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Continuous spectrum for a class of smooth mixing Schrödinger operators

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Abstract. We give the first example of a smooth volume preserving mixing dynamical system such that the discrete Schrödinger operators on the line defined with a potential generated by this system and a Hölder sampling function have almost surely a continuous spectrum.

1. *Introduction*

Given a dynamical system (Ω, T, μ), a sampling function $V : \Omega \to \mathbb{R}$ and a base point $x \in \Omega$, we define the one-dimensional (1D) Schrödinger operator generated by (Ω, T, μ), V and x as the operator on $\ell^2(\mathbb{Z})$,

$$(H_{T,V,x}u)_n = u_{n+1} + u_{n-1} + V(T^n x)u_n. \qquad (*)$$

A general fact in spectral theory of 1D Schrödinger operators is that randomness of the potential is a source of localization of the spectrum. Thus, an ergodic dynamical system with randomness features has in general a localized pure point spectrum.

The goal of this paper is to prove the following result that we state informally here before we give its exact statement at the end of this section.

THEOREM. *There exist smooth volume preserving and mixing dynamical systems such that the associated 1D Schrödinger operators with Hölder potentials have, for almost every base point, no eigenvalues in their spectrum.*

Let us first recall some of the instances where randomness yields pure point spectra. The most famous example is Anderson's model: the dynamical system is the Bernoulli shift $(\mathbb{R}^{\mathbb{Z}}, \sigma, \mu^{\mathbb{Z}})$, where μ is a probability measure supported on $\mathbb{R}$; the sampling function $V : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is defined by $V(x) = x_0$, where $x = \cdots x_{-1}x_0x_1 \cdots$. It is well

known that if $\mu = r(x)dx$ with r bounded and compactly supported, then for $\mu^{\mathbb{Z}}$-almost every $x \in \mathbb{R}^{\mathbb{Z}}$, the operator $H_{\sigma,V,x}$ has pure point spectrum and the related eigenfunctions are localized; see for example [**7**, **8**, **14**, **16**]. Another kind of example is given by Bourgain and Schlag [**6**]. In their example, the dynamical system is $(\mathbb{T}^2, A, \text{Leb})$, where $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a hyperbolic toral automorphism; the sampling function $V(x) = \lambda F(x)$ with $F \in C^1(\mathbb{T}^2)$ non-constant and $\int F = 0$. For $\delta > 0$ small and the coupling constant $\lambda > 0$ sufficiently small, they established Anderson localization on some subinterval I_0 of $[-2 + \delta, -\delta] \cup [\delta, 2 - \delta]$. We note that $(A, \mathbb{T}^2, \text{Leb})$ is mixing since A is hyperbolic.

An interesting question is to understand how much randomness is needed to ensure localization. For example, recently much interest in Schrödinger operators generated by the skew-shift was in part motivated by the study of the transition from complete localization to continuous spectra as randomness of the potential decreases.

It is a known fact indeed that quasi-periodic potentials often display absolutely continuous spectra for small coupling. The skew-shift dynamics are on one hand related to the quasi-periodic dynamics, but present on the other hand some mixing (in the fibers) and parabolic features (see [**13**]). The skew-shift T_α is defined on $\mathbb{T}^2$ by $T_\alpha(x, y) = (x + \alpha, y + x)$, where $\alpha \in \mathbb{R}$ is irrational. When α is Diophantine and the sampling function is regular, it is expected that localization holds almost surely for arbitrary non-zero coupling constant (see [**4**, **5**] for results in that direction).

We note that the system $(\mathbb{T}^2, T_\alpha, \text{Leb})$ is strictly ergodic, but not weakly mixing. Thus, the above expectation aims at establishing localization under a weak randomness hypothesis.

Our approach in the present work is a dual one, since we will construct a smooth mixing system for which the associated Schrödinger operators with any Hölder potential have no eigenvalues for almost all base points.

Our construction is a smooth reparametrization of a linear flow on $\mathbb{T}^3$ (see §3.2 for the definitions) that will combine mixing with the existence of super-recurrence times (see Definition 1) for almost every point. The strong recurrence implies a Gordon property on the potential, that in turn yields absence of a pure point in the spectrum (see §2 below).

The construction of the reparametrized flow follows the construction in [**10**] of a reparametrization of a linear flow on $\mathbb{T}^3$ that is mixing but has a purely singular maximal spectral type. Indeed, the singularity of the maximal spectral type in [**10**] was due to the existence of very strong periodic approximations on parts of the phase space that have a slowly decaying measure.

