ISOLATED ELLIPTIC FIXED POINTS FOR SMOOTH HAMILTONIANS

BASSAM FAYAD AND MARIA SAPRYKINA

ABSTRACT. We construct on $\mathbb{R}^{2d}$, for any $d \geq 3$, smooth Hamiltonians having an elliptic equilibrium with an arbitrary frequency, that is not accumulated by a positive measure set of invariant tori. For $d \geq 4$, the Hamiltonians we construct have not any invariant torus of dimension d. Our examples are obtained by a version of the successive conjugation scheme *à la* Anosov-Katok.

INTRODUCTION

KAM theory (after Kolmogorov Arnol'd and Moser) asserts that generically an elliptic fixed point of a Hamiltonian system is stable in a probabilistic sense, or *KAM-stable: the fixed point is accumulated by a positive measure set of invariant Lagrangian tori.* In classical KAM theory, an elliptic fixed point is shown to be KAM-stable under the hypothesis that the frequency vector at the fixed point is non resonant (or just sufficiently non-resonant) and that the Hamiltonian is sufficiently smooth and satisfies the Kolmogorov non degeneracy condition that involves its Hessian matrix at the fixed point. Further development of the theory allowed to relax the non degeneracy condition. In [EFK1] KAM-stability was established for non resonant elliptic fixed points under the Rüssmann non-planarity condition on the Birkhoff normal form of the Hamiltonian.

The problem is more tricky if no non-degeneracy conditions are imposed on the Hamiltonian. In the analytic setting, no examples are known of an elliptic fixed point with a non-resonant frequency ω_0 that is not KAM-stable or Lyapunov unstable (none of these two properties implies the other).

It was conjectured by M. Herman in his ICM98 lecture [H1] that for analytic Hamiltonians, KAM-stability holds in the neighborhood of a an elliptic fixed point if its frequency vector is assumed to be

Date: October 5, 2016.

The first author is supported by ANR BEKAM and ANR GeoDyM and the project BRNUH. This work was accomplished while the first author was affiliated to the Laboratorio Fibonacci of the Scuola Normale Superiore di Pisa.

Diophantine. The conjecture is known to be true in two degrees of freedom [R], but remains open in general. Partial results were obtained in [EFK1] and [EFK2].

Below analytic regularity, Herman proved that KAM-stability of a Diophantine equilibrium holds without any twist condition for smooth Hamiltonians in 2 degrees of freedom (see [H2], [FK] and [EFK2]). In his ICM98 lecture [H1, §3.5], he announced that KAM-stability of Diophantine equilibria does not hold for smooth Hamiltonians[1] in four or more degrees of freedom, without giving any clew about the counter-examples he had in mind. He also announced that nothing was known about KAM-stability of Diophantine equilibria for smooth Hamiltonians in three degrees of freedom.

In this note, we settle this problem by constructing examples of smooth Hamiltonians for any $d \geq 3$ having non KAM-stable elliptic equilibria with arbitrary frequency. We now state our results more precisely.

Let $\omega_0 \in \mathbb{R}^d$ and let

$$(*) \quad \begin{cases} H(x,y) = \langle \omega_0, r \rangle + \mathcal{O}^3(x,y) \\ r = (r_1, \ldots, r_d), \quad r_j = \frac{1}{2}(x_j^2 + y_j^2) \end{cases}$$

be a smooth function defined in a neighborhood of $(0,0)$. The Hamiltonian system associated to H is given by the vector field $X_H = (\partial_y H, -\partial_x H)$, namely

$$\begin{cases} \dot{x} = \partial_y H(x,y) \\ \dot{y} = -\partial_x H(x,y). \end{cases}$$

The flow of X_H denoted by Φ_H^t has an elliptic fixed point at the origin with frequency vector ω_0.

In [EFK2], it was shown that for any $\omega_0 \in \mathbb{R}^d, d \geq 4$, it is possible to construct C^∞ (Gevrey) Hamiltonians H with a smooth invariant torus, on which the dynamics is the translation of frequency ω_0, that is not accumulated by a positive measure of invariant tori. In this note we adapt the latter construction to the context of elliptic equilibria and we extend it to the three degrees of freedom case.

