Analytic uniquely ergodic volume preserving maps on odd spheres

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Abstract

We construct examples of volume-preserving uniquely ergodic (and hence minimal) real-analytic diffeomorphisms on odd-dimemsional spheres.

1 Introduction

There is a general belief that topology of a manifold M with some low-dimensional exceptions, does not influence ergodic properties of volume-preserving dynamical systems on M and that restriction on topological properties of systems with strong recurrence, say, come only from algebraic and differential topology rather than from dynamics.

There are two aspects here: (i) the *smooth realization problem* that asks what isomorphism types or properties of measure-preserving transformations or flows appear for volume-preserving dynamical systems on a compact manifold and (ii) the *phase space dependence*: given an isomorphism type or property (measurable or topological) that appear in a smooth dynamical system on a compact manifold M describe the class of manifolds that allow a system of the same kind.

We do not discuss the smooth realization problem here. It is enough to mention that, while the only known restriction is finiteness of entropy (and it is not specific to systems preserving a smooth measure and true for any Borel measure), very few systems that are naturally not smooth have been shown to allow a smooth realization, e.g. certain translation on the infinite-dimensional tori, see [2], and certain unpublished constructions.

More is known about the phase space dependence. For example, using a surjective continuous map diffeomorphic on the interior from the closed disc onto an arbitrary compact manifold (closed or with boundary) of the same dimension one shows that every system that can be realized on the disc $\mathbb{D}^n$ an is sufficiently

flat at the boundary can be realized on an arbitrary n-dimensional manifold. This was used to show existence of a zero entropy ergodic diffeomorphisms and flows [1] and Bernoulli transformations [3].

Beyond Bernoulli case and its simple concatenations with zero entropy examples, smooth positive entropy examples are few and far between, see [10, 12]. In the zero entropy setting however there is versatile *approximation by conjugation* method originally introduced in [2] and sometimes called Anosov-Katok method. We use this method in the present work and introduce basic scheme in Section 2.3. For a detailed modern overview of the method see [5]. In the discrete time case as the starting point the method requires an effective smooth action of the circle (for ergodic properties) or free or at least locally free action of the circle (for properties involving behavior of all orbits such as minimality or unique ergodicity). Similarly for the continuous time constructions an action of the two-dimenssional torus on the ambient manifold is needed.

After a long lull following original development in the late 1960s – mid 1970s this methods enjoyed a lively resurrection during the last decade or so. As examples of important advances during that period one should mention a multiple frequency" version of the method that allows to produce new classes of mixing examples [4, 5] and realization of any circle rotation with a Liouvillean rotation number as a C^∞ volume preserving diffeomorphism of any compact manifold with a circle action [6].

Applicability of the approximation by conjugation method critically depends on construction of successive conjugating diffeomorphisms with prescribed behavior. This conjugacies are invariably very large in the appropriate topologies but they should lie in the space; e.g. all derivatives for the map and its inverse must exist although they may be very large. What is required from those conjugacies is controlled behavior in a large mart of the phase space. In the smooth category such constructions are readily available since maps defined on various parts of the space can be glued together.[1] However the situation changes drastically when one passes to the real-analytic category. The most basic property required to start the construction in a particular class is transitivity of the action by diffeomorphisms of that class on pairs of points. In the setting of a real-analytic manifold M this means existence of diffeomorphisms $H_{x,y}$ for any pair $x, y \in M$ such that $H_{x,y}x = y$ such that both $H_{x,y}$ and their inverses extend to a fixed complex neighborhood of M. We are not aware of such a fact for close manifolds but for manifolds with boundaries or for a restricted version, say requiring that H fixes a point z that is excluded from the construction there are obvious difficulties. Those are situations that appear for example in the most basic cases where effective analytic action of the circle exists: the disc $\mathbb{D}^2$ and the two-dimensional sphere S^2. Accordingly the following basic question is still open:

Question 1. *Does there exist a real analytic area preserving diffeomorphism of $\mathbb{D}^2$ or $\mathbb{S}^2$ that is ergodic and has zero entropy?*

[1]Notice however difficulties of the global character that appear in the symplectic versions of the method, see [8].

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Other properties such as ergodicity and closeness to the identity, almost minimality, almost unique ergodicity, etc. are not available on the disc or the sphere in the real analytic category.[2]

In this paper we consider the most basic situation where such a transitive family commuting with a free action of the circle is present, namely the odd-dimensional spheres. A construction of volume-preserving uniquely ergodic real-analytic diffeomorphisms on $\mathbb{S}^{2n+1}$ was outlines in [5]. In the present paper we give complete proofs.

Let us emphasize that spheres are considered to present the method in a succinct way. Existence of a transitive family with large domain of analyticity is the key. For example, our results extend fairly straightforwardly to the case of compact Lie groups, the setting is explained in Section 2.2. An even more general setting is possible; it will appear in a subsequent paper.

2 Formulation of the result and outline of proof

2.1 Notations.

We will consider the standard embedding of the sphere $\mathbb{S}^{2n-1}$ into $\mathbb{R}^{2n}$ and the standard complexification $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$. The vector-field defined in Euclidean coordinates as $v_0(x_1, ..., x_{2n}) = 2\pi(x_2, -x_1, ..., x_{2n}, x_{2n-1})$ defines a linear action of the circle $\mathbb{S}^1$ which we will denote by $\phi_t, t \in \mathbb{R}$, $\phi_1 = \text{Id}$. In Euclidean coordinates $\phi_t(x_1, ..., x_{2n}) = (\cos(2\pi t)x_1 + \sin(2\pi t)x_2, -\sin(2\pi t)x_1 + \cos(2\pi t)x_2, ..., \cos(2\pi t)x_{2n-1} + \sin(2\pi t)x_{2n}, -\sin(2\pi t)x_{2n-1} + \cos(2\pi t)x_{2n})$. We will use the same notations v_0 and φ^t for extensions to $\mathbb{C}^{2n}$ or its subsets. We will call a function on $\mathbb{S}^{2n-1}$ entire if it extends to a holomorphic function on $\mathbb{C}^{2n}$. We say that the map is in C_Δ^ω if it extends to a holomorphic function in the ball $B_\Delta := \{z \in \mathbb{C}^{2n} : |z| \leqslant \Delta\}$. We then use the notation $h \in C_\infty^\omega$ if h is entire. A map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ is said to be C_Δ^ω if its coordinate functions are C_Δ^ω. A diffeormorphism f : $\mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ is C_Δ^ω if both f and f^{-1} are C_Δ^ω. Invertible linear maps are obviously entire differomorphisms. Notice that product of entire diffeomorphisms is an entire diffeomorphism so that entire diffeomorphisms form a group that we will denote $\text{Ent}(\mathbb{S}^{2n-1})$. Its subgroup of entire diffeomorphisms preserving Lebesque measure λ is denoted by $\text{Ent}(\mathbb{S}^{2n-1}, \lambda)$. A homeomorphism h of a compact metric space X is uniquely ergodic if it has only one invariant Borel probability measure. If h preserves a measure with full support (nonempty open sets have positive measure) then unique ergodicity implies minimality (every orbit is dense). Unique ergodicity is equivalent to a uniform distribution property: time averages of any continuous function converge uniformly to a constant which is then equal to the integral with respect to the invariant probability measure.