Note that for continuous potentials, the behavior of the Schrödinger operator above the skew-shifts is completely different from the one described in the above conjecture. Indeed, Boshernitzan and Damanik showed that for a typical skew-shift and a generic continuous sampling function, the associated Schrödinger operators have no eigenvalues for almost all base points [**2**]. In [**3**], they generalized this result to skew-shifts on the torus $\mathbb{T}^3$ with a frequency in a residual set. They also showed that their approach cannot be carried out for skew-shifts in dimension larger than or equal to 4.

The approach of [**2**, **3**] is to introduce a recurrence property called 'repetition property', that, when checked for a given dynamical system (at some or all orbits depending on the system) implies a Gordon property on the generic continuous potential (above some or all

orbits depending on the system). Our approach is inspired by theirs, but in our case the Gordon property follows directly from super-recurrence, for all Hölder potentials. In their case, especially since they include almost every frequency of the skew-shift, recurrence is not very strong and the 'repetition property' they prove is only sufficient to guarantee continuous spectrum for a generic continuous potential. Note also that in [**2**], the 'repetition property' and its spectral consequences are shown to hold for almost every point, for almost every interval exchange. It is well known that almost every interval exchange transformation is weakly mixing [**1**], but it is never mixing [**12**].

We will now give the precise statement of our construction. We start with the definition of super-recurrence.

Let (Ω, T) be a topological dynamical system with Ω a compact metric space and T a homeomorphism.

Definition 1. Assume that $x \in \Omega$. If there exist $\alpha > 1$ and an integer sequence $k_n \uparrow \infty$ such that

$$d(T^{k_n}x, x) \leq \exp(-k_n^{\alpha}),$$

then we say that x is super-recurrent with recurrent exponent α.

If μ is an invariant ergodic measure of T, we say that the system (Ω, T, μ) is super-recurrent if μ-almost every $x \in \Omega$ is super-recurrent.

As mentioned above, our examples will be a reparametrization of a minimal translation flow on the three torus by a smooth function Φ. As will be recalled in §3, such flows are uniquely ergodic for a measure equivalent to the Haar measure with density $1/\phi$. We denote by μ the Haar measure on the torus and by μ_ϕ the measure with density $1/\phi$. Note that reparametrizations of linear flows always have zero topological entropy.

THEOREM 1. *There exist* $(\alpha, \alpha') \in \mathbb{R}^2$ *and a smooth reparametrization* $\phi \in C^\infty(\mathbb{T}^3, \mathbb{R}_+^*)$ *of the translation flow* $T_{t(\alpha,\alpha',1)}$ *such that the resulting flow is mixing, for its unique ergodic invariant probability measure* μ_ϕ*, and* μ*-almost every* $x \in \mathbb{T}^3$ *is super-recurrent for its time-one map* T.

As a consequence, for every Hölder continuous potential $V : \mathbb{T}^3 \to \mathbb{R}$*, the operator* $H_{T,V,x}$ *has purely continuous spectrum for* μ*-almost every* $x \in \mathbb{T}^3$.

Remark 1. It is easy to see that super-recurrence for almost every point implies the metric repetition property of [**2**]. Hence, for a generic continuous function V and μ-almost every x, the operator $H_{T,V,x}$ has continuous spectrum.

The frequencies α and α' are specially chosen super Liouville numbers, so that $T_{t_n(\alpha,\alpha',1)}$ is very close to the identity for some sequence $t_n \to \infty$ (see §3.5). Naturally, the very strong periodic approximations of the linear flow are lost after time change, otherwise mixing would not be possible. However, one can choose the reparametrization in such a way that along a sequence of times $t_n \to \infty$, the very strong almost periodic behavior of the translation flow still appears on a set of small measure ϵ_n. If now ϵ_n decreases, but not too rapidly, say $\epsilon_n \sim 1/n$, then by a Borel–Cantelli argument, most of the points on the torus will be strongly recurrent along a subsequence of the sequence t_n.

2. *Super-recurrence and continuous spectrum*

To show why super-recurrence implies the absence of a point part in the spectrum, we just have to show that it implies the Gordon condition, which we now state.

The Gordon condition. A bounded function $V : \mathbb{Z} \to \mathbb{R}$ is called a Gordon potential if there are positive integers $k_n \to \infty$ such that

$$\max_{1 \le l \le k_n} |V(l) - V(l \pm k_n)| \le n^{-k_n}$$

for any $n \ge 1$.