It is a common knowledge that creating instability in the neighborhood of a fixed point is more delicate than in the context of invariant tori, mainly because the action angle coordinates are singular in the neighborhood of the axes $\{r_i = 0\}$. For instance, when all the coordinates of ω_0 are of the same sign, the fixed elliptic point is Lyapunov

[1]Herman actually raised the problem in the very related context of symplectic maps.

stable, while it is easy to produce examples of diffusive and isolated invariant tori for any resonant frequency vector, even in the analytic category (see [S]).

Definition 1. We say that Φ_H^t is *diffusive* if given any $A > 0$ there exists p and $t_1, t_2 \in \mathbb{R}$ such that $|\Phi_H^{t_1}(p)| \leq A^{-1}$ and $|\Phi_H^{t_2}(p)| \geq A$.

Obviously, if the flow is diffusive, the origin is not Lyapunov stable.

Theorem A. *For any $\omega_0 \in \mathbb{R}^d, d \geq 4$, there exists $H \in C^\infty(\mathbb{R}^{2d})$ as in $(*)$, such that Φ_H^t has no invariant torus of dimension d. More precisely, the manifolds $\{r_i = 0\}$ for $i \leq d$, are foliated by invariant tori of dimension $\leq d-1$ and all other obits accumulate on these manifolds or at infinity.*

Moreover, if the coordinates of ω_0 are not all of the same sign, then Φ_H^t is diffusive.

In the case $d = 3$, our examples will have invariant Lagrangian tori of maximal dimension (equal to 3) that accumulate the origin, but only for r_3 in a countable set.

Theorem B. *For any $\omega_0 \in \mathbb{R}^3$, there exists $H \in C^\infty(\mathbb{R}^6)$ as in $(*)$, and a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of real numbers such that $\lim_{n\to-\infty} a_n = 0$ and $\lim_{n\to+\infty} a_n = +\infty$, such that the manifolds $\{r_3 = a_n\}$, as well as $\{r_i = 0\}$ for $i \leq 3$, are foliated by invariant tori and such that all other obits accumulate on these manifolds or at infinity.*

Moreover, if the coordinates of ω_0 are not all of the same sign, then Φ_H^t is diffusive.

Remark 1. The same construction can be carried out for invariant quasi-periodic tori and gives examples of KAM-unstable tori with arbitrary frequency in 3 degrees of freedom.

Remark 2. Our examples are obtained by a successive conjugation scheme *à la* Anosov-Katok [AK], and the flows that we obtain are rigid in the sense that their iterates along a subsequence of time converges to identity in the C^∞ topology.

Remark 3. In case all the coordinates of ω_0 are of the same sign, there are naturally no diffusive orbits since the equilibrium is Lyapunov stable.

In the case where not all the components of ω_0 are of the same sign and $d \geq 3$, Douady gave in [D] examples of elliptic fixed points with diffusive trajectories. However, his construction, that produces actually examples with an arbitrarily chosen Birkhoff normal form at the fixed point, does not overrule KAM-stability.

1. Notations

–A vector $\omega_0 \in \mathbb{R}^d$ is said to be non-resonant if for any $k \in \mathbb{Z}^d \setminus \{0\}$ we have that $|\langle k, \omega \rangle| \neq 0$, where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product.
–A vector $\omega_0 \in \mathbb{R}^d$ is said to be Diophantine if there exist $N > 0$ and $\gamma > 0$ such that for any $k \in \mathbb{Z}^d \setminus \{0\}$ we have that $|\langle k, \omega_0 \rangle| \geq \gamma \|k\|^{-N}$.
–A non-resonant vector ω_0 is said to be Liouville if it is not Diophantine.
–We denote by $\mathcal{O}_H^T(p)$ the orbit of length T of the point p by the Hamiltonian flow of H. The full orbit of p is denoted by $\mathcal{O}_H^\infty(p)$.
–The notation of type $\{r_i < A\}$ should be understood as $\{(r, \theta) \mid r_i < A\}$.
–We shall say, with a slight abuse of notation, that $\mathcal{O}_H^\infty(p)$ accumulates on $\{r_i = \infty\}$ for some $i = 1, \ldots, d$ if $\mathrm{proj}_{r_i} \mathcal{O}_H^{t_j}(p) \to \infty$ over a sequence of times (t_j).

