between general regular observables or characteristic functions of regular sets such as balls or squares. Note that even a lower bound on the decay of correlations is not sufficient to preclude absolute continuity of the maximal spectral type. However, an approach based on slowly coalescent periodic approximations as in [10] may be explored in the aim of proving that the spectrum is purely singular.

Plan of the paper

In Section 3 we first give the formal definition of the relevant special flows and we describe an L^2-dense set of observables that we will work with. These are smooth coboundaries vanishing on some neighborhood of the boundary of the singularity and of the base of the special flow.

The proof that the flow $T^t_{\alpha,\varphi}$ has an absolutely continuous maximal spectral type follows by a standard argument from Theorem 3 that states that the Fourier transforms of the spectral measures of functions in our special dense set are square-integrable.

The proof of Theorem 3 splits in two parts. We consider a time $t \in [q_n, q_{n+1}]$ for some $n \in \mathbb{N}$. We further consider intervals of time of the type $t \in [l^{21/20}, (l+1)^{21/20}] \subset [q_n, q_{n+1}]$.

First, a decay faster than $t^{-1/2-\varepsilon}$ for some $\varepsilon > 0$ is established outside a bad set $\mathscr{B}_l$ of measure less than $t^{-1/2+\varepsilon}$. This result is stated as Proposition 3.2. Second, the squared correlations on the bad set $\mathscr{B}_l$ are controlled on average for $t \in [l^{21/20}, (l+1)^{21/20}]$. This is the content of Proposition 3.3.

Section 4 is devoted to the proof of general stretching estimates for the Birkhoff sums of the ceiling function.

In Section 5 the bad set $\mathscr{B}_l$ is constructed and the stretching properties outside this set are stated. This is the content of Propositions 5.3, 5.4 and 5.5.

Section 6 explains the derivation of correlation decay estimates from uniform stretching of Birkhoff sums. The main results of Section 6.1 are Corollary 2 that describes the fast decay of order at least $t^{-1/2-\varepsilon}$ on the good intervals that partition the complement of the bad set, and Corollary 3 that describes the decay of order $t^{-1/2+\varepsilon}$ on general intervals (with the bad set $\mathscr{B}_l$ included). Corollary 2 will directly yield the proof of Proposition 3.2 on fast decay outside $\mathscr{B}_l$, given in Section 6.2, while Corollary 3 is crucial in the bootstrap argument that yields the averaged decay on the set $\mathscr{B}_l$ of Proposition 3.3, given in Section 6.3.

Finally, in Section 7, we complete the proof of Theorem 2 and prove that the spectral type of Kochergin flows is Lebesgue with countable multiplicity. The proof that the spectral type is not just absolutely continuous, but indeed equivalent to the Lebesgue measure, is based on a new criterion for countable Lebesgue spectrum of smooth flows (Theorem 6) and on the construction of an arbitrary number of observables, localized on an arbitrarily long flow-box, which have given arbitrary correlation functions on a finite, but arbitrary long, time interval. The control of the correlation functions beyond this time is guaranteed by the estimates on correlation decay obtained in Sections 5 and 6.

The outline of the construction of the observables comes from the proof of the Lebesgue maximal spectral type for time changes of horocycle flows [16] . However, again in contrast with the case of time-changes of horocycle flows, whose phase space has dimension 3, for this approach to work in the case of surface flows, that is, in dimension 2, it is crucial that the constant in the estimates on the square integrals of correlations satisfy good bounds in terms of the smooth norms of the functions. For this reason, we will estimate carefully this dependence throughout the paper.

3 Decay of correlations for smooth coboundaries

3.1 Special flows above rotations of the circle.

Let $R_\alpha : \mathbb{T} \to \mathbb{T}$, $R_\alpha(\theta) = \theta + \alpha$ mod 1, where $\alpha \in \mathbb{T}$ is an irrational number with the sequence of denominators $(q_n)_{n=1}^{+\infty}$ and let $\varphi \in L^1(\mathbb{T}, \mathscr{B}, \lambda_{\mathbb{T}})$ be a strictly positive function. We denote by $d_{\mathbb{T}}$ the distance on the circle. We recall that the special flow $T^t := T^t_{\alpha,\varphi}$ constructed above R_α and under φ is the flow defined almost everywhere by

$$\begin{array}{rcl} (\mathbb{T}\setminus\{0\})\times\mathbb{R}/\sim & \to & (\mathbb{T}\setminus\{0\})\times\mathbb{R}/\sim \\ (\theta,s) & \to & (\theta,s+t), \end{array}$$

where $\sim$ is the identification, defined on $(\mathbb{T}\setminus\{0\})\times\mathbb{R}$,

$$(\theta, s+\varphi(\theta)) \sim (R_\alpha(\theta), s)\,.$$

Equivalently (see Figure 2), this special flow is defined for $t+s \geqslant 0$ (with a similar definition for negative times) by

$$T^t(\theta,s) = (\theta + N(\theta,s,t)\alpha, t+s-S_{N(\theta,s,t)}\varphi(\theta)),$$

where $N(\theta,s,t)$ is the unique integer such that

$$0 \leqslant t+s-S_{N(\theta,s,t)}\varphi(\theta) \leqslant \varphi(\theta + N(\theta,s,t)\alpha),$$

and

$$S_n\varphi(\theta) = \begin{cases} \varphi(\theta)+\ldots+\varphi(R_\alpha^{n-1}\theta) & \text{if} \quad n>0 \\ 0 & \text{if} \quad n=0 \\ -(\varphi(R_\alpha^n\theta)+\ldots+\varphi(R_\alpha^{-1}\theta)) & \text{if} \quad n<0. \end{cases}$$

3.2 Density of smooth coboundaries.

From now on we will fix $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and φ as in the statement of Theorem 2, that is: $\alpha \in D_{\log,\xi}$ as in (H_1) and $\varphi \in \mathscr{C}_\eta$ as in Definition 2.1, for some $\xi \in (0,1/10)$ and $\eta \in (0, \min(1/1000, \xi))$.

We recall the notations $M = \{(\theta,s) \in (\mathbb{T}\setminus\{0\})\times\mathbb{R} \,:\, s \leqslant \varphi(\theta)\}$ for the configuration space of the flow $T^t_{\alpha,\varphi}$ and μ for the measure equal to the restriction to M of the product of the Haar measures $\lambda_{\mathbb{T}}$ on the circle $\mathbb{T}$ and $\lambda_{\mathbb{R}}$ on the real line $\mathbb{R}$.

For a given $\zeta > 0$, let us denote

$$M_\zeta := \{(\theta,s) \in M \,:\, d_{\mathbb{T}}(\theta,0) > \zeta, \zeta < s < \varphi(\theta)-\zeta\}. \tag{1}$$

We recall that f is called a smooth coboundary over the flow $T^t_{\alpha,\varphi}$ if there exists a smooth function ϕ such that, for any $a<b$,

$$\int_a^b f(u,t)dt = \phi(u,b) - \phi(u,a).$$

The function ϕ is called the *transfer function* of f. The space of smooth coboundaries is dense in the subspace $L^2_0(M,\mu) \subset L^2(M,\mu)$ of zero average functions, provided $T^t_{\alpha,\varphi}$ is ergodic (which is always the case if α is irrational). Because of the singularity, we need a slightly more precise statement. For $\zeta > 0$, we denote by $\mathscr{F}_\zeta$ the set of smooth coboundaries f over $(T^t_{\alpha,\varphi}, M)$ such that the transfer function ϕ is supported inside M_ζ. Let $\mathscr{F} = \cup_{\zeta>0}\mathscr{F}_\zeta$.

Proposition 3.1. *The subspace $\mathscr{F}$ is dense in $L^2_0(M,\mu)$.*

Proof. Let us prove that the orthogonal space $\mathscr{F}^\perp \subset L^2_0(M,\mu)$ contains only the zero function. In fact, every function $g \in L^2_0(M,\mu)$, which belongs to the orthogonal space $\mathscr{F}^\perp \subset L^2_0(M,\mu)$, is by definition orthogonal to the Lie derivative along the flow of every smooth function with support contained in M_ζ for some $\zeta > 0$. It follows that for every $t>0$ the function $g\circ T^t_{\alpha,\varphi} - g$ is orthogonal to all smooth functions with support in M_ζ, for every $\zeta > 0$, hence it is orthogonal to all square-integrable functions, as the space of smooth functions with support contained in M_ζ for some $\zeta > 0$ is dense in $L^2(M,\mu)$. It follows that for any $t>0$, the function $g\circ T^t_{\alpha,\varphi} - g$ vanishes, hence g is invariant and constant by the ergodicity of the flow. As g has zero average, it is equal to the zero function. $\square$

3.3 Square integrable correlations for smooth coboundaries.

The proof that the flow $T^t_{\alpha,\varphi}$ has an absolutely continuous maximal spectral type follows by a standard argument from Proposition 3.1 and the following result.

Theorem 3. *For positive $\xi < 1/10$ and $\eta < \min(1/1000, \xi)$, let $(T^t_{\alpha,\varphi}, M, \mu)$ be the special flow over the rotation of angle $\alpha \in D_{\log,\xi}$ (see equation (H_1)) and under a ceiling function $\varphi \in \mathscr{C}_\eta$ (see Definition 2.1). Suppose that for some $\zeta > 0$, $f \in \mathscr{F}_\zeta$ and g is a smooth function on M supported in M_ζ, then the correlation function*

$$\mathscr{C}_{f,g}(t) := \int_M f(T^t_{\alpha,\varphi}(x))g(x)d\mu\,, \quad \textit{for all } t \in \mathbb{R}\,,$$

belongs to the space $L^2(\mathbb{R}, d\lambda_{\mathbb{R}})$.

All through the paper, the symbols $C_{f,g}$, $C'_{f,g}$, $C''_{f,g}$ will denote positive constants depending only on the C^1 norms of $f \in \mathscr{F}$ and $g \in C^1(M)$, and on the C^1 norm of the transfer function ϕ for $f \in \mathscr{F}$. They may change value from line to line. Theorem 3 immediately follows from

Theorem 4. *For every $f \in \mathscr{F}$ and $g \in C^1_0(M_\zeta)$ there exists a constant $C_{f,g} > 0$ such for all $l \in \mathbb{N}$, we have*

$$\int_{l^{21/20}}^{(l+1)^{21/20}} \left| \int_M f(T^t_{\alpha,\varphi}(x))g(x)d\mu \right|^2 dt < C_{f,g}\, l^{-1-\frac{\eta}{100}}.$$

We remark that although Theorem 4 only implies that correlations are square-integrable on the positive real line $\mathbb{R}^+$, Theorem 3 still follows by applying Theorem 4 to both the flow and the reverse flow, which is still a Kochergin flow in the same class, satisfying the same hypotheses on the roof function and on the rotation number.

Notation: For $l \in \mathbb{N}$, we denote $l_0 = l^{21/20}$, $l_1 = (l+1)^{21/20}$ and we let $n \in \mathbb{N}$ be the unique positive integer such that

$$q_n \leqslant l_0 < q_{n+1}\,. \tag{2}$$

These conventions will be followed throughout the paper. We remark that, as a consequence of the Diophantine condition (H_1), for l sufficiently large, we also have

$$q_n \leqslant l_1 < 2q_{n+1}\,. \tag{3}$$

As explained earlier, the proof of Theorem 4 relies on two steps: first, we construct a bad set $\mathscr{B}_l$ of measure less than to $t^{-1/2+\varepsilon}$ for some $\varepsilon > 0$ and we show a decay of correlations faster than $t^{-1/2-\varepsilon}$ outside $\mathscr{B}_l$; second, the correlations on $\mathscr{B}_l$ are controlled on average for $t \in [l^{21/20}, (l+1)^{21/20}]$. The two steps are summarized by the following two propositions.

Proposition 3.2. *There exists a set $\mathscr{B}_l \subset M$, $\mu(\mathscr{B}_l) < q_n^{-1/2+6\eta}$ such that for every $t \in [l_0, l_1]$, we have*

$$\left| \int_{M \setminus \mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu \right| < C_{f,g}\, t^{-1/2-\frac{\eta}{6}}.$$

Proposition 3.3. *We have*

$$\int_{l_0}^{l_1} \left| \int_{\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu \right| dt < C_{f,g}\, \frac{(l_1 - l_0)\mu(\mathscr{B}_l)}{q_n^{20\eta}}.$$

Let us show how these two propositions imply Theorem 4, and therefore Theorem 3 and the first part of Theorem 2, that is the absolute continuity of the spectrum.

Proof of Theorem 4. Let $F(t) := \frac{1}{\mu(\mathscr{B}_l)}\left|\int_{\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu\right|$ and let $G_l := \{t \in [l_0,l_1] \,:\, F(t) \geqslant q_n^{-7\eta}\}$. By Markov's inequality and Proposition 3.3, we have

$$|G_l| \leqslant C_{f,g}\frac{l_1-l_0}{q_n^{13\eta}}.$$

By splitting the integration below into G_l and G_l^c, and since $\mu(\mathscr{B}_l) < q_n^{-1/2+6\eta}$, and recalling that $C_{f,g}$ are positive constants depending only on f and g which may change value from line to line, we get

$$\begin{aligned}\int_{l_0}^{l_1}\left|\int_{\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu\right|^2 dt &\leqslant C_{f,g}\mu(\mathscr{B}_l)^2\,|G_l| + \mu(\mathscr{B}_l)^2 q_n^{-14\eta}(l_1-l_0)\\ &\leqslant C_{f,g}\frac{(l_1-l_0)\mu(\mathscr{B}_l)^2}{q_n^{13\eta}}\\ &\leqslant C_{f,g}\frac{(l_1-l_0)}{q_n^{1+\eta}} \leqslant C_{f,g}\frac{(l_1-l_0)}{q_{n+1}^{1+\eta/2}} \leqslant C_{f,g}\frac{(l_1-l_0)}{l_0^{1+\eta/2}}\\ &\leqslant C_{f,g}\frac{l^{1/20}}{l^{21/20(1+\eta/2)}} < C_{f,g}l^{-1-\frac{\eta}{2}}.\end{aligned}$$

Using this and Proposition 3.2, we have

$$\begin{aligned}\int_{l^{21/20}}^{(l+1)^{21/20}}\left|\int_M f(T^t_{\alpha,\varphi}(x))g(x)d\mu\right|^2 dt \leqslant 2C_{f,g}\int_{l_0}^{l_1} t^{-1-\frac{\eta}{3}}dt+&\\ 2\int_{l_0}^{l_1}\left|\int_{\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu\right|^2 dt \leqslant C_{f,g}l^{-1-\frac{\eta}{10}},&\end{aligned}$$

which finishes the proof of Theorem 4. □

The proofs of Propositions 3.2 and 3.3 will occupy the Sections 3, 4, 5, and will be concluded in Sections 6.2 and 6.3, respectively. The key estimates on the decay of correlations that we will gather in the Sections 3–5 will also be instrumental in the proof of the countable Lebesgue spectrum property that will be carried out in Section 7.

4 Stretching of Birkhoff sums

We recall that we have fixed $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and φ such that $\alpha \in D_{\log,\xi}$ (as in (H_1)) and $\varphi \in \mathscr{C}_\eta$ as in Definition 2.1, for some $\xi \in (0,1/10)$ and $\eta \in (0,\min(1/1000,\xi))$.

We collect in this section some important technical estimates of the Birkhoff sums of the ceiling function φ above the rotation R_α on $\mathbb{T}$. The estimates gathered here, although rather elementary to obtain, will be essential in establishing the uniform stretch properties that are the key in our proof of absolutely continuous spectrum of the Kochergin flow. The source of stretching of the Birkhoff sums is the passage of the orbit near the singularity. Hence it is natural to focus on the closest visit near the origin of an orbit segment of a point on the base.

We will adopt the following notation: for any $x \in M$ and $N \in \mathbb{N}$, we let

$$x_{min}^{N} = \min_{0 \leqslant j < N} d(\bar{x} + j\alpha, 0).$$

Before we state the stretching estimates, let us give a brief heuristic explanation of their content as well as their proof. Let us start with the second derivative. Suppose that we are interested in the Birkhoff sums $S_N \varphi''(x,s)$ for some point $(x,s) \in M$ and some $N \in \mathbb{N}$, N of order $t > 0$.[3] WLOG, let us suppose that $s = 0$. We know that, up to a multiplicative constant, φ'' is equivalent to $|\theta|^{-3+\eta}$ near the origin. Assume, very roughly, that $x_{min}^{N} \sim 1/N$ and that the next best visits to the origin are $2/N, 3/N, \ldots$. Since $1/2^{3-\eta} + 1/3^{3-\eta} + \ldots$ is summable, we see that $S_N(\varphi'')(x,0)$ is comparable to its leading term $S_N(\varphi'')(x,0) \approx (x_{min}^{N})^{-3+\eta} \approx N^{3-\eta}$. If we replace the exponent $3-\eta$ by $2-\eta$ we see that the same rationale apply to the sums of φ'.[4]

Of course this "even" distribution of the orbits does not hold for every $x \in M$ and $N \in \mathbb{N}$. However, bythe Diophantine condition (H_1), and the fact that we can discard a set of $x \in M$ of measure square summable in $t \in \mathbb{R}$, we can obtain a control of the second derivative of order $N^{3-\eta}$ up to some logarithmic correction.

This kind of estimate can be achieved by several computational methods. The one we opted for is the following. We split φ'' as a sum of two parts: $\bar{\varphi}_r''$, which is the restriction of φ'' in a $1/q_r$ neighborhood of the origin for an adequately chosen $r \in \mathbb{N}$ (as a function of the position of t relative to the denominators of α), and the difference function $\varphi'' - \bar{\varphi}_r''$. On the one hand, the latter is a bounded variation function so that we can apply the Denjoy-Koksma bounds to estimate its Birkhoff sums, when N is a denominator of α. Because of the Diophantine condition (H_1), we can then extend these estimates to general integers $N \in \mathbb{N}$ by the Ostrowski expansion of $N = \sum_{j=0}^{r} a_j q_j$, where $0 \leqslant a_j \leqslant \frac{q_{j+1}}{q_j}$. On the other hand, the Birkhoff sum $S_N \bar{\varphi}_r''$ accounts for the closest visit to the origin.

The lemmas and corollaries gathered in this section are all based on this approach, and since their proofs are sometimes technical and do not influence the global reading of the paper we relegate them to Appendix A.

Throughout this section we suppose fixed $l_0 = l^{21/20}$, $l_1 = (l+1)^{21/20}$, for $l \in \mathbb{N}$, and the unique integer n such that $q_n \leqslant l_0 < q_{n+1}$ (see formula (2)). We will be interested in Birkohff sums of the ceiling function $\varphi \in \mathscr{C}_\eta$ and of its derivatives at times $t \in [l_0, l_1]$.

We recall the notations $M = \{(\theta,s) \in \mathbb{T} \times \mathbb{R} \,:\, s \leqslant \varphi(\theta)\}$, and for any $\zeta > 0$,

$$M_\zeta = \{(\theta,s) \in M \,:\, d_{\mathbb{T}}(\theta,0) > \zeta, \zeta < s < \varphi(\theta) - \zeta\}.$$

For every $x \in M$ we will denote by $\bar{x} \in \mathbb{T}$ its first coordinate and, for any $t \in \mathbb{R}$, we will denote the first coordinate of $T_{\alpha,\varphi}^{t}(x) \in M$ by $\bar{T}_{\alpha,\varphi}^{t}(x)$.

Similarly, for any horizontal interval $I \subset M$, we will denote $\bar{I} \subset \mathbb{T}$ its vertical projection and by $\lambda(I)$ its (horizontal) Lebesgue measure, that is, the Lebesgue measure $\lambda_{\mathbb{T}}(\bar{I})$.

[3] By the unique ergodicity of the base rotation, the orbit of any point of M up to a time $t > 0$ under the special flow crosses the base of M a number of times at most of order t.

[4] This heuristic of course does not apply to φ (because the exponent at 0 of φ is $1-\eta < 1$), but unique ergodicity simply implies in this case that $S_N(\varphi)$ is of order N. Note also that sometimes, because in the Birkhoff sums of φ' there might be cancellations due to close visits to the origin from one or the other of its sides, we prefer to use lower bounds on $S_N(\varphi'')$ to give a lower bound on $|S_N(\varphi')|$. This is the case for example in Lemma 4.3 of Section 4.2.

4.1 Partial partition of M into horizontal intervals

To estimate the decay of correlations, we want to construct an almost complete partition of M into small horizontal intervals on which we have a fine control on the Birkhoff sums of the ceiling function φ and its derivatives, corresponding to times $t \in [l_0, l_1]$. To this end we want the following properties of the partial partition. First, the intervals must be sufficiently small so that the Birkhoff sums above them are smooth functions (the intervals must not meet the singularity before time l_1). Second, the intervals must not be too small because we want to see the stretching effect of the ceiling function above them. Third, the intervals must not go too close to the origin, otherwise they will spend too much time there, which is bad for the estimates on their uniform distribution under the flow. Finally, we need to make sure that it is possible to cover most of the set M with the intervals of the partial partition. We now define such a partition, which naturally involves the Diophantine condition and the asymptotics of the singularity at the origin.

Let $q_k \in [q_n \log^{15} q_n, q_n \log^{20} q_n]$ (such a q_k exists, for large $k \in \mathbb{N}$, by the Diophantine assumptions on α, because the ratio of the endpoints of this interval diverges faster than the ratio $q_{n+1}/q_n \leqslant \ln^{1+\xi} q_n$) and consider the partition $\mathscr{I}_k$ of $\mathbb{T}$ into intervals with endpoints $\{-i\alpha\}_{i=0}^{q_k-1}$.

By definition, the length $\lambda(\bar{I})$ of every interval $\bar{I} \in \mathscr{I}_k$ satisfies the inequalities

$$\|q_{k-1}\alpha\| \leqslant \lambda(\bar{I}) \leqslant \|q_{k-1}\alpha\| + \|q_k\alpha\|$$

(where $\|\cdot\|$ denotes the distance from the integer lattice).

Let $\mathscr{I}_{n,k} \subset \mathscr{I}_k$ denote the subset of intervals $\bar{I} \in \mathscr{I}_k$ such that $\bar{I} \cap [-q_n^{-3/5}, q_n^{-3/5}] = \emptyset$[5].

Define the family $\mathscr{W}_{n,k}$ of horizontal intervals

$$\mathscr{W}_{n,k} := \{\bar{I} \times \{s\} \ : \ \bar{I} \in \mathscr{I}_{n,k} \text{ and } 0 \leqslant s \leqslant \min_{\theta \in \bar{I}} \varphi(\theta)\}. \tag{4}$$

Let $W_{n,k} := \cup \mathscr{W}_{n,k} \subset M$ denote the union of all elements of the family $\mathscr{W}_{n,k}$.

Notice that $M_\zeta \subset W_{n,k}$ for $\zeta > q_n^{-3/5}$, and that since $t \leqslant l_1 \leqslant 2q_{n+1}$, by (3), and $\varphi > C_\varphi^{-1}$, for some $C_\varphi > 1$, for sufficiently large $n \in \mathbb{N}$ we have

$$N_t := \sup_{x \in M} N(x,t) \leqslant 2C_\varphi q_{n+1} \leqslant q_n \ln^{1+\xi} q_n \ll q_k.$$

By the definition of the partition $\mathscr{I}_k$ and of the partial partition $\mathscr{W}_{n,k}$ in formula (4), since $N_t \ll q_k$ and $q_n^{-3/5} \ll q_{k-1}^{-1}$, we have

$$0 \notin \bigcup_{i=0}^{N_t} R_\alpha^i(\bar{I}). \tag{5}$$

As a consequence of (5) the Birkhoff sum $S_{N(x,t)}(\varphi)$ is (twice) differentiable on I, for every $x \in I$ and $t \leqslant l_1$. This fact will be used repeatedly in the proofs.

Note also, that for any $t \in [l_0, l_1]$ we can focus in our estimates on intervals that are in $\mathscr{W}_{n,k}$ since the measure of the discarded intervals is then a square summable function of time. In addition, since the observables we consider vanish on intervals I such that $T_{\alpha,\varphi}^t(I) \subset W_{n,k}^c$ we can also discard such intervals and work with intervals $I \in \mathscr{W}_{n,k}$ such that $T_{\alpha,\varphi}^t(I) \cap W_{n,k} \neq \emptyset$.

[5] The choice of the exponent $3/5$ is a trade-off between two requirements. It is chosen away from $1/2$ to make sure that the measure of the discarded part remains square summable as a function of $t \in [l_0, l_1]$. At the same time, the exponent is chosen not too large to avoid visits of the orbit too close to the origin before time t.

In fact, we will restrict our analysis to points $x \in I \cap T_{\alpha,\varphi}^{-t}(W_{n,k})$. The reason is that we want to discard points whose orbit segment at time $t > 0$ ends with an ineffective long tail that gets "trapped" near the singularity and result in a lack of equidistribution. The removal of this orbits from our analysis is possible since, as we have observed, the measure of the sets $T^{-t}W_{n,k}$ (recall that $W_{n,k}$ is defined for $t \in [l_0, l_1]$) is a square summable function of time.

4.2 Birkhoff sums estimates

We start by gathering some estimates on the Birkhoff sums of the ceiling function $\varphi \in \mathscr{C}_\eta$ and of its derivatives in terms of the corresponding times of the flow and of the closest visit to the singularity. The proofs of these estimates, involving as we mentioned earlier Denjoy-Koksma type estimates, will be given in Appendix A.

In this section the symbol $C_\varphi, C'_\varphi, C''_\varphi > 1$ will denote constants depending only on the ceiling function $\varphi \in \mathscr{C}_\eta$ and which are not necessarily the same throughout the exposition.

Lemma 4.1. *There exists $r_\varphi \in \mathbb{N}$ and $C_\varphi > 1$ such that for every $r \geqslant r_\varphi$, every $x \in M$ and every $N \in [q_r, q_{r+1}]$, we have*

$$\varphi\left(x_{min}^N\right) + \frac{1}{2}q_r \leqslant S_N(\varphi)(\bar{x}) \leqslant \varphi\left(x_{min}^N\right) + 2q_{r+1}\,, \tag{6}$$

For $N \in \mathbb{N}$ and for every $x \in M$, we have

$$|S_N(\varphi')(\bar{x})| < C_\varphi (S_N(\varphi)(\bar{x}))^{\frac{2-\eta}{1-\eta}}\,, \tag{7}$$

and

$$|S_N(\varphi'')(\bar{x})| > C_\varphi^{-1}(S_N(\varphi)(\bar{x}))^{3-\eta}\log^{-3} N\,. \tag{8}$$

As a consequence, we have that for every $x \in M \cap (\mathbb{T} \times \{s\})$ and every $t \in \mathbb{R}$,

$$|S_{N(x,t)}(\varphi')(\bar{x})| < C_\varphi (s^{\frac{2-\eta}{1-\eta}} + t^{\frac{2-\eta}{1-\eta}}) \tag{9}$$

and

$$|S_{N(x,t)}(\varphi'')(\bar{x})| > C_\varphi^{-1}(t + s - \varphi(\bar{x} + N(x,t)\alpha))^{3-\eta}\log^{-3} N(x,t). \tag{10}$$

Uniform stretching of the Birkhoff sums on some interval allows to treat the image of the interval under the flow as if it was a long affine line, which allows for precise estimates of the correlations decay. In practice, such a control requires a lower bound on the derivatives of the Birkhoff sums of the ceiling function and an upper bound on their second derivatives.

A necessary condition for the estimates is a uniform upper bound on the *distortion*, that is, on the ratio between the second derivative and the square of the first derivative of the Birkhoff sums (see formula (17) and Definition 5.1 below).

We therefore collect below, in Lemmas 4.2 and 4.3, some lower bounds on the first derivative and upper bound on the second derivative of the Birkhoff sums of the ceiling function, which will be crucial in the sequel. The proofs of these lemmas are given in Appendix A.