The Gordon condition ensures that the 1D Schrödinger operator on $\ell^2(\mathbb{Z})$ with potential V has no eigenvalues [**11**].

Assume that $V : \Omega \to \mathbb{R}$ is continuous. Define

$$V_x(n) = V(T^n x), \quad x \in \Omega, n \in \mathbb{Z}.$$

We now assume that M is a smooth compact manifold and T is a C^1 diffeomorphism of M. Let d be the Riemann metric on M. Then we have the following simple consequence of super-recurrence.

PROPOSITION 1. *If $x \in M$ is super-recurrent and $V : M \to \mathbb{R}$ is Hölder, then V_x is a Gordon potential.*

Proof. Since T is C^1, T is Lipschitz. Let $L > 1$ be the Lipschitz constant. Assume that V is β-Hölder with Hölder constant C_1. Let $\alpha > 1$ be the recurrent exponent of x. Let $\{k_n : n \ge 1\}$ be the sequence related to x. By taking a subsequence, we can assume that $k_n \ge n$. For $1 \le l \le k_n$, we have

$$\begin{aligned} |V_x(l) - V_x(l \pm k_n)| &= |V(T^l x) - V(T^{l \pm k_n} x)| \\ &\le C_1 d(T^l x, T^{l \pm k_n} x)^\beta \\ &\le C_1 [L^l d(x, T^{\pm k_n} x)]^\beta \\ &\le C_1 [L^{2k_n} e^{-k_n^\alpha}]^\beta \\ &= C_1 \exp(-\beta(k_n^\alpha - 2k_n \ln L)) \\ &\le n^{-k_n} \end{aligned}$$

as soon as n is big enough. By the definition, V_x is a Gordon potential. □

Hence, the second part of Theorem 1 follows from the first part of Theorem 1 and Proposition 1 and the above-mentioned spectral consequence of the Gordon condition on the potentials.

We now proceed to the construction of the reparametrized flow.

3. *Super-recurrent mixing flows*

We start with some notation and reminders on reparametrizations and special flows.

3.1. *Translation flows on the torus.* The translation flow on $\mathbb{T}^n$ of the vector $\alpha \in \mathbb{R}^n$ is the flow arising from the constant vector field $X(x) = \alpha$. We denote this flow by $\{R_{t\alpha}\}$. When the numbers $1, \alpha_1, \ldots, \alpha_n$ are rationally independent, i.e. none of them is a rational combination of the others, $\{R_{t\alpha}\}$ is uniquely ergodic for the Haar measure μ on the torus. In this case we say that it is an irrational flow.

3.2. *Reparametrized flows.* If ϕ is a strictly positive smooth real function on $\mathbb{T}^n$, we define the reparametrization of $\{R_{t\alpha}\}$ with velocity ϕ as the flow given by the vector field $\phi(x)\alpha$, that is, by the system

$$\frac{dx}{dt} = \phi(x)\alpha.$$

The new flow has the same orbits as $\{R_{t\alpha}\}$ and preserves a measure equivalent to the Haar measure given by the density $1/\phi$. Moreover, if $\{R_{t\alpha}\}$ is uniquely ergodic, then so is the reparametrized flow (see [**15**]).

3.3. *Special flows.* The reparametrizations of linear flows can be viewed as special flows above toral translations. We give the formal definition.

Definition 2. Given a Lebesgue space L, a measure-preserving transformation T on L and an integrable strictly positive real function φ defined on L, we define the special flow over T and under the *ceiling function* φ by inducing on $L \times \mathbb{R}/\sim$, where $\sim$ is the identification $(x, s + \varphi(x)) \sim (T(x), s)$, the action of

$$\begin{aligned} L \times \mathbb{R} &\to L \times \mathbb{R} \\ (x, s) &\to (x, s + t). \end{aligned}$$

If T preserves a unique probability measure λ, then the special flow will preserve a unique probability measure that is the normalized product measure of λ on the base and the Lebesgue measure on the fibers.

We will be interested in special flows above minimal translations $R_{\alpha,\alpha'}$ of the two torus and under smooth functions $\varphi(x, y) \in C^\infty(\mathbb{T}^2, \mathbb{R}_+^*)$ that we will denote by $T^t_{\alpha,\alpha',\varphi}$. We denote $M_\varphi = \{(z, s) : z \in \mathbb{T}^2, s \in [0, \varphi(z))\}$. We will still denote by μ the product of the Haar measure of $\mathbb{T}^2$ with the normalized Lebesgue measure on the line.