2. Orbits accumulating the axis and diffusive orbits

2.1. **Two degrees of freedom.** As discussed earlier, in 2 degrees of freedom it follows from the Last Geometric Theorem of Herman (see [H2], [FK] and [EFK2]) that if $\omega_0 \in \mathbb{R}^2$ is Diophantine then if $H \in C^\infty(\mathbb{R}^4)$ is as in $(*)$, then the origin is KAM-stable for Φ_H^t.

We will be interested in constructing close to integrable non KAM-stable (and diffusive if the frequency vector satisfies $\omega_{0,1}\omega_{0,2} < 0$) examples in 2 degrees of freedom when the frequency is Liouville. Consider

$$H_0^2(r) = \langle \omega_0, r \rangle. \tag{2.1}$$

Let $\mathcal{U}_2$ be the set of symplectomorphisms U such that $U(r, \theta) = (r, \theta)$ near the axes $\{r_i = 0\}$, $i = 1, 2$, as well as for $|r|$ sufficiently large. Consider the class of conjugates of H_0^2 by the elements of $\mathcal{U}_2$, and denote its C^∞-closure by $\bar{\mathcal{H}}_2$.

Theorem 1. *Let $H_0^2(r) = \langle \omega_0, r \rangle$ for a Liouville vector $\omega_0 \in \mathbb{R}^2$. Let $\mathcal{D}$ be the set of Hamiltonians $H \in \bar{\mathcal{H}}_2$ such that for each $p \in \mathbb{R}^4$ we have that $\mathcal{O}_H^\infty(p)$ accumulates on at least one of the following sets: $\{|r| = \infty\}$, $\{r_1 = 0\}$ or $\{r_2 = 0\}$.*

If $\omega_{0,1}\omega_{0,2} < 0$, then we assume moreover that for $H \in \mathcal{D}$, for every $A > 0$, there exists $p' \in \mathbb{R}^4$ such that $\mathcal{O}_H^\infty(p')$ intersects both $\{|r| \leq A^{-1}\}$ and $\{|r| \geq A\}$.

Then $\mathcal{D}$ contains a dense (in the C^∞ topology) $\mathcal{G}^\delta$ subset of $\bar{\mathcal{H}}_2$.

Remark 4. Note that for $H \in \mathcal{D}$ and for $i = 1, 2$, the sets $\{r_i = 0\}$ are foliated by invariant tori of dimension 1 on which the dynamics is the rotation by angle $\omega_{0,j}$ $j \in \{1, 2\} \setminus \{i\}$. These are the only invariant tori for H.

Remark 5. The construction can be extended to any degrees of freedom $d \geq 2$ and any Liouville vector $\omega \in \mathbb{T}^d$.

2.2. **Four degrees of freedom and higher.** We consider $d = 4$, the case $d \geq 5$ being similar. Fix $\omega_0 \in \mathbb{R}^4$. To prove Theorem A we will use the same technique of construction that serves in the Liouville 2 degrees of freedom construction.

We will first introduce a completely integrable flow with a fixed point at the origin of frequency ω_0 following [EFK2]. It will have the form

$$H_0^4(r) = \langle \omega(r_4), r \rangle = \langle \omega_0 + f(r_4), r \rangle, \tag{2.2}$$

where we use action coordinates $r_j(x,y) = (x_j^2 + y_j^2)/2$ and where

$$f(r_4) = (f_1(r_4), f_2(r_4), f_3(r_4), 0)$$

with f defined as follows.

We call a sequence of intervals (open or closed or half-open) $I_n = (a_n, b_n) \subset (0, \infty)$ an increasing cover of the half line if:

(1) $\lim\limits_{n \to -\infty} a_n = 0$
(2) $\lim\limits_{n \to +\infty} a_n = +\infty$
(3) $a_n \leq b_{n-1} < a_{n+1} \leq b_n$.