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[2]The original Bernoulli construction on the sphere or the disc from [9] can be carried out in the real-analytic category with proper adjustments; see [7, 11].

2.2 Formulation of the result.

Fix $n \in \mathbb{N}$. For $\Delta > 0$ and $f, g \in C_\Delta^\omega(\mathbb{S}^{2n-1})$ we denote

$$|f - g|_\Delta = \max\left\{\max_{z \in B_\Delta} |f(z) - g(z)|, \max_{z \in B_\Delta} |f^{-1}(z) - g^{-1}(z)|\right\}$$

Theorem 1. *For any $t_0 \in [0,1]$ and any $\varepsilon > 0$, $\Delta > 0$, there exists a uniquely ergodic volume preserving diffeomorphism $f \in C_\Delta^\omega(\mathbb{S}^{2n-1})$ that satisfies*

$$|f - \varphi^{t_0}|_\Delta < \varepsilon$$

Furthermore, the diffeomorphism f is obtained as a limit in the C_Δ^ω norm of entire maps of the form $F_n = H_n \circ \varphi^{t_n} \circ H_n^{-1}$, $H_n \in \mathrm{Ent}(\mathbb{S}^{2n-1}, \lambda)$.

Remark 1. *The starting point of our argument is existence of a double transitive family of entire diffeomorphisms, commuting with the S^1 action, namely rotations. Our argument works whenever such a family exists with modifications that are essentially notational. Examples are compact connected Like groups and some of their homogeneous spaces.*

Here are more details on the compact Lie group setting. Let G be a compact connected Lie group with probability Haar measure χ. We denote by l_g and r_g correspondingly the left and right translation on G by the element $g \in G$.

The group G can be embedded into $SO(N, \mathbb{R})$ as a subgroup defined by polynomial equations in the matrix coefficients. Without loss of generality we may assume that the image of G is Zariski dense in $SO(N, \mathbb{R})$. We consider the standard coordinate embeddings $SO(N, \mathbb{R}) \hookrightarrow \mathbb{R}^{n^2} \hookrightarrow \mathbb{C}^{n^2}$.

We will call a function on G entire if it extends to a holomorphic function in $\mathbb{C}^{n^2}$. Entire maps and diffeormorphisms of G are defined as in the previos section. Left and right translations are given by linear maps in matrix coordinates and thus extend to invertible linear maps of $\mathbb{C}^{n^2}$ and are thus entire diffeomorphisms. We use the same notations for the extensions.

2.3 The approximation by conjugation construction scheme

We will use the *approximation by conjugation* method sometimes called Anosov–Katok method which was originally introduced in [2].

Without loss of generality we can assume that t_0 is rational, say $t_0 = P_0/Q_0$ where P_0 and Q_0 are relatively prime integers.

We will construct the desired diffeomorphism f inductively, as $\lim_{n\to\infty} F_n$ with $F_{-1} = \varphi^{t_0}$. Each diffeomorphism F_n, $n \geq 0$, will be conjugate via an entire diffeomorphism H_n to a rational element of the action φ with rapidly increasing periods:

$$F_n = H_n \circ \varphi^{P_{n+1}/Q_{n+1}} \circ H_n^{-1}$$

The conjugating diffeomorphisms are defined inductively as[3]

$$H_n = H_{n-1} \circ h_n, \quad \text{with} \quad h_n \circ \varphi^{P_n/Q_n} = \varphi^{P_n/Q_n} \circ h_n$$

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[3]In our case, we will actually have instead of the equality that $h_n \circ \varphi^{P_n/Q_n} \circ h_n^{-1}$ is close in the C_Δ^ω norm to φ^{P_n/Q_n} (see section 2.4 below).

Thus, at the nth step of the construction the parameters are the diffeomorphism $h_n \in \text{Ent}(\mathbb{S}^{2n-1}, \lambda)$ and the rational $t_{n+1} = P_{n+1}/Q_{n+1}$. First one chooses the diffeomorphism h_n to make *all orbits* of the $\mathbb{S}^1$ action φ_n defined by

$$\varphi_n = H_n \circ \varphi \circ H_n^{-1} = H_{n-1} \circ h_n \circ \varphi \circ h_n^{-1} \circ H_{n-1}^{-1} \qquad (1)$$

distributed in an *equivalent way to Lebesgue measure*, in the sense that Birkhoff averages of continuous functions along the φ_n action become as n tends to ∞ proportional with a fixed ratio distortion to the Lebesgue averages of these functions. This will be sufficient to guarantee unique ergodicity of the limit map.

Naturally, H_n, although entire, is likely to have large derivatives, and in particular to be very large on B_Δ. Thus, t_{n+1} has to be chosen with a sufficiently large denominator Q_{n+1} to make the orbits of the finite subgroup

$$H_n \circ \varphi^{kt_{n+1}} \circ H_n^{-1}, \ \ k = 0, \dots Q_{n+1} - 1$$

of the action φ_n approximate the continuous orbits of φ_n sufficiently well to maintain the uniform distribution almost without any loss of precision. Moreover, for the convergence of the construction in the analytic norms, observe that the $\mathbb{S}^1$ action φ_n is entire (since H_n, H_n^{-1} and φ are entire), thus on every compact subset of $\mathbb{C}^{2n}$, $h_n \varphi^{t_{n+1}} h_n^{-1} \to \varphi^{t_n}$ if $t_{n+1} \to t_n$. Hence the latter further constraint on the choice of t_{n+1} will guarantee closeness on B_Δ between F_{n+1} and F_n, and between their inverses.