In all the sequel we will use the following notation, for $m \in \mathbb{N}$,

$$\varphi_m(z) := S_m\varphi(z) = \sum_{l=0}^{m-1} \varphi(z + l(\alpha, \alpha')).$$

With this notation, given $t \in \mathbb{R}_+$, we have for $\xi \in M_\varphi$, $\xi = (z, s)$,

$$T^t\xi = (R_{\alpha,\alpha'}^{N(t,s,z)}(z), t + s - \varphi_{N(t,s,z)}(z)), \tag{3.1}$$

where $N(t, s, z)$ is the largest integer m such that $t + s - \varphi_m(x) \geq 0$, that is, the number of fibers covered by (z, s) during its motion under the action of the flow until time t.

3.4. *Mixing.* We also recall the definition of mixing for a measure-preserving flow: a flow $\{T^t\}$ preserving a measure ν on M is said to be mixing if, for any measurable subsets A and B of M, one has

$$\lim_{t\to\infty} \nu(T^t A \cap B) = \nu(A)\nu(B).$$

By standard equivalence between special flows and reparametrizations (see for example [**9**]), Theorem 1 follows from the following theorem.

THEOREM 2. *There exist a vector $(\alpha, \alpha') \in \mathbb{R}^2$ and a smooth strictly positive function φ defined over $\mathbb{T}^2$ such that the special flow $T^t_{\alpha,\alpha',\varphi}$ is mixing and μ-almost every $\xi \in M_\varphi$ is super-recurrent for $T^1_{\alpha,\alpha',\varphi}$.*

We will now undertake the construction of the special flow $T^t_{\alpha,\alpha',\varphi}$, following the same steps as [**10**]. We will first choose a special translation vector on $\mathbb{T}^2$; then we will give two criteria on the Birkhoff sums of the special function φ above $R_{\alpha,\alpha'}$ that will guarantee mixing and super-recurrence, respectively. Finally, we build a smooth function φ satisfying these criteria.

3.5. *Choice of the translation on $\mathbb{T}^2$.* We start with a quick reminder on continued fractions. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. There exists a sequence of rationals $\{p_n/q_n\}_{n\in\mathbb{N}}$, called the convergents of α, such that

$$\|q_{n-1}\alpha\| < \|k\alpha\| \quad \text{for all } k < q_n \tag{3.2}$$

(where $\|x\| := \min\{|x - n| : n \in \mathbb{Z}\}$) and, for any n,

$$\frac{1}{q_n(q_n + q_{n+1})} \le (-1)^n\left(\alpha - \frac{p_n}{q_n}\right) \le \frac{1}{q_n q_{n+1}}. \tag{3.3}$$

We recall also that any irrational number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ has a writing in continued fractions

$$\alpha = [a_0, a_1, a_2, \ldots] = a_0 + 1/(a_1 + 1/(a_2 + \cdots)),$$

where $\{a_i\}_{i\ge1}$ is a sequence of integers ≥ 1, $a_0 = [\alpha]$. Conversely, any sequence $\{a_i\}_{i\in\mathbb{N}}$ corresponds to a unique number α. The convergents of α are given by the a_i in the following way:

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \quad \text{for } n \ge 2,\ p_0 = a_0,\ p_1 = a_0a_1 + 1,\\ q_n &= a_n q_{n-1} + q_{n-2} \quad \text{for } n \ge 2,\ q_0 = 1,\ q_1 = a_1. \end{aligned}$$

Following [**17**] and as in [**9**], we take α and α' satisfying

$$q'_n \ge e^{(q_n)^5}, \tag{3.4}$$

$$q_{n+1} \ge e^{(q'_n)^5}. \tag{3.5}$$

Vectors $(\alpha, \alpha') \in \mathbb{R}^2$ satisfying (3.4) and (3.5) are obtained by an adequate choice of the sequences $a_n(\alpha)$ and $a_n(\alpha')$. Moreover, it is easy to see that the set of vectors satisfying (3.4) and (3.5) is a continuum; cf. [**17**, Appendix 1].

3.6. *Mixing criterion.* We will use the criterion on mixing for a special flow $T^t_{\alpha,\alpha',\varphi}$ studied in [**9**]. It is based on the uniform stretch of the Birkhoff sums $S_m\varphi$ of the ceiling function above the x or the y direction alternatively depending on whether m is far from the q_n or from the q'_n. From [**9**, Propositions 3.3–3.5], we have the following sufficient mixing criterion. We denote by $\{x\} \in [0, 1)$ the fractional part of a real number x.