Proposition 1. [EFK2] *Let $(\omega_{0,1}, \omega_{0,2}, \omega_{0,3}) \in \mathbb{R}^3$ be fixed. For every $\epsilon > 0$ and every $s \in \mathbb{N}$, there exist an increasing cover (I_n) of $(0, \infty)$ and functions $f_i \in C^\infty(\mathbb{R}, [0,1])$, $i = 1, 2, 3$, such that $\|f_i\|_s < \epsilon$ and $f_i(0) = 0$, and*

- *For each $n \in \mathbb{Z}$, the functions f_1 and f_2 are constant on I_{3n} :*
$$f_{1|I_{3n}} \equiv \bar{f}_{1,n}, \quad f_{2|I_{3n}} \equiv \bar{f}_{2,n}$$
- *For each $n \in \mathbb{Z}$, the functions f_1 and f_3 are constant on I_{3n+1} :*
$$f_{1|I_{3n+1}} \equiv \bar{f}_{1,n}, \quad f_{3|I_{3n+1}} \equiv \bar{f}_{3,n}$$
- *For each $n \in \mathbb{Z}$, the functions f_2 and f_3 are constant on I_{3n-1} :*
$$f_{2|I_{3n-1}} \equiv \bar{f}_{2,n}, \quad f_{3|I_{3n-1}} \equiv \bar{f}_{3,n-1}$$
- *The vectors $(\bar{f}_{1,n} + \omega_{0,1}, \bar{f}_{2,n} + \omega_{0,2})$, $(\bar{f}_{1,n} + \omega_{0,1}, \bar{f}_{3,n} + \omega_{0,3})$ and $(\bar{f}_{2,n} + \omega_{0,2}, \bar{f}_{3,n} + \omega_{0,3})$ are Liouville.*

If $\omega_{0,1}\omega_{0,2} < 0$, we ask that $\epsilon > 0$ be sufficiently small so that $\omega_1(r_4)\omega_2(r_4) < 0$ for every r_4.

Remark 6. It follows that f_1, f_2, f_3 are $\mathcal{C}^\infty$-flat at zero.

Notice that, as a consequence of Proposition 1, for $r_4 \in I_n$ two of the coordinates of $(f_1(r_1) + \omega_1, f_2(r_1) + \omega_2, f_3(r_1) + \omega_3)$ are constant and form a Liouville vector. This is why we will be able to use a similar construction as in the two dimensional Liouville case.

Let $\mathcal{U}_4$ be the set of exact symplectic diffeomorphisms of $\mathbb{R}^8$ with the following properties: $U(r,\theta) = (r,\theta)$ in the neighborhood of the axes $\{r_i = 0\}$, $i = 1, \ldots, 4$, as well as for $|r|$ sufficiently large, and $U(r,\theta) = (R,\Theta)$ satisfies $R_4 = r_4$. Let $\mathcal{H}_4$ be the set of Hamiltonians of the form $H_0^4 \circ U$, $U \in \mathcal{U}_4$. Finally we denote $\bar{\mathcal{H}_4}$ the closure in the C^∞ topology of $\mathcal{H}_4$.

We denote

$$\widetilde{I_n} = \mathbb{R}^3 \times I_n \times \mathbb{T}^4.$$

For $H \in \mathcal{H}_4$ the flow Φ_H^t leaves r_4 invariant. In particular, for $U \in \mathcal{U}_4$ we have $U(\widetilde{I_n}) = \widetilde{I_n}$ for any $n \in \mathbb{Z}$. We shall show how to make arbitrarily small perturbations of H_0^4 inside $\mathcal{H}_4$ that create oscillations of the corresponding flow in two of the three directions r_1, r_2, r_3. These perturbations will actually be compositions inside $\mathcal{H}_4$ by exact symplectic maps obtained from suitably chosen generating functions.

Iterating the argument gives a construction by successive conjugations scheme similar to [AK]. The difference here is that the conjugations will be applied in a "diagonal" procedure to include more and more intervals I_n into the scheme. Rather than following this diagonal scheme which would allow to define the conjugations explicitly at each step, we will actually adopt a $\mathcal{G}^\delta$-type construction (see [FH]) that makes the proof much shorter and gives slightly more general results.

Theorem 2. *Let $H_0^4(r) = \langle \omega(r_4), r \rangle$ be as in (2.2). Let $\mathcal{D}$ be the set of Hamiltonians $H \in \bar{\mathcal{H}_4}$ such that for each $p \in \mathbb{R}^8$ such that $r_i(p) \neq 0$ for every $i = 1, \ldots, 4$, there exists $i \in \{1, 2, 3\}$ such that $\mathcal{O}_H^\infty(p)$ accumulates on at least one of the sets $\{r_i = \infty\}$ and $\{r_i = 0\}$.*

If $\omega_{0,1}\omega_{0,2} < 0$, we assume moreover that for $H \in \mathcal{D}$, there exists for every $A > 0$, $p' \in \mathbb{R}^8$ such that $\mathcal{O}_H^\infty(p')$ intersects both $\{|r| \leq A^{-1}\}$ and $\{|r| \geq A\}$.

instead of $\{r_1 \geq A\} \cap \{r_2 \geq A\}$

Then $\mathcal{D}$ contains a dense (in the C^∞ topology) $\mathcal{G}^\delta$ subset of $\bar{\mathcal{H}_4}$.