Since there are no other restrictions on the choice of t_{n+1} the only essential part of the inductive step is the consturction of the diffeomorphism h_n. It is here where the difficuties of the analytic case are very obvious. Since those maps are very large in the real domain control of the complexification presents great problems. A natural approach inspired by the smooth case would be to construct smooth maps first and then to make some kind of approximation (by polynomials or other special classes of functions) to guarantee analyticity in a large domain. The problem however remains since even if such a general approximation procedure works the inverses would have singularities close to the real domain and the construction collapses. Thus it is necessary to find conjugating diffeomorphisms of a special form which may be inverted more or less explicitely to guarantee analyticity of both maps and their inverses in a large compelx domain. We now proceed to showing how to do this in the specific case in question.

2.4 Making $\mathbb{S}^1$ orbits uniformly distributed along a transitive torus action on the sphere

The action φ, which may of course be considered as a subgroup of the orthogonal group $SO(2n)$,has a large centralizer in $SO(2n)$. An easy way to see it is to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via the map $(x_1, x_2, \dots, x_{2n-1}, x_{2n}) \to (x_1 + ix_2, \dots x_{2n-1} + ix_{2n})$ In the complex coordinates φ becomes scalar action $\varphi^t(z_1, \dots, z_n) = (\exp 2\pi it z_1, \dots, \exp 2\pi it z_n)$. The unitary group $U(n)$ commutes with φ. For

our puposes it is useful to notice that already the special unitary group $SU(n)$ which has finite intersection with φ, already acts transitively of the sphere $\mathbb{S}^{2n-1}$.

Assume we are given a collection of one-parameter compact subgroups of period one $k_0, \ldots, k_N$ acting transitively on $\mathbb{S}^{2n-1}$. Given any t_n we want to construct $h_n \in \mathrm{Ent}(\mathbb{S}^{2n-1}, \lambda)$ and t_{n+1} such that $\bar{F}_n = h_n \circ \varphi^{t_{n+1}} \circ h_n^{-1}$ is arbitrarily close to φ^{t_n} and such that the arcs of orbits of $\bar{F}_n$ of length Q_{n+1} for any $x \in \mathbb{S}^{2n-1}$, that we denote $\mathcal{O}(\bar{F}_n, Q_{n+1}, x)$, are distributed with high precision in the same way as the $\mathbb{T}^{N+2}$ orbit $\varphi^t k_0^{s_0} \ldots k_{N+1}^{s_{N+1}} y$, $(t, s_0, \ldots, s_{N+1}) \in \mathbb{T}^{N+2}$ for some $y \in \mathbb{S}^{2n-1}$ that depends on x. The latter distribution is equivalent to Lebesgue measure. The precision with which the orbits $\mathcal{O}(\bar{F}_n, Q_{n+1}, x)$ become distributed as the $\mathbb{T}^{N+1}$ transitive action can be made so high that even after application of the conjugacy H_{n-1} it still holds that the orbits $\mathcal{O}(F_n, Q_{n+1}, x)$ are distributed in an equivalent way to the Lebesgue measure.

The construction of h_n and $\bar{F}_n$ is itself done using a finite number of successive conjugations of periodic times of the φ action. This is the main ingredient in the construction and we now describe it.

We start with $p_0/q_0 = \alpha_0 = t_n$. We consider an entire function ψ_0 that is constant on any k_0 orbit but such that $\psi_0(\varphi^t(\cdot))$ depends wildly on t and $\psi_0(\varphi^{\alpha_0}(\cdot)) = \psi_0(\cdot)$ (the translation groups k_i are chosen so that such functions do exist and are simple to produce). Then if we let $g_0 = k_0^{\psi_0}$ we get that $g_0 \circ \varphi^{\alpha_0} \circ g_0^{-1} = \varphi_{\alpha_0}$. As a consequence of our choices, we observe that for $\alpha_1 = p_1/q_1$ sufficiently close to α_0 we have that $f_0 = g_0 \circ \varphi^{\alpha_1} \circ g_0^{-1}$ is close to $f_{-1} = \varphi^{\alpha_0}$ while due to the twisting of ψ_0 under the φ^t action the orbits of f_0 will be distributed as the continuous $\mathbb{T}^2$ orbits $\varphi^s k_0^{t_0}$, $(s, t_0) \in \mathbb{T}^2$.

In the same way we introduce $g_1 = k_1^{\psi_1}$ that commutes with φ^{α_1} and then choose α_2 sufficiently close to α_1 so that $f_1 = g_0 g_1 \varphi^{\alpha_2} g_1^{-1} g_0^{-1}$ is close to f_1 while the orbits of f_1 are distributed as the $\mathbb{T}^3$ orbits $\varphi^s k_0^{t_0} k_1^{t_1}$, $(s, t_0, t_1) \in \mathbb{T}^3$. We then follow this induction until we obtain α_{N+1} and $f_N = g_0 \ldots g_N \varphi^{\alpha_{N+1}} g_N^{-1} \ldots g_0^{-1}$ such that f_N is close to $f_{-1} = \varphi^{t_n}$ while its orbits are distributed as the $\mathbb{T}^{N+2}$ orbits of $\varphi^s k_0^{t_0} \ldots k_N^{t_N}$, $(s, t_0, \ldots, t_N) \in \mathbb{T}^{N+2}$. Thus we let $t_{n+1} = \alpha_{N+1}$, $h_n = g_0 \ldots g_N$, and $\bar{F}_n = f_N$ and the step n construction is accomplished.

Actually, in the above scheme we omitted an extra difficulty that is related to the control of *every* orbit that is necessary for unique ergodicty. Namely the points x for which the orbit $\mathcal{O}(f_0, x, q_1)$ is well distributed are those for which the ψ_0 twist is effective and this excludes a small measure set of points (suppose for example that ψ_0 depends only on the coordinate z_1, then $\psi_0(\varphi^t(z))$ does not depend on t for points z such that $z_1 = 0$). To overcome this difficulty a certain number of additional conjugacies $k_{N+1}, \ldots, k_M$ must be applied to make sure that each point is affected by the twist in all the directions $k_0, \ldots, k_N$. A consequence of this extra difficulty is that equi-distribution of different points will happen at different times and for different indices in the maps f_l, $l \in [N, M]$.