For any interval $I \in \mathscr{W}_{n,k}$, we therefore introduce the notation

$$u_I := \sup_{t\in[l_0,l_1]} \sup_{x\in I} |S_{N(x,t)}(\varphi'')(\bar{x})|. \tag{11}$$

Lemma 4.2 below, proved in Appendix A, states that if u_I is sufficiently large, namely if $u_I \geqslant q_n^{3-\eta}\log^9 q_n$, then we have strong stretching for every $t \in [l_0,l_1]$ on all of $I_t = I \cap T_{\alpha,\varphi}^{-t}(W_{n,k})$. The reason for this is the following: $u_I \geqslant q_n^{3-\eta}\log^9 q_n$ implies that $x_{0_{min}}^{N(x_0,t_0)} \leqslant 1/(q_n\log^3 q_n)$ for some $(x_0,t_0) \in I \times [l_0,l_1]$ (as will be seen in Sublemma 1 in Appendix A). From there we can deduce that, for any t and any $x \in I_t$, we have that $x_{min}^{N(x,t)} \leqslant 2/(q_n\log^3 q_n)$. This can be explained as follows. Let $j < N(x_0,t_0)$ be such that $x_{0_{min}}^{N(x_0,t_0)}$ is achieved by $\bar{x}_0 + j\alpha$. If $N(x,t) \geqslant j$, then $x_{min}^{N(x,t)} \leqslant \|\bar{x}+j\alpha\| \leqslant \|\bar{x}_0+j\alpha\| + \|\bar{x}-\bar{x}_0\| \leqslant 2/(q_n\log^3 q_n)$ (since $\bar{x}$ and $\bar{x}_0$ are $1/(q_n\log^{15} q_n)$ close). If on the contrary $N(x,t) < j < N(x_0,t_0)$, then $|S_{N(x,t)}\varphi(\bar{x}) - S_{N(x_0,t_0)}\varphi(\bar{x})| \geqslant q_n^{1-\eta}$, which falls in contradiction with the assumption that $|t-t_0| \leqslant l_1 - l_0 \leqslant (2q_{n+1})^{1/21}$. The link between stretching and $x_{min}^{N(x,t)}$ is then recovered from Lemma 4.1 and its proof.

Lemma 4.2. *There exists a constant $C_\varphi > 1$ such that the following holds. Let $I \in \mathscr{W}_{n,k}$. If $u_I \geqslant q_n^{3-\eta}\log^9 q_n$, then for every $t \in [l_0,l_1]$ and for every $x \in I \cap T_{\alpha,\varphi}^{-t}(W_{n,k})$, we have*

$$x_{min}^{N(x,t)} \leqslant \frac{2}{q_n\log^3 q_n} \tag{12}$$

and

$$|S_{N(x,t)}(\varphi')(\bar{x})| \geqslant C_\varphi^{-1}|\varphi'(x_{min}^{N(x,t)})| \quad \text{and} \quad |S_{N(x,t)}(\varphi'')(\bar{x})| \leqslant C_\varphi|\varphi''(x_{min}^{N(x,t)})|. \tag{13}$$

Next, Lemma 4.3, also proved in Appendix A, deals with those intervals of the partial partition $\mathscr{W}_{n,k}$ where the condition $u_I \geqslant q_n^{3-\eta}\log^9 q_n$ does not hold. It exploits the lower bound on $S_{N(x,t)}(\varphi'')(\bar{x})$, given by formula (10) of Lemma 4.1, to give an affine approximation of $S_{N(x,t)}(\varphi')$ based on the intermediate value theorem. As we explained earlier, it is sometimes convenient to infer lower bounds on $S_{N(x,t)}(\varphi')$ from lower bounds on $S_{N(x,t)}(\varphi'')$, which are are easier to obtain since the sign of φ'' is the same on both sides of the singularity. This is what we do in the next lemma and in its immediate corollary.

Lemma 4.3. *Let $t \in [l_0,l_1]$, and let $x_0, x \in W_{n,k}$ with $|\bar{x}-\bar{x}_0| \geqslant q_n^{-3/2+2\eta}$ satisfy the conditions that $T_{\alpha,\varphi}^t(x_0), T_{\alpha,\varphi}^t(x) \in W_{n,k}$ and*

$$|S_{N(x_0)}(\varphi')(\bar{x}_0)| \leqslant q_n^{7/4+\eta} \quad \text{and} \quad |S_{N(x_0)}(\varphi'')(\bar{x}_0)| \leqslant q_n^{3-\eta}\log^{10} q_n. \tag{14}$$

Then, for some $A_{x,x_0} \geqslant q_n^{3-\eta}(\log^5 q_n)^{-1}$, we have

$$|S_{N(x)}(\varphi')(\bar{x}) - S_{N(x_0)}(\varphi')(\bar{x}_0) - A_{x,x_0}(\bar{x}-\bar{x}_0)| \leqslant \frac{A_{x,x_0}}{10}|\bar{x}-\bar{x}_0|.$$

Lemma 4.3 has the following straightforward consequence.

Corollary 1. *Let $I \in \mathscr{W}_{n,k}$. For some $x_0 \in I \cap T_{\alpha,\varphi}^{-t}(x)(W_{n,k})$, assume that*

$$|S_{N(x_0)}(\varphi')(\bar{x}_0)| < 3q_n^{3/2+\eta} \quad \text{and} \quad |S_{N(x_0)}(\varphi'')(\bar{x}_0)| < q_n^{3-\eta}\log^{10} q_n\,.$$

Then for every $x \in I$ such that $|\bar{x}-\bar{x}_0| \geqslant q_n^{-3/2+3\eta}$ either $T_{\alpha,\varphi}^t(x) \in W_{n,k}^c$, or (if $T_{\alpha,\varphi}^t(x) \in W_{n,k}$)

$$|S_{N(x)}(\varphi')(\bar{x})| \geqslant \frac{q_n^{3-\eta}}{2\log^5 q_n}|\bar{x}-\bar{x}_0|. \tag{15}$$

We end this section by giving a consequence of the Diophantine condition on the discrepancies of the base rotation relative to intervals.

Lemma 4.4. *There exists a constant $C(\alpha) > 0$ such that the following holds. Let $\bar{J} \subset \mathbb{T}$ be an interval. Then for every $N \in \mathbb{N}$ and every $\theta \in \mathbb{T}$*

$$|S_N(\chi_{\bar{J}})(\theta) - N\lambda(J)| \leqslant 2C(\alpha)^{-1} \log^{2+\xi} N\,.$$

Proof. Notice that by Denjoy-Koksma inequality, for every $j \in \mathbb{N}$ and $\theta \in \mathbb{T}$, we have

$$|S_{q_j}(\chi_{\bar{J}})(\theta) - q_j\lambda(J)| \leqslant 2. \tag{16}$$

To conclude, we write $N = \sum_{j=0}^{r} a_j q_j$, where $0 \leqslant a_j \leqslant \frac{q_{j+1}}{q_j}$ (it is called Ostrowski expansion of N) use the cocycle identity, the bound in (16) for $j = r, r-1, \ldots, 0$ and the fact that by our Diophantine condition $a_j \leqslant C(\alpha)^{-1}(\log q_j)^{1+\xi}$ for all $j \in \mathbb{N}$. □

5 Refined stretching estimates. Construction of the bad set $\mathscr{B}_l$

In this section we will refine the stretching estimates over the horizontal intervals $I \in \mathscr{W}_{n,k}$, where $\mathscr{W}_{n,k}$ is the partial partition introduced in Section 4.1, see (4). The section contains preparatory material for the estimates on decay of correlations which will be carried out in Section 6.

In Section 5.1 (see Definition 5.1) we introduce a family of horizontal intervals, subintervals of the intervals of $\mathscr{W}_{n,k}$, which we call *good*. In Section 6.1, Corollary 2, we will see that the decay of correlations on good intervals can be bounded above by a square integrable function of time. Unfortunately, because of the cancellations due to the symmetry of the ceiling function at the singularity, the complementary set of the family of good intervals has a non square-summable measure and therefore cannot be neglected.

In Sections 5.2 and 5.3, we construct a complementary set of the union of good intervals, which we call the *bad set*, and analyze its properties. This technical part of the paper can be skipped in a first reading. What has to be retained for reading through the rest of the paper are Definitions 5.1 and 5.2 of good intervals and complete towers, and Propositions 5.3 and 5.4 which give the definition and the properties of the bad set, and of its complement, the good intervals.

In Section 6.3, we will prove that the decay of correlations on the bad set can be controlled in average over the time interval $[l_0, l_1]$. The argument will require estimates of the measure of the bad set and of its geometry. The goal of Section 5.3 is to carry out these preliminary estimates.

For the sake of the reader, before we give the precise statements, we give an outline of the construction of the bad set and of our strategy to derive correlation estimates from control of its geometry. Recall that we are interested in times $t \in [l_0, l_1]$, where $l_0 = l^{21/20}$, $l_1 = (l+1)^{21/20}$ for a fixed $l \in \mathbb{N}$, and we let n be the unique integer such that $q_n \leqslant l_0 < q_{n+1}$.

Let J be a small horizontal interval in M of size $o(1/q_n)$. Uniform stretching at time t over J can roughly be measured by the distortion ratio $S_N(\varphi'')/S_N(\varphi')^2$ at any point x of J and at the discrete time $N = N(x,t)$. When the distortion ratio is less than $t^{-1/2-\varepsilon}$, for some $\varepsilon > 0$, then we can say that the interval is good, because the decay of correlations of coboundaries on this interval is bounded above by a square summable function of time. This is the content of Proposition 5.4 and of Sections 6.1 and 6.2.

Because of our standing assumptions (H_1) on the rotation number α and $(H_{2.1})$–$(H_{2.3})$ on the singularity of the ceiling function, we can see that, roughly speaking, each interval I of $\mathscr{W}_{n,k}$[6] can be decomposed into at most three subintervals, all *good* for all $t \in [l_0, l_1]$, except possibly for a *bad* subinterval that has length at most $q_n^{-3/2+\varepsilon}$ (see Properties (B_3) and (B_4) of Proposition 5.3). In the estimate of the length of the bad interval, the fact that $1-\eta$ is away from $1/2$ is essential.

Now, the bad set is a collection of strips above q_k small intervals of length at most $q_n^{-3/2+\varepsilon}$, which are $1/q_k$ well-distributed on the base (see Properties (B_2) and (B_3) of Proposition 5.3). Being bad is a property that propagates for a time interval that is relatively small compared to t. Hence we can divide the bad set into a finite collection $\{U_1, \dots, U_m\}$ of Rokhlin towers for the flow of height $h = q_n^{3/5}$ (see Property (B_1) of Proposition 5.3, and Figure 6).

The important information that we gain from decomposing the bad set into this collection of towers is that, for any $t \in [l_0, l_1]$, we can view $T^t_{\alpha,\varphi}(U_i)$ as well as a Rokhlin tower of the flow of height h (see Property (B_5) of Proposition 5.3). Indeed, this will allow to decompose $T^t_{\alpha,\varphi}(U_i)$ into horizontal slices, each a union of horizontal intervals (of size less than $q_n^{-3/2}$) almost uniformly distributed inside an interval of size $q_n^{-1/4}$, see Figure 7 (this decomposition will be studied in Definition 6.5 and the lemmas that follow it).

In turn, the latter property allows to see why we have square summable averaged decay of correlations on each of the towers U_i. To understand this, note first that since the measure of the bad set is already small (not square summable small, but almost so, as explicitly stated in Property (B_2) of Proposition 5.3), it is sufficient to prove a weak estimates on the average decay of correlations on the sets U_i. More precisely, we prove in Proposition 6.4, that for every $i \in \{1, \dots, m\}$, we have

$$(*) \qquad \int_{l_0}^{l_1} \left| \int_{U_i} f(T^t_{\alpha,\varphi}(x)) g(x) d\mu \right| dt < C_{f,g} \frac{(l_1 - l_0)\mu(U_i)}{q_n^{20\eta}}.$$

Note that the mixing rate $q_n^{-20\eta}$ here is sufficient due to the fact that $\mu(\cup U_i) = \mu(\mathscr{B}_l) \leqslant q_n^{-1/2+6\eta}$ as required in the construction of the bad set $\mathscr{B}_l$ (see Property B_2 of Proposition 5.3).

Let us explain the heuristics in the proof of the key mixing estimate $(*)$.

Assume indeed that for some time t^*, the extreme opposite of fast mixing happens on U_i, that is $T^{t^*}_{\alpha,\varphi} \sim \mathrm{Id}$ on U_i, for example at $t^* = q_n$.[7] To give an idea of how $(*)$ is proved we need to see why for $\tau \geqslant (l_1 - l_0)^{1/2}$, we have that $\left| \int_{U_i} f(T^{t^*+\tau}_{\alpha,\varphi}(x)) g(x) d\mu \right| < C_{f,g}\mu(U_i) q_n^{-20\eta}$. This follows from the fact that $T^{t^*}_{\alpha,\varphi}(U_i)$ can be decomposed into horizontal slices of diameter $q_n^{-1/4}$ inside which the trace of U_i is uniformly distributed, and from the mixing estimates of $T^{\tau}_{\alpha,\varphi}$ at rate $\tau^{-1/4}$ on the intervals of length $q_n^{-1/4}$, as obtained in Corollary 3.[8]

[6] Recall that $q_k \in [q_n \log^{15} q_n, q_n \log^{20} q_n]$ and that the projection of I on the base is an interval of the dynamical partition with endpoints $\{-i\alpha\}_{i=0}^{q_k-1}$.

[7] This assumption in fact cannot hold on all of U_i because near the boundary of bad intervals, the uniform stretching that holds on the adjacent good intervals still holds approximately. However, in the bulk of the bad intervals this scenario is possible and is indeed responsible for insufficient mixing estimates at times like q_n (or $2q_n$). It is to deal with this kind of rigidity, due to symmetry cancellations in the ceiling function, that we have to resort to averaged decay.

[8] A mixing rate $\tau^{-1/4}$ is far from the rate we established on good intervals, and holds on all intervals, as proved below in Section 6.1, Corollary 3.

5.1 Good intervals and the bad set

For a horizontal interval $I \in \mathscr{W}_{n,k}$, we denote $I_t := I \cap T_{\alpha,\varphi}^{-t}(W_{n,k})$. If $I_t \neq \emptyset$, the quantity that measures uniform stretching on I is the ratio

$$S_I^t := \inf_{x \in I_t} \frac{(S_{N(x,t)}(\varphi')(\bar{x}))^2}{S_{N(x,t)}(\varphi'')(\bar{x})}, \tag{17}$$

where we set $S_I^t = +\infty$ whenever $I_t = \emptyset$.

Definition 5.1 (Good intervals)**.** *An subinterval $J = [u,v] \subset I \in \mathscr{W}_{n,k}$ is called* good *if for every $t \in [l_0, l_1]$, at least one of the following holds: either*

$$S_J^t \geqslant t^{\frac{1}{2}+2\varepsilon} \tag{18}$$

or for some choice of $x^* \in I$ and for every $x \in J_t := J \cap T_{\alpha,\varphi}^{-t}(W_{n,k})$, we have

$$|S_{N(x,t)}(\varphi'')(\bar{x})| < q_n^{3-\eta} \log^9 q_n \text{ and } |S_{N(x,t)}(\varphi')(\bar{x})| \geqslant \frac{1}{2} q_n^{3/2+\eta} + \frac{1}{2} \frac{q_n^{3-\eta}}{\log^6 q_n} |\bar{x} - \bar{x}^*|. \tag{19}$$

When estimates (18) *or* (19) *hold for a given t, we say that J is t-good.*

In what follows $I \in \mathscr{W}_{n,k}$ will be a horizontal interval and $h = q_n^{3/5}$. Then we know that the iterates $R_\alpha^i(\bar{I})$ for $i = 0, \ldots, h$ are all disjoint and do not contain 0.

Recall the definition of u_I given in (11):

$$u_I := \sup_{t \in [l_0, l_1]} \sup_{x \in I} |S_{N(x,t)}(\varphi'')(\bar{x})|.$$

Moreover whenever $I_t := I \cap T_{\alpha,\varphi}^{-t}(W_{n,k}) \neq \emptyset$, we define

$$r_I^t = \inf_{x \in I_t} |S_{N(x,t)}(\varphi')(\bar{x})| \tag{20}$$

(if $I_t = \emptyset$ we may define $r_I^t = +\infty$). We also let

$$r_I = \inf_{t \in [l_0, l_1]} r_I^t. \tag{21}$$

For any $\bar{I} \in \mathscr{I}_{n,k}$, let I_φ denote the vertical strip

$$I_\varphi := \{(\theta, s) \in M : \theta \in \bar{I}, 0 \leqslant s \leqslant \min_{\theta \in \bar{I}} \varphi(\theta)\}.$$

Definition 5.2 (Complete towers)**.** *Fix a horizontal interval $I \subset M \cap (\mathbb{T} \times \{s\})$ centered at z and a number $h > 0$. A complete tower of 'height' h above the interval I is the set:*

$$\bigcup_{i=0}^{N(z,h)} (R_\alpha^i(\bar{I}))_\varphi \setminus \cup_{t=0}^{s} T_{\alpha,\varphi}^t(\bar{I} \times \{0\}).$$

We now describe the bad set for correlations $\mathscr{B}_l$ (see Figure 6).

Proposition 5.3. *There exists a set $\mathscr{B}_l \subset M$ with the following properties:*

(B_1) *$\mathscr{B}_l = U_1 \cup \cdots \cup U_m$ where U_i are disjoint complete towers with heights $h = q_n^{3/5}$ over horizontal intervals $B_i \subset W_{n,k}$ with horizontal measure $\lambda(B_i) = 2q_n^{-3/2+5\eta}$;*

(B_2) *$\mu(\mathscr{B}_l) \leqslant q_n^{-1/2+6\eta}$;*

(B_3) *for every interval $I \in \mathscr{W}_{n,k}$, we have $I = J_1 \sqcup J_2 \sqcup I_{bad}$ where either $I \cap \mathscr{B}_l = \emptyset$ and I_{bad} and J_2 are empty, or I_{bad} is a level of some U_i and J_1, J_2 are intervals. When I_{bad} is not empty, we denote by x_{bad} its center.*

(B_4) *for every interval $I \in \mathscr{W}_{n,k}$ and every $t \in [l_0, l_1]$, we have one of the following*

$(B_4.i)$ *$r_I^t \geqslant q_n^{3/2+\eta}$,*

$(B_4.ii)$ *$r_I^t < q_n^{3/2+\eta}$, $I_{bad} \neq \emptyset$, $u_I \leqslant q_n^{3-\eta} \log^9 q_n$ and for every $x \in J_1 \sqcup J_2$ s.t. $T^t_{\alpha,\varphi} x \in W_{n,k}$*

$$|S_{N(x,t)}(\varphi')(\bar{x})| \geqslant \frac{q_n^{3-\eta}}{\log^6 q_n} |\bar{x} - \bar{x}_{bad}|$$

(B_5) *For every $t \in [l_0, l_1]$, for every $i \in [1, m]$, there exists a complete tower $\mathscr{T}_{t,i}$ over an interval $B_{t,i} = [\theta_{t,i} - q_n^{-3/2+5\eta}, \theta_{t,i} + q_n^{-3/2+5\eta}] \times \{s_{t,i}\} \subset M$ of height $h_{t,i} \geqslant q_n^{3/5-1/50}$ such that*

$$\mu\big(\big(T^t_{\alpha,\varphi}(U_i) \triangle \mathscr{T}_{t,i}\big) \cap M_\zeta\big) \leqslant q_n^{-1+10\eta}.$$

Proposition 5.4. *In the decomposition $I = J_1 \sqcup J_2 \sqcup I_{bad}$ of (B_3), we have that J_1 and J_2 are good.*

Proof of Proposition 5.4. Let $t \in [l_0, l_1]$. If $r_I^t < q_n^{3/2+\eta}$ then (19) holds on J_1 and J_2 (with $x^* = x_{bad}$) by Proposition 5.3, part $(B_4.ii)$, since for $x \in J_1 \cup J_2$ we have that $|\bar{x} - \bar{x}_{bad}| \geqslant q_n^{-3/2+5\eta}$.

Now, if $r_I^t \geqslant q_n^{3/2+\eta}$, then we will actually establish that all of I is t-good (which in particular implies the conclusion of Proposition 5.4 in this case):

Lemma 5.5. *For any $t \in [l_0, l_1]$, if $r_I^t \geqslant q_n^{3/2+\eta}$, then I is t-good.*

Proof of Lemma 5.5. **Case 1:** $u_I \geqslant q_n^{3-\eta} \log^9 q_n$**.**

In this case we do not use the assumption $r_I^t \geqslant q_n^{3/2+\eta}$. By Lemma 4.2 there exist constants $C_\varphi, C'_\varphi > 1$ such that, for l_0 sufficiently large, for every $t \in [l_0, l_1]$ and every $x \in I \cap T^{-t}_{\alpha,\varphi}(W_{n,k})$,

$$S_I^t = \inf_{x \in I_t} \frac{\big(S_{N(x,t)}(\varphi')(\bar{x})\big)^2}{|S_{N(x,t)}(\varphi'')(\bar{x})|} \geqslant C_\varphi^{-1} \frac{\varphi'(x_{min}^{N(x,t)})^2}{\varphi''(x_{min}^{N(x,t)})}$$
$$\geqslant (C'_\varphi)^{-1} \Big(\frac{1}{x_{min}^{N(x,t)}}\Big)^{2(2-\eta)-(3-\eta)} \geqslant \Big(\frac{1}{x_{min}^{N(x,t)}}\Big)^{2/3} \geqslant q_n^{2/3} \geqslant t^{3/5}.$$

The last inequality holds because of $t < q_{n+2}$ and the Diophantine assumptions on α. This shows that I satisfies (18) and hence finishes the proof of Lemma 5.5 in this case.

Case 2: $u_I < q_n^{3-\eta}\log^9 q_n$.

Notice first that if $r_I^t \geqslant q_n^{7/4+\frac{\eta}{2}}$ (see (20) for the definition of r_I^t), then either $x \in T_{\alpha,\varphi}^{-t}(W_{n,k}^c)$ or

$$S_I^t = \inf_{x\in I_t} \frac{(S_{N(x,t)}(\varphi')(\bar{x}))^2}{S_{N(x,t)}(\varphi'')(\bar{x})} \geqslant \frac{q_n^{7/2+\eta}}{q_n^{3-\eta}\log^9 q_n} \geqslant q_n^{1/2+\eta} \geqslant t^{1/2+\varepsilon},$$

where the last inequality again holds because of $t < q_{n+2}$ and assumptions on α. Therefore (18) holds for I and the proof is finished in this case .

Let us consider only $x \in I$ such that $T^t_{\alpha,\varphi}(x) \in W_{n,k}$. If $r_I^t < q_n^{7/4+\eta/2}$, let $x_0 \in I$ be such that $|S_{N(x_0,t)}(\varphi')(\bar{x}_0)| = r_I^t$. Let us assume WLOG that $S_{N(x_0,t)}(\varphi')(\bar{x}_0) > 0$. Then by Lemma 4.3, whenever $\bar{x} \geqslant \bar{x}_0 + q_n^{-3/2+2\eta}$, we have

$$|S_{N(x,t)}(\varphi')(\bar{x})| \geqslant \frac{q_n^{3-\eta}}{2\log^5 q_n}|\bar{x}-\bar{x}_0| \geqslant \frac{2q_n^{3-\eta}}{\log^6 q_n}|\bar{x}-\bar{x}_0|. \tag{22}$$

If $\bar{x} < \bar{x}_0 - \frac{1}{q_n^{3/2-2\eta}}$, then $S_{N(x,t)}(\varphi')(\bar{x}) < 0$. Indeed, otherwise by Lemma 4.3 we have

$$0 \leqslant S_{N(x,t)}(\varphi')(\bar{x}) < S_{N(x_0,t)}(\varphi')(\bar{x}_0) + \frac{q_n^{3-\eta}}{2\log^5 q_n}(\bar{x}-\bar{x}_0) \leqslant S_{N(x_0,t)}(\varphi')(\bar{x}_0) - q_n,$$

which is a contradiction with the choice of x_0. Therefore we have $S_{N(x,t)}(\varphi')(\bar{x}) < 0$ and, by Lemma 4.3 and by the definition of x_0, we derive

$$|S_{N(x,t)}(\varphi')(\bar{x})| \geqslant \frac{q_n^{3-\eta}}{4\log^5 q_n}|\bar{x}-\bar{x}_0| \geqslant \frac{2q_n^{3-\eta}}{\log^6 q_n}|\bar{x}-\bar{x}_0|. \tag{23}$$

Then by (22) and (23) and since $r_I^t \geqslant q_n^{3/2+\eta}$, we get that (19) is satisfied with $x^* := x_0$. This finishes the proof in Case 2. and Lemma 5.5 is established. □

The proof of Proposition 5.4 is hence finished. □

5.2 Construction of the bad set $\mathscr{B}_l$

Recall that the partition $\mathscr{I}_k$ is given by two towers i.e. disjoint sets of the form $\{B+i\alpha\}_{i=0}^{q_k}$ and $\{C+i\alpha\}_{i=0}^{q_{k-1}}$ where B, C are intervals around 0 of length $\|q_{k-1}\alpha\|, \|q_k\alpha\|$ respectively. Denote $D_1 = B+\alpha, D_2 = C+\alpha$ (the shift comes from the fact that we want to stay away from the singularity). The following construction works for $D = D_1, D_2$. We will present it for the tower above $D = D_1$, the other case being analogous. Consider a complete tower $\mathscr{D}$ of height $H_k = q_k - 1$ over D. Notice that $\mathscr{D} \cap W_{n,k}$ is a union of horizontal intervals of length $\lambda(D)$. Moreover there is a natural order on horizontal intervals in $\mathscr{D} \cap W_{n,k}$ (coming from the order on $\mathscr{D}$): each interval in $\mathscr{D} \cap W_{n,k}$ is of the form $D(h)$ for some $0 \leqslant h \leqslant H_k$ (with $D(0) = D$).

Let $0 \leqslant h_1 \leqslant H_k$ be the smallest real number such that $D(h_1) \subset \mathscr{D} \cap W_{n,k}$ and $r_{D(h_1)} \leqslant 2q_n^{3/2+\eta}$. Let $t_1 \in [l_0, l_1]$ and $x_1 := (\theta_1, s_1) \in D(h_1)$ be such that

$$T^{t_1}_{\alpha,\varphi}x_1 \in W_{n,k} \text{ and } S_{N(\theta_1,t_1)}(\varphi')(\theta_1) \leqslant 2q_n^{3/2+\eta}.$$

Let U_1 be the complete tower of height $q_n^{3/5}$ over

$$B_1 := \Big([-q_n^{-3/2+5\eta} + \theta_1, \theta_1 + q_n^{-3/2+5\eta}] \times \{s_1\}\Big) \cap \mathscr{D}.$$

Let k_2 be the largest number such that $D(k_2) \subset \mathscr{D} \cap W_{n,k}$.

Now inductively let $H_k \geqslant h_i \geqslant k_i$ be the smallest real number such that $D(h_i) \subset \mathscr{D} \cap W_{n,k}$ and $r_{D(h_i)} \leqslant 2q_n^{3/2+\eta}$. Let $t_i \in [l_0, l_1]$ and $x_i := (\theta_i, s_i) \in D(h_i)$ be such that

$$T^{t_i}_{\alpha,\varphi} x_i \in W_{n,k} \text{ and } S_{N(\theta_i,t_i)}(\varphi')(\theta_i) \leqslant 2q_n^{3/2+\eta}. \tag{24}$$

We define U_i to be the complete tower of height $q_n^{3/5}$ over

$$B_i := \Big([-q_n^{-3/2+5\eta} + \theta_i, \theta_i + q_n^{-3/2+5\eta}] \times \{s_i\}\Big) \cap \mathscr{D}.$$

We continue this procedure until the last possible $h_m \leqslant H_k$ is defined.