Proof that Theorem 2 implies Theorem A. Note that for $H \in \mathcal{D}$ and $i = 1, \ldots, 4$, the set $\{r_i = 0\}$ is foliated by invariant tori of dimension 3 on which the dynamics is that of the integrable Hamiltonian H_0^4. We want to show that these are the only invariant tori of H. Indeed, let $p \in \mathbb{R}^8$ such that $r_i(p) \neq 0$ for every $i = 1, \ldots, 4$. Since the orbit of p accumulates on the axis or at infinity, it cannot lie on an invariant compact set.

Note now that if ω_0 does not have all its coordinates of the same sign, we can assume that $\omega_{0,1}\omega_{0,2} < 0$ by possibly renaming the variables. Hence the second part of Theorem A follows form the second part of Theorem 2. □

2.3. **Three degrees of freedom.** The construction of Theorem B for $d = 3$ will be similar to the case $d = 4$ but with this difference that we cannot count anymore on an invariant action variable r_4 that will now play the role of a parameter. Instead, one of the action coordinates that are involved in the diffusion should also play the role of the parameter. We choose this variable to be r_3 and assume without loss of generality that if the coordinates of ω_0 are not all of the same sign then $\omega_{0,1}\omega_{0,2} < 0$.

We fix a sequence of intervals $I_n = [a_{n-1}, a_n] \subset (0,\infty)$, $n \in \mathbb{Z}$, such that $\lim_{n\to-\infty} a_n = 0$, and $\lim_{n\to+\infty} a_n = +\infty$, and let $\widetilde{I}_n = \mathbb{R}_+^2 \times I_n \times \mathbb{T}^3$. We introduce a completely integrable flow with a fixed point at the origin by

$$H_0^3(r) = \langle \omega(r_3), r \rangle = \langle \omega_0 + f(r_3), r \rangle, \tag{2.3}$$

where $f = (f_1, f_2, f_3)$ is as in Proposition 1 with r_4 replaced by r_3, and we use action-angle coordinates as above.

Let $\mathcal{U}_3$ be the set of symplectomorphisms U such that $U(r,\theta) = (r,\theta)$ near the axes $\{r_i = 0\}$, $i = 1,2,3$, as well as near the sets $\{r_3 = a_n\}$, $n \in \mathbb{Z}$, and for $|r|$ sufficiently large. Consider the class of conjugates of H_0^3 by the elements of $\mathcal{U}_3$, and denote its C^∞-closure by $\bar{\mathcal{H}}_3$.

Theorem 3. *Let $H_0^3(r) = \langle \omega(r_3), r \rangle$ be as in (2.3). Let $\mathcal{D}$ be the set of Hamiltonians $H \in \bar{\mathcal{H}}_3$ such that for each $p \in \mathbb{R}^6$ satisfying $r_i(p) \neq 0$ for every $i = 1,2,3$ and $r_3(p) \notin \{a_n\}_{n\in\mathbb{Z}}$ we have that $\mathcal{O}_H^\infty(p)$ accumulates on at least one of the following sets: $\{r_1 = \infty\}$, $\{r_2 = \infty\}$, $\{r_1 = 0\}$, $\{r_2 = 0\}$, $\cup_{n\in\mathbb{Z}}\{r_3 = a_n\}$.*

If $\omega_{0,1}\omega_{0,2} < 0$, then we assume moreover that for $H \in \mathcal{D}$, there exists for every $A > 0$, $p' \in \mathbb{R}^6$ such that $\mathcal{O}_H^\infty(p')$ intersects both $\{|r| \leq A^{-1}\}$ and $\{r_1 \geq A\} \cap \{r_2 \geq A\}$.