3 Proof of Theorem 1

3.1 Criterion for unique ergodicity. Reduction to the main induction step.

Definition 1. *Given $C > 0$ and $\varepsilon > 0$, a finite set $\mathcal{O}$ is said to be (C, ε)-uniformly distributed on a manifold X if for any ball $B \subset X$ of radius ε we have that $\#(\mathcal{O} \cap B)/\#\mathcal{O} \in (\lambda(B)/C, C\lambda(B))$.*

Definition 2. *A finite collection of one-parameter compact subgroups of period 1: $k_0, \ldots, k_N \in SU(n)$ is said to have a transitive action on X if for all $x, y \in X$ there exists $t_0, \ldots, t_N \in [0, 1)$ such that $y = k_0^{t_0} \ldots k_N^{t_N} x$.*

Let $X := \mathbb{S}^{2n-1}$. In the sequel we will obtain and fix a finite collection of one-parameter compact subgroups of period 1: $k_0, \ldots, k_N \in SU(n)$ whose action is transitive on X.

Definition 3. *A finite set $\mathcal{O}$ is said to be ε-uniformly distributed along $k_0, \ldots, k_L$ and x if for any ball B of radius ε in $[0, 1]^L$ we have that $\#(\mathcal{O} \cap \bar{k}^B x)/\#\mathcal{O} \in ((1-\varepsilon)\mathrm{Leb}(B), (1+\varepsilon)\mathrm{Leb}(B))$. We used the notation $\bar{k}^B x := \{y = k_0^{t_0} \ldots k_L^{t_L} x : (t_0, \ldots, t_L) \in B\}$.*

Proposition 1. *There exists $C_0 > 0$ such that for any $\varepsilon > 0$, there exists $\eta > 0$ such that for any $x \in X$ we have the following : If a finite set $\mathcal{O}$ is η-uniformly distributed along $\bar{k}_N = k_0, \ldots, k_N$ and x, then $\mathcal{O}$ is (C_0, ε)-uniformly distributed on X.*

Proof. The proof is straightforward by transitivity and periodicity of the action by $k_0, \ldots, k_N$ and compactness of X. $\qquad\square$

We now state a general criterion that we will use to prove unique ergodicity of a volume preserving transformations f. For $x \in X$ and $m \in \mathbb{N}$, we denote the arcs of orbits $\mathcal{O}(f, M, x) := \{f^m(x) : m = 1, \ldots, M\}$.

Proposition 2. *Let f be a volume preserving homeomorphism on X. Assume that there exists $C > 0$ such that the following holds : For any $\varepsilon > 0$ and any $x \in X$, there exists $M \in \mathbb{N}$, for which $\mathcal{O}(f, M, x)$ is (C, ε)-uniformly distributed on X. Then f is uniquely ergodic.*

Proof. The assumption implies that given any continuous function $\psi : X \to \mathbb{C}$, and any $x \in X$, there exists a sequence $M_n \to \infty$ such that

$$\frac{1}{M_n} \sum_{i=1}^{M_n - 1} \psi(f^i x) \in \left(\int_X \psi(z) d\lambda(z)/C', C' \int_X \psi(z) d\lambda(z) \right)$$

with $C' = 2C$. It follows that all the invariant probability measures by f are equivalent to Lebesgue, whence unique ergodicty. $\qquad\square$

We can now state what we will request at a given step of our construction to guarantee unique ergodicity of the limiting transformation.

Proposition 3. *If for any $t_0 \in \mathbb{T}$ and any $\varepsilon > 0$ and $\Delta > 0$ there exists $\tau \in \mathbb{T}$ and a diffeomorphism $h \in Ent(X, \lambda)$ such that the entire diffeomorphism $f = h \circ \varphi^\tau \circ h^{-1}$ satisfies the following*

- *$|f - \varphi^{t_0}|_\Delta < \varepsilon$*

- *There exists $M \in \mathbb{N}$ with the property that for every $x \in X$, there exist $y \in X$ and $M'(x) < M$ such that $\mathcal{O}(f, M', x)$ is ε-uniformly distributed along $k_0, \ldots, k_N$ and y*

Then it is possible to construct a transformation that satisfies the requirements of Theorem 1.

Proof. Assume that sequences $M_n \in \mathbb{N}$, t_n and $H_n \in Ent(X, \lambda)$ have been constructed such that $H_{-1} = \mathrm{Id}$ and $F_n = H_n \circ \varphi^{t_{n+1}} \circ H_n^{-1}$ satisfies
$(\mathcal{H}_n)$: $|F_n - F_{n-1}|_\Delta < \varepsilon/2^n$ for any $n \geqslant 0$; and for any $0 \leqslant j \leqslant n$, and any $x \in X$, there exists $M'_j(x) < M_j$ such that $\mathcal{O}(F_n, M'_j, x)$ is $(C_0, 1/(j+1))$-uniformly distributed in X.

Clearly the first step $n = 0$ follows from Proposition 1 and the assumption. At step n, given H_n, observe that there exists ε_n such that if h_{n+1} and t_{n+2} and M_{n+1} are such that $\bar{f}_{n+1} = h_{n+1} \circ \varphi^{t_{n+2}} \circ h_{n+1}^{-1}$ satisfies that for any $x \in X$ there exists y and $\bar{M}'_{n+1}(x) < M_{n+1}$ such that $\mathcal{O}(\bar{f}_{n+1}, \bar{M}'_{n+1}, x)$ is ε_n-uniformly distributed along $k_0, \ldots, k_N$ and y, then, using Proposition 1, $F_{n+1} = H_n \circ \bar{f}_{n+1} \circ H_n^{-1}$ satisfies that $\mathcal{O}(F_{n+1}, M'_{n+1}(x), x)$ is $(C_0, 1/(n+2))$-uniformly distributed in X for any $x \in X$ (we took $M'_{n+1}(x) = \bar{M}'_{n+1}(H_n^{-1}(x)) < M_{n+1}$). In addition we can require due to our assumption that $\bar{f}_{n+1}$ be arbitrarily close to $\varphi^{t_{n+1}}$. As a consequence of the latter F_{n+1} will be arbitrarily close to F_n and the requirements of $(\mathcal{H}_{n+1})$ will hold if we take $H_{n+1} = H_n \circ h_{n+1}$.