Let us define

$$\mathscr{B}_l := \bigcup_{1\leqslant i\leqslant m} U_i. \tag{25}$$

Now, (B_1) and (B_3) follow by construction (notice that the top of U_i is below the base of U_{i+1}). Moreover, by Lemma 1 we get that $S_{q_k-1}(\varphi)(\alpha) \leqslant cq_{k+1}$, hence (B_2) follows from

$$\mu(\mathscr{B}_l) \leqslant S_{q_k}(\varphi)(\alpha)\lambda(B_i) \leqslant \frac{1}{q_n^{1/2-6\eta}}.$$

It remains to prove (B_4) and (B_5), which will be the subject of the next section.

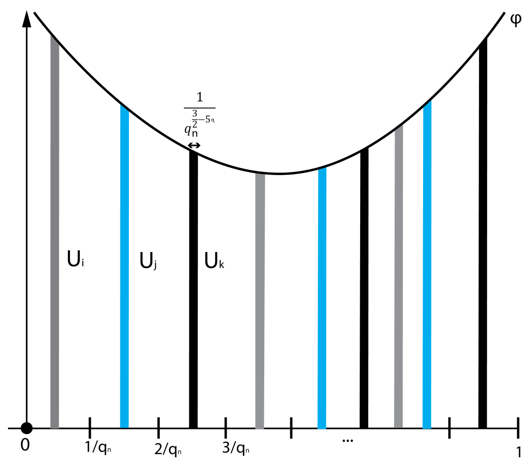

Figure 6: The set $\mathscr{B}_l$ is a union of complete towers U_i.

5.3 Proving the properties of the bad set

In this section we give the proofs of (B_4) and (B_5) in Proposition 5.3

Proof of (B_4). Fix $t \in [l_0, l_1]$. By the construction of $\mathscr{B}_l$, whenever for a partition interval $I \in \mathscr{W}_{n,k}$ we have $r^t_I \leqslant q_n^{3/2+\eta}$, then

$$I \cap \mathscr{B}_l = I_{bad},$$

where I_{bad} is a level of some U_i. In fact, otherwise $I \cap \mathscr{B}_l = \emptyset$ and by construction, Properties (B_3) and (B_4), we have $r_I > 2q_n^{3/2+\eta}$, where r_I is as in (21). Therefore we need to show $(B_4.ii)$ for $I \in \mathscr{W}_{n,k}$ such that $I_{bad} \neq \emptyset$ and $r_I^t < q_n^{3/2+\eta}$. Then, by definition, there exists $x_I^t \in I$ such that

$$T_{\alpha,\varphi}^t(x_I^t) \in W_{n,k} \text{ and } S_{N(x_I^t,t)}(\varphi')(\bar{x}_I^t) \leqslant q_n^{3/2+\eta}. \tag{26}$$

Notice that we have

$$u_I < q_n^{3-\eta} \log^9 q_n. \tag{27}$$

Indeed, if not, then by Lemma 4.2 we would get by (12) and (13) that $S_{N(x_I^t,t)}(\varphi')(\bar{x}_I^t) \geqslant q_n^{2-\eta}$, which is a contradiction with (26).

Notice that by (26) and (27), the assumptions of Corollary 1 are satisfied with $x_0 = x_I^t$. Therefore, for every $x \in I$ such that $T_{\alpha,\varphi}^t(x) \in W_{n,k}$ and $|x - x_I^t| \geqslant q_n^{-3/2+3\eta}$, we have

$$|S_{N(x,t)}(\varphi')(\bar{x})| \geqslant \frac{q_n^{3-\eta}}{2\log^5 q_n} |\bar{x} - \bar{x}_I^t|. \tag{28}$$

We claim that

$$|\bar{x}_I^t - \bar{x}_{bad}| \leqslant q_n^{-3/2+4\eta}. \tag{29}$$

Now, (28), (27) and (29) will finish the proof of $(B_4.ii)$ since for $x \in J_1 \sqcup J_2 = I \setminus I_{bad}$, we have that $|\bar{x} - \bar{x}_I^t| \geqslant |\bar{x} - \bar{x}_{bad}| - |\bar{x}_I^t - \bar{x}_{bad}| \geqslant q_n^{-3/2+3\eta}$.

Thus it only remains to show our claim (29). By construction of the U_i's, for some $h > 0$, we can write

$$x_{bad} = T_{\alpha,\varphi}^h x_i.$$

Moreover, since U_i is a complete tower of height $q_n^{3/5}$ and $T_{\alpha,\varphi}^h x_i \in U_i$, we have that

$$h \leqslant S_{N(x_i,q_n^{3/5})}(\varphi)(\bar{x}_i) + \varphi(\bar{x}_i + N(x_i,h)\alpha).$$

Since $x_i \in W_{n,k}$, we get by the definition of special flow

$$S_{N(x_i,q_n^{3/5})}(\varphi)(\bar{x}_i) \leqslant 2q_n^{3/5}.$$

Moreover, since $T_{\alpha,\varphi}^h x_i \in W_{n,k}$, we have

$$\varphi(\bar{x}_i + N(x_i,h)\alpha) \leqslant 2q_n^{3/5}.$$

By putting together the above bounds, we get

$$h < 2q_n^{3/5+1/50}. \tag{30}$$

Let $m_i := \max(t_i,t)$. We will show that

a. $T_{\alpha,\varphi}^{m_i}(T_{\alpha,\varphi}^h x_i), T_{\alpha,\varphi}^{m_i}(x_I^t) \in W_{n,k}$;

b. $|S_{N(x_I^t,m_i)}(\varphi')(\bar{x}_I^t)| \leqslant 2q_n^{3/2+\eta}$;

c. $|S_{N(T_{\alpha,\varphi}^h x_i,m_i)}(\varphi')(\bar{T}_{\alpha,\varphi}^h x_i)| \leqslant 5q_n^{3/2+\eta}$.

The above properties will give (29) (and hence $(B4.ii)$), since if $|\bar{T}^h_{\alpha,\varphi}x_i - \bar{x}^t_I| \geqslant q_n^{-3/2+4\eta}$ then by (27) and a., b., the assumptions of Corollary 1 are satisfied with $x_0 = x^t_I$, $x = T^h_{\alpha,\varphi}x_i$ but then c. is in contradiction with estimate (15) stated there. It remains then to show a., b., c.

For a. we notice that by (24) and (26) we have $T^{t_i}_{\alpha,\varphi}x_i, T^t_{\alpha,\varphi}x^t_I \in W_{n,k}$. Moreover, by the immediate bound $|m_i - t| \leqslant l_1 - l_0 < q_n^{1/10}$ and by (30), we have the estimate

$$0 \leqslant m_i - t, m_i - t_i + h \leqslant 2q_n^{3/5+1/50} + q_n^{1/10} \leqslant 3q_n^{3/5+1/50}, \tag{31}$$

from which we derive that

$$\{T^{m_i}_{\alpha,\varphi}(T^h_{\alpha,\varphi}x_i), T^{m_i}_{\alpha,\varphi}(x^t_I)\} = \{T^{m_i-t_i+h}_{\alpha,\varphi}(T^{t_i}_{\alpha,\varphi}x_i), T^{m_i-t}_{\alpha,\varphi}(T^t_{\alpha,\varphi}x^t_I)\} \subset W_{n,k}.$$

This gives a.

For b. we first notice that since $T^t_{\alpha,\varphi}(x^t_I) \in W_{n,k}$ and $|m_i - t| \leqslant l_1 - l_0 < q_n^{1/10}$, by (9), we have

$$S_{N(x^t_I, m_i-t)}(\varphi')(\bar{T}^t_{\alpha,\varphi}(x^t_I))| \leqslant q_n^{3/2+\eta}$$

and by (26), $|S_{N(x^t_I,t)}(\varphi')(x^t_I)| \leqslant q_n^{3/2+\eta}$. By the cocycle identity, we then have

$$|S_{N(x^t_I,m_i)}(\varphi')(x^t_I)| \leqslant |S_{N(x^t_I,t)}(\varphi')(\bar{x}^t_I)| + |S_{N(x^t_I,m_i-t)}(\varphi')(\bar{T}^t_{\alpha,\varphi}(x^t_I))| \leqslant 2q_n^{3/2+\eta}.$$

This gives b.

For c., by cocycle identity, (24), (31) and (9) (for $T^{t_i}_{\alpha,\varphi}(x_i) \in W_{n,k}$), we get

$$|S_{N(x_i,m_i+h)}(\varphi')(\bar{x}_i)| \leqslant |S_{N(x_i,t_i)}(\varphi')(\bar{x}_i)| + |S_{N(T^{t_i}_{\alpha,\varphi}(x_i),m_i+h-t_i)}(\varphi')(\bar{T}^{t_i}_{\alpha,\varphi}(x_i))| \leqslant 2q_n^{3/2+\eta}. \tag{32}$$

Since $x_i \in W_{n,k}$, by (30) and (9), we have

$$|S_{N(x_i,h)}(\varphi')(\bar{x}_i)| \leqslant 2q_n^{3/2+\eta}. \tag{33}$$

Finally from the cocycle identity, (32) and (33) we conclude that

$$|S_{N(T^h_{\alpha,\varphi}x_i,m_i)}(\varphi')(\bar{T}^h_{\alpha,\varphi}x_i)| \leqslant |S_{N(x_i,m_i+h)}(\varphi')(\bar{x}_i)| + |S_{N(x_i,h)}(\varphi')(\bar{x}_i)| \leqslant 5q_n^{3/2+\eta}.$$

This finishes the proof of c. and hence also $(B4.ii)$.

Proof of (B_5).

Let s_i be such that $x_i \in D(h_i) \subset \mathbb{T} \times \{s_i\}$ ($D(h_i)$ is the base of U_i). Let $t^* \in [t, t-1]$ be such that for $z_{t,i} = (\theta_{t,i}, s_{t,i}) := T^{t^*}_{\alpha,\varphi}x_i$ we have

$$B_{t,i} := [\theta_{t,i} - \frac{1}{q_n^{3/2-5\eta}}, \theta_{t,i} + \frac{1}{q_n^{3/2-5\eta}}] \times \{s_{t,i}\} \subset M,$$

Let $h_{t,i} := S_{N(x_i,q_n^{3/5})}(\varphi)(x_i) - s_i - (t^* - t)$ and let $\mathscr{T}_{t,i}$ be the complete tower of height $h_{t,i}$ over $B_{t,i}$. Notice that $s_i \leqslant q_n^{3/5(1-\eta)}$ and (by (6)) $S_{N(x_i,q_n^{3/5})}(\varphi)(x_i) \geqslant q_n^{3/5}\log^{-10} q_n$, hence $h_{t,i} \geqslant q_n^{3/5-1/50}$.

The difference between $U_i \cap M_\zeta$ and $T^t_{\alpha,\varphi}(\mathscr{T}_{t,i}) \cap M_\zeta$ will come from the stretching of Birkhoff sums of the top and at the base of $\mathscr{T}_{t,i}$ and from the difference $|t^* - t| \leqslant 1$. The measure of the symmetric difference between the two sets is twice the maximal stretching times the measure of the base of $\mathscr{T}_{t,i}$. First let us estimate the maximal stretch.

For any $z \in B_{t,i}$ there exists $\xi_i \in [\bar{z}, \theta_{t,i}]$ such that

$$|S_{N(\theta_{t,i},t)}(\varphi)(\bar{z}) - S_{N(\theta_{t,i},t)}(\varphi)(\theta_{i,t})| \leqslant |S_{N(\theta_{t,i},t)}(\varphi')(\xi_i)||\bar{z} - \theta_{t,i}| \tag{34}$$

Since $t < q_{n+1}$, it follows that for $j = 0, \ldots, N(\theta_{t,i}, t) - 1$, we have

$$\theta_{t,i} + j\alpha \notin [-\frac{1}{q_n \log^{100} q_n}, \frac{1}{q_n \log^{100} q_n}]$$

and since $|\xi_i - \theta_{t,i}| < q_n^{-3/2+5\eta}$, it follows that for $j = 0, \ldots, N(\theta_{t,i}, t) - 1$, we have

$$\xi_i + j\alpha \notin \left[-\frac{1}{2q_n \log^{100} q_n}, \frac{1}{2q_n \log^{100} q_n}\right].$$

By the above condition and by (6), we derive from (34) the bound

$$|S_{N(\theta_{t,i},t)}(\varphi)(\bar{z}) - S_{N(\theta_{t,i},t)}(\varphi)(\theta_{t,i})| \leqslant q_n^{1/2+3\eta}.$$

Therefore,

$$\mu((T^t_{\alpha,\varphi}(U_i) \triangle \mathscr{T}_{t,i}) \cap M_\zeta) \leqslant \lambda(B_i)(4q_n^{1/2+3\eta} + |t - t^*|) \leqslant q_n^{-1+10\eta}.$$

This finishes the proof of $(B5)$ and hence also of Proposition 5.3.

6 From stretching of Birkhoff sums to decay of correlations

6.1 Uniform stretching of Birkhoff sums and correlations

The aim of this section is to prove Corollary 2, which establishes a square-summable upper bound for the decay of correlations on *good intervals*, and Corollary 3, which establishes a power-like estimate of correlations decay, with exponent slightly lower than $1/2$, hence not square-summable, but which holds for general intervals. As we explained in the introduction of Section 5, Corollary 3 will be important in the estimation of the averaged decay of correlations on the bad set in Section 6.3. To read through the rest of the paper, only Corollary 2 and Corollary 3 will be needed.

We will adopt below the following notation.

For all $f \in \mathscr{F}$ with transfer function ϕ and $g \in C^1(M)$, let

$$\mathscr{N}_0(f,g) := \|\phi\|_0 \|g\|_0 \quad \text{and} \quad \mathscr{N}_1(f,g) := (\|f\|_0 + \|\phi\|_0)\|g\|_1 + (\|f\|_1 + \|\phi\|_1)\|g\|_0, \tag{35}$$

where $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively denote the C^0 and the C^1 norm. Moreover, we will denote by the letter C_φ a generic constant which depends only on on the rotation number α and on the ceiling function $\varphi \in \mathscr{C}_\eta$.

In all what follows I denotes any interval of the partition of $\mathscr{W}_{n,k}$ defined in Section 5.

Let us recall the following notation (see (20)). For any interval $J \subset I$ denote

$$r^t_J := \inf_{x \in J_t} |S_{N(x,t)}(\varphi')(\bar{x})|,$$

where $J_t := J \cap T^{-t}_{\alpha,\varphi}(W_{n,k})$ ($r^t_J = +\infty$ if $J_t = \emptyset$).

Our main technical result in this section is the following relation between uniform stretching of the Birkhoff sums and decay of correlations.

Proposition 6.1. *For any interval $J = [z,w] \times \{s\} \subset I$, we have the following estimate:*

$$\left| \int_{\bar{J}} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta - p(z,w) \right| \leqslant C_\varphi \left\{ \mathscr{N}_0(f,g)\frac{\lambda(J)}{S^t_J} + \mathscr{N}_1(f,g)\frac{\lambda(J)}{r^t_J} \right\},$$

where $p(z,w) = \frac{g(z,s)\phi(T^t_{\alpha,\varphi}(z,s))}{S_{N(z,t)}(\varphi')(z)} - \frac{g(w,s)\phi(T^t_{\alpha,\varphi}(w,s))}{S_{N(w,t)}(\varphi')(w)}$.

To prove Proposition 6.1, we will need the following lemma that contains the main estimate on the correlation of coboundaries, based on the stretching of the Birkhoff sums of the roof function. The proof is based on a change of variable that transforms the correlations integral along a small horizontal interval into an integral on a long stretched interval. The fact that one of the observables in the correlations integral is a coboundary is crucial in the argument.

Let $J_* := [u,v] \times \{s\} \subset J$ be such that $v - u \leqslant t^{-10}$.

Lemma 6.2. *Let $r^t_u := -S_{N(u,t)}(\varphi')(u)$. For all $f \in \mathscr{F}$ and for all $g \in C^1_0(M)$ and for all $t > 0$ we have*

$$\left| \int_{\bar{J}_*} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta - \Delta(J_*,t) \right| \leqslant C_\varphi \mathscr{N}_1(f,g)\frac{\lambda(J_*)}{r^t_I}, \tag{36}$$

where $\mathscr{N}_1(f,g)$ is the norm defined in formula (35) *and*

$$\Delta(J_*,t) := \frac{1}{r^t_u}\left[g(v,s)\phi(T^t_{\alpha,\varphi}(v,s)) - g(u,s)\phi(T^t_{\alpha,\varphi}(u,s)) \right].$$

Proof. Let $I \in \mathscr{W}_{n,k}$, with $I \subset \mathbb{T} \times \{s\}$, be a horizontal interval as in Section 5. Let $J_* = [u,v] \subset I$ such that $v_0 - u_0 \leqslant t^{-10}$. If $T^{-t}_{\alpha,\varphi}J_* \subset W^c_{n,k}$ then Lemma 6.2 holds trivially. We use the notation

$$T^t_{\alpha,\varphi}(u,s) = (\tilde{u},\tilde{s}) = (u + N(u,t)\alpha, t + s - S_{N(u,t)}(\varphi)(u)),$$

where $0 \leqslant \tilde{s} \leqslant \varphi(u + N(u,t)\alpha)$. We also denote $\tilde{v} = v + N(u,t)\alpha$.

In the remainder of this proof we will denote for simplicity the integer $N(u,t)$ by N. We will suppose that $r^t_u = -S_N(\varphi')(u) \geqslant r^t_I \geqslant 0$, the case where $r^t_I < 0$ being similar. Let us also denote

$$B^t_I := \sup_{\theta \in I} S_N(\theta)(\varphi'').$$

We will use the notation $X = O(Y)$ if there exists a constant $C > 0$ such that $X \leqslant CY$.

We have for $\theta \in [0, \lambda(J_*)]$ that $T^t_{\alpha,\varphi}(u+\theta,s) = (\tilde{u}+\theta, \tilde{s} + S_N(\varphi)(u) - S_N(\varphi)(u+\theta))$. By the intermediate value theorem, since $r^t_I \ll \lambda(J_*)^{-1}$, we have

$$\begin{aligned}\int_{\bar{J}_*} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta &= \int_0^{\lambda(J_*)} f(\tilde{u}+\theta, \tilde{s}+S_N(\varphi)(u) - S_N(\varphi)(u+\theta))g(u+\theta,s)d\theta \\ &= g(u,s)\int_0^{\bar{\lambda}(J_*)} f(\tilde{u}+\theta, \tilde{s}+S_N(\varphi)(u) - S_N(\varphi)(u+\theta))d\theta + O(\|f\|_0\|g\|_1\frac{\lambda(J_*)}{r^t_I}).\end{aligned}$$

Now, since $S_N(\varphi)(u) - S_N(\varphi)(u+\theta) \ll 1$ we also have

$$\int_0^{\lambda(J_*)} f(\tilde{u}+\theta, \tilde{s}+S_N(\varphi)(u) - S_N(\varphi)(u+\theta))d\theta$$
$$= \int_0^{\lambda(J_*)} f(\tilde{v}, \tilde{s}+S_N(\varphi)(u) - S_N(\varphi)(u+\theta))d\theta + O(\|f\|_1 \frac{\lambda(J_*)}{r_I^t}),$$

and by the definition of B_I^t, we have $|S_N(\varphi)(u) - S_N(\varphi)(u+\theta) - r_u^t\theta| \leqslant B_I^t\theta^2$. Therefore,

$$\int_{\bar{J}_*} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta = g(u,s)\int_0^{\lambda(J_*)} f(\tilde{v},\tilde{s}+r_u^t\theta)d\theta$$
$$+O(\|f\|_0\|g\|_1\frac{\lambda(J_*)}{r_I^t}) + O(\|f\|_1\|g\|_0\frac{\lambda(J_*)}{r_I^t}).$$

For simplicity let us denote $w(f,g) := \|f\|_0\|g\|_1 + \|f\|_1\|g\|_0$. A change of variable then gives

$$\int_{\bar{J}_*} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta = \frac{1}{r_u^t}g(u,s)\int_0^{r_u^t\lambda(J_*)} f(\tilde{v},\tilde{s}+\theta)d\theta + O(w(f,g)\frac{\lambda(J_*)}{r_I^t})$$
$$= \frac{1}{r_u^t}g(u,s)\left[\phi(\tilde{v},\tilde{s}+r_u^t\lambda(J_*)) - \phi(\tilde{v},\tilde{s})\right] + O(w(f,g)\frac{\lambda(J_*)}{r_I^t})$$

but $T^t_{\alpha,\varphi}(v,s) = (\tilde{v},\tilde{s}+S_N(\varphi)(u) - S_N(\varphi)(v)) = (\tilde{v},\tilde{s}+r_u^t\lambda(J_*)+\mathscr{E})$ with $\mathscr{E} \leqslant B_I^t\lambda(J_*)^2$, hence

$$\int_{\bar{J}_*} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta = \frac{1}{r_u^t}g(u,s)\left[\phi(T^t_{\alpha,\varphi}(v,s)) - \phi(\tilde{v},\tilde{s})\right]$$
$$+O(w(f,g)\frac{\lambda(J_*)}{r_I^t} + \|g\|_0\|\phi\|_1\frac{\lambda(J_*)}{r_I^t})$$
$$= \frac{1}{r_u^t}\left[g(v,s)\phi(T^t_{\alpha,\varphi}(v,s)) - g(u,s)\phi(T^t_{\alpha,\varphi}(u,s))\right]$$
$$+O(\mathscr{N}_1(f,g)\frac{\lambda(J_*)}{r_I^t}),$$

which is precisely formula (36). □

Proof of Proposition 6.1. Since the proof is symmetric for $t > 0$ and $t < 0$, from now on we will assume that $t > 0$. If $T^t_{\alpha,\varphi}(J) \subset W^c_{n,k}$, then Proposition 6.1 holds trivially. We assume for definiteness that $-S_{N(u,t)}(\varphi')(u) \geqslant r_J^t$ on J. Let us decompose J into finitely many subintervals $J = \bigcup_{i=1}^m J_i$ such that $J_i = [u_i, u_{i+1}) \times \{s\}$ with $|u_{i+1} - u_i| \leqslant t^{-10}$, and so that $N(\cdot,t)$ is constant on each J_i.

Then

$$\int_{\bar{J}} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta = \sum_{i=1}^m \int_{J_i} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta = \sum_{i=1}^m \Delta(J_i,t) + \mathscr{E}, \tag{37}$$

where, by (36)

$$\mathscr{E} \leqslant \mathscr{N}_1(f,g)\frac{\lambda(J)}{r_J^t}.$$

Notice that if $T^t_{\alpha,\varphi}(J_i) \subset W^c_{n,k}$ then the corresponding integral in (37) is 0. Therefore we only have to consider those J_i for which $T^{-t}_{\alpha,\varphi}(J_i) \not\subseteq W^c_{n,k}$. By enumeration let us assume that this is the case for all J_i.

Let us denote $r_i^t := -S_{N(u_i,t)}(\varphi')(u_i)$ and $\Theta_i := g(u_i,s)\phi(T^t_{\alpha,\varphi}(u_i,s))$. We then have

$$\begin{aligned}
|\sum_{i=1}^{m}\Delta(J_i,t)-p(z,w)| &= |\sum_{i=1}^{m}\frac{1}{r_i^t}(\Theta_{i+1}-\Theta_i)-p(z,w)| \\
&= |\frac{1}{r_m^t}\Theta_{m+1}-\frac{1}{r_1^t}\Theta_1+\sum_{i=1}^{m-1}\left(\frac{1}{r_i^t}-\frac{1}{r_{i+1}^t}\right)\Theta_{i+1}-p(z,w)| \\
&= |\sum_{i=1}^{m-1}\left(\frac{1}{r_i^t}-\frac{1}{r_{i+1}^t}\right)\Theta_{i+1}| \leqslant \|\phi\|_0\|g\|_0\left(\frac{1}{r_J^t}+\sum_{i=1}^{m-1}\frac{|r_{i+1}^t-r_i^t|}{r_{i+1}^t r_i^t}\right).
\end{aligned}$$

To estimate the quantity $\sum_{i=1}^{m-1}\frac{|r_{i+1}^t-r_i^t|}{r_{i+1}^t r_i^t}$, by the choice of $(u_i)_{i=1}^m$ (since $N(\cdot,t)$ is constant on J_i) and $u_{i+1}-u_i \leqslant t^{-10}$, we get

$$|r_{i+1}^t-r_i^t| \leqslant 2B_i^t\lambda(J_i)$$

where $B_i^t := S_{N(u_i,t)}(\varphi'')(u_i)$. To conclude the argument, we notice that (since $u_{i+1}\sim u_i$)

$$\sum_{i=1}^{m-1}\frac{B_i^t\lambda(J_i)}{r_{i+1}^t r_i^t} \leqslant \frac{\lambda(J)}{S_J^t}.$$

This, by (37), finishes the proof of Proposition 6.1

□

Proposition 6.1 has the following corollaries that allow us to deal with the decay of correlations, on good intervals for Corollary 2, and on general intervals for Corollary 3. In the corollaries below, $C_{\alpha,\varphi}$ again denotes a global positive constant which depends only on the rotation number α and on the ceiling function φ. It may be different in each corollary.