PROPOSITION 2. (Mixing criterion) *Let (α, α') be as in §3.5 and $\varphi \in C^2(\mathbb{T}^2, \mathbb{R}^*_+)$. If for every $n \in \mathbb{N}$ sufficiently large, we have a set I_n equal to* $[0, 1]$ *minus a finite number of intervals whose lengths converge to zero such that:*

- $m \in [e^{2(q_n)^4}/2, 2e^{2(q'_n)^4}] \implies |D_x S_m\varphi(x, y)| \geq (m/e^{(q_n)^4})(q_n/n)$ *for any* $y \in \mathbb{T}$ *and* $\{q_n x\} \in I_n$*;*
- $m \in [e^{2(q'_n)^4}/2, 2e^{2(q_{n+1})^4}] \implies |D_y S_m\varphi(x, y)| \geq (m/e^{(q'_n)^4})(q'_n/n)$ *for any* $x \in \mathbb{T}$ *and* $\{q'_n y\} \in I_n$*,*

then the special flow $T^t_{\alpha,\alpha',\varphi}$ is mixing.

3.7. *Super-recurrence criterion.* We give now a condition on the Birkhoff sums of φ above $R_{\alpha,\alpha'}$ that is sufficient to ensure super-recurrence for $T^1_{\alpha,\alpha',\varphi}$.

PROPOSITION 3. (Super-recurrence criterion) *If for n sufficiently large, we have for any x such that $1/n^2 \leq \{q_n x\} \leq 1/n - 1/n^2$ and for any $y \in \mathbb{T}$,*

$$|S_{q_n q'_n}\varphi(x, y) - q_n q'_n| \leq \frac{1}{e^{(q_n q'_n)^2}}, \tag{3.6}$$

then μ-almost every $z = (x, y, s) \in M_\varphi$ is super-recurrent for the time-one map of the special flow $T^t_{\alpha,\alpha',\varphi}$ as in Definition 2.

Proof. Denote by T^t the flow $T^t_{\alpha,\alpha',\varphi}$ and let $t_n = q_n q'_n$. From (3.1), we have that

$$T^{t_n}(x, y, s) = (x + t_n\alpha, y + t_n\alpha', s + t_n - S_{t_n}\varphi(x, y)).$$

From (3.3), (3.4) and (3.5), we get that $\|t_n\alpha\|, \|t_n\alpha'\| \leq 1/e^{t_n^3}$. Now, for x such that $1/n^2 \leq \{q_n x\} \leq 1/n - 1/n^2$, for any $y \in \mathbb{T}$ and for $s \in [0, \varphi(x, y))$, we have from (3.6) that $z = (x, y, s)$ satisfies (for the Euclidean distance) $d(z, T^{t_n} z) \leq 2/e^{t_n^2}$.

Now, the set $\mathcal{C}_n = \{x \in \mathbb{T} : 1/n^2 \leq \{q_n x\} \leq 1/n - 1/n^2\}$ has Lebesgue measure larger than $1/2n$. As q_n increases very fast, we have that the sets $\mathcal{C}_n$ are almost independent, from which it follows by Borel–Cantelli-type lemmas that Lebesgue almost every $x \in \mathbb{T}$ belongs to infinitely many of the $\mathcal{C}_n$. Thus, almost every $z \in M_\varphi$ is super-recurrent. □

3.8. *Choice of the ceiling function φ.* Let (α, α') be as above and define

$$f(x, y) = 1 + \sum_{n\geq 2} X_n(x) + Y_n(y),$$

where

$$X_n(x) = \frac{1}{e^{(q_n)^4}} \cos(2\pi q_n x), \tag{3.7}$$

$$Y_n(y) = \frac{1}{e^{(q'_n)^4}} \cos(2\pi q'_n y). \tag{3.8}$$

Using Proposition 2, we proved in [**9**] that the flow $T^t_{\alpha,\alpha',f}$ is mixing (the proof will be recalled below). In order to keep this criterion valid but have in addition the conditions of Proposition 3 satisfied. we modify the ceiling function in the following way.