The limiting diffeomorphism $f = \lim F_n$ thus satisfies that $\mathcal{O}(f, M'_j(x), x)$ is $(C_0, 1/(j+1))$-uniformly distributed in X. The unique ergodicity criterion of Proposition 2 being satisfied by f, Theorem 1 follows. $\square$

It only remains to prove the main inductive step given by Proposition 3. Before we do this we shall introduce now the special translations that we will use in order to move the orbits transversally to the φ-action.

3.2 A special family of translations

For any $q \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, we define $\psi_{i,q}(z) = \mathrm{Re}(z_i^q)$ and $\chi_{i,q}(z) = \mathrm{Re}((z_1 - z_i)^q)$. Clearly $\psi_{i,q}$ and $\chi_{i,q}$ are entire, since they are polynomials in the variables $x_1, \ldots, x_{2n}$. A crucial property is that $g \circ \varphi^{p/q} = g$ for $g = \psi_{i,q}$ or $\chi_{i,q}$.

The translations we will use are

$$\xi_i^s(z_1, \ldots, z_n) = (z_1, \ldots, z_{i-1}, e^{i2\pi s} z_i, z_{i+1}, \ldots, z_n)$$
$$\tau_i^s(z_1, \ldots, z_n) = (z_{1,s}, z_2, \ldots, z_{i-1}, z_{i,s}, z_{i+1}, \ldots, z_n)$$
$$z_{1,s} = 1/2 \left((e^{i2\pi s} + 1) z_1 + (e^{i2\pi s} - 1) z_i \right)$$
$$z_{i,s} = 1/2 \left((e^{i2\pi s} - 1) z_1 + (e^{i2\pi s} + 1) z_i \right)$$

Note that under the action by τ_i^s we have that $z_{1,s} + z_{2,s} = e^{i2\pi s}(z_1 + z_2)$ while $z_{1,s} - z_{2,s} = (z_1 - z_2)$. This is crucial to insure that $\left(\xi_i^{A\psi_{j,q}}\right)^{-1} = \xi_i^{-A\psi_{j,q}}$ and the similar property for $\tau_i^{A\chi_{i,q}}$. Also, as a consequence of our definitions we have that for any $A > 0$

$$\xi_i^{A\psi_{j,q}} \varphi_{p/q} z = \varphi_{p/q} \xi_i^{A\psi_{j,q}} z, \forall j \neq i$$
$$\tau_i^{A\chi_{i,q}} \varphi_{p/q} z = \varphi_{p/q} \tau_i^{A\chi_{i,q}} z$$

Observe finally that $\xi_i^{A\psi_{j,q}}$ and $\xi_i^{-A\psi_{j,q}}$ are entire maps as well as $\tau_i^{A\chi_{i,q}}$ and $\tau_i^{-A\chi_{i,q}}$.

Proposition 4. *Let $k_0 = \xi_1, k_1 = \tau_2, k_2 = \xi_2, k_3 = \xi_3, \ldots, k_n = \xi_n, k_{n+1} = \tau_2, k_{n+2} = \xi_2, k_{n+3} = \tau_3, k_{n+4} = \xi_3, \ldots, k_{3n-2} = \tau_n, k_{3n-1} = \xi_n, k_{3n} = \tau_2, k_{3n+1} = \xi_2, k_{3n+2} = \tau_3, k_{3n+3} = \xi_3, \ldots, k_{5n-4} = \tau_n, k_{5n-4} = \xi_n$. Then the sequence $k_0, \ldots, k_{5n-4}$ is transitive.*

Remark 2. *The translation $k_1 = \tau_2$ is not necessary in making the sequence transitive but will be useful later when we will build the conjugacy to make sure that the shear along the z_1 direction is triggered.*

The proof of Proposition 4 will be an immediate consequence of lemma 2 below.

Lemma 1. *Fix $j = 1, \ldots, n$ and $(z_1, \ldots, z_n) \in X$. For any ρ_1, ρ_j such that $\rho_1^2 + \rho_j^2 = |z_1|^2 + |z_j|^2$ there exist $t, s \in [0,1]^2$ such that $z' = \tau_j^s \xi_j^t z$ satisfies $|z_1'| = \rho_1$ and $|z_j'| = \rho_j$.*

Proof. Let $z_j = r_j e^{i2\pi\theta_j}$. We have that

$$z_1' = e^{i\pi s}\left(\cos(\pi s) r_1 e^{i2\pi\theta_1} - i\sin(\pi s) r_2 e^{i2\pi\theta_j} e^{i2\pi t}\right)$$

hence if we choose $t + \pi/2 + \theta_j = \theta_1[2\pi]$ and $\tan(\pi s) = r_1/r_2$ we get that $z_1' = 0$ hence $|z_j'|^2 = r_1^2 + r_j^2$. Since

$$z_j' = e^{i\pi s}\left(-i\sin(\pi s) r_1 e^{i2\pi\theta_1} + \cos(\pi s) r_2 e^{i2\pi\theta_j} e^{i2\pi t}\right)$$

it is also possible to choose t and s such that $z_j' = 0$. By continuity any value between 0 and $r_1^2 + r_j^2$ is possible for $|z_j'|^2$ and the lemma is proved. $\square$

Lemma 2. *Given any $\rho_1, \ldots, \rho_n$ such that $\rho_1^2 + \ldots + \rho_n^2 = 1$ and any $z \in X$, there exist $t_1, \ldots, t_{4n-4}$ such that*

$$z' = \tau_2^{t_{4n-4}} \xi_2^{t_{4n-5}} \ldots \tau_n^{t_{2n}} \xi_n^{t_{2n-1}} \tau_2^{t_{2n-2}} \xi_2^{t_{2n-3}} \ldots \tau_n^{t_2} \xi_n^{t_1} z$$

satisfies $|z_j'| = \rho_j$ for every $j = 1, \ldots, n$.