Corollary 2. *For every good interval J, we have*

$$\left|\int_{\bar{J}} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta\right| \leqslant C_{\alpha,\varphi}(\mathcal{N}_0(f,g)q_n^{-1}+\mathcal{N}_1(f,g)q_n^{-2})t^{-1/2-\frac{\eta}{4}}. \tag{38}$$

Proof. Assume $J\cap T^{-t}_{\alpha,\varphi}(W_{n,k})\neq\emptyset$ (otherwise the LHS is 0) and let first (18) hold in the definition of a good interval (see Definition 5.1). Notice that for $x\in T^{-t}_{\alpha,\varphi}(W_{n,k})$, $S_{N(x,t)}(\varphi'')(\bar{x})\geqslant q_n^{3-10\eta}$ (see (10)) and hence by (18), $1/r_J^t \leqslant q_n^{-3/2-4\varepsilon} \leqslant t^{-1/2-2\varepsilon}\lambda(J)$. Moreover,

$$p(z,w) \leqslant C_{\alpha,\varphi}\mathcal{N}_0(f,g)/r_J^t \leqslant C_{\alpha,\varphi}\mathcal{N}_0(f,g)t^{-1/2-\varepsilon}\lambda(J).$$

An application of Proposition 6.1 for J finishes the proof in this case. If (19) holds, define $J_{weak} := [x^*-q_n^{-3/2+2\eta}, x^*+q_n^{-3/2+2\eta}]\cap J$. Notice that by (19),

$$r^t_{J_{weak}} \geqslant q_n^{3/2+\eta} \quad\text{and}\quad S^t_{J_{weak}} \geqslant q_n^{\frac{5\eta}{2}}.$$

So by Proposition 6.1 for J_{weak}, we have

$$\left|\int_{\bar{J}_{weak}} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta\right| \leqslant C_{\alpha,\varphi}(\mathcal{N}_0(f,g)q_n^{-1}+\mathcal{N}_1(f,g)q_n^{-2})t^{-1/2-\frac{\eta}{4}}.$$

Therefore it remains to show (38) with $J \setminus J_{weak}$. Let $J = [z,w] \times \{s\}$ and let $J \setminus J_{weak} = J' \cup J''$, so that $z \in J'$ (unless $J' = \emptyset$) and $w \in J''$ (unless $J'' = \emptyset$). We will show (38) for J' and J''. We will apply the same procedure to both J' and J'', therefore we will explain the argument only in the case of J''. Let $m \in \mathbb{N}$ be the unique positive integer s.t. $2^m \leqslant q_n^{3/2-2\eta}(w-x^*) \leqslant 2^{m+1}$. Let us consider the intervals $J''_i := [w_i, w_{i+1}] \times \{s\} = [x^* + (w-x^*)/2^{i+1}, x^* + (w-x^*)/2^i] \times \{s\} \cap J''$, where $i = 0, \ldots, m$. Then $J'' = \bigcup_{i=0}^m J''_i$ (notice that J_m may be degenerated). Consider only those J''_i for which $T^{-t}_{\alpha,\varphi}(J''_i) \nsubseteq W^c_{n,k}$. By enumeration assume this is the case for all $i = 0, \ldots, m$. By (19) we have

$$r^t_{J''} \geqslant q_n^{3/2+\frac{\eta}{2}}. \tag{39}$$

Moreover by (19), for every J''_i, we have

$$\sup_{x \in J''_i} |S_{N(x)}(\varphi'')(\bar{x})| \leqslant q_n^{3-\eta} \log^9 q_n \quad \text{and} \quad \inf_{x \in J''_i} S_{N(x)}(\varphi')(\bar{x}) \geqslant \frac{q_n^{3-\eta}(w-x^*)}{2^{i+2}\log^5 q_n}.$$

Therefore, we have the following estimate:

$$\sum_{i=0}^m \frac{\lambda(J''_i)}{S^t_{J''_i}} \leqslant \frac{\log^{20} q_n}{q_n^{3-\eta}} \sum_{i=0}^m \frac{2^{2i+4}}{(w-x^*)^2} \lambda(J''_i) \leqslant \frac{8\log^{20} q_n}{(w-x^*) q_n^{3-\eta}} 2^{m+1} \leqslant \frac{1}{q_n^{3/2+\frac{\eta}{2}}} \leqslant t^{-1/2-\frac{\eta}{3}} \lambda(J). \tag{40}$$

Notice that by the definition of the function $p(z,w)$ (see Proposition 6.1), we have $p(w_0, w_{m+1}) = \sum_{i=0}^m p(w_i, w_{i+1})$. By Proposition 6.1 for J''_i, $i = 0, \ldots, m$ and by (39), (40), we derive

$$\left| \int_{\bar{J}''} f(T^t_{\alpha,\varphi}(\theta,s)) g(\theta,s) d\theta \right| \leqslant |p(w_0, w_{m+1})| + \left| \int_{\bigcup J''_i} f(T^t_{\alpha,\varphi}(\theta,s)) g(\theta,s) d\theta - p(w_0, w_{m+1}) \right|$$
$$\leqslant C_{\alpha,\varphi} \{ \mathcal{N}_0(f,g)\lambda(J) + \mathcal{N}_1(f,g)\lambda(J)^2 \} t^{-1/2-\frac{\eta}{4}}.$$

The same estimate is true for J'. This completes the proof of Corollary 2. □

Moreover, we also have the following crucial corollary for the bootstrap argument in Section 6.3. Recall that the integers $l, l_0, l_1, n \in \mathbb{N}$, the partial partition $\mathscr{W}_{n,k}$ and its union $W_{n,k} = \cup \mathscr{W}_{n,k}$ are chosen as in Section 5.

Corollary 3. *For every interval $\bar{I} \in \mathscr{I}_k$ and for all $s \in \mathbb{R}^+$ such that $I := \bar{I} \times \{s\} \subset M$, for all $t \in [l_0, l_1]$, we have*

$$\left| \int_{\bar{I}} f(T^t_{\alpha,\varphi}(\theta,s)) g(\theta,s) d\theta \right| \leqslant C_{\alpha,\varphi} \{ \mathcal{N}_0(f,g)\lambda(I) + \mathcal{N}_1(f,g)\lambda(I)^2 \} t^{-1/2+6\eta}.$$

Proof. If $I \cap W^c_{n,k} \neq \emptyset$, then $I \subset M^c_\zeta$ hence (LHS) is 0. If $I \in \mathscr{W}_{n,k}$ then let $I = J_1 \sqcup J_2 \sqcup I_{bad}$ as in Proposition 5.4. We apply Corollary 2 to J_1 and J_2 together with the estimates

$$\lambda(I) \geqslant q_n \log^{-20} q_n \quad \text{and} \quad \lambda(I_{bad}) < \frac{1}{q_n^{3/2-2\eta}}.$$

For the interval I_{bad} we estimate the integral by the uniform norm of the integrand times the measure $\lambda(I_{bad})$ of the domain of integration. □

6.2 Summable decay on good intervals. Proof of Proposition 3.2

We now explain how the results of Section 6.1 imply Proposition 3.2.

In fact, we prove a more general statement that will be relevant in Section 7.3, to complete the proof that the spectrum is Lebesgue with countable multiplicity.

Proposition 6.3. *For every set E, measurable with respect to the partial partition $\mathscr{W}_{n,k}$ (see* (4) *for its definition), we have*

$$\left|\int_{E\setminus\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu\right| < C_{\alpha,\varphi}\{\mathscr{N}_0(f,g)\mu(E)+\mathscr{N}_1(f,g)\mu(E)^2\}t^{-1/2-\frac{\eta}{5}}.$$

Proof. Since $g=0$ on $M^c_\zeta \supset W^c_{n,k}$, we have

$$\left|\int_{E\setminus\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu\right| = \left|\int_{(E\cap W_{n,k})\setminus\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu\right|.$$

By Fubini, it is enough to show that, for every interval $I\in\mathscr{W}_{n,k}$, we have

$$\left|\int_{\bar{I}\setminus\bar{I}_{bad}} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta\right| \leqslant C_{\alpha,\varphi}\{\mathscr{N}_0(f,g)\lambda(I)+\mathscr{N}_1(f,g)\lambda(I)^2\}t^{-1/2-\varepsilon},$$

where the subinterval I_{bad} is as in Proposition 5.4. It is then enough to apply Corollary 2 (to the subintervals J_1 and J_2) together with the lower bound $\lambda(I)\geqslant q_n\log^{-20}q_n$.

Proposition 6.3 is thus proved, and Proposition 3.2 immediately follows, as among the properties of the bad set (see Proposition 5.3) we have the bound $\mu(\mathscr{B}_l)\leqslant q_n^{-1/2+6\eta}$. □

6.3 Averaged decay on the bad set. Proof of Proposition 3.3

Notice that, as the bad set $\mathscr{B}_l$ decomposes by (25) as the union of the towers $U_1,\dots,U_m$, Proposition 3.3 follows from corresponding estimates on each tower U_i, as in the proposition below. We are therefore interested in the decay of $\int_{l_0}^{l_1}\left|\int_{U_i} f(T^t_{\alpha,\varphi}(x))g(x)d\mu\right|dt$. Our strategy will be to exploit the decay of correlations between different times of $[l_0,l_1]$ by applying the Cauchy-Schwarz-Hölder inequality to the above integral of correlations. As we have already mentioned, the difficulty that we have to overcome is given by the existence of times t^* such that $T^{t^*}_{\alpha,\varphi}$ is close to the identity on the bad set. The use of the Cauchy-Schwarz-Hölder inequality allows to exploit the mixing properties for times $t\in[l_0,l_1]$ that are far from t^*, as seen in formulas (41) and (42).

Let $C_{f,g}$ denote a positive constant which depends on the functions $f\in\mathscr{F}$ and $g\in C^1_0(M)$ only through the quantities $\mathscr{N}_0(f,g)$ and $\mathscr{N}_1(f,g)$.

Proposition 6.4. *For every $i\in\{1,\dots,m\}$, we have*

$$\int_{l_0}^{l_1}\left|\int_{U_i} f(T^t_{\alpha,\varphi}(x))g(x)d\mu\right|dt < C_{f,g}\frac{(l_1-l_0)\mu(U_i)}{q_n^{20\eta}}.$$

Proof. Fix $i\in\{1,\dots,m\}$. Let $A:=\{t\in[l_0,l_1] \ : \ \int_{U_i} f(T^t_{\alpha,\varphi}(x,s))g(x,s)dx>0\}$. Let $\rho(t)=1$ if

$t \in A$ and $\rho(t) = -1$ if $t \in [l_0, l_1] \setminus A$. Then, by Cauchy-Schwarz (Hölder) inequality, we have

$$\int_{l_0}^{l_1} \left| \int_{U_i} f(T^t_{\alpha,\varphi}(x)) g(x) d\mu \right| dt = \int_{U_i} \left(\int_{l_0}^{l_1} \rho(t) f(T^t_{\alpha,\varphi}(x)) dt \right) g(x) d\mu$$
$$\leqslant \left(\int_{U_i} \left(\int_{l_0}^{l_1} \rho(t) f(T^t_{\alpha,\varphi}(x)) dt \right)^2 d\mu \right)^{1/2} \left(\int_{U_i} g(x)^2 d\mu \right)^{1/2}$$
$$\leqslant \|g\|_0 \mu(U_i)^{1/2} \left(\int_{U_i} \left(\int_{l_0}^{l_1} \rho(t) f(T^t_{\alpha,\varphi}(x)) dt \right)^2 d\mu \right)^{1/2}.$$

Moreover we have

$$\left(\int_{U_i} \left(\int_{l_0}^{l_1} \rho(t) f(T^t_{\alpha,\varphi}(x)) dt \right)^2 d\mu \right) \leqslant \|f\|_0^2 (l_1 - l_0)^{3/2} \mu(U_i)$$
$$+ \left(\int_{U_i} \left(\int_{l_0}^{l_1} \left(\int_{r \in [l_0, l_1]\,:\, |r-t| \geqslant (l_1 - l_0)^{1/2}} \rho(r)\rho(t) f(T^t_{\alpha,\varphi}(x)) f(T^r_{\alpha,\varphi}(x)) dr \right) dt \right) d\mu \right).$$

Therefore, to finish the proof of Proposition 6.4 it is enough to show that there exists a constant $C > 0$ such that, for every $t \leqslant r$ with $t, r \in [l_0, l_1]$ s.t. $|t - r| \geqslant (l_1 - l_0)^{1/2}$, we have

$$\left| \int_{T^t_{\alpha,\varphi}(U_i)} f(x) f(T^{r-t}_{\alpha,\varphi}(x)) d\mu \right| \leqslant C \mathscr{N}_1(f, f) \frac{\mu(U_i)}{q_n^{40\eta}}. \tag{41}$$

Note that $t^* := r - t \in [q_n^{1/41}, q_n^{1/19}]$. Let us then fix such a $t^* \in [q_n^{1/41}, q_n^{1/19}]$. Following the notation of Section 5 we then let $l^* = [t^*]$ and n^* be the unique integer such that $q_{n^*} \leqslant l^* < q_{n^*+1}$.

Let k^* be any integer such that $q_{k^*} \in [q_{n^*} \log^{15} q_{n^*}, q_{n^*} \log^{20} q_{n^*}]$. It follows by construction that we have $q_{k^*} \in [q_n^{1/41}, q_n^{1/19} \log^{20} q_n]$

Observe now that by Corollary 3 there exists a constant $C > 0$ such that, for any interval $\bar{I} \in \mathscr{I}_{k^*}$ and for all $s \in \mathbb{R}^+$ such that $I := \bar{I} \times \{s\} \subset M$, we have

$$\left| \int_{\bar{I}} f(T^{t^*}_{\alpha,\varphi}(\theta, s)) f(\theta, s) d\theta \right| \leqslant C\{\mathscr{N}_0(f, f) + \mathscr{N}_1(f, f)\lambda(I)\} \frac{\lambda(I)}{q_n^{1/100}}. \tag{42}$$

Thus, it only remains to be seen that the integral in (41) decomposes into integrals over the sets of the form $T^t_{\alpha,\varphi}(U_i) \cap I, \bar{I} \in \mathscr{I}_{k^*}$, and that each is roughly equal to the product of $\lambda(U_i \cap I)/\lambda(I)$ times the integral in (42). This is what we will now derive from Proposition 5.3, namely from the property that $T^t_{\alpha,\varphi}(U_i)$ is almost equal to the tower $\mathscr{T}_{t,i}$ of (B_5). In fact, by properties (B_1), (B_2) in Proposition 5.3, we have the bound $m \leqslant q_n^{2/5+\eta}$, hence by property (B_5) we conclude that

$$\sum_{i=1}^{m} \mu(\mathscr{T}_{t,i} \triangle T^t_{\alpha,\varphi}(U_i)) \leqslant q_n^{-3/5+15\eta}. \tag{43}$$

The intersection of each tower $\mathscr{T}_{t,i}$ with I is a regular union of equally separated small intervals (see Figure 7). In this situation the interpolation between the integrals is possible. However, explicitly carrying out this argument is a bit tedious and technical, as it relies on the following definition of quantitative uniform distribution of a union of intervals inside a larger one, and on the two lemmas that follow it. Lemma 6.7 in particular is a simple observation on the geometry of the repartition of a thin Rokhlin tower of the flow $T^t_{\alpha,\varphi}$ in the manifold M, whose technical proof can be skipped in a first reading. We introduce the following

Definition 6.5. *Let $\nu,\gamma \in (0,1)$. We will say that a collection $\mathscr{S} := K_1 \sqcup \ldots \sqcup K_H \subset \mathbb{T} \times \{s\}$ of pairwise disjoint horizontal intervals of equal lengths is (ν,γ)-uniformly distributed in the interval I if there exists a decomposition of I into a disjoint union of $L \leqslant \gamma H$ intervals $I_1,\ldots,I_L$ of equal length $\ell \in [\nu, 2\nu]$ such that, for all $j \in [1,L]$, we have*

$$\#\{i \in [1,H] : K_i \subset I_j\} \in [(1-\gamma)\frac{H}{L}, (1+\gamma)\frac{H}{L}].$$

This definition is useful in the following straightforward lemma.

Lemma 6.6. *If $\mathscr{S}$ and I are as in Definition 6.5, then for any C^1 real function G defined over the interval $I := \bar{I} \times \{s\}$, we have*

$$\left| \int_{\mathscr{S}\cap\bar{I}} G(\theta,s)d\theta - \frac{\lambda(\mathscr{S}\cap I)}{\lambda(I)} \int_{\bar{I}} G(\theta,s)d\theta \right| \leqslant C\,(\nu\|G\|_1 + \gamma\|G\|_0)\,\lambda(\mathscr{S}\cap I).$$

Lemma 6.7. *For any complete tower $\mathscr{T}$ of height $h \geqslant q_n^{3/5-1/50}$ above any horizontal interval of the the form $B_{\mathscr{T}} = [-q_n^{-3/2+5\eta} + \theta_{\mathscr{T}}, \theta_{\mathscr{T}} + q_n^{-3/2+5\eta}] \times \{s_{\mathscr{T}}\}$, we have the following:*

- (I_1) *if $N(\theta_{\mathscr{T}},h) \leqslant q_n^{1/3}$, then $\mu(\mathscr{T} \cap M_\zeta) \leqslant q_n^{1/2-3/5}\mu(\mathscr{T})$;*
- (I_2) *if $N(\theta_{\mathscr{T}},h) \geqslant q_n^{1/3}$, then for any $\bar{I} \in \mathscr{I}_{k^*}$ such that $I := \bar{I} \times \{s\} \subset M_\zeta$, the set $\mathscr{T} \cap I$ is contained in a collection of disjoint intervals of equal size $(q_n^{-1/4}, q_n^{-1/100})$-uniformly distributed in the interval I.*

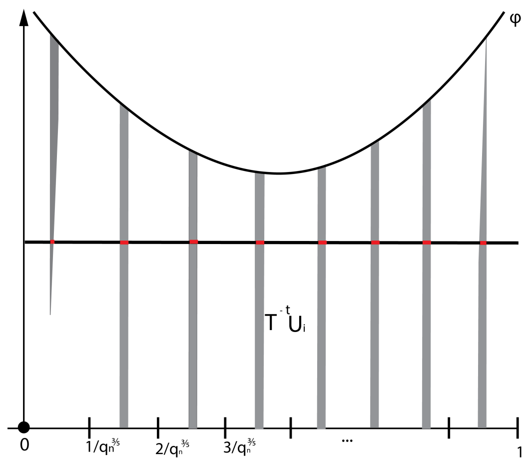

Figure 7: The image of the set U_i under the flow. The intersection with any horizontal interval is a union of equispaced intervals.

Before proving Lemma 6.7, we show how it implies (41). By (43), it suffices to show that there exists a constant $C > 0$ such that

$$\left| \int_{\mathscr{T}_{t,i}} f(x) f(T^{t^*}_{\alpha,\varphi}(x) d\mu \right| \leqslant C\mathscr{N}_1(f,f)\frac{\mu(\mathscr{T}_{t,i})}{q_n^{50\eta}}. \tag{44}$$

If (I_1) holds, then since f is supported on M_ζ we have

$$\left|\int_{\mathscr{T}_{t,i}} f(x)f(T^{t^*}_{\alpha,\varphi}(x))d\mu\right| \leqslant \|f\|_0^2\,\mu(\mathscr{T}_{t,i}\cap M_\zeta) \leqslant \|f\|_0^2\,\frac{\mu(\mathscr{T}_{t,i})}{q_n^{1/10}}, \tag{45}$$

hence the proof is finished in this case. Notice that by Fubini's theorem (44) follows from the following claim: there exists a constant $C>0$ such that, for any $I := \bar I \times \{s\}$ with $\bar I \in \mathscr{I}_{k^*}$, we have

$$\left|\int_{\overline{\mathscr{T}_{t,i}\cap I}} f(\theta,s)f(T^{t^*}_{\alpha,\varphi}(\theta,s))d\theta\right| \leqslant C\{\mathscr{N}_0(f,f)+\mathscr{N}_1(f,f)\lambda(I)\}\frac{\lambda(\mathscr{T}_{t,i}\cap I)}{q_n^{50\eta}}. \tag{46}$$

In fact, the above bound is stronger than what we need to prove the absolute continuity of the spectrum. The precise dependence of the constant on the function f and on the interval $I\in\mathscr{I}_{k^*}$ will be crucial in the proof, in Section 7.3, that the spectrum is Lebesgue with countable multiplicity .

Now, if $I\subset M_\zeta^c$ then the integral in (46) is zero. Notice that, since $t^*\leqslant q_n^{1/19}$, by Corollary 4.1 the function $G: I\to\mathbb{R}$ defined as $G(\cdot) = f(\cdot)f(T^{t^*}_{\alpha,\varphi}(\cdot))$ satisfies $\|G\|_1\leqslant q_n^{1/8}\|f\|_0\|f\|_1$, thus (I_2) and Lemma 6.6 imply that

$$\left|\int_{\overline{\mathscr{T}_{t,i}\cap I}} G(\theta,s)d\theta - \frac{\lambda(\mathscr{T}_{t,i}\cap I)}{\lambda(I)}\int_{\bar I} G(\theta,s))d\theta\right| \leqslant C\|f\|_0\{\|f\|_0+\|f\|_1\lambda(I)\}\frac{\lambda(\mathscr{T}_{t,i}\cap I)}{q_n^{1/200}},$$

and therefore (46) follows from (42). The proof of the derivation of the bound in (41) from Lemma 6.7 is complete.

It only remains to give the

Proof of Lemma 6.7. Let us first consider the case $N := N(\theta_{\mathscr{T}},h)\geqslant q_n^{1/3}$. Let $\{K_1,\dots,K_H\}$ be the smallest collection of disjoint intervals of equal length such that

$$I\cap\mathscr{T}\subset K_1\sqcup K_2\sqcup\cdots\sqcup K_H.$$

Notice that for every $i\in\{1,\dots,H\}$, the interval $\bar K_i$ is centered at the point $\theta_{\mathscr{T}}+k_i\alpha$, for some $k_i\in[0,N]$. In fact, there is an injective map from the set of $k\in[0,N]$ such that $\theta_{\mathscr{T}}+k\alpha\in\bar I$ to the collection of intervals $\{K_1,\dots,K_H\}$ which misses at most 2 intervals. By Lemma 4.4 for $\bar J=\bar I$ and $\theta=\theta_{\mathscr{T}}$, we have

$$|H-N\lambda(I)|\leqslant 2+2C^{-1}\log N^{2+\xi}\,. \tag{47}$$

Let us then divide I into equal intervals $I_1,\dots,I_L$ of equal length $\ell\in[q_n^{-1/4},2q_n^{-1/4}]$ and let us consider $I_j\subset I$. The map from the set $\{i\in[1,H]: K_i\subset I_j\}$ to the set of $k\in[0,N]$ such that $\theta_{\mathscr{T}}+k\alpha\in\bar I_j$, which sends every interval $\bar K_i$ to its center, is injective and misses at most 2 elements. From Lemma 4.4 for $\bar J=\bar I_j$ and $\theta=\theta_{\mathscr{T}}$, it follows that

$$|\#\{i\in[1,H]: K_i\subset I_j\}-N\lambda(I_j)|\leqslant 2+2C^{-1}\log N^{2+\xi}\,. \tag{48}$$

Notice that since $I\in\mathscr{I}_k$ by the bound (47), it follows that $H\geqslant q_n^{1/3-1/20}$ and by construction we have $L\leqslant q_n^{1/4-1/40}$, hence in particular $H/L\geqslant q_n^{1/12-1/40}$. We then derive the estimate

$$|N\lambda(I_j)-\frac{H}{L}| = |\frac{N\lambda(I)}{L}-\frac{H}{L}|\leqslant\frac{2+C^{-1}\log N^{2+\xi}}{L}\leqslant q_n^{-1/10}\frac{H}{L},$$

which in turn by the bound (48) implies that

$$\#\{i \in [1,H] : K_i \subset I_j\} \in \left[(1-q_n^{-1/100})\frac{H}{L}, (1+q_n^{-1/100})\frac{H}{L}\right].$$

This shows that the collection $\mathscr{S} = K_1 \sqcup \cdots \sqcup K_H$ is $(q_n^{-1/4}, q_n^{-1/100})$-uniformly distributed in I. The proof of Lemma 6.7 is finished in case (I_2).

Assume now that $N(\theta_{\mathscr{T}}, h) \leqslant q_n^{1/3}$. Notice that, since the height of the complete tower $\mathscr{T}$ is $h \geqslant q_n^{3/5-1/10}$, we have

$$S_{N(\theta_{\mathscr{T}},h)+1}(\varphi)(\theta_{\mathscr{T}}) \geqslant q_n^{3/5-1/50}.$$

But then

$$\mu(\mathscr{T} \cap M_\zeta) \leqslant q_n^{1/3}\zeta^{-1}\lambda(B_{\mathscr{T}}) \leqslant q_n^{1/2-3/5}q_n^{3/5-1/50}\lambda(B_{\mathscr{T}}) \leqslant q_n^{1/2-3/5}\mu(\mathscr{T}).$$

This finishes the proof of Lemma 6.7. □

□

7 Countable Lebesgue Spectrum

In this section we prove a general criterion for establishing the countable Lebesgue spectral property for smooth flows with square-integrable correlations of smooth coboundaries. From our criterion we derive that our Kochergin flows have countable Lebesgue spectrum, thereby completing the proof of our main result, Theorem 2 (the precise formulation of Theorem 1). We also derive that time-changes of horocycle flows have countable Lebesgue spectrum, thereby completing the proof of the Katok-Thouvenot conjecture (see [26], Conjecture 6.8). In fact, it was proved in [16] that smooth time changes of the horocycle flow have Lebesgue maximal spectral type, but the multiplicity question was left open.

The section will be divided in three parts. In the first Section 7.1, we give in Theorem 5 an abstract Criterion for Infinite Lebesgue Spectrum (CILS), that guarantees infinite Lebesgue multiplicity for a strongly continuous group of unitary operators on a separable Hilbert space having an absolutely continuous spectral type.

To guarantee that the multiplicity of the Lebesgue component in the spectrum is at least $n+1$, for some $n \geqslant 0$, the criterion requires, for any given positive measure and bounded subset C of the real line, the construction of $n+1$- functions such that the $(n+1)\times(n+1)$ matrix of Fourier transforms of their square-integrable mutual correlations has maximal rank equal to $n+1$ on C. The latter would indeed contradict that all the (equivalence classes of the) spectral measures in the decreasing spectral decomposition be zero on C starting from the $n+1^{\text{st}}$ measure.

An equivalent way of presenting the hypotheses of the CILS, is to require the existence, for any $n \geqslant 0$, of $n+1$ functions, such that each function is almost orthogonal to the cyclic space of any other one, and such that the spectral measures of the functions can be chosen to be not too small on any fixed bounded measurable set of $\mathbb{R}$.

In Section 7.2, we state in Theorem 6 a criterion that guarantees infinite Lebesgue multiplicity for a flow, based on the control of the decay of correlations for functions supported on tall flow-boxes with an arbitrarily thin base. When mixing between such functions is effectively obtained at times that compares to the height of the flow-boxes, it is then possible to construct the functions as in the CILS and conclude infinite Lebesgue multiplicity. Indeed, we show in the same section how the hypotheses of Theorem 6 immediately imply the hypotheses of the abstract criterion in Theorem 5.

In Section 7.3, elaborating on the mixing estimates of Sections 5 and 6, we show that Kochergin flows (with a sufficiently degenerate singularity) typically satisfy the hypotheses of our criterion, thus completing the proof of Theorem 2 (hence of Theorem 1), our main result.

We also explain how to derive from [16] that smooth time changes of horocycle flows satisfy the hypothesis of Theorem 6, our criterion for infinite Lebesgue multiplicity for flows. Since by the results of [16] the maximal spectral type is Lebesgue, we conclude that the smooth time changes of horocycle flows also have a Lebesgue spectrum with infinite multiplicity.

7.1 The Criterion for Infinite Lebesgue Spectrum (CILS)

Our criterion for countable Lebesgue spectrum of smooth flows is based on the following abstract criterion for strongly continuous one-parameter unitary groups on separable Hilbert spaces.

Let $\mathscr{F}: L^2(\mathbb{R}, dt) \to L^2(\mathbb{R}, d\tau)$ denote the Fourier transform, given by the formula

$$\mathscr{F}(f)(\tau) = \int_{\mathbb{R}} f(t) e^{-2\pi\imath t\tau} dt\,, \quad \text{for all } f \in L^2(\mathbb{R}, dt)\,.$$

Theorem 5. *Let $\{\phi_{\mathbb{R}}\}$ be a strongly continuous one-parameter unitary group on a separable Hilbert space H with absolutely continuous spectrum. For a fixed $n \in \mathbb{N}$, let us assume that for every compact set $C \subset \mathbb{R} \setminus \{0\}$ of positive Lebesgue measure there exists $\varepsilon_{n,C} > 0$ such that the following holds. For every $\varepsilon \in (0, \varepsilon_{n,C})$ there exist vectors $f_1, \dots, f_{n+1} \in H$ such that*

$$\begin{aligned} &\|\langle f_i \circ \phi_t, f_j \rangle\|_{L^2(\mathbb{R},dt)} \leqslant \delta_{ij} + \varepsilon\,, \quad \textit{for all } i,j \in 1, \dots, n+1\,; \\ &\|\prod_{i=1}^{n+1} \mathscr{F}(\langle f_i \circ \phi_t, f_i \rangle)\|_{L^{\frac{2}{n+1}}(C)} > (n+1)!(1+\varepsilon)^n \varepsilon\,. \end{aligned}$$

Then the spectral type of $\{\phi_{\mathbb{R}}\}$ is Lebesgue with multiplicity at least $n+1$.

The proof of the theorem is based on the following lemma.