- We keep $Y_n(y)$ unchanged.
- We replace $X_n(x)$ by a trigonometric polynomial $\tilde{X}_n$ with integral zero, that is, essentially equal to 0 for $\{q_n x\} < 1/n$ and whose derivative has its absolute value bounded from below by $q_n/e^{(q_n)^4}$ for $\{q_n x\} \in [0,1]\backslash\bigcup_{j=0}^{4}[j/4 - 2/n,\ j/4 + 2/n]$. The first two properties of $\tilde{X}_n$ will yield Criterion 3 while the lower bound on the absolute value of its derivative will ensure Criterion 2.

More precisely, the following proposition enumerates some properties that we will require on $\tilde{X}_n$ and its Birkhoff sums, and that will be sufficient for our purposes.

PROPOSITION 4. *Let (α, α') be as in §3.5. There exists a sequence of trigonometric polynomials $\tilde{X}_n(x)$ satisfying:*

(1) $\int_{\mathbb{T}} \tilde{X}_n(x)\,dx = 0$;
(2) *for any $r \in \mathbb{N}$, for every $n \geq N(r)$, $\|\tilde{X}_n\|_{C^r} \leq 1/(e^{(q_n)^4/2})$;*
(3) *for $\{q_n x\} \leq 1/n$, $|\tilde{X}_n(x)| \leq 1/e^{(q_n q'_n)^4}$;*
(4) *for $\{q_n x\} \in [2/n, \frac{1}{4} - 2/n] \cup [\frac{3}{4} + 2/n, 1 - 2/n]$, we have $\tilde{X}'_n(x) \geq q_n/e^{(q_n)^4}$, as well as for $\{q_n x\} \in [\frac{1}{4} + 2/n, \frac{1}{2} - 2/n] \cup [\frac{1}{2} + 2/n, \frac{3}{4} - 2/n]$, we have $\tilde{X}'_n(x) \leq -q_n/e^{(q_n)^4}$;*
(5) $\|S_{q_n} \sum_{l \leq n-1} \tilde{X}_l\|_{C^0} \leq 1/e^{(q_n q'_n)^4}$;
(6) *for any $m \in \mathbb{N}$, $\|S_m \sum_{l \leq n-1} \tilde{X}'_l\|_{C^0} \leq q_n$.*

Before we prove this proposition, let us show how it allows us to produce the example of Theorem 2. Define for some $n_0 \in \mathbb{N}$

$$\varphi(x, y) = 1 + \sum_{n=n_0}^{\infty} \tilde{X}_n(x) + Y_n(y) \tag{3.9}$$

that is of class C^∞ from Property (2) of $\tilde{X}_n$ and from the definition of Y_n in (3.8). From (2) again, we can choose n_0 sufficiently large so that φ is strictly positive. Furthermore, we have the following theorem.

THEOREM 3. *Let $(\alpha, \alpha') \in \mathbb{R}^2$ be as in §3.5 and φ be given by (3.9). Then the special flow $T^t_{\alpha,\alpha',\varphi}$ satisfies the conditions of Propositions 2 and 3 and hence the conclusion of Theorem 2.*

Proof. The second part of Proposition 2 is valid exactly as in [**9**] since Y_n has not been modified. We sketch its proof for completeness. For $m \in \mathbb{N}$, we have that

$$S_m Y_n(y) = \mathrm{Re}\left(\frac{Y(m, n)}{e^{(q'_n)^4}} e^{i2\pi q'_n y}\right)$$

with

$$Y(m, n) = \frac{1 - e^{i2\pi m q'_n \alpha'}}{1 - e^{i2\pi q'_n \alpha'}}.$$

It follows from (3.2)–(3.5) that

$$|Y(m,k)| \le m \quad \text{for all } k \in \mathbb{N}^*, \tag{3.10}$$

$$|Y(m,k)| \le q_n' \quad \text{for all } k < n, \tag{3.11}$$

$$|Y(m,n)| \ge \frac{2}{\pi} m \quad \text{for all } m \le \frac{q_{n+1}'}{2}. \tag{3.12}$$

Let now $m \in [e^{2(q_n')^4}/2,\, 2e^{2(q_{n+1})^4}]$ and y be such that $3/n \le \{q_n' y\} \le 1/2 - 2/n$. Then (3.3) and (3.4) imply that for any $0 \le l \le m$, $2/n \le \{q_n'(y + l\alpha')\} \le 1/2 - 1/n$. Hence, (3.12) implies that

$$|S_m Y_n'(y)| \ge \frac{8mq_n'}{ne^{(q_n')^4}}.$$

Using (3.10) and (3.11) to bound $\| \sum_{k>n} S_m Y_k' \|$ and $\| \sum_{k<n} S_m Y_k' \|$, respectively, we obtain that $|S_m \sum_{k\ge 1} Y_k'(y)| \ge mq_n'/ne^{(q_n')^4}$. The case $1/2 + 3/n \le \{q_n' y\} \le 1 - 2/n$ is treated similarly.