Proof. Using lemma 1 repeatedly we obtain first $t_1, \ldots, t_{2n-2}$ such that $\bar{z} = \tau_2^{t_{2n-2}} \xi_2^{t_{2n-3}} \ldots \tau_n^{t_2} \xi_n^{t_1} z$ satisfies $\bar{z}_j = 0$ for every $j = 2, \ldots, n$. Next we choose t_{2n-1} and t_{2n} such that $\bar{z}^{(2)} = \tau_n^{t_{2n}} \xi_n^{t_{2n-1}} \bar{z}$ satisfies $|\bar{z}_n^{(2)}| = \rho_n$: this is possible by lemma 1 since $1 = |\bar{z}_1|^2 + |\bar{z}_n|^2 \geqslant \rho_n^2$. We proceed inductively so that at each step $j \leqslant n - 2$ we have that $|\bar{z}_l^{(j)}| = \rho_l$ for $n - j \leqslant l \leqslant n$ and $\bar{z}_l^{(j)} = 0$ for $1 < l < n - j$. Indeed, since $|\bar{z}_1^{(j)}|^2 + |\bar{z}_{n-j-1}^{(j)}|^2 = 1 - \rho_n^2 - \ldots - \rho_{n-j}^2 \geqslant \rho_{n-j-1}^2$, we can apply lemma 1 and choose $t_{2n+2(j+1)-1}$ and $t_{2n+2(j+1)}$ such that $\bar{z}^{(j+1)} = \tau_{n-j-1}^{t_{2n+2(j-2)}} \xi_{n-j-1}^{t_{2n+2(j-2)-1}} \bar{z}^{(j)}$ satisfies $|\bar{z}_{n-j-1}^{(j+1)}| = \rho_{n-j-1}$. Since τ_{n-j-1} and ξ_{n-j-1} leave the rth coordinates intact for $r \neq 1$ and $r \neq n - j - 1$ we still have $|\bar{z}_l^{(j+1)}| = \rho_l$ for $n - j \leqslant l \leqslant n$. $\qquad\blacksquare$

Proof of Proposition 4. Lemma 2 implies that with an adequate choice of $s_{n+1}, \ldots, s_{5n-4}$ one can obtain arbitrary moduli for the coordinates of $\bar{z} = k_{n+1}^{s_{n+1}} \ldots k_{5n-4}^{s_{5n-4}} z$. Next, with an adequate choice of $s_0, \ldots, s_n$, such that $s_1 = 0$ we can further fix the arguments of $z' = k_0^{s_0} \ldots k_n^{s_n} \bar{z} = \xi_1^{s_0} \xi_2^{s_2} \xi_3^{s_3} \ldots \xi_n^{s_n} \bar{z}$, and the proof of Proposition 4 is complete. $\qquad\blacksquare$

3.3 The inductive step of the successive conjugation construction

The novelty in our construction is that the each step of the successive conjugacy is itself constructed through an inductive procedure that allows to saturate all the directions of the transitive sequence of rotations.

We further expand our transitive sequence of k_i's by introducing $k_{5n-3} = \tau_n, \ldots, k_{6n-5} = \tau_2$, $k_{6n-4} = \xi_2$.

We let $M = 6n - 4$.

For a choice (to be determined later) of sequences $A_0, \ldots, A_M$ and $q_0, \ldots, q_M$, we let

$$
\begin{aligned}
g_0 &= k_0^{A_0 \psi_{2,q_0}}, & k_0 &= \xi_1 \\
g_1 &= k_1^{A_1 \chi_{2,q_1}}, & k_1 &= \tau_2 \\
g_2 &= k_2^{A_2 \psi_{1,q_2}}, & k_2 &= \xi_2 \\
g_3 &= k_3^{A_3 \psi_{1,q_3}}, & k_3 &= \xi_3 \\
&\cdots \\
g_n &= k_n^{A_n \psi_{1,q_n}}, & k_n &= \xi_n
\end{aligned}
$$

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next, we let

$$g_{n+1} = k_{n+1}^{A_{n+1}\chi_2, q_{n+1}}, \quad k_{n+1} = \tau_2$$
$$g_{n+2} = k_{n+2}^{A_{n+2}\psi_1, q_{n+2}}, \quad k_{n+1} = \xi_2$$
$$\dots$$
$$g_{5n-5} = k_{5n-5}^{A_{5n-5}\chi_2, q_{5n-5}}, \quad k_{5n-5} = \tau_n$$
$$g_{5n-4} = k_{5n-4}^{A_{5n-4}\psi_1, q_{5n-4}}, \quad k_{5n-4} = \xi_n$$

and finally

$$g_{5n-3} = k_{5n-3}^{A_{5n-3}\chi_2, q_{5n-3}}, \quad k_{5n-3} = \tau_n$$
$$\dots$$
$$g_{6n-5} = k_{6n-5}^{A_{6n-5}\chi_2, q_{6n-5}}, \quad k_{6n-5} = \tau_2$$
$$g_{6n-4} = k_{6n-4}^{A_{6n-4}\psi_1, q_{6n-4}}, \quad k_{6n-4} = \xi_2$$

We define for each $l \leqslant M$, $G_l = g_0 \circ \dots \circ g_l$.

Definition 4. *We say that z is $(m; a_1, \dots, a_s; \nu)$-transversal if for any $i \neq m$ we have for $\lambda = 0$ and $\lambda = 1$*

$$\mathrm{Leb}\{t_1, \dots, t_s : |\lambda(a_1^{t_1} \dots a_s^{t_s} z)_i - (a_1^{t_1} \dots a_s^{t_s} z)_m| < \nu\} < C\nu$$

where C is a constant that does not depend on z or ν.

Notice that if z is such that $|z_1| > \eta$ then for any $\nu > 0$ sufficiently small

$$\mathrm{Leb}\{t : |(\xi_j^t z)_j - z_1| < \nu\} < C\nu,$$

a property that we denote by z is $((1, j), \xi_j; \nu)$-transversal.