Lemma 7.1. *Let H be a separable Hilbert space and let $H^{(n)} := \oplus_{k=1}^n H_k \subset H$ denote an orthogonal, invariant decomposition into cyclic subspaces of a strongly continuous one-parameter unitary group $\{\phi_{\mathbb{R}}\}$ with absolutely continuous spectrum. Let $f_1, \dots, f_{n+1} \in H^{(n)}$ be vectors such that the correlations functions $\langle f_i \circ \phi_t, f_j \rangle \in L^2(\mathbb{R}, dt)$. We have*

$$\det(\mathscr{F}\langle f_i \circ \phi_t, f_j \rangle) = 0 \quad \textit{almost everywhere.}$$

Proof. Let us begin to illustrate the argument in the case $n = 1$. Then we can assume (up to a unitary equivalence) that there is a function $m \in L^1(\mathbb{R}, d\tau)$ such that $f_1, f_2 \in L^2(\mathbb{R}, m d\tau)$. We therefore assume $f_1 = f_1(\tau)$, and $f_2 = f_2(\tau) \in L^2(\mathbb{R}, m d\tau)$. The flow acts on $L^2(\mathbb{R}, m d\tau)$ by multiplication by $e^{2\pi\imath\tau t}$. So we have

$$\begin{aligned} \mathscr{F}(\langle f_i \circ \phi_t, f_i \rangle) &= m|f_i|^2\,, \quad \text{for } i = 1,2\,, \\ \mathscr{F}(\langle f_1 \circ \phi_t, f_2 \rangle) &= m f_1 \bar{f}_2\,. \end{aligned}$$

We have the identity between functions in $L^1(\mathbb{R})$:

$$\begin{aligned}\mathscr{F}(\langle f_1\circ\phi_t, f_1\rangle)\mathscr{F}(\langle f_2\circ\phi_t, f_2\rangle) &= m^2|f_1|^2|f_2|^2 = (mf_1\bar{f}_2)(mf_2\bar{f}_1)\\ &= \mathscr{F}(\langle f_1\circ\phi_t, f_2\rangle)\mathscr{F}(\langle f_2\circ\phi_t, f_1\rangle)\,.\end{aligned}$$

In the general case, let f_{ik} denote the projection of the vector f_i on the cyclic space H_k, for all $i\in\{1,\dots,n+1\}$ and $k\in\{1,\dots,n\}$. Since the cyclic spaces are invariant and orthogonal, for all $i,j\in\{1,\dots,n+1\}$ we have

$$\langle f_i\circ\phi_t, f_j\rangle = \sum_{k=1}^{n}\langle f_{ik}\circ\phi_t, f_{jk}\rangle\,.$$

Since the spaces H_k are cyclic and the group has absolutely continuous spectrum, there exist functions $m_1,\dots,m_k\in L^1(\mathbb{R},d\tau)$ and, for each $k\in\{1,\dots,n\}$ and all $i\in\{1,\dots,n+1\}$, there exist functions $u_{ik}\in L^2(\mathbb{R},m_k d\tau)$ such that for all $i,j\in\{1,\dots,n+1\}$ we have

$$\mathscr{F}\langle f_i\circ\phi_t, f_j\rangle = \sum_{k=1}^{n} u_{ik}\bar{u}_{jk}m_k\,.$$

By the above formula, every column of the $(n+1)\times(n+1)$ matrix $(\mathscr{F}\langle f_i\circ\phi_t, f_j\rangle(\tau))$ can be written as the sum of n vectors as follows. For each $j\in\{1,\dots,n+1\}$, we have

$$\begin{pmatrix}\mathscr{F}\langle f_1\circ\phi_t, f_j\rangle(\tau)\\ \dots\\ \mathscr{F}\langle f_{n+1}\circ\phi_t, f_j\rangle(\tau)\end{pmatrix} = \sum_{k=1}^{n} m_k(\tau)\begin{pmatrix}u_{1k}(\tau)\bar{u}_{jk}(\tau)\\ \dots\\ u_{(n+1)k}(\tau)\bar{u}_{jk}(\tau)\end{pmatrix} = \sum_{k=1}^{n} m_k(\tau)\bar{u}_{jk}(\tau)\begin{pmatrix}u_{1k}(\tau)\\ \dots\\ u_{(n+1)k}(\tau)\end{pmatrix}.$$

Since the matrix $(\mathscr{F}\langle f_i\circ\phi_t, f_j\rangle(\tau))$ is $(n+1)\times(n+1)$, its determinant is a sum of determinants of matrices containing at least two columns proportional to the same vector $(u_{1k}(\tau),\dots,u_{(n+1)k}(\tau))$. This proves that the determinant vanishes almost everywhere. The argument is complete.

□

Proof of Theorem 5. Let $\oplus_{n\in\mathbb{N}}H_n$ denote an orthogonal, invariant decomposition into cyclic subspaces such that, for all $n\in\mathbb{N}$, we have $H_n\approx L^2(\mathbb{R},\mu_n)$ with

$$\mu_1 := m_1(\tau)d\tau \gg \mu_2 := m_2(\tau)d\tau \gg \dots \gg \mu_n := m_n(\tau)d\tau \gg \dots$$

Let $\{f_i\}$ be a sequence of vectors in H and, for each $i,j\in\mathbb{N}$, let f_{ij} denote the orthogonal projection onto H_j. Since the spaces H_k are $\phi_{\mathbb{R}}$-invariant and mutually orthogonal, we have, for all $i,j\in\mathbb{N}$,

$$\langle f_i\circ\phi_t, f_j\rangle = \sum_{k\in\mathbb{N}}\langle f_{ik}\circ\phi_t, f_{jk}\rangle\,,$$

hence after taking the Fourier transform

$$\mathscr{F}(\langle f_i\circ\phi_t, f_j\rangle) = \sum_{k\in\mathbb{N}}\mathscr{F}(\langle f_{ik}\circ\phi_t, f_{jk}\rangle)\,.$$

Let us assume by contradiction that the spectrum is Lebesgue with multiplicity at most n. Then there exists a compact set $C\subset\mathbb{R}\setminus\{0\}$ of positive Lebesgue measure such that

$$m_{n+1}(\tau) = m_{n+2}(\tau) = \dots = 0\,,\qquad \text{for all } \tau\in C\,.$$

Let $f_1,\dots,f_{n+1}\in H$ be vectors given by the assumptions of the theorem Let $\bar f_1,\dots,\bar f_{n+1}\in H^{(n)}:= H_1\oplus\cdots\oplus H_n$ denote, respectively, the orthogonal projections of vectors $f_1,\dots,f_{n+1}\in H$ onto the subspace $H^{(n)}$. Since for each $k\in\mathbb{N}$ the subspace H_k is cyclic, the Fourier transform of the correlation $\langle f_{ik}\circ\phi_t, f_{ik}\rangle$ is absolutely continuous (as a density) with respect to the measure μ_k on $\mathbb{R}$. Hence we derive, for all $i,j\in\{1,\dots,n+1\}$ the identity

$$\mathscr{F}(\langle f_i\circ\phi_t, f_j\rangle)(\tau)=\mathscr{F}(\langle \bar f_i\circ\phi_t,\bar f_j\rangle)(\tau)\,,\qquad \text{for almost all } \tau\in C\,.$$

It follows that, by Lemma 7.1, for all $i,j\in\{1,\dots,n+1\}$, we have that

$$\det(\mathscr{F}(\langle f_i\circ\phi_t, f_j\rangle)(\tau)=\det(\mathscr{F}(\langle \bar f_i\circ\phi_t,\bar f_j\rangle)(\tau)=0\,,\qquad \text{for almost all } \tau\in C\,. \tag{49}$$

By Hölder inequality, for any $p>1$ the product of $n+1$ functions in $L^p(\mathbb{R})$ belongs to $L^{\frac{p}{n+1}}(\mathbb{R})$ and we have

$$\|\prod_{i=1}^{n+1} g_i\|_{L^{\frac{p}{n+1}}(\mathbb{R})}\leqslant \prod_{i=1}^{n+1}\|g_i\|_{L^p(\mathbb{R})}\,. \tag{50}$$

Since the determinant of a $(n+1)\times(n+1)$ matrix is a polynomial of degree $n+1$ in the entries of the matrix, the determinant of the matrix $(\mathscr{F}\langle f_i\circ\phi_t, f_j\rangle)$ belongs to $L^{\frac{2}{n+1}}(\mathbb{R})$.

By the assumptions on the vectors $f_1,\dots,f_{n+1}$ we have

$$\|\mathscr{F}\langle f_i\circ\phi_t, f_j\rangle\|_{L^2(C,d\tau)}\leqslant\|\mathscr{F}\langle f_i\circ\phi_t, f_j\rangle\|_{L^2(\mathbb{R},d\tau)}=\|\langle f_i\circ\phi_t, f_j\rangle\|_{L^2(\mathbb{R},dt)}\leqslant\delta_{ij}+\varepsilon\,,$$

hence, by formula (49), by the expansion of the determinant, and by the estimate in formula (50),

$$\begin{aligned}\|\prod_{i=1}^{n+1}\mathscr{F}(\langle f_i\circ\phi_t, f_i\rangle)\|_{L^{\frac{2}{n+1}}(C)}&=\|\det\mathscr{F}(\langle f_i\circ\phi_t, f_j\rangle)-\prod_{i=1}^{n+1}\mathscr{F}(\langle f_i\circ\phi_t, f_i\rangle)\|_{L^{\frac{2}{n+1}}(C)}\\&\leqslant(n+1)!(1+\varepsilon)^n\varepsilon\,.\end{aligned} \tag{51}$$

However, by assumption we also have

$$\|\prod_{i=1}^{n+1}\mathscr{F}(\langle f_i\circ\phi_t, f_i\rangle)\|_{L^{\frac{2}{n+1}}(C)}>(n+1)!(1+\varepsilon)^n\varepsilon\,,$$

a contradiction with the upper bound in formula (51). The argument is thus complete. □

We give now a version of the CILS that is well adapted to derive countable Lebesgue spectrum from mixing estimates for Kochergin flows and horocycle flows (that is, from Theorem 6 below).

Corollary 4. *Let us assume that for every $n\in\mathbb{N}$, for any even functions $\omega_1,\dots,\omega_{n+1}\in\mathscr{S}(\mathbb{R})$ (the Schwartz space), and for any any $\varepsilon>0$, there exists vectors $f_1,\dots,f_{n+1}\in H$ such that, for all $i,j\in\{1,\dots,n+1\}$, we have*

$$\|\langle f_i\circ\phi_t, f_j\rangle-\frac{d^2}{dt^2}\omega_i*\omega_i(t)\delta_{ij}\|_{L^2(\mathbb{R})}\leqslant\varepsilon\,.$$

Then the spectral type of the strongly continuous one-parameter unitary group $\phi_{\mathbb{R}}$ is Lebesgue with countable multiplicity.

Proof. Let C be a given compact subset of $\mathbb{R}\setminus\{0\}$ of positive Lebesgue measure. By the Lebesgue density theorem, it is not restrictive to assume that there exists an interval $[a,b]$ with $0<a<b$ such that $\mathrm{Leb}(C\cap[a,b])\geqslant(b-a)/2$. The case when $C\cap\mathbb{R}^+=\emptyset$ is similar. Let $\chi_C:\mathbb{R}\to[0,1]$ denote any smooth odd function with compact support in $[-2b,-a/2]\cup[a/2,2b]$ such that $\chi_C^2\equiv 1$ on $[-b,-a]\cup[a,b]$. For all $i\in\{1,\dots,n+1\}$ let ω_i be the function determined by the identity

$$\mathscr{F}(\omega_i)(\tau)=\frac{1}{\tau}\frac{\chi_C(\tau)}{\|\chi_C^2\|^{1/2}_{L^2(\mathbb{R})}}\,,\qquad \text{for all } i\in\{1,\dots,n+1\}\,.$$

The functions ω_i are all even, and we can take $f_1,\dots,f_{n+1}$ as in the statement of the corollary. We then verify that the hypotheses of Theorem 5 hold. In fact, we have

$$\begin{aligned}\|\mathscr{F}(\langle f_i\circ\phi_t,f_j\rangle)&-\mathscr{F}(\frac{d^2}{dt^2}\omega_i*\omega_i)\delta_{ij}\|_{L^2(\mathbb{R})}\\&=\|\mathscr{F}(\langle f_i\circ\phi_t,f_j\rangle)-\frac{\chi_C^2}{\|\chi_C^2\|_{L^2(\mathbb{R})}}\delta_{ij}\|_{L^2(\mathbb{R})}\leqslant\varepsilon\,,\end{aligned}$$

hence in particular

$$\|\mathscr{F}(\langle f_i\circ\phi_t,f_j\rangle)\|_{L^2(\mathbb{R})}\leqslant\delta_{ij}+\varepsilon\,,\qquad\text{for all } i,j\in\{1,\dots,n+1\}\,.$$

By the construction and by the Hölder inequality bound of formula (50) we have

$$\|\prod_{i=1}^{n+1}\mathscr{F}(\langle f_i\circ\phi_t,f_i\rangle)-\left(\frac{\chi_C^2}{\|\chi_C^2\|_{L^2(\mathbb{R})}}\right)^{n+1}\|_{L^{\frac{2}{n+1}}(\mathbb{R})}\leqslant 2^n(1+\varepsilon)^{n-1}\varepsilon\,,$$

hence by convexity we derive that

$$\begin{aligned}\|\prod_{i=1}^{n+1}\mathscr{F}(\langle f_i\circ\phi_t,f_i\rangle)\|^{\frac{2}{n+1}}_{L^{\frac{2}{n+1}}(C)}&\geqslant\|\left(\frac{\chi_C^2}{\|\chi_C^2\|_{L^2(\mathbb{R})}}\right)^{n+1}\|^{\frac{2}{n+1}}_{L^{\frac{2}{n+1}}(C)}-[2^n(1+\varepsilon)^{n-1}\varepsilon]^{\frac{2}{n+1}}\\&\geqslant\left(\frac{\|\chi_C^2\|_{L^2(C)}}{\|\chi_C^2\|_{L^2(\mathbb{R})}}\right)^2-[2^n(1+\varepsilon)^{n-1}\varepsilon]^{\frac{2}{n+1}}\,.\end{aligned}$$

From the above estimate we conclude that

$$\|\prod_{i=1}^{n+1}\mathscr{F}(\langle f_i\circ\phi_t,f_i\rangle)\|_{L^{\frac{2}{n+1}}(C)}>(n+1)!(1+\varepsilon)^n\varepsilon\,,$$

for all $\varepsilon>0$ such that

$$\left(\frac{\|\chi_C^2\|_{L^2(C)}}{\|\chi_C^2\|_{L^2(\mathbb{R})}}\right)^2>[2^n(1+\varepsilon)^{n-1}\varepsilon]^{\frac{2}{n+1}}+(n+1)!(1+\varepsilon)^n\varepsilon\,.$$

By Theorem 5 it follows that the strongly continuous one-parameter unitary group $\{\phi_{\mathbb{R}}\}$ has Lebesgue spectrum with multiplicity at least n. Since $n\in\mathbb{N}$ is arbitrary, it has Lebesgue spectrum with countable multiplicity. $\square$

7.2 Decay of correlations and infinite Lebesgue multiplicity.

As explained in the introduction of this section, we now give a criterion based on decay of correlations that allows to construct the functions that as required in the CILS to guarantee infinite Lebesgue multiplicity. The idea is to guarantee mixing between functions supported on tall flow-boxes with thin base J after a time that is comparable to the height of the flow-boxes. Indeed by fixing such a flow-box with base $J \subset M$ and height $T_J > 0$, we can choose functions supported on this flow-box in an arbitrary way so as to guarantee the satisfaction of the CILS conditions up to some finite time comparable to T_J. After time T_J it is the effective mixing between functions supported on such flow-boxes that insures the complete satisfaction of the CILS conditions.

One additional technical point is that our mixing estimates only hold for coboundaries, hence we have to define corresponding classes of functions supported on tall and thin flow-boxes. For Kochergin flows, one extra technical difficulty is that mixing is effectively controlled only away from the singularity (and for technical reasons related to our proof, away from the ceiling function). Hence the family of functions we need to consider are not just supported on tall flow-boxes with thin bases, but also have to vanish on a small measure set inside these flow-boxes that correspond to a small neighborhood of the origin (and of the ceiling function). The latter difficulty is not present in our application of the CILS to time changes of horocycle flows.

Let $\{T^t\}$ be a smooth aperiodic flow on a smooth manifold M, preserving a smooth volume form of finite total volume. For any given transverse embedded closed multi-dimensional interval $J \subset M$, let T_J be the maximal real number $T > 0$ such that the map

$$F_J^T(x,t) = T^t(x,0)\,, \text{ for all } (x,t) \in J \times (-T,T)\,,$$

is a flow-box for the flow $\{T^t\}$. The flow-box $F_J := F_J^{T_J}$ will be called a *maximal flow-box* over the (basis) interval $J \subset M$. Since the flow $\{T^t\}$ has no periodic orbits, for any $T_0 > 0$ there exists an interval J such that $T_J > T_0$.

Let $\mathscr{M} := \{M_\zeta | \zeta \in (0,1)\}$ be a fixed family of open subsets of M such that $\cap_{\zeta>0} M_\zeta^c$ is a closed subset M_0 of zero-measure. In the case of Kochergin flows this is the family introduced in formula (1) of Section 3. Given a flow-box F_J^T, we define, for any $\zeta > 0$, the set $S_\zeta^T(J) \subset \mathbb{R}$ as follows

$$S_\zeta^T(J) := \{t \in (-T,T) \,:\, T^t(J) \cap M_\zeta^c = \emptyset\}\,.$$

By definition we have that $S_\zeta^T(J)$ is an open subset (which in general may be empty).

In the sequel, we will focus our attention on very long and thin maximal flow boxes, that spend most of the initial time away from the bad sets M_ζ^c, for $\zeta > 0$ sufficiently small. This motivates the following definition.

Definition 7.2. *A family $\Phi = \{F_J\}$ of maximal flow-boxes is called* admissible *if for every $T > 0$ and $\nu > 0$, there exist $N \in \mathbb{N}$ and $\tau > 0$, and $\zeta > 0$ such that for all maximal flow-boxes $F_J \in \Phi$ with $T_J > \tau$, the set $S_\zeta^T(J)$ has at most N connected components, and*

$$Leb((-T,T) \setminus S_\zeta^T(J)) \leqslant \nu\,. \tag{52}$$

Remark 1. *It should be noted that, in the special case when the bases of the maximal flow-boxes of a family $\Phi = \{F_J\}$ form a decreasing sequence $\{J\}$ of intervals with respect to the inclusion, then in order to establish that Φ is admissible it is enough to verify the conditions for all the* *degenerate flow-boxes (the orbits segments) over the singleton equal to the intersection of all of their bases. Our construction below of admissible families of maximal flow-boxes for Kochergin flows (in Section 7.3) is based on this principle.*

For any $k \in \mathbb{N} \setminus \{0\}$, and for any constants $C > 0, \zeta > 0$, we define $G_k(J,T,C,\zeta)$ to be the set of all functions $\psi_J \in C_0^\infty(J \times (-T,T))$ defined as follows. Let $\chi_J \in C_0^\infty(J)$ be any smooth function such that $\int_J \chi_J^2 d\lambda = 1$, with $C^s(J)$ norm bounded above by $C/\lambda(J)^{s+1/2}$ for all $s \in \{0,\dots,k\}$, and let $\psi \in C_0^\infty(S_\zeta^T(J))$ is any smooth function with C^k norm on $\mathbb{R}$ bounded above by C. We can now define the functions supported on flow-boxes that we will be working with.

Definition 7.3. *Given a flow-box F_J^T and constants $C, \zeta > 0$, we define $\mathscr{G}_k(F_J^T, C, \zeta)$ to be the class of all functions $g_J \in C^\infty(M)$, defined on the range R_J^T of the flow-box map F_J^T as*

$$(g_J \circ F_J^T)(x,t) := \chi_J(x)\psi(t), \quad \text{if } (x,t) \in J \times (-T,T),$$

for any $\psi \in G_k(J,T,C,\zeta)$, and defined as $g_J := 0$ on $M \setminus R_J^T$.

The class $\mathscr{F}_k(F_J^T, C, \zeta) \subset \mathscr{G}_k(F_J^T, C, \zeta)$ consists of all functions $f_J \in \mathscr{G}_k(F_J^T, C, \zeta)$ which are derivatives along the flow.

We can now state our general criterion, based on correlation decay, for countable Lebesgue spectrum.

Theorem 6. *Let $\{T^t\}$ be a smooth, aperiodic, volume-preserving ergodic flow with absolutely continuous spectrum on a smooth manifold M of finite total volume. Assume that there exists an admissible family of maximal flow-boxes $\Phi := \{F_J\}$ for the flow, such that $\inf \lambda(J) = 0$ (hence $\sup T_J = +\infty$) and there exists $k \in \mathbb{N} \setminus \{0\}$ such that given any $T > 0$, $C > 0$ and $\zeta > 0$, for any family $\{(f_J, g_J)\}$ of pair of functions such that $f_J \in \mathscr{F}_k(F_J^T, C, \zeta)$ and $g_J \in \mathscr{G}_k(F_J^T, C, \zeta)$ we have*

$$\inf_{F^J \in \Phi} \int_{\mathbb{R} \setminus [-T_J, T_J]} |\langle f_J \circ T^t, g_J \rangle|^2 dt = 0.$$

Then the flow $\{T^t\}$ has countable Lebesgue spectrum.

We will derive the criterion from Corollary 4. Since we only control the decay of correlations for functions in the classes $\mathscr{F}_k(F_J^T, C, \zeta)$ and $\mathscr{G}_k(F_J^T, C, \zeta)$, we first prove below a simple approximation lemma to approximate the target even functions $\omega_1, \dots, \omega_{n+1} \in \mathscr{S}(\mathbb{R})$ of Corollary 4 by (even) functions supported inside sets of the type $S_\zeta^T(J)$. For technical reasons that will appear below in the proof of the approximation lemma, we prefer to first symmetrize the set $S_\zeta^T(J)$ and consider instead functions supported in $S_\zeta^T(J) \cap (-S_\zeta^T(J))$.

Lemma 7.4. *Let $\Phi = \{F_J\}$ be* admissible. *Then, for every $k \in \mathbb{N}$, $\varepsilon > 0$, for every even function $\omega \in \mathscr{S}(\mathbb{R})$, there exist $\tau > 0$ such that for every $T \geqslant \tau$, there exists $\zeta > 0$ such that if J is such that $T_J > T$, there exists an even function $\psi \in C_0^\infty(-T,T)$ such that $\frac{d\psi}{dt} \in C_0^\infty(S_\zeta^T(J) \cap (-S_\zeta^T(J)))$ with C^k norm bounded above by a constant $C := C(k, \varepsilon, \omega, T) > 0$ (crucially) independent of the flow-box $F_J \in \Phi$, such that*

$$\|\frac{d^2}{dt^2}(\psi * \psi) - \frac{d^2}{dt^2}(\omega * \omega)\|_{L^2(\mathbb{R})} < \varepsilon.$$

Proof. By properties of convolution we can write

$$\begin{aligned}\frac{d^2}{dt^2}(\psi*\psi)-\frac{d^2}{dt^2}(\omega*\omega) &= \frac{d\psi}{dt}*\frac{d\psi}{dt}-\frac{d\omega}{dt}*\frac{d\omega}{dt}\\ &= (\frac{d\psi}{dt}+\frac{d\omega}{dt})*(\frac{d\psi}{dt}-\frac{d\omega}{dt}),\end{aligned}$$

hence, by Young's convolution inequality, we have

$$\|\frac{d^2}{dt^2}(\psi*\psi)-\frac{d^2}{dt^2}(\omega*\omega)\|_{L^2(\mathbb{R})} \leqslant \|\frac{d\psi}{dt}+\frac{d\omega}{dt}\|_{L^1(\mathbb{R})}\|\frac{d\psi}{dt}-\frac{d\omega}{dt}\|_{L^2(\mathbb{R})}. \tag{53}$$

It is therefore enough to construct functions ψ such that $\frac{d\psi}{dt}$ are L^2 approximations of the function $\frac{d\omega}{dt}$ with bounded L^1 norm, and are supported in $S^T_\zeta(J)$.

By the definition of an admissible family of maximal flow-boxes, for every $T>0$ and $\nu>0$, there exist $\tau,\zeta>0$ and $N\in\mathbb{N}$ such that for all maximal flow-boxes $F_J\in\Phi$ with $T_J>\tau$ we have that $J\subset M_\zeta$, the symmetric set $S^T_\zeta(J)\cap(-S^T_\zeta(J))=I_1\cup\ldots\cup I_N$ is a union of (at most) N open intervals and $\mathrm{Leb}((-T,T)\setminus I_1\cup\ldots\cup I_N)\leqslant\nu/4$. Let then $\{I'_1,\ldots,I'_N\}$ be a symmetric family of closed subintervals such that $I'_i\subset I_i$, for each $i\in\{1,\ldots,N\}$, and

$$\mathrm{Leb}((-T,T)\setminus(I'_1\cup\ldots\cup I'_N))\leqslant\nu/2.$$

In order to control the norm of higher derivatives of the function ψ, we also choose the family $\{I'_i\}$ such that, for all $i\in\{1,\ldots,N\}$,

$$\mathrm{dist}(\partial I_i,\partial I'_i)\geqslant\frac{\nu}{10N}.$$

We claim that there exists an even function $\psi\in C^\infty_0((-T,T))$ such that

- $\frac{d\psi}{dt}$ is an odd smooth function supported inside $S^T_\zeta(J)\cap(-S^T_\zeta(J))$,
- $\frac{d\psi}{dt}=\frac{d\omega}{dt}$ on $(I'_1\cup\cdots\cup I'_N)\cap[-T+\nu/4,T-\nu/4]$,
- $|\frac{d\psi}{dt}(t)|\leqslant|\frac{d\omega}{dt}(t)|$, for all $t\in\mathbb{R}$,
- $\|\frac{d\psi}{dt}\|_{C^k(\mathbb{R})}\leqslant C_k\|\omega\|_{C^{k+1}(\mathbb{R})}(\frac{\nu}{10N})^{-k}$, for all $k\geqslant 1$

(with $C_k>0$ a constant depending only on $k\in\mathbb{N}$).

It follows from formula (53) and by Hölder inequality that we have

$$\|\frac{d^2}{dt^2}(\psi*\psi)-\frac{d^2}{dt^2}(\omega*\omega)\|_{L^2(\mathbb{R})}\leqslant 2\|\frac{d\omega}{dt}\|_{L^1(\mathbb{R})}\left(2\|\frac{d\omega}{dt}\|_{C^0(\mathbb{R})}\nu^{1/2}+\|\frac{d\omega}{dt}\|_{L^2(\mathbb{R}\setminus(-T,T))}\right).$$

Hence for every $\omega\in\mathscr{S}(\mathbb{R})$ and for every $\varepsilon>0$, there exists $T_\varepsilon>0$ and $\nu_\varepsilon>0$ such that, for all $T>T_\varepsilon$ and all $\nu<\nu_\varepsilon$, we have

$$\|\frac{d^2}{dt^2}(\psi*\psi)-\frac{d^2}{dt^2}(\omega*\omega)\|_{L^2(\mathbb{R})}<\varepsilon.$$

We have thus reduced the proof of the lemma to that of the above claim.