Then $\mathcal{D}$ contains a dense (in the C^∞ topology) $\mathcal{G}^\delta$ subset of $\bar{\mathcal{H}}_3$

Proof that Theorem 3 implies Theorem B. Note that the axes and the sets $\{r_3 = a_n\}$ are foliated by invariant tori. The orbit of $p \in \mathbb{R}^6$ satisfying $r_i(p) \neq 0$ for every $i = 1,2,3$ and $r_3(p) \notin \{a_n\}_{n\in\mathbb{Z}}$ cannot accumulate on any of these sets if it lies on an invariant torus. Hence the only invariant tori for Φ_H^t are those foliating the axis and $\cup_{n\in\mathbb{Z}}\{r_3 = a_n\}$. The second part of Theorem B follows clearly from the second part of Theorem 3. □

3. Proof for the case $d = 2$

All our constructions will be derived from the main building block with two dimensional Liouville frequencies. The construction is summarised in the following Proposition 2 from which Theorem 1 will easily follow.

For $A > 0$, denote

$$R(A) := [A^{-1}, A] \times [A^{-1}, A], \quad \widetilde{R}(A) = R(A) \times \mathbb{T}^2.$$

We define the "margins" by:

$$M(A) = \{r_1 > A\} \cup \{r_2 > A\} \cup \{r_1 < A^{-1}\} \cup \{r_2 < A^{-1}\}.$$

We shall refer to the individual sets of the above union as *margin sets.*

Proposition 2. *For any Liouville vector $\omega = (\omega_1, \omega_2)$, any $\epsilon > 0, s \in \mathbb{N}$, $A_0 > 0$, and any symplectic map V that is identity outside $\widetilde{R}(A_0)$ we have that for any $A > A_0$ there exist $U \in \mathcal{U}$ and $T > 0$ with the following properties for $H = H_0^2 \circ U^{-1} \circ V^{-1}$:*

1. *U =Id in the complement of $\widetilde{R}(2A)$,*
2. $\|H - H_0^2 \circ V^{-1}\|_s < \epsilon$,
3. *For any $P \in \widetilde{R}(A)$ we have: $\mathcal{O}_H^T(P)$ intersects $M(A)$.*
4. *Moreover, if $\omega_1\omega_2 < 0$, then there exists $p' \in \mathbb{R}^4$ such that $\mathcal{O}_H^T(p')$ intersects both $\cap_{i=1}^2 \{r_i < 2A^{-1}\}$ and $\cup_{i=1}^2 \{r_i > A\}$.*

Proof of Theorem 1. For $n, A, T \in \mathbb{N}^*$, let

$$\mathcal{D}(A, T) := \left\{ H \in \bar{\mathcal{H}}_2 \mid \forall P \in \widetilde{R}(A), \mathcal{O}_H^T(P) \text{ intersects } M(A) \right\}.$$

It is clear that $\mathcal{D}(A, T)$ are open subsets of $\bar{\mathcal{H}}_2$ in any C^s topology. Proposition 2 (1)–(3) implies that $\cup_{T \in \mathbb{N}^*} \mathcal{D}(A, T)$ is dense in $\bar{\mathcal{H}}_2$ in any C^s topology. Hence the following set $\bar{\mathcal{D}} \subset \mathcal{D}$ is a dense G^δ set (in any C^s topology) in $\bar{\mathcal{H}}_2$

$$\bar{\mathcal{D}} = \bigcap_{A \in \mathbb{N}^*} \bigcup_{T \in \mathbb{N}^*} \mathcal{D}(A, T).$$

In case $\omega_1\omega_2 < 0$ we just have to add to the definition of $\mathcal{D}(A, T)$ the existence of a point $p' \in \mathbb{R}^4$ such that $\mathcal{O}_H^T(p')$ intersects both $\cap_{i=1}^2 \{r_i < 2A^{-1}\}$ and $\cup_{i=1}^2 \{r_i > A\}$. The density of $\cup_{T \in \mathbb{N}^*} \mathcal{D}(A, T)$ then follows from (1)–(4) of Proposition 2. □

The rest of this section is devoted to the proof of Proposition 2. The idea is to construct a conjugacy U that "wiggles" the invariant tori of H_0^2 and makes them accumulate on the margin sets. Since V in the statement of the proposition is assumed to be Identity outside

$\widetilde{R}(A)$ it will be possible to conclude from there that the Hamiltonian $H = H_0^2 \circ U^{-1} \circ V^{-1}$ satisfies the requirements of the proposition.