Proposition 5. *Given any $\alpha_0 = p_0/q_0 \in [0, 1)$, any $\eta > 0$, $\Delta > 0$ and $\varepsilon > 0$, there exist sequences $A_0, \dots, A_M$, $\alpha_1 = p_1/q_1, \dots, \alpha_{M+1} = p_{M+1}/q_{M+1}$, and $\varepsilon^{100} = \varepsilon_0 \geqslant \varepsilon_1 \geqslant \dots \geqslant \varepsilon_M$ such that if we denote $f_l = G_l \varphi^{\alpha_{l+1}} G_l^{-1}$, $f_{-1} = \varphi^{\alpha_0}$ we have*

1. *$\left| f_{l+1}^i - f_l^i \right|_\Delta < \varepsilon_0/2^{l+2}, \quad \forall |i| \leqslant q_{l+1}$ and $M - 1 \geqslant l \geqslant -1$.*

2. *For any $M > l > 2$ and any $M \geqslant L \geqslant l$, and any $\mathcal{O}$ such that $\mathcal{O}$ is ε_l-uniformly distributed along $\varphi, k_l, \dots, k_L, z$ and if z is $(1; k_l, \dots, k_L; \varepsilon_l)$-transversal then $g_{l-1}\mathcal{O}$ is ε_{l-1}-uniformly distributed along $\varphi, k_{l-1}, \dots, k_L, z$ and z is $(1; k_{l-1}, k_l, \dots, k_L; \varepsilon_{l-1})$-transversal.*

3. *(Case $l = 2$). For any $M \geqslant L \geqslant 2$, and any $\mathcal{O}$ such that $\mathcal{O}$ is ε_2-uniformly distributed along $\varphi, k_2, \dots, k_L, z$ and if z is $(1; k_2, \dots, k_L; \varepsilon_2)$-transversal then $g_1\mathcal{O}$ is ε_1-uniformly distributed along $\varphi, k_1, \dots, k_L, z$ and z is $(2; k_1, k_2, \dots, k_L; \varepsilon_1)$-transversal (the difference form the previous property is in the change of the transverse coordinate from 1 to 2)).*

4. *(Case $l = 1$). If $\mathcal{O}$ is ε_1-uniformly distributed along $\varphi, k_1, \ldots, k_L, z$ and z is $(2; k_1, k_2, \ldots, k_L; \varepsilon_1)$-transversal, then $g_0\mathcal{O}$ is ε_0-uniformly distributed along $\varphi, k_0, \ldots, k_L, z$.*

5. *If $|z_1| > \eta$ then $g_{6n-4}(\mathcal{O}(\varphi^{\alpha_{6n-3}}, q_{6n-3}, z))$ is ε_{6n-4}-uniformly distributed along φ, k_{6n-4} and z; and z is $((1,2); k_{6n-4}; \varepsilon_{6n-4})$-transversal.*

6. *If $\mathcal{O}$ is ε_{6n-4}-uniformly distributed along φ, k_{6n-4} and z and if z is $((1,2), k_{6n-4}; \varepsilon_{6n-4})$-transversal then $g_{6n-5}\mathcal{O}$ is ε_{6n-5}-uniformly distributed along $\varphi, k_{6n-5}, k_{6n-4}$ and z is $(1, k_{6n-5}, k_{6n-4}; \varepsilon_{6n-5})$-transversal.*

7. *If for some $n \geqslant j \geqslant 2$ we have that $|z_1 - z_j| > \eta$ then $g_{6n-j-3}(\mathcal{O}(\varphi^{\alpha_{6n-j-2}}, q_{6n-j-2}, z))$ is ε_{6n-j-3}-uniformly distributed along k_{6n-j-3} and z; and z is $(1; k_{6n-j-3}; \varepsilon_{6n-j-3})$-transversal.*

Proof of Theorem 1. Before we prove Proposition 5 we show how it implies that $f_M = G_M \varphi^{\alpha_{M+1}} G_M^{-1}$ satisfies the requirements of Proposition 3, from where Theorem 1 would follow.

First of all, it follows from 1) of Proposition 5 and a choice of α_0 such that $|\alpha_0 - t_0| < \varepsilon_0/2$ that $|f_M - \varphi^{t_0}|_\Delta < \varepsilon_0$.

Given any $x \in X$, we claim that there exists $N \leqslant l \leqslant M$ such that $\mathcal{O}(f_l, q_{l+1}, x)$ is ε_0-uniformly distributed along $\{k_0, \ldots, k_l\}$ and some $y \in X$. Due to 1) of Proposition 5 this is sufficient to yield a similar property for f_M (if we replace ε_0 by ε). But uniform distribution along $\{k_0, \ldots, k_l\}$ yields *a fortiori* uniform distribution along $\{k_0, \ldots, k_N\}$ and hence the requirements of Proposition 3 will be satisfied.

To prove our claim, we first need the following immediate lemma.

Lemma 3. *Define $\eta = 1/4^{n+1}$. Then, given any $\bar{z} \in X$, then either $|\bar{z}_1| > \eta$ or there exists $j \in [2, n]$ such that $z = g_{6n-j-2} \ldots g_{6n-4}\bar{z}$ satisfies $|z_1 - z_j| > \eta$.*

Proof of lemma 3. Define $\eta_j = 4^j/4^{n+1}$ for $j = 1, \ldots, n$. If $|\bar{z}_1| \leqslant \eta_1$ while $|\bar{z}_2| \geqslant \eta_2$ then since $z = g_{6n-4}\bar{z}$ satisfies $z_1 = \bar{z}_1$ and $|z_2| = |\bar{z}_2|$ we get that $|z_1 - z_2| \geqslant \eta_2 - \eta_1 \geqslant 2\eta_1$. We now apply a similar argument for $j > 2$.

We first show by induction on j that if $|\bar{z}_i| \leqslant \eta_i$ for $i = 1, \ldots, j$ then $z' = \tau_j^{t_j} \ldots \tau_2^{t_2} \xi_2^{t_1} \bar{z}$ satisfies $|z'_1| \leqslant 2\eta_j$ for any choice of $t_1, \ldots, t_j$. Indeed if we suppose the latter true up to j and assume in addition that $|\bar{z}_{j+1}| \leqslant \eta_{j+1}$ then $z := \tau_{j+1}^{t_{j+1}} z'$ satisfies $z_1 = 1/2\left((e^{i2\pi t_{j+1}} + 1)z'_1 + (e^{i2\pi t_{j+1}} - 1)\bar{z}_{j+1}\right) \leqslant 2\eta_j + \eta_{j+1} \leqslant 2\eta_{j+1}$ (we used that $z'_{j+1} = \bar{z}_{j+1}$).