In order to prove the claim we consider an even function $\phi_\nu\in C^\infty_0((-T,T))$ such that

- $\phi_\nu(t) \in [0,1]$, for all $t \in \mathbb{R}$,
- $\phi_\nu(t) = 0$, for all $t \notin S^T_\zeta(J) \cap (-S^T_\zeta(J)) = I_1 \cup \cdots \cup I_N$,
- $\phi_\nu(t) = 1$, for all $t \in (I'_1 \cup \cdots \cup I'_N) \cap [-T+\nu/4, T-\nu/4]$,
- $\|\phi_\nu\|_{C^k(\mathbb{R})} \leqslant C_k(\frac{\nu}{10N})^{-k}$,

and then we define

$$\psi(t) = \begin{cases} \int_{-T}^t \phi_\nu(s) \frac{d\omega}{dt}(s) ds\,, & \text{for all } t \in (-T,T)\,, \\ 0\,, & \text{for all } t \in \mathbb{R} \setminus (-T,T)\,. \end{cases}$$

The function $\psi \in C_0^\infty(-T,T)$ since the function $\phi_\nu \frac{d\omega}{dt}$ is odd, hence all of its primitives are even, and has compact support in $(-T,T)$, so that ψ is the unique primitive which vanishes on the complement of $(-T,T)$. Finally, it is straightforward to verify that ψ satisfies all the properties listed in the claim. □

Proof of Theorem 6. Let us fix $\varepsilon > 0$ and any given number $n+1 \in \mathbb{N} \setminus \{0\}$ of even Schwartz functions $\omega_1, \ldots, \omega_{n+1} \in \mathscr{S}(\mathbb{R})$. Let $\Phi = \{F_J\}$ be an admissible family of maximal flow-boxes.

By Lemma 7.4 there exist T, $\tau > 0$ (large) and $\zeta > 0$ (small) such that if J is such that $T_J > \tau$, there exist even functions $\psi_i \in C_0^\infty((-T,T)), i = 1, \ldots, n+1$, such that $\frac{d\psi_i}{dt} \in C_0^\infty(S^T_\zeta(J))$ with C^{k+1} norm uniformly bounded above by a constant $C' := C'(k, \varepsilon, \omega_1, \ldots, \omega_{n+1}, T) > 0$ such that

$$\|\frac{d^2}{dt^2}(\psi_i * \psi_i) - \frac{d^2}{dt^2}(\omega_i * \omega_i)\|_{L^2(\mathbb{R})} < \varepsilon/2\,. \tag{54}$$

Let now $\chi_J^{(1)}, \ldots, \chi_J^{(n+1)} \in C^\infty(J)$ be functions such that

$$\int_J \chi_J^{(i)} \chi_J^{(j)} d\lambda = \delta_{ij}\,, \quad \text{for all } i,j \in \{1, \ldots, n+1\}\,,$$

with C^s norm bounded above by $C''/\lambda(J)^{s+1/2}$ on J for all $s \in \{0, \ldots, k\}$ (this is possible provided that the constant C'' is taken to be larger than some constant that only depends on n).

Let $C > \max\{C', C''\}$. For every $i \in \{1, \ldots, n+1\}$, let $f_J^{(i)} \in \mathscr{F}_k(F_J^T, C, \zeta)$ be the function defined on the range R_J^T of the flow-box map F_J^T as

$$f_J^{(i)} \circ F_J^T(x,t) := \chi_J^{(i)}(x) \frac{d}{dt} \psi_i(t)\,, \quad \text{if } (x,t) \in J \times (-T,T)\,,$$

and defined as $f_J^{(i)} = 0$ on $M \setminus R_J^T$.

We then compute the correlations. Let $T_J/2 > \max\{T, \tau/2\}$. For all $t \in [-T_J, T_J]$ we have (since the functions $\psi_1, \ldots, \psi_{n+1}$ are all even)

$$\begin{aligned}\langle f_J^{(i)} \circ T^t, f_J^{(j)} \rangle &= \int_J \int_{-T}^T \chi_J^{(i)}(x) \chi_J^{(j)}(x) \frac{d\psi_i}{dt}(\sigma+t) \frac{d\psi_j}{dt}(\sigma) d\sigma dx \\ &= (\frac{d\psi_i}{dt} * \frac{d\psi_j}{dt})(t)\, \delta_{ij} = \frac{d}{dt^2}(\psi_i * \psi_j)(t) \delta_{ij}\,.\end{aligned}$$

and by the assumption of the theorem, if $\lambda(J)$ is small enough, for every $i, j \in \{1,\dots,n+1\}$ we have:

$$\|\langle f_J^{(i)} \circ T^t, f_J^{(j)} \rangle\|_{L^2(\mathbb{R}\setminus[-T_J,T_J])} \leqslant \varepsilon/2 .$$

Note that, since the functions ψ_i are supported in $[-T,T]$ and $T < T_J/2$, we also have

$$\frac{d}{dt^2}(\psi_i * \psi_j)(t)\delta_{ij} = 0, \text{ for } t \in \mathbb{R}\setminus[-T_J,T_J]. \tag{55}$$

By putting together formulas (54)–(55), it follows that if $\lambda(J)$ is small enough (hence T_J is large enough), the functions $f_J^{(i)}$, with $i \in \{1,\dots,n+1\}$, satisfy the assumptions of Corollary 4:

$$\|\langle f_J^{(i)} \circ T^t, f_J^{(j)} \rangle - \frac{d^2}{dt^2}(\omega_i * \omega_j)\delta_{ij}\|_{L^2(\mathbb{R})} \leqslant \varepsilon .$$

It follows then by Corollary 4 that, under the hypotheses of the theorem, the flow $\{T^t\}$ has countable Lebesgue spectrum, hence the argument is completed. □

7.3 CILS for Kochergin flows and time changes of horocycle flows

We prove below that the hypotheses of Theorem 6 are verified for Kochergin flows $\{T^t_{\alpha,\varphi}\}$.

Let $\mathscr{M} = \{M_\zeta | \zeta > 0\}$ denote the family introduced in formula (1) of Section 3.

Theorem 7. *For every Kochergin flow $\{T^t_{\alpha,\varphi}\}$ with $\alpha \in D_{\log,\xi}$, $\xi < 1/10$, there exists an admissible family of maximal flow-boxes $\{F_J\}$, over a decreasing sequence $\{J\}$ of intervals satisfying $\lim\lambda(J) \to 0^+$ (hence $T_J \to +\infty$), such that the following holds. For any $T > 0$, for any C and $\zeta > 0$, for any sequence of pair of functions $\{(f_J, g_J)\}$ such that $f_J \in \mathscr{F}_1(F_J^T, C, \zeta)$ and $g_J \in \mathscr{G}_1(F_J^T, C, \zeta)$ we have*

$$\lim_{\lambda(J)\to 0^+} \int_{\mathbb{R}\setminus[-T_J,T_J]} |\langle f_J \circ T^t_{\alpha,\varphi}, g_J \rangle|^2 dt = 0 .$$

Proof. We consider the following family of maximal flow boxes. We fix θ_0 that is not in the orbit of 0 by R_α on the circle and take any sequence of basis intervals $\{J\} \subset \{J_m\}$ with

$$J_m := [\theta_0 - \frac{1}{10q_m}, \theta_0 + \frac{1}{10q_m}] \times \{0\}, \quad \text{for all } m \in \mathbb{N}.$$

Since the sequence $\{J_m\}$ is decreasing with respect to the inclusion, it is immediate to prove that the family $\{F^{J_m}\}$ of maximal flow-boxes is admissible by verifying that the conditions hold for all the degenerate flow-boxes (orbit segments) $F^T_{J_\infty} := \{T^t_{\alpha,\varphi}(\theta_0, 0) | t \in [-T,T]\}$ over the degenerate interval $J_\infty := \{(\theta_0, 0)\} \subset M$. Indeed, by the definition of the family $\{M_\zeta\}$ (in formula (1) of Section 3) for any $T > 0$ and $\nu > 0$ there exists $\zeta > 0$ such that $(\theta_0, 0) \in M_\zeta$, the orbit segment $F^T_{J_\infty}$ does not intersect the interval $[-\zeta, \zeta] \times \{0\}$. Therefore the set $S^T_\zeta(J_\infty) = F^T_{J_\infty}$ is equal to a finite union of $N := N_T$ intervals such that $\text{Leb}((-T,T) \setminus S^T_\zeta(J_\infty)) \leqslant (2T)(2\zeta)$, hence the estimate in formula (52) of the definition of an admissible family of maximal flow-boxes holds if $4T\zeta < \nu$.

We also observe that by construction there exists a constant $c > 0$ such that $T_{J_m} \geqslant cq_m$, for all $m \in \mathbb{N}$, which implies that for any $J \in \{J_m\}$ we have

$$\lambda(J) \geqslant \frac{c}{5T_J} . \tag{56}$$

For any $J \in \{J_m\}$, let us fix $T \in (0, T_J^{1/2})$. We want to prove a bound on the correlations for any pair of functions $f_J \in \mathscr{F}_1(F_J^T, C, \zeta)$ and $g_J \in \mathscr{G}_1(F_J^T, C, \zeta)$, where C is a fixed constant, and derive the vanishing of the limit in the statement of the theorem. In the sequel the symbols C', C'' will denote generic universal constants independent of J (but dependent on $T > 0$), and that also depend on the constant C in the classes of functions $\mathscr{F}_1(F_J^T, C, \zeta)$ and $\mathscr{G}_1(F_J^T, C, \zeta)$. Let then $f_J \in \mathscr{F}_1(F_J^T, C, \zeta)$ and $g_J \in \mathscr{G}_1(F_J^T, C, \zeta)$. By Definition 7.3, the functions f_J and g_J are given by

$$f_J \circ F_J^T(x,t) = \phi_J(x,t) = \chi_J(x)\phi(t) \quad \text{ and } \quad g_J \circ F_J^T(x,t) = \psi_J(x,t) = \chi_J(x)\psi(t)$$

on R_J^T, with $\phi, \psi \in G_1(J, T, C, \zeta)$ and ϕ a derivative, and $f_J = g_J = 0$ on $M \setminus \mathbb{R}_J^T$.

Since the function g_J is supported on the range R_J^T of the flow-box map F_J^T it is enough to prove bounds on

$$\int_{R_J^T} f_J \circ T^t_{\alpha,\varphi}(x) g_J(x) d\mu\,.$$

Let then $|t| \geqslant T_J$. WLOG we can assume $t > 0$ since the argument for $t < 0$ is similar. As in Section 6.2 we split the estimate into two parts: the integral over the complement of the bad set $\mathscr{B}_l$ (see formula (25) in Section 5.2 for its definition), and the integral over the bad set.

Claim 1. *There exists $C > 0$ such that, for some $\varepsilon > 0$ and for all $J \in \{J_m\}$, we have (recalling that T is fixed and $T_J \to +\infty$)*

$$|\int_{R_J^T \setminus \mathscr{B}_l} f_J \circ T^t_{\alpha,\varphi}(x) g_J(x) d\mu| \leqslant C t^{-1/2-\varepsilon}\,. \tag{57}$$

Proof. Let us recall the partitions $\mathscr{I}_k$ of $\mathbb{T}$ into intervals with endpoints $\{-i\alpha\}_{i=0}^{q_k-1}$ and $\mathscr{W}_{n,k}$ introduced (see formula (4)) at the beginning of Section 4. By the assumption that $t \geqslant T_J$, by formula (56) there exists a constant $c > 0$ such that $t \geqslant c/2\lambda(J)$. It follows that there exists a constant $C > 0$ and, for all $J \in \{J_m\}$, there exists a product set $E^T_{J,k}$, measurable with respect to the partial partition $\mathscr{W}_{n,k}$, with base $\bar{E}^T_{J,k}$ measurable with respect to the partition $\mathscr{I}_k$, such that $R_J^T \subset E^T_{J,k}$ and we have

$$\mu(E^T_{J,k}) \leqslant C\mu(R_J^T) = CT\lambda(J)\,.$$

By construction there exists a constant $C' > 0$ such that

$$\mathscr{N}_0(f_J, g_J) = \|f_J\|_0 \|g_J\|_0 \leqslant \frac{C'}{\lambda(J)} \|\phi\|_0 \|\psi\|_0\,;$$
$$\mathscr{N}_1(f_J, g_J) = (\|f_J\|_0 + \|\phi_J\|_0)\|g_J\|_1 + (\|f_J\|_1 + \|\phi_J\|_1)\|g_J\|_0 \leqslant \frac{C'}{\lambda(J)^2} \|\phi\|_2 \|\psi\|_1\,.$$

Hence it follows from Proposition 6.3 that

$$\left| \int_{E^T_{J,k} \setminus \mathscr{B}_l} f_J(T^t_{\alpha,\varphi}(x,s)) g_J(x,s) d\mu \right| < C' \left(CT \|\phi\|_0 \|\psi\|_0 + (CT)^2 \|\phi\|_2 \|\psi\|_1 \right) t^{-1/2-\varepsilon}\,.$$

The bound in formula (57) is therefore proved and the proof of Claim 1 is completed. □

It remains to estimate the integral on the bad set $\mathscr{B}_l \cap R_J^T$. Let $t \in [l^{21/20}, (l+1)^{21/20}]$ with $l \in \mathbb{N}$. Let us recall the notation $l_0 = l^{21/20}$, $l_1 = (l+1)^{21/20}$ and let $n \in \mathbb{N}$ be the unique natural number such that $q_n < l_0 < q_{n+1}$. Let then $k \in \mathbb{N}$ be such that $q_k \in [q_n \log^{15} q_n, q_n \log^{20} q_n]$.

Claim 2. *There exists a constant $C > 0$ such that, for all $l_0 > T_J$, we have*

$$\int_{l_0}^{l_1} \left| \int_{\mathscr{B}_l \cap R_J^T} f_J(T_{\alpha,\varphi}^t(x)) g_J(x) d\mu \right| dt \leqslant C \frac{l_1 - l_0}{q_n^{1/2+15\eta}}.$$

Proof. We follow the proof of Proposition 6.4 in Section 6.3. Let

$$A_J := \{t \in [l_0, l_1] \, : \, \int_{\mathscr{B}_l} f_J(T_{\alpha,\varphi}^t(x)) g_J(x) d\mu > 0\}$$

and let $\rho_J(t) = 1$ if $t \in A_J$ and $\rho_J(t) = -1$ if $t \in [l_0, l_1] \setminus A_J$. Let F_J^T denote as above a flow-box map and let $R_J^T \subset M$ denote its range. Then, by Cauchy-Schwarz (Hölder) inequality, we have

$$\begin{aligned}\int_{l_0}^{l_1} \left| \int_{\mathscr{B}_l} f_J(T_{\alpha,\varphi}^t(x)) g_J(x) d\mu \right| dt &= \int_{\mathscr{B}_l \cap R_J^T} \left(\int_{l_0}^{l_1} \rho_J(t) f_J(T_{\alpha,\varphi}^t(x)) dt \right) g_J(x) d\mu \\ &\leqslant \left(\int_{\mathscr{B}_l \cap R_J^T} \left(\int_{l_0}^{l_1} \rho_J(t) f_J(T_{\alpha,\varphi}^t(x)) dt \right)^2 d\mu \right)^{1/2} \left(\int_{\mathscr{B}_l \cap R_J^T} g_J(x)^2 d\mu \right)^{1/2} \\ &\leqslant \|g_J\|_0 \mu(\mathscr{B}_l \cap R_J^T)^{1/2} \left(\int_{\mathscr{B}_l} \left(\int_{l_0}^{l_1} \rho(t) f_J(T_{\alpha,\varphi}^t(x)) dt \right)^2 d\mu \right)^{1/2}. \quad (58)\end{aligned}$$

We split the auto-correlation integral on the RHS of formula (58), as follows:

$$\begin{aligned}&\int_{\mathscr{B}_l} \left(\int_{l_0}^{l_1} \rho_J(t) f_J(T_{\alpha,\varphi}^t(x)) dt \right)^2 d\mu \\ &\quad = \left(\int_{\mathscr{B}_l} \left(\int_{l_0}^{l_1} \left(\int_{r \in [l_0,l_1] \, : \, |r-t| \leqslant 10T} \rho(r)\rho(t) f_J(T_{\alpha,\varphi}^t(x)) f_J(T_{\alpha,\varphi}^r(x)) dr \right) dt \right) d\mu \right) \\ &\quad + \left(\int_{\mathscr{B}_l} \left(\int_{l_0}^{l_1} \left(\int_{r \in [l_0,l_1] \, : \, |r-t| \geqslant 10T} \rho(r)\rho(t) f_J(T_{\alpha,\varphi}^t(x)) f_J(T_{\alpha,\varphi}^r(x)) dr \right) dt \right) d\mu \right). \quad (59)\end{aligned}$$

By invariance of the measure, since f_J is supported on R_J^T, we can write

$$\begin{aligned}\int_{\mathscr{B}_l} f_J(T_{\alpha,\varphi}^t(x)) f_J(T_{\alpha,\varphi}^r(x)) d\mu &= \int_{T_{\alpha,\varphi}^t(\mathscr{B}_l)} f_J(x) f_J(T_{\alpha,\varphi}^{r-t}(x)) d\mu \\ &= \int_{T_{\alpha,\varphi}^t(\mathscr{B}_l) \cap R_J^T} f_J(x) f_J(T_{\alpha,\varphi}^{r-t}(x)) d\mu. \end{aligned} \quad (60)$$

hence we have the immediate estimate

$$\begin{aligned}&\left(\int_{\mathscr{B}_l} \left(\int_{l_0}^{l_1} \left(\int_{r \in [l_0,l_1] \, : \, |r-t| \leqslant 10T} \rho(r)\rho(t) f_J(T_{\alpha,\varphi}^t(x)) f_J(T_{\alpha,\varphi}^r(x)) dr \right) dt \right) d\mu \right) \\ &\qquad \leqslant 20T \|f_J\|_0^2 (l_1 - l_0) \mu(T_{\alpha,\varphi}^t(\mathscr{B}_l) \cap R_J^T). \quad (61)\end{aligned}$$

We then have the following crucial fact. For any $(r,t) \in [l_0, l_1]^2$ such that $10T \leqslant |r-t| \leqslant T_J/10$, either $x_t' := T_{\alpha,\varphi}^t(x) \in R_J^T$ or $T_{\alpha,\varphi}^r(x) = T_{\alpha,\varphi}^{r-t}(x_t') \in R_J^T$ (but not both), hence, by taking into account that the function f_J is supported in R_J^T, we have

$$\begin{aligned}&\left(\int_{\mathscr{B}_l} \left(\int_{l_0}^{l_1} \left(\int_{r \in [l_0,l_1] \, : \, |r-t| \geqslant 10T} \rho(r)\rho(t) f_J(T_{\alpha,\varphi}^t(x)) f_J(T_{\alpha,\varphi}^r(x)) dr \right) dt \right) d\mu \right) \\ &\quad = \left(\int_{\mathscr{B}_l} \left(\int_{l_0}^{l_1} \left(\int_{r \in [l_0,l_1] \, : \, |r-t| \geqslant T_J/10} \rho(r)\rho(t) f_J(T_{\alpha,\varphi}^t(x)) f_J(T_{\alpha,\varphi}^r(x)) dr \right) dt \right) d\mu \right). \quad (62)\end{aligned}$$

By formula (60), our goal is now to estimate for $r-t \geqslant T_J/10$ the integral

$$\int_{\mathscr{B}_l} f_J(T^t_{\alpha,\varphi}(x)) f_J(T^r_{\alpha,\varphi}(x)) d\mu = \int_{T^t_{\alpha,\varphi}(\mathscr{B}_l)\cap R^T_J} f_J(x) f_J(T^{r-t}_{\alpha,\varphi}(x)) d\mu\,.$$

Let then $t^* = r-t$ (which without loss of generality we can assume positive) and recall the notation established in Section 6.3: let $l^* = [t^*]$ and n^* to be the unique integer such that $q_{n^*} \leqslant l^* < q_{n^*+1}$. Let k^* be any integer such that $q_{k^*} \in [q_{n^*}\log^{15} q_{n^*}, q_{n^*}\log^{20} q_{n^*}]$. We recall that by construction we have $q_{k^*} \in [q_n^{1/41}, q_n^{1/19}\log^{20} q_n]$. By the lower bound in formula (56), since $t^* \geqslant T_J/10$, it follows that $\lambda(J) \geqslant 1/q_{k^*}$ and that, for any interval $\bar{I} \in \mathscr{I}_{k^*}$, we have

$$\lambda(I) \leqslant 1/q_{k^*} \leqslant 1/q_n^{1/41}\,.$$

We recall that the set $\mathscr{B}_l$ was defined (in formula (25) of Section 5.2) as a union of a finite number of disjoint complete towers $U_1,\dots,U_m$. By property (B_5) of Proposition 5.3, for every $t \in [l_0, l_1]$ there exist complete towers $\mathscr{T}_{t,1},\dots,\mathscr{T}_{t,m}$, which are approximations of the images $T^t_{\alpha,\varphi}(U_1),\dots,T^t_{\alpha,\varphi}(U_m)$ respectively. Indeed, by taking into account that $\lambda(J) \geqslant 1/q_{k*} \geqslant 1/q_n^{1/18}$ we have form the measure bound (43) that

$$\frac{\sum_{i=1}^m \mu(\mathscr{T}_{t,i} \triangle T^t_{\alpha,\varphi}(U_i))}{\lambda(J)} \leqslant \frac{1}{q_n^{3/5-1/18-15\eta}} \leqslant \frac{1}{q_n^{1/2+45\eta}}\,.$$

Hence

$$\begin{aligned}\left|\int_{T^t_{\alpha,\varphi}(\mathscr{B}_l)\cap R^T_J} f_J(x) f_J(T^{t^*}_{\alpha,\varphi}(x)) d\mu\right| &\leqslant \sum_{i=1}^m \left|\int_{T^t_{\alpha,\varphi}(U_i)\cap R^T_J} f_J(x) f_J(T^{t^*}_{\alpha,\varphi}(x)) d\mu\right| \\ &\leqslant \sum_{i=1}^m \left|\int_{\mathscr{T}_{t,i}\cap R^T_J} f_J(T^{t^*}_{\alpha,\varphi}(x)) f_J(x) d\mu\right| + \frac{\sum_{i=1}^m \mu(\mathscr{T}_{t,i} \triangle T^t_{\alpha,\varphi}(U_i))}{\lambda(J)} \\ &\leqslant \sum_{i=1}^m \left|\int_{\mathscr{T}_{t,i}\cap R^T_J} f_J(T^{t^*}_{\alpha,\varphi}(x)) f_J(x) d\mu\right| + \frac{1}{q_n^{1/2+45\eta}} \qquad (63)\end{aligned}$$

and we can focus now on estimating $\sum_{i=1}^m \left|\int_{\mathscr{T}_{t,i}\cap R^T_J} f_J(T^{t^*}_{\alpha,\varphi}(x)) f_J(x) d\mu\right|$. Following Lemma 6.7, we distinguish two cases. In the first case we have $N(\theta_{t,i}, h_{t,i}) \leqslant q_n^{1/3}$. By the bound in (45) we then have

$$\left|\int_{\mathscr{T}_{t,i}\cap R^T_J} f_J(x) f_J(T^{t^*}_{\alpha,\varphi}(x)) d\mu\right| \leqslant \|f_J\|_0^2 \frac{\mu(\mathscr{T}_{t,i})}{q_n^{1/10}} \leqslant \frac{C}{q_n^{1/10}} \frac{\mu(\mathscr{T}_{t,i})}{\lambda(J)}. \qquad (64)$$

In the second case we have $N(\theta_{t,i}, h_{t,i}) \geqslant q_n^{1/3}$. From the bound in (46), for all $I := \bar{I} \times \{s\}$ with $\bar{I} \in \mathscr{I}_{k^*}$ we have

$$\left|\int_{\overline{\mathscr{T}_{t,i}\cap I}} f_J(T^{t^*}_{\alpha,\varphi}(\theta,s)) f_J(\theta,s) d\theta\right| \leqslant C\{\mathscr{N}_0(f_J,f_J) + \mathscr{N}_1(f_J,f_J)\lambda(I)\} \frac{\lambda(\mathscr{T}_{t,i}\cap I)}{q_n^{50\eta}}. \qquad (65)$$

By the lower bound (56) it follows that $\lambda(J) \geqslant 1/q_{k^*}$, hence there exists a product set E^T_{J,k^*}, measurable with respect to the partial partition $\mathscr{W}_{n,k}$, with base $\bar{E}^T_{J,k^*}$ measurable with respect to the partition $\mathscr{I}_{k^*}$, such that $R^T_J \subset E^T_{J,k^*}$ and we have

$$\mu(E^T_{J,k^*}) \leqslant C\mu(R^T_J) = C' T \lambda(J)\,.$$

Thus, from the bound in formula (65) we derive the estimate

$$\left|\int_{\mathscr{T}_{t,i}\cap R_J^T} f_J(x) f_J(T^{t^*}_{\alpha,\varphi}(x))d\mu\right| \leqslant \frac{C}{q_n^{50\eta}}\frac{\mu(\mathscr{T}_{t,i}\cap R_J^T)}{\lambda(J)}\,. \tag{66}$$

By formulas (64) and (66), and summing over all towers of $\mathscr{B}_l$, that is, over $i \in \{1,\ldots,m\}$, we conclude

$$\sum_{i=1}^m \left|\int_{\mathscr{T}_{t,i}\cap R_J^T} f_J(T^{t^*}_{\alpha,\varphi}(x)) f_J(x)d\mu\right| \leqslant \frac{C}{q_n^{1/10}}\frac{\mu(\cup_{i=1}^m \mathscr{T}_{t,i})}{\lambda(J)} + \frac{C'}{q_n^{50\eta}}\frac{\mu(\cup_{i=1}^m \mathscr{T}_{t,i}\cap R_J^T)}{\lambda(J)}\,. \tag{67}$$

We then estimate the terms on the RHS of formula (67). By property (B_2) in Proposition 5.3 we have $\mu(\mathscr{B}_l) \leqslant q_n^{-1/2+6\eta}$, and by the measure bound in formula (43), we derive

$$\mu(\cup_{i=1}^m \mathscr{T}_{t,i}) \leqslant \mu(\mathscr{B}_l) + \sum_{i=1}^m \mu(\mathscr{T}_{t,i} \triangle T^t_{\alpha,\varphi}(U_i)) \leqslant q_n^{-1/2+6\eta} + q_n^{-3/5+15\eta}\,.$$

and taking into account that $\lambda(J) \geqslant 1/q_{k*} \geqslant q_n^{-1/18}$ we have

$$\frac{1}{q_n^{1/10}}\frac{\mu(\cup_{i=1}^m \mathscr{T}_{t,i})}{\lambda(J)} \leqslant \frac{1}{q_n^{1/10-1/18}}\left(\frac{1}{q_n^{1/2-6\eta}} + \frac{1}{q_n^{3/5-15\eta}}\right) \leqslant \frac{1}{q_n^{1/2+45\eta}}\,, \tag{68}$$

By the equidistribution properties of the base rotation under the Diophantine assumption on the rotation number, there exist constants $C, C' > 0$ such that

$$\max\left(\mu(\mathscr{B}_l \cap R_J^T), \mu(\cup_{i=1}^m \mathscr{T}_{t,i}\cap R_J^T)\right) \leqslant C\mu(R_J^T)\frac{\log^2 q_n}{q_n^{1/2-4\eta}} \leqslant C'\frac{\lambda(J)}{q_n^{1/2-5\eta}}\,, \tag{69}$$

so that we also have

$$\frac{1}{q_n^{50\eta}}\frac{\mu(\cup_{i=1}^m \mathscr{T}_{t,i}\cap R_J^T)}{\lambda(J)} \leqslant \frac{1}{q_n^{1/2+45\eta}}\,. \tag{70}$$

Thus, by formulas (63), (67), (68) and (70), we derive that

$$\left|\int_{T^t_{\alpha,\varphi}(\mathscr{B}_l)\cap R_J^T} f_J(x) f_J(T^{t^*}_{\alpha,\varphi}(x))d\mu\right| \leqslant \frac{C}{q_n^{1/2+45\eta}}\,. \tag{71}$$

At this point, by formulas (59), (60), (61), (62) and (71) (by integration over $(r,t) \in [l_0, l_1]^2$ with $|r-t| \geqslant T_J/10$) we derive that there exist constants $C, C' > 0$ such that

$$\int_{\mathscr{B}_l}\left(\int_{l_0}^{l_1} \rho_J(t) f_J(T^t_{\alpha,\varphi}(x))dt\right)^2 d\mu \leqslant \frac{C}{q_n^{1/2+45\eta}}(l_1-l_0)^2 + C'\frac{\mu(T^t_{\alpha,\varphi}(\mathscr{B}_l)\cap R_J^T)}{\lambda(J)}(l_1-l_0)\,. \tag{72}$$

By taking into account that $\lambda(J) \geqslant 1/q_{k*} \geqslant 1/q_n^{1/18}$ and that that $l_1 - l_0 \geqslant q_n^{80\eta}$, by the measure bound (43) and the equidistribution estimate (69) we then have

$$\begin{aligned}\frac{\mu(T^t_{\alpha,\varphi}(\mathscr{B}_l)\cap R_J^T)}{\lambda(J)} &\leqslant \left(\frac{\mu(\cup_{i=1}^m \mathscr{T}_{t,i}\cap R_J^T)}{\lambda(J)} + q_n^{1/18}\sum_{i=1}^m \mu(\mathscr{T}_{t,i}\triangle T^t_{\alpha,\varphi}(U_i))\right) \\ &\leqslant \frac{C}{q_n^{1/2-5\eta}} + \frac{q_n^{1/18}}{q_n^{3/5-15\eta}} \leqslant \frac{C'}{q_n^{1/2+45\eta}}(l_1-l_0)\,,\end{aligned}$$

hence from formula (72) we derive the following bound on self-correlations:

$$\int_{\mathscr{B}_l}\left(\int_{l_0}^{l_1}\rho_J(t)f_J(T^t_{\alpha,\varphi}(x))dt\right)^2 d\mu \leqslant C\frac{(l_1-l_0)^2}{q_n^{1/2+45\eta}}. \tag{73}$$

Finally, from formulas (58), (69) and (73) we derive the bound

$$\begin{aligned}\int_{l_0}^{l_1}\left|\int_{\mathscr{B}_l} f_J(T^t_{\alpha,\varphi}(x))g_J(x)d\mu\right|dt \\ \leqslant \|g_J\|_0\mu(\mathscr{B}_l\cap R_J^T)^{1/2}\left(\int_{\mathscr{B}_l}\left(\int_{l_0}^{l_1}\rho_J(t)f_J(T^t_{\alpha,\varphi}(x))dt\right)^2 d\mu\right)^{1/2} \\ \leqslant C\left(\frac{\mu(\mathscr{B}_l\cap R_J^T)}{\lambda(J)}\right)^{1/2}\frac{(l_1-l_0)}{q_n^{1/4+20\eta}} \leqslant C'\frac{(l_1-l_0)}{q_n^{1/2+15\eta}}.\end{aligned}$$

Claim 2 is therefore proved. □

From Claim 2, together with the immediate estimate

$$\left|\int_{\mathscr{B}_l\cap R_J^T} f_J(T^t_{\alpha,\varphi}(x))g_J(x)d\mu\right| \leqslant C\frac{\mu(\mathscr{B}_l\cap R_J^T)}{\lambda(J)}$$

and the bound (69), we derive our final estimate on the bad set, that is, as soon as $l_0 \geqslant T_J$,

$$\int_{l_0}^{l_1}\left|\int_{\mathscr{B}_l\cap R_J^T} f_J(T^t_{\alpha,\varphi}(x))g_J(x)d\mu\right|^2 dt \leqslant C\frac{l_1-l_0}{q_n^{1+\eta}} \leqslant C'\frac{l_1-l_0}{l_0^{1+\eta/2}} \leqslant \frac{C''}{l^{1+\eta/3}}. \tag{74}$$

The statement of Theorem 7 then follows from the estimates in formulas (57) and (74). □

The hypotheses of the criterion for countable Lebesgue spectrum stated in Theorem 6 also hold for all smooth time-changes of a horocycle flow $\{h^t\}$ on the unit tangent bundle M of a compact hyperbolic surface, as we will explain below.