One has to observe however that since we want $H_0^2 \circ U^{-1}$ to be very close to H_0^2, the "wiggling" should take place almost inside the energy levels of H_0^2. In particular, one does not get diffusion in the case $\omega_1\omega_2 > 0$ because in that case the energy lines are compact segments (see Figure 1). To be more precise, fix $\omega = (\omega_1, \omega_2)$ and define the energy line

$$E_p := \{(r_1, r_2) \in \mathbb{R}_+^2 : \omega_1 r_1 + \omega_2 r_2 = \omega_1 r_1(p) + \omega_2 r_2(p)\}, \quad \widetilde{E}_p = E_p \times \mathbb{T}^2.$$

Clearly, $\widetilde{E}_p$ has the form $\{H_0^2 = const.\}$, and is invariant under the flow of H_0^2 (with the same fixed ω). For $p = (r, \theta)$, let

$$\mathcal{T}(p) = \{r\} \times \mathbb{T}^2$$

denote the flat torus passing through p. This is the invariant torus of H_0^2 passing through p. Let $H = H_0^2 \circ U^{-1}$ for a symplectic transformation U. Then $U(\mathcal{T}(p))$ is the invariant torus of H passing through the point $U(p)$. The main ingredient in the proof of Proposition 2 is the following lemma in which we construct a symplectic map U such that $U(\mathcal{T}(p))$ will "wiggle" inside $\widetilde{E}_p$ essentially for a large set of starting points p.

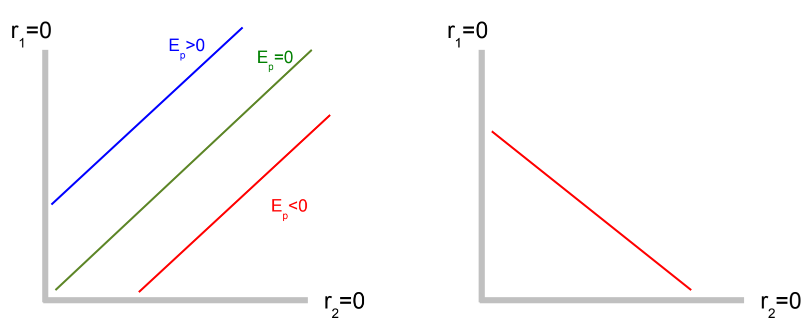

FIGURE 1. The diffusion lines are unbounded in the case $\omega_1\omega_2 < 0$ (the figure to the left) and bounded in the case $\omega_1\omega_2 > 0$.

Lemma 7. *For any Liouville vector $\omega = (\omega_1, \omega_2)$, any $\epsilon > 0, s \in \mathbb{N}$, $A_0 > 0$, and any symplectic map V that is identity outside $\widetilde{R}(A_0)$ we have that for any $A > A_0$ there exist $U \in \mathcal{U}$ and $T > 0$ with the following properties:*

(1) *$U =Id$ in the complement of $\widetilde{R}(2A)$,*
(2) $\|H_0^2 \circ U^{-1} \circ V^{-1} - H_0^2 \circ V^{-1}\|_s < \epsilon$
(3) *For any $p \in \widetilde{R}(A)$, the torus $V \circ U(\mathcal{T}(p))$ intersects two of the margin sets, and this intersection is ϵ–close to $\widetilde{E}_p \cap M(A)$.*
(4) *For any $p \in \mathbb{R}_+^2 \times \mathbb{T}^2$, $V \circ U(\mathcal{T}(p))$ intersects at least one of the margin sets, and this intersection is ϵ–close to $\widetilde{E}_p \cap M(A)$.*

Proof of Proposition 2. Let U be as in Lemma 7. The conditions (1) and (2) of the proposition follow directly from (1) and (2) of Lemma 7. The invariant torus of $H = H_0^2 \circ U^{-1} \circ V^{-1}$ passing through a point $P = V \circ U(p)$ has the form $V \circ U(\mathcal{T}(p))$. It follows from (4) of Lemma 7 that the latter torus intersects $M(A)$. Since ω is irrational, the orbit of p is dense on $\mathcal{T}(p)$. As a consequence, the orbit of P under the flow of H is dense on $V \circ U(\mathcal{T}(p))$, and conclusion (3) of the proposition follows.

shouldn't we switch (3) and (4) in lemma 7 to be consistent with Proposition 2?