Now, if to the contrary we suppose that $|\bar{z}_i| \leqslant \eta_i$ for $i = 1, \ldots, j$ while $|\bar{z}_{j+1}| \geqslant \eta_{j+1}$ and use the same notations as above for z' and z then since $z_{j+1} - z_1 = z'_{j+1} - z'_1 = \bar{z}_{j+1} - z'_1$ we get that $\|z_{j+1} - z_1\| \geqslant \eta_{j+1} - 2\eta_j = 2\eta_j$.

But since $\sum_{i=1}^n |\bar{z}|^2 = 1$, we have that if $|\bar{z}_1| \leqslant \eta$, then there necessarily exists a $j \in [1, n-1]$ such that $|\bar{z}_i| \leqslant \eta_i$ for $i = 1, \ldots, j$ while $|\bar{z}_{j+1}| \geqslant \eta_{j+1}$. This finishes the proof of the lemma. $\square$

Back to the requirements of Proposition 3, given $x \in X$ we let $\bar{z} = G_{6n-4}^{-1}x$. Then we have two alternatives

• If $|\bar{z}_1| > \eta$, we prove that $\mathcal{O}(f_{6n-4}, q_{6n-3}, x)$ is ε-uniformly distributed along $\{k_0, \ldots, k_{6n-4}\}$ and $\bar{z}$. Indeed, this amounts to proving that $G_{6n-4}(\mathcal{O}(\varphi^{\alpha_{6n-3}}, q_{6n-3}, \bar{z}))$ is ε_0-uniformly distributed along $\{k_0, \ldots, k_{6n-4}\}$ and $\bar{z}$. To obtain the latter, we apply 5) of Proposition 5, then 6), and then 2) inductively until we finish with 3) then 4).

• If $|\bar{z}_1| \leqslant \eta$, then for j as in Lemma 3 we let $z = g_{6n-j-2} \ldots g_{6n-4}\bar{z} = G_{6n-j-3}^{-1}x$, and we prove that $\mathcal{O}(f_{6n-j-3}, q_{6n-j-2}, x)$ is ε_0-uniformly distributed along $\{k_0, \ldots, k_{6n-j-3}\}$ and z. Indeed, it is sufficient to prove that $G_{6n-j-3}(\mathcal{O}(\varphi^{\alpha_{6n-j-2}}, q_{6n-j-2}, z))$ is ε_0-uniformly distributed along $\{k_0, \ldots, k_{6n-j-3}\}$ and z. Since $|z_1 - z_j| > \eta$, we can apply 6) of Proposition 5 and then 2) inductively until we finish with 3) then 4).

The proof of Theorem 1 is hence completed. $\qquad\square$

Proof of Proposition 5. Proposition 5 is proved by a finite induction of which the main building block is provided by the following straightforward fact:

For any $\varepsilon > 0$ and any $a_1, \ldots, a_s \in \{k_0, \ldots, k_M\}$ there exists $A > 0$ and $\varepsilon' > 0$ such that: if $g = a^{Av_1, q}$ with $q \geqslant Q$ and $(a, v) = (\tau_j, \chi)$ or $(a, v) = (\xi_j, \psi)$ and if $\mathcal{O}$ is ε'-uniformly distributed along $\varphi, a_1, \ldots, a_s, z$ and if z is $(1; a_1, \ldots, a_s; \varepsilon')$-transversal then $g\mathcal{O}$ is ε-uniformly distributed along $\varphi, a, a_1, \ldots, a_s, z$ and z is $(1; a, a_1, \ldots, a_s; \varepsilon)$-transversal.

The latter as we will see will be useful for the proof of 2) of Proposition 5. Similar statements are valid and serve for proving 3)–7) of the proposition.

We describe now how the finite induction is carried out to prove Proposition 5. Assume that we are given A_i for $i \leqslant 6n-5$ and α_i, ε_i for $i \leqslant 6n-4$. Then we choose A_{6n-4} sufficiently large and ε' such that if $\mathcal{O}$ is ε'-uniformly distributed along φ and z and if $|z_1| > \eta$ then $g_{6n-4}\mathcal{O}$ is ε_{6n-4}-uniformly distributed along φ, k_{6n-4} and z is $((1,2), k_{6n-4}; \varepsilon_{6n-4})$-transversal.

Now, we choose α_{6n-3} such that 1) of proposition 5 holds with $l = 6n-5$ and $\mathcal{O}(\varphi^{6n-3}, q_{6n-3}, z)$ is ε'-uniformly distributed along φ and z. Hence (5) of proposition 5 holds.

Next, given A_i for $i \leqslant 6n-6$ and α_i, ε_i for $i \leqslant 6n-5$, we choose A_{6n-5} sufficiently large and ε_{6n-4} such that if $\mathcal{O}$ is ε_{6n-4}-uniformly distributed along φ, k_{6n-4} and z and if z is $((1,2), k_{6n-4}; \varepsilon_{6n-4})$-transversal then $g_{6n-5}\mathcal{O}$ is ε_{6n-5}-uniformly distributed along $\varphi, k_{6n-5}, k_{6n-4}$ and z is $(1, k_{6n-5}, k_{6n-4}; \varepsilon_{6n-5})$-transversal. Then we choose α_{6n-4} to guarantee 1) of proposition 5 with $l = 6n-6$. We can also ask from our choice of A_{6n-5} and ε_{6n-4} and α_{6n-4} that (7) of proposition 5 holds.

We can continue inductively : for l decreasing from $l = 6n-5$ to $l = 3$, we assume given A_i for $i \leqslant l-2$ and α_i, ε_i for $i \leqslant l-1$, we choose A_{l-1} and ε_l such that 2) of proposition 5 holds, then we choose α_l such that 1) of proposition 5 holds, that is $\left| f_{l-1}^i - f_{l-2}^i \right|_\Delta < \varepsilon_0/2^l, \quad \forall |i| \leqslant q_l$. For $l \geqslant 5n-2$ we also ask that (7) of proposition 5 holds.

For $l = 2$, we choose A_1 and ε_2, then α_2 such that 3) of proposition 5 holds and $\left| f_1^i - f_0^i \right|_\Delta < \varepsilon_0/4, \quad \forall |i| \leqslant q_2$. To finish, we choose A_0 and ε_1, then α_1 such that 4) of proposition 5 holds and $\left| f_0^i - f_{-1}^i \right|_\Delta < \varepsilon_0/2, \quad \forall |i| \leqslant q_1$.

$\qquad\square$

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