Let $\mathscr{M} = \{M_\zeta | \zeta > 0\}$ denote in this case the trivial family such that $M_\zeta = M$ for all $\zeta > 0$. For such a trivial family, all families of flow-boxes are admissible and, for all $T > 0$ and $C > 0$, the families of functions $\mathscr{F}_k(F_J^T, C, \zeta) := \mathscr{F}_k(F_J^T, C)$ and $\mathscr{G}_k(F_J^T, C, \zeta) := \mathscr{G}_k(F_J^T, C)$, introduced in Definition 7.3, are independent of $\zeta > 0$.

Proposition 7.5. *For any sufficiently smooth time change $\{h^t_\varphi\}$ of a horocycle flow $\{h^t\}$, there exists a family $\Phi := \{F_J\}$ of maximal flow-boxes with $\inf\lambda(J) = 0$ (and $\sup T_J = +\infty$), such that for any $T > 0$ and $C > 0$, and for any family of pairs of functions $\{(f_J, g_J)\}$ such that $f_J \in \mathscr{F}_1(F_J^T, C)$ and $g_J \in \mathscr{G}_1(F_J^T, C)$ we have*

$$\inf_{F^J\in\Phi}\int_{\mathbb{R}\setminus[-T_J,T_J]} | < f_J \circ h^t_\varphi, g_J > |^2 dt = 0.$$

Proof. The estimates required to prove this assertion are carried out in Section 6.3 of [16] where the authors prove that sufficiently smooth time-changes of the horocycle flow have Lebesgue maximal spectral type (see the Remark 2 below about the smoothness assumptions). The base of the flow-boxes are 2-dimensional intervals of uniform (fixed) size in the geodesic direction and arbitrarily small size in the direction of the complementary horocycle. Such flow-boxes and the test functions of the classes $\mathscr{F}_1(F_J^T, C)$ and $\mathscr{G}_1(F_J^T, C)$ are introduced in Lemma 28 of [16] (where the relevant estimates on their derivatives are proved). The key estimates on correlations of functions in the classes $\mathscr{F}_1(F_J^T, C)$ and $\mathscr{G}_1(F_J^T, C)$ (supported on flow-boxes) are stated in [16] as formulas (40) and (41) in the proof of Lemma 28. In fact, estimates on correlations follow from those formulas by taking into account the formula of Lemma 9 of [16], which reduces estimates on correlations for the time changes to estimates on curvilinear integrals along push-forwards of geodesic arcs. $\square$

Remark 2. *The relevant estimates on correlations from [16] (see in particular Lemma 24) are stated for time-change functions of L^2 Sobolev regularity $r > 11/2$. However, the argument that establishes that the maximal spectral type is Lebesgue, which also implies the hypotheses of our criterion for countable Lebesgue spectrum, hold under the milder hypothesis that the time-change function is C^1 and C^2 in the geodesic direction.*

We are now ready to derive our conclusions. By Theorem 3, Theorem 6, and Theorem 7 we derive the completion of the proof of our main result:

Corollary 5. *For $\alpha \in D_{\log,\xi}$, $\xi < 1/10$, the dynamical system $(T^t_{\alpha,\varphi}, M, \mu)$ has Lebesgue spectral type with countable multiplicity.*

By Theorem 25 of [16], §6.3, Theorem 6 of Section 7.2, and Proposition 7.5, we derive a similar result for smooth time changes of the horocycle flow, thereby completing the proof of the Katok-Thouvenot conjecture ([26], Conjecture 6.8):

Corollary 6. *Any flow obtained by a sufficiently smooth time change from a horocycle flow has Lebesgue spectral type with countable multiplicity.*

A Birkhoff sums estimates

The goal of this appendix is to prove Lemmas 4.1, 4.2 and 4.3 of Section 4.2.

We start with two intermediate results on the control of Birkhoff sums of the ceiling function $\varphi \in \mathscr{C}_\eta$ and its derivatives in terms of the closest visit to the singularity.

In this section the symbol C_φ, C'_φ, $C''_\varphi > 1$ will denote constants depending only on the ceiling function $\varphi \in \mathscr{C}_\eta$ and which is not necessarily the same throughout the exposition.

Sublemma 1. *There exists $r_\varphi \in \mathbb{N}$ and $C_\varphi > 1$ such that for every $r \geqslant r_\varphi$, every $x \in M$ and every $N \in [q_r, q_{r+1}]$, we have*

$$|S_N(\varphi')(\bar{x}) - \varphi'\left(x^N_{min}\right)| \leqslant C_\varphi q_{r+1}^{2-\eta}, \tag{75}$$

and

$$\varphi''\left(x^N_{min}\right) \leqslant S_N(\varphi'')(\bar{x}) \leqslant \varphi''\left(x^N_{min}\right) + C_\varphi q_{r+1}^{3-\eta}. \tag{76}$$

Sublemma 2. *For sufficiently large $N \in \mathbb{N}$ and for all $x \in M$, we have*

$$|S_N(\varphi')(\bar{x})| \leqslant |\varphi'(x^N_{min})| + C_\varphi N^{2-\eta} \log^{(1+2\xi)(2-\eta)} N. \tag{77}$$

If $N \leqslant K q_{r+1}$, then for all $x \in M$ we have

$$S_N(\varphi'')(\bar{x}) \leqslant K\varphi''\left(x^N_{min}\right) + KC_\varphi q_{r+1}^{3-\eta}. \tag{78}$$

Proof of Sublemma 1. We first prove (75). Let χ_{r+1} denote the characteristic function of $I_{r+1} = [-\frac{1}{4}q_{r+1}^{-1}, \frac{1}{4}q_{r+1}^{-1}]$, and define $\psi_r := (1-\chi_{r+1})|\varphi'|$. Note that $(H_{2.1})$–$(H_{2.2})$ imply that

$$\mathrm{Var}(\psi_r) \leqslant C'_\varphi q_{r+1}^{2-\eta} \quad \text{and} \quad \int_{\mathbb{T}} \psi_r d\lambda_{\mathbb{T}} \leqslant C'_\varphi q_{r+1}^{1-\eta}.$$

By Denjoy-Koksma inequality, since $\psi_r \geqslant 0$ and $N \leqslant q_{r+1}$, we have

$$S_N(\psi_r)(\bar{x}) \leqslant S_{q_{r+1}}(\psi_r)(\bar{x}) \leqslant q_{r+1} \int_{\mathbb{T}} \psi_r d\lambda_{\mathbb{T}} + \mathrm{Var}(\psi_r) \leqslant C''_\varphi q_{r+1}^{2-\eta}.$$

Since $N \leqslant q_{r+1}$, the orbit $\{\bar{x} + j\alpha\}_{0\leqslant j<N}$ is at least $\frac{1}{2}q_{r+1}^{-1}$ separated, hence its intersection with I_{r+1} is either a singleton (which by definition would then realize x_{min}^N) or an empty set. Hence

$$\left|S_N(\varphi')(\bar{x}) - \varphi'(x_{min}^N)\right| \leqslant S_N(\psi_r)(\bar{x}) \leqslant C''_\varphi q_{r+1}^{2-\eta}.$$

Let us now prove (76). Notice that $(H_{2.3})$ implies that the function φ'' is positive for $x \in (0,c) \cup (1-c,1)$ for some fixed $c > 0$. Let y_{min}^N denote the point in the orbit $\{\bar{x} + j\alpha\}_{j\leqslant N} \setminus \{x_{min}^N\}$ that is closest to 0. Then

$$S_N(\varphi'')(\bar{x}) \geqslant \varphi''(x_{min}^N) + \varphi''(y_{min}^N) - N \cdot \sup_{\bar{x}\in[c,1-c]} |\varphi''(\bar{x})|.$$

Since $N \geqslant q_r$, it follows that the orbit $\{\bar{x} + j\alpha\}_{j\leqslant N}$ intersects the interval $[-\frac{4}{q_r}, \frac{4}{q_r}]$ at least twice. Therefore and by the diophantine assumption on α,

$$\varphi''(y_{min}^N) \geqslant c' q_r^{3-\eta} \geqslant q_{r+1}^2 \geqslant N^2 \geqslant N \cdot \sup_{\bar{x}\in[c,1-c]} |\varphi''(\bar{x})|,$$

and the lower bound is proved.

For the upper bound, we proceed analogously to the proof of (75). The only difference is that we now define $\psi_r(\cdot) := (1-\chi_{r+1}(\cdot))w(\cdot)$, where $w(x) := C_\varphi + \frac{C_\varphi}{x^{3-\eta}} + \frac{C_\varphi}{(1-x)^{3-\eta}}$ and $C_\varphi > 1$ is such that $\forall \bar{x} \in \mathbb{T} : |\varphi''(\bar{x})| < w(\bar{x})$ (the existence of C_φ follows from $(H_{2.1})$–$(H_{2.2})$). We then observe that

$$\mathrm{Var}(\psi_r) \leqslant C'_\varphi q_{r+1}^{3-\eta} \quad \text{and} \quad \int_{\mathbb{T}} \psi_r d\lambda_{\mathbb{T}} \leqslant C'_\varphi q_{r+1}^{2-\eta},$$

and the rest of the proof follow exactly the same lines of the proof of (75). □

Proof of Sublemma 2. We start with (77). By the Diophantine assumptions, if $q_r \leqslant N < q_{r+1}$, it follows that $N \log^{1+2\xi} N \geqslant q_{r+1}$ (for sufficiently large N). Hence (77) follows from (75).

Next we prove (78). We decompose $N = kq_{r+1} + m$, where $k \in [0, K-1]$ and $m < q_{r+1}$. We then use the cocycle identity to write

$$S_N(\varphi'')(\bar{x}) = \sum_{i=0}^{k-1} S_{q_{r+1}}(\varphi'')(\bar{x} + iq_{r+1}\alpha) + S_m(\varphi'')(\bar{x} + kq_{r+1}\alpha).$$

We now apply (76) to each of the terms $S_{q_{r+1}}(\varphi'')(\bar{x} + iq_{r+1}\alpha)$ and also to $S_m(\varphi'')(\bar{x} + kq_{r+1}\alpha)$. Equation (78) then follows by summation, since by definition, for every $i \in \{0, \ldots, k-1\}$,

$$(\bar{x} + iq_{r+1}\alpha)_{min}^{q_{r+1}} \geqslant x_{min}^N \quad \text{and} \quad (\bar{x} + kq_{r+1}\alpha)_{min}^m \geqslant x_{min}^N.$$

□

We are ready now to prove the lemmas of Section 4.2.

Proof of Lemma 4.1. We will give the proof of (6) first. It is similar to the proof of (75) and (76). For $r \in \mathbb{N}$, denote again by χ_r the characteristic function of the interval $I_r = [-\frac{1}{4}q_r^{-1}, \frac{1}{4}q_r^{-1}]$ and define $\psi_r := (1-\chi_{r+1})\varphi$. Note that by the assumption $(H_{2.1})$ we have $\mathrm{Var}(\psi_r) \leqslant C_\varphi q_{r+1}^{1-\eta}$, while $\int_{\mathbb{T}} \psi_r d\lambda_{\mathbb{T}} \leqslant \int_{\mathbb{T}} \varphi d\lambda_{\mathbb{T}} = 1$. By Denjoy-Koksma inequality, for r sufficiently large we thus obtain

$$S_{q_{r+1}}(\psi_r)(\bar{x}) \leqslant q_{r+1}\int_{\mathbb{T}} \psi_r d\lambda_{\mathbb{T}} + \mathrm{Var}(\psi_r) \leqslant q_{r+1}\int_{\mathbb{T}} \bar{\varphi}_r d\lambda_{\mathbb{T}} + C_\varphi q_{r+1}^{1-\eta} \leqslant 2q_{r+1}\,.$$

Note that the intersection of $\{\bar{x}+j\alpha\}_{0\leqslant j<q_{r+1}}$ with $[-\frac{1}{4}q_r^{-1}, \frac{1}{4}q_r^{-1}]$ is either a singleton (which by definition is then realizes the x_{min}^N) or an emptyset. Since $N \leqslant q_{r+1}$, we get

$$S_N(\varphi)(\bar{x}) \leqslant S_{q_{r+1}}(\varphi)(\bar{x}) \leqslant \varphi(x_{min}^N) + S_{q_{r+1}}(\psi_r)(\bar{x}) \leqslant \varphi(x_{min}^N) + 2q_{r+1}.$$

This gives the upper bound. For the lower bound, we let $\tilde{\psi}_r := (1-\chi_r)\varphi$. Notice that since $N \geqslant q_r$, there is at least one point in the orbit $\{\bar{x}+j\alpha\}_{0\leqslant j<N}$ that belongs to $[-q_r^{-1}, q_r^{-1}]$. In particular, by the definition of x_{min}^N it follows that $x_{min}^N \leqslant q_r^{-1}$. Hence,

$$S_N(\varphi)(\bar{x}) \geqslant \varphi(x_{min}^N) + S_N(\tilde{\psi}_r)(\bar{x}) \geqslant \varphi(x_{min}^N) + S_{q_r}(\tilde{\psi}_r)(\bar{x}). \tag{79}$$

Since $\int_{\mathbb{T}} \varphi d\lambda_{\mathbb{T}} = 1$, it follows that for r large enough, $\int_{\mathbb{T}} \tilde{\psi}_r d\lambda_{\mathbb{T}} \geqslant 3/4$. Moreover, $(H_{2.1})$ implies that $\mathrm{Var}(\tilde{\psi}_r) \leqslant C'_\varphi q_r^{1-\eta} < q_r/4$ (for r large enough). So by the Denjoy-Koksma inequality, if r is large enough,

$$S_{q_r}(\tilde{\psi}_r)(\bar{x}) \geqslant q_r\int_{\mathbb{T}} \tilde{\psi}_r d\lambda_{\mathbb{T}} - \mathrm{Var}(\tilde{\psi}_r) \geqslant \frac{3}{4}q_r - \mathrm{Var}(\tilde{\psi}_r) \geqslant \frac{1}{2}q_r\,.$$

This lower bound and (79) finish the proof of (6).

We pass now to the proof of (7). Let $r \in \mathbb{N}$ be unique such that $N \in [q_r, q_{r+1})$. By the Diophantine assumptions on α, it follows that for any $C > 0$ we have $Cq_{r+1}^{2-\eta} \leqslant [\frac{1}{3}q_r]^{\frac{2-\eta}{1-\eta}}$, for sufficiently large $N \in \mathbb{N}$. Moreover, by $(H_{2.1})$, $(H_{2.2})$, since $\eta < 1/1000$, there exist constants $C_\varphi, C'_\varphi > 1$ such that

$$|\varphi'(x_{\min}^N)| \leqslant C_\varphi(x_{min}^N)^{-2+\eta} = C_\varphi\left((x_{min}^N)^{-1+\eta}\right)^{\frac{2-\eta}{1-\eta}} \leqslant C'_\varphi \varphi(x_{\min}^N)^{\frac{2-\eta}{1-\eta}}.$$

This and (75), implies for sufficiently large $N \in \mathbb{N}$

$$|S_N(\varphi')(\bar{x})| < |\varphi'(x_{\min}^N)| + C_\varphi q_{r+1}^{2-\eta} \leqslant C''_\varphi(\varphi(x_{\min}^N) + \frac{1}{3}q_r)^{\frac{2-\eta}{1-\eta}} \leqslant C'''_\varphi(S_N(\varphi)(\bar{x}))^{\frac{2-\eta}{1-\eta}},$$

where the last inequality follows from (6). This finishes the proof of (7).

To prove (8), we note that, by the assumption $(H_{2.3})$, there exist $C_\varphi, C'_\varphi > 1$ such that

$$\varphi''(x_{min}^N) \geqslant C_\varphi^{-1}(x_{min}^N)^{-3+\eta} \geqslant (C'_\varphi)^{-1}(\varphi(x_{\min}^N))^{\frac{3-\eta}{1-\eta}}.$$

Moreover, since $N \geqslant q_r$ and $\xi > \eta$, there exist $C''_\varphi > 1$ such that

$$\varphi''(x_{min}^N) \geqslant (C''_\varphi)^{-1}q_r^{3-\eta} \geqslant (C''_\varphi)^{-1}q_{r+1}^{3-\eta}\log^{(-3+\eta)(1+\xi)} q_r \geqslant (C''_\varphi)^{-1}q_{r+1}^{3-\eta}\log^{-3-\xi} q_{r+1}.$$

Putting the two above together, (8) follows from (6) and (76).

Notice that (9) and (10) follow straightforward from (7) and (8) and the definition of special flows: $S_{N(x,t)}(\varphi)(\bar{x}) \leqslant t+s < S_{N(x,t)+1}(\varphi)(\bar{x})$.

□

Proof of Lemma 4.2. By the definition of u_I in (11), we know that there exist $x_0 \in I$ and $t_0 \in [l_0, l_1]$ such that $S_{N(x_0,t_0)}(\varphi'')(\bar{x}_0) \geqslant q_n^{3-\eta} \log^9 q_n$. Note that if $t \in [l_0, l_1]$ then

$$t \leqslant l_1 = (l+1)^{21/20} < 2l^{21/20} \leqslant 2l_0 \leqslant 2q_{n+1}$$

(see also formula (3)) . Moreover, for every $x \in M$, we have

$$2q_{n+1} > t \geqslant S_{N(x,t)}(\varphi)(\bar{x}) \geqslant C_\varphi^{-1} \cdot N(x,t), \tag{80}$$

and so $N(x,t) < 2C_\varphi q_{n+1}$. By the Diophantine condition on α, for any $C_\varphi > 0$

$$2C_\varphi q_{n+1}^{3-\eta} \leqslant q_n^{3-\eta} \log^5 q_n,$$

for n large enough. Therefore, by (78) (with $K = 2C_\varphi$), we get

$$\begin{aligned} q_n^{3-\eta} \log^9 q_n \leqslant S_{N(x_0,t_0)}(\varphi'')(\bar{x}_0) &\leqslant 2C_\varphi \cdot q_{n+1}^{3-\eta} + C'_\phi \varphi''(x_{min}^{N(x_0,t_0)}) \\ &< q_n^{3-\eta} \log^5 q_n + C'_\varphi \varphi''(x_{min}^{N(x_0,t_0)}), \end{aligned}$$

This implies that $\varphi''(x_{min}^{N(x_0,t_0)}) \geqslant (C''_\varphi)^{-1} q_n^{3-\eta} \log^9 q_n$ and consequently by $(H_{2.3})$ that there exists $j \in [0, N(x_0,t_0) - 1]$ such that

$$\bar{x}_0 + j\alpha \in [-\frac{1}{q_n \log^3 q_n}, \frac{1}{q_n \log^3 q_n}]. \tag{81}$$

Below, we will show the following

Sublemma 3. *For every $t \in [l_0, l_1]$ and every $x \in I \cap T_{\alpha,\varphi}^{-t}(W_{n,k})$, we have*

$$N(x,t) > j. \tag{82}$$

Let us first show how (82) implies (12) and (13). Since $\lambda(I) \leqslant 2/(q_n \log^{15} q_n)$ it follows by (82) that for every $t \in [l_0, l_1]$ and every $x \in I \cap T_{\alpha,\varphi}^{-t}(W_{n,k})$

$$x_{min}^{N(x,t)} \leqslant d(\bar{x} + j\alpha, 0) \leqslant d(\bar{x}_0 + j\alpha, 0) + \lambda(I) \leqslant \frac{2}{q_n \log^3 q_n}. \tag{83}$$

This gives (12). For (13), let $\ell := \ell(x,t)$ be the unique integer such that $q_\ell \leqslant N(x,t) \leqslant q_{\ell+1}$. Then by (80), $2C_\varphi q_{n+1} \geqslant N(x,t) \geqslant q_\ell$ and, by the Diophantine assumptions on α, it follows that, for n sufficiently large, we have

$$\begin{aligned} q_{\ell+1} &\leqslant C(\alpha)^{-1} \cdot q_\ell \log^{1+\xi} q_\ell \leqslant C(\alpha)^{-1}(2C_\varphi q_{n+1}) \log^{1+\xi}(2C_\varphi q_{n+1}) \\ &\leqslant 2C(\alpha)^{-2} C_\varphi (q_n \log^{1+\xi} q_n) \log^{1+\xi}(2C(\alpha)^{-1} C_\varphi q_n \log^{1+\xi} q_n) \leqslant C'(\alpha) C'_\varphi \cdot q_n \log^{2+3\xi} q_n. \end{aligned}$$

Therefore and by (83), for some $C''_\varphi > 0$ and for n sufficiently large,

$$|\varphi'(x_{min}^{N(x,t)})| \geqslant (C''_\varphi)^{-1} (q_n \log^3 q_n)^{2-\eta} \geqslant 2C_\varphi q_{\ell+1}^{2-\eta}$$

and analogously,

$$\varphi''(x_{min}^{N(x,t)}) \geqslant (C''_\varphi)^{-1} (q_n \log^3 q_n)^{3-\eta} \geqslant C_\varphi q_{\ell+1}^{3-\eta}.$$

By the above estimates, and by (75) and (76), we derive that

$$|S_{N(x,t)}(\varphi')(\bar{x})| \geqslant |\varphi'(x_{min}^{N(x,t)})| - C_\varphi q_{\ell+1}^{2-\eta} \geqslant \frac{1}{2}|\varphi'(x_{min}^{N(x,t)})|,$$

and

$$|S_{N(x,t)}(\varphi'')(\bar{x})| \leqslant \varphi''(x_{min}^{N(x,t)}) + C_\varphi q_{\ell+1}^{3-\eta} \leqslant 2\varphi''(x_{min}^{N(x,t)}).$$

This gives (13).

Therefore it only remains to give the following

Proof of of Sublemma 3. Notice that for $x \in I \cap T_{\alpha,\varphi}^{-t}(W_{n,k})$, (82) is equivalent to

$$N(x,t) \geqslant j \tag{84}$$

(since $T_{\alpha,\varphi}^{t}(x) = (\bar{x}+N(x,t)\alpha, s') \in W_{n,k}$ and $\bar{x}+j\alpha$ is to very close to the singularity, so $\bar{x}+j\alpha \notin W_{n,k} \cap \mathbb{T}$ by the definition of j). Notice also that if the lower bound

$$N(x,t_0) \geqslant j, \tag{85}$$

holds, then (84) follows for all $t \in [l_0, l_1]$. Indeed, otherwise,

$$S_{N(x,t_0)}(\varphi)(\bar{x}) - S_{N(x,t)}(\varphi)(\bar{x}) \geqslant \varphi(\bar{x}+j\alpha).$$

Moreover, since $T_{\alpha,\varphi}^{t}(x) \in W_{n,k}$, we get $\bar{x}+N(x,t)\alpha \notin [-q_n^{3/5}, q_n^{3/5}]$ and so

$$(4q_{n+1})^{1-\eta} \geqslant \varphi(\bar{x}+N(x,t)\alpha).$$

Finally, by (81) and the fact that $|\bar{x}-\bar{x}_0| \leqslant \lambda(I) < 2(q_n \log^{15} q_n)^{-1}$, we have

$$\varphi(\bar{x}+j\alpha) \geqslant 2q_n^{1-\eta}\log^2 q_n.$$

By the above and using that $|t-t_0| \leqslant l_1 - l_0 \leqslant (2q_{n+1})^{1/21}$, we have

$$\begin{aligned}(4q_{n+1})^{1-\eta} \geqslant \varphi(\bar{x}+N(x,t)\alpha) \geqslant t+s-S_{N(x,t)}(\varphi)(\bar{x}) \geqslant t-t_0+S_{N(x,t_0)}(\varphi)(\bar{x}) - S_{N(x,t)}(\varphi)(\bar{x}) \\ \geqslant t-t_0+\varphi(\bar{x}+j\alpha) \geqslant \varphi(\bar{x}+j\alpha) - (2q_{n+1})^{1/21} \geqslant q_n^{1-\eta}\log^2 q_n,\end{aligned}$$

where the last inequality holds by the Diophantine condition; a contradiction.