We now turn to the control of the Birkhoff sums in the x direction. Let $m \in [e^{2(q_n)^4}/2,\, 2e^{2(q_n')^4}]$ and x be such that $|\{q_n x\} - j/4| > 3/n$ for any positive integer $j \le 4$. For definiteness, assume that $\{q_n x\} \in [3/n,\, \frac{1}{4} - 3/n]$, the other cases being similar.

From (3.5), we get for any $0 \le l \le m$ that $2/n \le \{q_n(x + l\alpha)\} \le 1/4 - 2/n$; hence, by Property (4) of $\tilde{X}_n$,

$$|S_m \tilde{X}_n'(x)| \ge \frac{mq_n}{e^{(q_n)^4}}.$$

On the other hand, Properties (2) and (6) imply that

$$\begin{aligned} \|S_m \varphi' - S_m \tilde{X}_n'\| &\le q_n + m \sum_{l \ge n+1} \frac{1}{e^{(q_l)^4/2}} \\ &\le q_n + \frac{2m}{e^{(q_{n+1})^4/2}} \\ &= o\left(\frac{mq_n}{e^{(q_n)^4}}\right) \end{aligned}$$

for the current range of m. The criterion of Proposition 2 thus holds true.

Let now x be as in Proposition 3, that is, $1/n^2 \le \{q_n x\} \le 1/n - 1/n^2$. From (3.5), we have for any $l \le q_n q_n'$ that $0 \le \{q_n(x + l\alpha)\} \le 1/n$; hence, Property (3) implies that

$$|S_{q_n q_n'} \tilde{X}_n(x)| \le \frac{q_n q_n'}{e^{(q_n q_n')^4}} \le \frac{1}{e^{(q_n q_n')^3}}. \tag{3.13}$$

From Properties (2) and (5), we get for n sufficiently large,

$$\begin{aligned} \|S_{q_n q_n'} \sum_{l \ne n} \tilde{X}_l\| &\le \frac{q_n'}{e^{(q_n q_n')^4}} + q_n q_n' \sum_{l \ge n+1} \frac{1}{e^{(q_l)^4/2}} \\ &\le \frac{1}{e^{(q_n q_n')^3}}. \end{aligned} \tag{3.14}$$

On the other hand, it follows from the definition of convergents in §3.5 and (3.3) that for any $y \in \mathbb{T}$, for any $|j| < q'_n$, we have

$$|S_{q'_n} e^{i2\pi jy}| = \left| \frac{\sin(\pi j q'_n \alpha')}{\sin(\pi j \alpha')} \right| \leq \frac{2\pi j q'_n}{q'_{n+1}}, \tag{3.15}$$

which, using (3.4) and (3.5), yields for Y_l as in (3.8),

$$\| S_{q'_n} \sum_{l<n} Y_l \| \leq \frac{1}{e^{(q_n q'_n)^3}}, \tag{3.16}$$

while clearly

$$\| S_{q'_n} \sum_{l>n} Y_l \| \leq e^{-(q'_{n+1})^4/2} \leq \frac{1}{e^{(q_n q'_n)^3}} \tag{3.17}$$

and

$$\| S_{q'_n} Y_n \| \leq \frac{q'_n}{e^{(q'_n)^4}} \leq \frac{1}{e^{(q_n q'_n)^3}}. \tag{3.18}$$

Putting together (3.16)–(3.18) yields

$$\| S_{q_n q'_n} \sum_{l=n_0}^{\infty} Y_l \| \leq \frac{1}{2e^{(q_n q'_n)^2}}. \tag{3.19}$$

In conclusion, (3.6) follows from (3.13), (3.14) and (3.19). □

It remains to construct $\tilde{X}_n$ satisfying (1)–(6).

3.9. *Proof of Proposition 4.* Consider on $\mathbb{R}$ a C^∞ function, $0 \leq \theta \leq 1$, such that

$$\theta(x) = 0 \quad \text{for } x \in (-\infty, 0],$$
$$\theta(x) = 1 \quad \text{for } x \in [1, +\infty).$$

Then we define

$$\theta_n(x) := \theta\left(nq_n\left(x - \frac{1}{nq_n}\right)\right) - \theta\left(nq_n\left(x - \frac{1}{4q_n} + \frac{2}{nq_n}\right)\right).$$

Observe that

$$\theta_n(x) = \begin{cases} 1 & \text{for } x \in \left[\frac{2}{nq_n}, \frac{1}{4q_n} - \frac{2}{nq_n}\right], \\ 0 & \text{for } x \in \left[-\infty, \frac{1}{nq_n}\right] \cup \left[\frac{1}{4q_n} - \frac{1}{nq_n}, +\infty\right]. \end{cases}$$

We define on $\mathbb{R}$ the following functions:

$$U_n(x) = \int_{-\infty}^{x} \theta_n(u)\, du;$$

then

$$V_n(x) = U_n(x) - U_n\left(x - \frac{1}{4q_n}\right).$$

Observe that V_n is compactly supported inside $[0,\ 1/2q_n]$ and has derivative equal to 1 for $x \in J_n = [2/nq_n,\ 1/4q_n - 2/nq_n]$ and derivative equal to -1 on $1/4q_n + J_n$. Define now the zero-averaged function supported inside $[0,\ 1/q_n]$,

$$W_n(x) = V_n(x) - V_n\left(x - \frac{1}{2q_n}\right).$$

The derivative of W_n is constant and equal to 1 on $J_n \cup (3/4q_n + J_n)$ and constant and equal to -1 on $(1/4q_n + J_n) \cup (1/2q_n + J_n)$. Also, $W_n \equiv 0$ on $[0,\ 1/nq_n] \cup [1/q_n - 1/nq_n,\ 1/q_n]$.

We define the following function on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$\hat{X}_n(x) := \frac{q_n}{e^{(q_n)^4}} \sum_{k=0}^{q_n-1} W_n\left(x + \frac{k}{q_n}\right).$$

It is easy to check (1), (2), (3) and (4) of Proposition 4 for $\hat{X}_n$.

Now we consider the Fourier series of $\hat{X}_n(x) = \sum_{k\in\mathbb{Z}} \hat{X}_{n,k} e^{i2\pi kx}$ and let

$$\tilde{X}_n(x) := \sum_{k=-q_{n+1}+1}^{q_{n+1}-1} \hat{X}_{n,k} e^{i2\pi kx}.$$

From the order of the truncation and the C^r norms of $\hat{X}_n$, it is easy to deduce that for any $r \in \mathbb{N}$,

$$\|\tilde{X}_n - \hat{X}_n\|_{C^r} \le \frac{1}{e^{(q_n q'_n)^5}},$$

which allows us to check (1), (2), (3) and (4) for $\tilde{X}_n$.

Proof of Property (5). As for (3.15), using the definition of convergents in §3.5, we obtain for any $x \in \mathbb{T}$, and for any $|k| < q_n$,

$$|S_{q_n} e^{i2\pi kx}| \le \frac{2\pi k q_n}{q_{n+1}};$$

hence, for $\tilde{X}_l := \sum_{k=-q_{l+1}+1}^{q_{l+1}-1} \hat{X}_{l,k} e^{i2\pi kx}$ and $l \le n-1$, we have

$$\begin{aligned}\|S_{q_n}\tilde{X}_l\| &\le \frac{2\pi q_n^2}{q_{n+1}} \sum_{k=-q_{l+1}+1}^{q_{l+1}-1} |\hat{X}_{l,k}| \\ &\le \frac{4\pi q_n^3}{q_{n+1}} \|\hat{X}_l\|;\end{aligned}$$

thus, Property (5) follows. □

Proof of Property (6). For any $|k| < q_n$, we have

$$\begin{aligned} |S_m e^{i2\pi kx}| &= \left| \frac{\sin(\pi mk\alpha)}{\sin(\pi k\alpha)} \right| \\ &\le \frac{1}{|\sin(\pi k\alpha)|} \\ &\le q_n. \end{aligned}$$

Thus, for $l \le n-1$, we use that $|\hat{X}_{l,k}| \le 1/(2\pi|k|)^3 \|D_x^3 \hat{X}_l\|$ and get that

$$\begin{aligned} \|S_m \tilde{X}_l'\| &\le \sum_{k:1\le|k|<q_{l+1}} \frac{1}{(2\pi|k|)^2} q_{l+1} \|D_x^3 \hat{X}_l\| \\ &\le \frac{1}{12} q_{l+1} \|D_x^3 \hat{X}_l\|, \end{aligned}$$

from which Property (6) follows. □

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