Hence it remains to show (85). Let us assume by contradiction that $N(x,t_0) < j$ for some $x \in I \cap T_{\alpha,\varphi}^{-t}(W_{n,k})$. Then

$$\bigcup_{i=0}^{N(x,t_0)} R_\alpha^i(\bar{I}) \subset \bigcup_{i=0}^{j-1} R_\alpha^i(\bar{I}).$$

Let ℓ_j be unique integer such that $q_{\ell_j} \leqslant j < q_{\ell_j+1}$. Then $q_{\ell_j} \leqslant j \leqslant N(x_0,t_0) \leqslant 2C_\varphi q_{n+1}$ and so by the Diophantine assumption on α,

$$q_{\ell_j+1} \leqslant C(\alpha)^{-1} q_{\ell_j}\log^{1+\xi} q_{\ell_j} \leqslant 2C(\alpha)^{-1}C_\varphi q_{n+1}\log^{1+\xi}(2C_\varphi q_{n+1}) \leqslant 10^{-6} q_n \log^{2+3\xi} q_n.$$

In particular,

$$\bigcup_{i=0}^{j} R_\alpha^i(\bar{x}_0) \cap [-\frac{10}{q_n\log^{2+3\xi} q_n}, \frac{10}{q_n\log^{2+3\xi} q_n}] \subset \bigcup_{i=0}^{q_{\ell_j+1}} R_\alpha^i(\bar{x}_0) \cap [-\frac{1}{10q_{\ell_j+1}}, \frac{1}{10q_{\ell_j+1}}] \subset \{\bar{x}_0 + j\alpha\}.$$

Since $\lambda(\bar{I}) \leqslant (q_n \log^{15} q_n)^{-1}$ and $N(x,t_0) < j$, the above inclusion shows that

$$\bigcup_{i=0}^{N(x,t_0)} R_\alpha^i(\bar{I}) \cap [-\frac{1}{q_n \log^{2+3\xi} q_n}, \frac{1}{q_n \log^{2+3\xi} q_n}] = \emptyset. \tag{86}$$

Moreover, $N(x,t_0) \leqslant 2C_\varphi q_{n+1} \leqslant 2C_\varphi C(\alpha)^{-1} q_n \log^{1+\xi} q_n \leqslant q_n \log^{(1+2\xi)} q_n$ and so

$$N(x,t_0)^{2-\eta} \log^{(1+2\xi)(2-\eta)}(N(x,t_0)) \leqslant q_n^{2-\eta} \log^8 q_n.$$

By (86), for some $C_{\alpha,\varphi} > 0$,

$$\varphi(\bar{x} + N(x,t_0)\alpha) \leqslant C_{\alpha,\varphi}(q_n \log^{2+3\xi} q_n)^{1-\eta}. \tag{87}$$

Therefore, for every $\theta \in \bar{I}$, by (77) and (86) we have

$$\begin{aligned} |S_{N(x,t_0)}(\varphi')(\theta)| &< C_{\alpha,\varphi} q_n^{2-\eta} \log^{(2+3\xi)(2-\eta)} q_n \\ &\quad + C_\varphi N(x,t_0)^{2-\eta} \log^{(1+2\xi)(2-\eta)} N(x,t_0) \leqslant q_n^{2-\eta} \log^{10} q_n. \end{aligned} \tag{88}$$

Moreover, by the definition of special flow we have $S_{N(x,t_0)+1}(\varphi)(\bar{x}) \geqslant t_0 + s \geqslant S_{N(x_0,t_0)}(\varphi)(\bar{x}_0)$, hence by the bound in (87), by the cocycle identity, since $N(x_0,t_0) > j > N(x,t_0)$, there exists $\theta \in \bar{I}$ such that for some constant $C'_{\alpha,\varphi} > 1$ we have

$$\begin{aligned} C_{\alpha,\varphi}(q_n \log^{2+3\xi} q_n)^{1-\eta} &\geqslant \varphi(\bar{x} + N(x,t_0)\alpha) \geqslant S_{N(x,t_0)+1}(\varphi)(\bar{x}) - S_{N(x,t_0)}(\varphi)(\bar{x}) \\ &\geqslant S_{N(x_0,t_0)}(\varphi)(\bar{x}_0) - S_{N(x,t_0)}(\varphi)(\bar{x}) \geqslant S_{N(x_0,t_0)}(\varphi)(\bar{x}_0) - S_{N(x,t_0)}(\varphi)(\bar{x}_0) \\ &\quad + S_{N(x,t_0)}(\varphi)(\bar{x}_0) - S_{N(x,t_0)}(\varphi)(\bar{x}) \geqslant \varphi(\bar{x}_0 + j\alpha) - |S_{N(x,t_0)}(\varphi')(\theta)|\lambda(\bar{I}) \\ &\geqslant (C'_{\alpha,\varphi})^{-1}\left(q_n \log^3 q_n\right)^{1-\eta} - (q_n)^{1-\eta}, \end{aligned}$$

where in the last inequality we use, in this order, (81), (88) and $\lambda(\bar{I}) \leqslant (q_n \log^{15} q_n)^{-1}$. This yields a contradiction as $\xi < 1/10$. So (85) holds. Sublemma 3 is proved. □

This completes the proof of Lemma 4.2. □

Proof of Lemma 4.3. By the cocycle identity we have

$$S_{N(x)}(\varphi')(\bar{x}) = S_{N(x_0)}(\varphi')(\bar{x}) + S_{N(x)-N(x_0)}(\varphi')(\bar{x} + N(x_0)\alpha).$$

It follows that, for some $\theta \in [\bar{x}, \bar{x}_0]$,

$$S_{N(x)}(\varphi')(\bar{x}) - S_{N(x_0)}(\varphi')(\bar{x}_0) = S_{N(x_0)}(\varphi'')(\theta)(\bar{x} - \bar{x}_0) + S_{N(x)-N(x_0)}(\varphi')(\bar{x} + N(x_0)\alpha).$$

Define $A_{x,x_0} := S_{N(x_0)}(\varphi'')(\theta)$. We will prove below that

$$S_{N(x_0)}(\varphi'')(\theta) \geqslant \frac{q_n^{3-\eta}}{\log^5 q_n}, \tag{89}$$

and that

$$|S_{N(x)-N(x_0)}(\varphi')(\bar{x} + N(x_0)\alpha)| \leqslant \frac{A_{x,x_0}}{10}|\bar{x} - \bar{x}_0|. \tag{90}$$

Notice that (89) and (90) immediately imply the lemma.

Since $|S_{N(x_0)}(\varphi'')(\bar{x}_0)| \leqslant q_n^{3-\eta}\log^{10} q_n$, by (76) for $N = N(x_0)$ it follows that

$$\{\bar{x}_0,\ldots,\bar{x}_0+(N(x_0)-1)\alpha\}\cap[-\frac{1}{q_n\log^4 q_n},\frac{1}{q_n\log^4 q_n}] = \emptyset. \tag{91}$$

Notice that for x satisfying $T^t_{\alpha,\varphi}(x_0) \in W_{n,k}$, we have that the second coordinate of $T^t_{\alpha,\varphi}(x)$ is smaller than $q_n^{3/5+1/10}$, i.e.

$$0 \leqslant t+s-S_{N(x)}(\varphi)(\bar{x}_0) \leqslant q_n^{3/5+1/10}. \tag{92}$$

Using this for $x = x_0$, we have

$$q_n \leqslant l_0 \leqslant t \leqslant t+s \leqslant S_{N(x_0)}(\varphi)(\bar{x}_0)+q_n^{3/4}.$$

Therefore $S_{N(x_0)}(\varphi)(\bar{x}_0) \geqslant \frac{1}{2}q_n$. Let r be unique such that $q_r \leqslant N(x_0) < q_{r+1}$. By (91) and by (6) for $N = N(x_0)$, we have $q_{r+1} \geqslant \frac{q_n}{10}$. Since $N(x_0) \geqslant q_r$, the orbit $\{\theta + j\alpha\}_{0\leqslant j<N(x_0)}$ intersects the interval $[-q_r^{-1}, q_r^{-1}]$. In particular, $\theta_{min}^{N(x_0)} \leqslant q_r^{-1}$. Consequently, for some $C_\varphi > 1$

$$\varphi''(\theta_{min}^{N(x_0)}) \geqslant C_\varphi^{-1} q_r^{3-\eta} \geqslant C_\varphi^{-1} C(\alpha)^{(3-\eta)} q_{r+1}^{3-\eta} \log^{-(3-\eta)(1+\xi)} q_r \geqslant \frac{q_n^{3-\eta}}{\log^5 q_n}.$$

This, by the lower bound in (76) for $N = N(x_0)$ and $x = \theta$, implies the estimate (89).

We now turn to the proof of the estimate in (90). Since $T^t_{\alpha,\varphi}(x_0), T^t_{\alpha,\varphi}(x) \in W_{n,k}$, it follows that (92) holds for x_0 and x. Therefore, for some $\bar{z} \in [\bar{x}, \bar{x}_0]$,

$$4q_n^{3/4(1-\eta)} \geqslant 4q_n^{3/5+1/10} \geqslant$$

$$|(t+s-S_{N(x_0)}(\varphi)(\bar{x}_0)) - (t+s-S_{N(x)}(\varphi)(\bar{x}))| = |S_{N(x)}(\varphi)(\bar{x}) - S_{N(x_0)}(\varphi)(\bar{x}_0)| =$$

$$\Big|S_{N(x_0)}(\varphi')(\bar{x}_0)(\bar{x}-\bar{x}_0) + S_{N(x_0)}(\varphi'')(\bar{z})(\bar{x}-\bar{x}_0)^2 + S_{N(x)-N(x_0)}(\varphi)(\bar{x}+N(x_0)\alpha)\Big|.$$

Moreover, we have the following:

Sublemma 4. *If $S_{N(x_0)}(\varphi'')(\bar{x}_0) < q_n^{3-\eta}\log^{10} q_n$, then for every $z \in I$,*

$$S_{N(x_0)}(\varphi'')(\bar{z}) < 30q_n^{3-\eta}\log^{10} q_n.$$

Before we prove the sublemma, let us continue with the proof. By (14) and Sublemma 4,

$$|S_{N(x)-N(x_0)}(\varphi)(\bar{x}+N(x_0)\alpha)| \leqslant [4q_n^{3/4(1-\eta)} + q_n^{7/4+\eta}|\bar{x}-\bar{x}_0| + 30q_n^{3-\eta}\log^{10} q_n(\bar{x}-\bar{x}_0)^2],$$

so by Corollary 4.1, there exists a constant $C_\varphi > 1$ such that

$$|S_{N(x)-N(x_0)}(\varphi')(\bar{x}+N(x_0)\alpha)| \leqslant C_\varphi\Big[4q_n^{3/4(1-\eta)} + q_n^{7/4+\eta}|\bar{x}-\bar{x}_0| + 30q_n^{3-\eta}\log^{10} q_n(\bar{x}-\bar{x}_0)^2\Big]^{\frac{2-\eta}{1-\eta}}. \tag{93}$$

Moreover,

$$\Big[4q_n^{3/4(1-\eta)}+q_n^{7/4+\eta}|\bar{x}-\bar{x}_0|+30q_n^{3-\eta}\log^{10}q_n(\bar{x}-\bar{x}_0)^2\Big]^{\frac{2-\eta}{1-\eta}}$$
$$\leqslant 10\Big[(4q_n^{3/4(1-\eta)})^{\frac{2-\eta}{1-\eta}}+(q_n^{7/4+\eta}|\bar{x}-\bar{x}_0|)^{\frac{2-\eta}{1-\eta}}+(30q_n^{3-\eta}\log^{10}q_n(\bar{x}-\bar{x}_0)^2)^{\frac{2-\eta}{1-\eta}}\Big].$$

Since $\frac{2-\eta}{1-\eta}\leqslant 2+2\eta$ and $|\bar{x}-\bar{x}_0|\geqslant q_n^{-3/2+2\eta}$,

$$(4q_n^{3/4(1-\eta)})^{\frac{2-\eta}{1-\eta}}\leqslant(2q_n^{3/4(1-\eta)})^{2+2\eta}\leqslant 8q_n^{3/2(1-\eta^2)}\leqslant\frac{1}{10^6C_\varphi}\frac{q_n^{3-\eta}}{\log^5 q_n}|\bar{x}-\bar{x}_0|.$$

Moreover, since $(q_n\log^{15}q_n)^{-1}\geqslant q_k^{-1}\geqslant\lambda(I)\geqslant|\bar{x}-\bar{x}_0|$,

$$(q_n^{7/4+\eta}|\bar{x}-\bar{x}_0|)^{\frac{2-\eta}{1-\eta}}\leqslant(q_n^{7/4+\eta}|\bar{x}-\bar{x}_0|)^{2+2\eta}$$
$$\leqslant q_n^{(7/2+2\eta)(1+\eta)}|\bar{x}-\bar{x}_0|^{2+2\eta}\leqslant\frac{1}{10^6C_\varphi}\frac{q_n^{3-\eta}}{\log^5 q_n}|\bar{x}-\bar{x}_0|.$$

Finally, using again that $(q_n\log^{15}q_n)^{-1}\geqslant|\bar{x}-\bar{x}_0|$, we get

$$(30q_n^{3-\eta}\log^{10}q_n(\bar{x}-\bar{x}_0)^2)^{\frac{2-\eta}{1-\eta}}\leqslant\frac{1}{10^6C_\varphi}\frac{q_n^{3-\eta}}{\log^5 q_n}|\bar{x}-\bar{x}_0|.$$

Putting the estimates together in (93), we get

$$|S_{N(x)-N(x_0)}(\varphi')(\bar{x}+N(x_0)\alpha)|\leqslant 10\cdot 3\cdot 10^{-6}\frac{q_n^{3-\eta}}{\log^5 q_n}|\bar{x}-\bar{x}_0|.$$

hence the estimate in (90) is proved.

We just have to give the following

Proof of Sublemma 4. We know that $N(x_0)\leqslant 2C_\varphi q_{n+1}$. Let r be unique integer such that we have $q_r\leqslant N(x_0)<q_{r+1}$. Then by the Diophantine assumption on α, since $\xi<1/10$, it follows that

$$q_{r+1}\leqslant q_r\log^{1+2\xi}q_r\leqslant N(x_0)\log^{1+2\xi}N(x_0)$$
$$\leqslant 2C_\varphi q_{n+1}\log^{1+2\xi}(2C_\varphi q_{n+1})\leqslant q_n\log^{(1+3\xi)+(1+2\xi)}q_n\leqslant q_n\log^{5/2}q_n.$$

If $S_{N(x_0)}(\varphi'')(\bar{z})\geqslant 30q_n^{3-\eta}\log^{10}q_n$, by (76) and the above bound on q_{r+1}, it follows that (see $(H_{2.3})$)

$$\frac{11}{10}\cdot\frac{\varphi_0}{(2-\eta)}\Big(z_{min}^{N(x_0)}\Big)^{-3+\eta}\geqslant 29q_n^{3-\eta}\log^{10}q_n.$$

Hence

$$z_{min}^{N(x_0)}\leqslant\Big(\frac{11}{10}\cdot\frac{\varphi_0}{(2-\eta)}\Big)^{\frac{1}{3-\eta}}\cdot\Big(29q_n^{3-\eta}\log^{10}q_n\Big)^{\frac{1}{-3+\eta}}.$$

However, since $x,z\in I$ and $\lambda(I)<(q_n\log^{15}q_n)^{-1}$, we get

$$x_{min}^{N(x_0)}\leqslant z_{min}^{N(x_0)}+q_n^{-1}\log^{-15}q_n\leqslant\Big(\frac{12}{10}\cdot\frac{\varphi_0}{(2-\eta)}\Big)^{\frac{1}{3-\eta}}\cdot\Big(29q_n^{3-\eta}\log^{10}q_n\Big)^{\frac{1}{-3+\eta}}.$$

Applying (76) for x_0 and $N(x_0)$ (and using $(H_{2.3})$), we would get

$$S_{N(x_0)}(\varphi'')(\bar{x}_0) \geqslant \left(\frac{9}{10}\cdot\frac{\varphi_0}{(2-\eta)}\right)\cdot\left(x_{min}^{N(x_0)}\right)^{-3+\eta} \geqslant \frac{9}{12}29q_n^{3-\eta}\log^{10}q_n \geqslant 2q_n^{3-\eta}\log^{10}q_n,$$

a contradiction. □

Lemma 4.3 is thus proved. □

References

[1] V. Arnol'd, *Topological and ergodic properties of closed 1-forms with incommensurable periods*, Funktsionalnyi Analiz i Ego Prilozheniya, **25**, no. 2 (1991), 1–12. (Translated in: Functional Analysis and its Applications, **25**, no. 2, 1991, 81–90).

[2] A. Avila and G. Forni, *Weak mixing for interval exchange tranformations and translation flows*, Ann. of Math. **165** (2007), 637-664.

[3] A. Avila and M. Viana, *Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture*, Acta Math. **198** (2007), 1–56.

[4] J. Chaika and A. Wright, *A smooth mixing flow on a surface with non-degenerate fixed points*, J. Amer. Math. Soc. **32** (2019), 81–117.

[5] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic theory,* Grundlehren der Math. Wissenschaften **245** (1982) 486 pp., Springer, New York.

[6] D. Dolgopyat and B. Fayad, *Limit Theorems for toral translations*, Proceedings of Symposia in Pure Mathematics, Volume **89**, 2015.

[7] D. Dolgopyat and Y. Pesin, *Every compact manifold carries a completely hyperbolic diffeomorphism* , Erg. Th. & Dynam. Sys. **22** (2002), 409–437.

[8] B. Fayad, *Analytic mixing reparametrizations of irrational flows*, Erg. Th., Dynam. Sys. **22** (2002) 437–468.

[9] B. Fayad, *Polynomial decay of correlations for a class of smooth flows on the two torus*, Bull. SMF **129** (2001), 487–503.

[10] B. Fayad, *Smooth mixing diffeomorphisms and flows with singular spectra*, Duke Math. J. **132**, no. 2 (2006), 371–391.

[11] B. Fayad and A. Katok, *Mixed spectrum reparametrizations of linear flows on* $\mathbb{T}^2$*,* Moscow Math. J., **1**, (2001), 521–537.

[12] B. Fayad and A. Kanigowski, *On multiple mixing for a class of conservative surface flows*, Inv. Math. **203** (2) (2016), 555–614.

[13] L. Flaminio and G. Forni, *Orthogonal powers and Möbius conjecture for smooth time changes of horocycle flows*, ERA **26** (2019), 16–23.

[14] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus.* Ann. of Math. **155** (2002), no. 1, 1–103.

[15] G. Forni and A. Kanigowski, Multiple mixing and disjointness for time changes of bounded-type Heisenberg nilflows, preprint, arXiv:1810.13319v1.

[16] G. Forni and C. Ulcigrai, *Time-Changes of Horocycle Flows*, J. Mod. Dynam. **6** (2012), 251–273.

[17] H. Furstenberg, *The unique ergodicity of the horocycle flow*, in Recent Advances in Topological Dynamics (New Haven, Conn., 1972), Lecture Notes in Math. **318**, Springer, Berlin, 1973, 95–115.

[18] I. M. Gelfand, S. V. Fomin, *Geodesic fiows on mani/olds 0/ constant negative curvature,* Uspekhi mat. nauk, 47,1 (1952,), 118-137.

[19] M. Guenais and F. Parreau, *Eigenvalues of transformations arising from irrational rotations and step functions*, preprint, arXiv:math/0605250.

[20] B. Host, *Mixing of all orders and pairwise independent joinings of system with singular spectrum*, Israel J. Math. **76** (1991), 289-298.

[21] H. Hu, Y. Pesin and A. Talitskaya, *Every Compact Manifold Carries a Hyperbolic Bernoulli Flow* in "Modern Dynamical Systems and Applications", Cambridge Univ. Press, 2004, 347–358.

[22] A. Kanigowski, J. Kułaga-Przymus and C. Ulcigrai, *Multiple mixing and parabolic divergence in smooth area-preserving flows on higher genus surfaces,* Journal of European Mathematical Society (to appear).

[23] A. Kanigowski, M. Lemańczyk and C. Ulcigrai, *On disjointness properties of some parabolic flows*, preprint, arXiv:1810.11576v1.

[24] A. B. Katok, *Spectral properties of dynamical systems with an integral invariant on the torus.* Dokl. Akad. Nauk SSSR **223** (1975), 789–792.

[25] A. B. Katok, *Bernoulli diffeomorphisms on surfaces*, Ann. Math. **110** (1979), 529-547.

[26] A. B. Katok and J.-P. Thouvenot, *Spectral properties and combinatorial constructions in ergodic theory*, Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, 649–743.

[27] K. M. Khanin and Ya. G. Sinai, *Mixing for some classes of special flows over rotations of the circle*, Funktsionalnyi Analiz i Ego Prilozheniya, **26**, no. 3 (1992), 1–21 (Translated in: Functional Analysis and its Applications, **26**, no. 3, 1992, 155–169).

[28] D. Kleinbock, N. Shah and A. Starkov, *Dynamics of subgroup actions on homogeneous spaces of lie groups and applications to number theory*, Chapter 11 of Handbook of Dynamical Systems, Volume 1, Part A, 2002, Pages 813-930.

[29] A V. Kochergin, *On the absence of mixing in special flows over the rotation of a circle and in flows on a two-dimensional torus*, Dokl. Akad. Nauk SSSR **205** (1972), 949-952.

[30] A. V. Kochergin, *Mixing in special flows over a shifting of segments and in smooth flows on surfaces*, Mat. Sb. (N.S.) , **96** (**138**) (1975), 471–502.

[31] A. V. Kochergin, *Nonsingular saddle points and the absence of mixing*, Mat. Zametki, **19** (3) (1976), 453–468 (Translated in: Math. Notes, 19:3: 277-286.)

[32] A. V. Kochergin *Nondegenerate fixed points and mixing in flows on a two-dimensional torus* I: Sb. Math. **194** (2003) 1195–1224; II: Sb. Math. **195** (2004) 317–346.

[33] A. V. Kochergin, *Nondegenerate saddles and the absence of mixing in flows on surfaces*, Tr. Mat. Inst. Steklova **256** (2007), 252–266.

[34] A. V. Kochergin, *Causes of stretching of Birkhoff sums and mixing in flows on surfaces*, in *Dynamics, Ergodic Theory and Geometry* (B. Basselblatt Editor), Cambridge University Press, 2010.

[35] A. N. Kolmogorov, *On dynamical systems with an integral invariant on the torus*, Doklady Akad. Nauk SSSR **93** (1953), 763–766.

[36] A. N. Kolmogorov, *Théorie générale des systèmes dynamiques et mécanique classique*, (French) Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, (1957), 315–333.

[37] A. N. Kolmogorov, *New metric invariant of transitive dynamical systems and automorphisms of Lebesgue spaces*, Topology, ordinary differential equations, dynamical systems, Collection of review articles. 2. On the occasion of the 50th anniversary of the institute, Trudy Mat. Inst. Steklov., **169**, 1985, 94–98; Proc. Steklov Inst. Math., **169** (1986), 97–102.

[38] D. V. Kosygin and Ya. G. Sinai, *From Kolmogorov's Work on entropy of dynamical systems to Non-uniformly hyperbolic dynamics*, In: Charpentier É., Lesne A., Nikolski N. K. (eds) Kolmogorov's Heritage in Mathematics. Springer, Berlin, Heidelberg, 2007.

[39] M. Kontsevich, *Lyapunov exponents and Hodge theory*. The mathematical beauty of physics (Saclay, 1996), 318–332, Adv. Ser. Math. Phys., 24, World Sci. Publishing, River Edge, NJ, 1997.

[40] M. Lemańczyk, *Spectral Theory of Dynamical Systems*, Encyclopedia of Complexity and Systems Science, Springer, 2009.

[41] M. Lemańczyk, *Sur l'absence de mélange pour des flots spéciaux au-dessus d'une rotation irrationnelle*, Colloq. Math. **84/85** (2000), 29–41.

[42] B. Marcus, *Unique ergodicity of the horocycle flow: variable negative curvature case*. Conference on Ergodic Theory and Topological Dynamics (Kibbutz Lavi, 1974). Israel J. Math. **21**, 1975, no. 2-3, 133–144.

[43] B. Marcus, *Ergodic properties of horocycle flows for surfaces of negative curvature*, Ann. of Math. (2) **105**, 1977, 81–105.

[44] S. P. Novikov, *The Hamiltonian formalism and a multivalued analogue of Morse theory*, Uspekhi Mat. Nauk **37** (1982), no. 5 (227), 3–49.

[45] S. P. Novikov, *The semiclassical electron in a magnetic field and lattice. Some problems of low dimensional "periodic" topology*, Geometric & Functional Analysis (GAFA) **5** (2) (1995), 434–444.

[46] O. S. Parasyuk, *Flows of horocycles on surfaces of constant negative curvature* (in Russian), Uspekhi Mat. Nauk **8**, no. 3, 1953, 125–126.

[47] Y. Pesin, *General Theory of Smooth Hyperbolic Dynamical Systems* in Dynamical Systems, Ergodic Theory and Applications, Editors: Sinai, Ya. G. (Ed.), Springer 2000.

[48] M. Ratner, *Horocycle flows, joinings and rigidity of products*, Ann. of Math. **118** (1983), 277–313.

[49] M. Ratner, *Rigidity of horocycle flows*, Ann. of Math. **115** (1982), 597-614.

[50] M. Ratner, *Horocycle flows are loosely Bernoulli*, Israel J. Math. **31** (1978), 122-132.

[51] D. Ravotti, *Quantitative mixing for locally Hamiltonian flows with saddle loops on compact surfaces*, Ann. Henri Poincaré **18** (12) (2017), 3815–3861.

[52] D. Scheglov, *Absence of mixing for smooth flows on genus two surfaces*, J. Mod. Dynam. **3** (2009), no. 1, 13–34.

[53] M. D. Shklover, *On dynamical systems on the torus with continuous spectrum* , Izv. Vuzov **10** (1967), 113–124.

[54] L. D. Simonelli, *Absolutely Continuous Spectrum for Parabolic Flows/Maps*, Disc. & Cont. Dynam. Sys. - A, **38** (1) (2018), 263–292.

[55] Ya. G. Sinai, *Dynamical systems with Lebesgue spectrum of countable multiplicities*, Izvestia Akad Nauk SSSR, **25, 6** (1961), 899-924.

[56] Ya. G. Sinai, *Ergodic Theory*, in Kolmogorov Selected Works III, Information Theory and the Theory of Algorithms. Authors: Kolmogorov, Andrei N., Editors: Shiryaev, Albert N. (Ed.), Springer, 1993.

[57] R. Tiedra de Aldecoa, *Spectral Analysis of Time-Changes of the Horocycle Flow*, J. Mod. Dynam. **6** (2) (2012), 275–285.

[58] R. Tiedra de Aldecoa, *Commutator Methods for the Spectral Analysis of Uniquely Ergodic Dynamical Systems*, Ergod. Th. Dynam. Sys. **35** (03) (2015), 944–967.

[59] C. Ulcigrai, *Mixing of asymmetric logarithmic suspension flows over interval exchange transformations*, Ergod. Th. Dyn. Sys. **27** (2007), 991–1035

[60] C. Ulcigrai, *Weak mixing for logarithmic flows over interval exchange transformations*, J. Mod. Dynam. **3** (2009),35–49.

[61] C. Ulcigrai, *Absence of mixing in area-preserving flows on surfaces*, Ann. of Math. **173** (2011), 1743–1778.

[62] A. Zorich, *Asymptotic flag of an orientable measured foliation on a surface*. Geometric study of foliations (Tokyo, 1993), 479–498, World Sci. Publishing, River Edge, NJ, 1994.

[63] A. Zorich, *Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents*. Ann. Inst. Fourier **46** (1996), no. 2, 325–370.

[64] A. Zorich, *Deviation for interval exchange transformations*. Erg. Th. Dynam. Sys. **17** (1997), no. 6, 1477–1499.