Let $\omega_1 \cdot \omega_2 < 0$. In this case, the line E_p has a positive slope of $\Omega = -\omega_1/\omega_2$. Take a point p such that $\max_{i=1,2}\{r_i(p)\} = 2A^{-1}$, and E_p passes through $r = (A^{-1}, A^{-1})$. Since $p \in \widetilde{R}(A)$, the torus $U(\mathcal{T}(p))$ intersects two margin sets close to the two components of $E_p \cap M(A)$. In particular, it contains points in $\{r_1 < 2A^{-1}\} \cap \{r_2 < 2A^{-1}\}$, as well as in $\{r_1 > A\} \cup \{r_2 > A\}$ (this corresponds to $\widetilde{E}_p$ close to 0 in Figure 1). The same is true for the torus $V \circ U(\mathcal{T}(p))$ since V preserves the margin sets. Of course, this condition holds for an open set of starting points p. Now, conclusion (4) of the proposition holds for any $p' \in V \circ U(\mathcal{T}(p))$ with p as above. □

We now turn to the proof of Lemma 7. We will build the conjugacy U by superposing a large number of very slightly wiggling symplectomorphisms which we now present.

Sublemma 8. *For any Liouville vector ω, for any $A > 1$, $s > 0$, $Q > 0$ and $\varepsilon > 0$, there exists an integer vector $q = (q_1, q_2)$, $|q| > Q$, and a symplectic map $u \in C^\infty$ with the following properties:*

- *$u =Id$ in the complement of $\widetilde{R}(2A)$.*
- $\|H_0^2 \circ u^{-1} - H_0^2\|_s < \varepsilon$,

- *For any* $p = (r, \theta) \in \widetilde{R}(A)$*, the image* $(R, \Theta) = u(r, \theta)$ *satisfies*[2]

$$R_1 = r_1(1 + br_2 \cos(2\pi \langle q, \Theta\rangle)), \quad R_2 = r_2(1 + b(q_2/q_1)r_1 \cos(2\pi \langle q, \Theta\rangle)) \tag{3.4}$$

where $b = (10A)^{-3}$*, and*

$$|\Theta - \theta|_0 < 10^{-3}|q|^{-1}. \tag{3.5}$$

These expressions mean that any small ball gets stretched by u both in r_1- and r_2-direction with a (possibly small [3]) amplitude of order $br_1r_2 \geq bA^{-2}$, and large frequency $|q|$. U will be a composition of a large number of such functions u_j, $j = 1, \dots, N$, constructed with the same A and b, but decaying ε_j and growing $|q_j|$.

Here is a heuristic idea. Vector q plays two important roles. On the one hand, having large $|q|$, we get high frequency of oscillation for R_1 and good closeness between θ and Θ (see the formulas above).

On the other hand, the key estimate $\|H_0^2 \circ u^{-1} - H_0^2\|_s < \varepsilon$ needs $\langle q, \omega\rangle < \varepsilon/pol(q)$, where $pol(q)$ is a polynomial of q. It is here that we use the assumption of ω being Liouville. It guarantees that there exists a q with sufficiently large components providing the desired smallness of $\langle q, \omega\rangle$.

Proof of Sublemma 8. Let $G(t) \in C^\infty$ be a monotone cut-off function such that $|G'(t)| \leq 2$ for all t and

$$G(t) = \begin{cases} 0, & \text{for } t \leq 0, \\ 1, & \text{for } t > 1. \end{cases} \tag{3.6}$$

Denote by $1/A$ the "margin size", and by b the scaling constant: $b = (10A)^{-3}$. Assume without loss of generality that $\Omega = -\omega_1/\omega_2 \in (0, 1]$. Define $g(t) = G(2At - 1) - G(t - A)$. In this case $|g'(t)| \leq 4A$ for all t, and

$$g(t) = \begin{cases} 0, & \text{for } t \leq (2A)^{-1} \text{ or } \ t \geq A + 1, \\ 1, & \text{for } t \in [A^{-1}, A]. \end{cases}$$

Given two integers q_1 and q_2, whose choice will be specified later, we define the symplectic map $u : (r, \theta) \mapsto (R, \Theta)$ by a generating function

$$S(r, \Theta) = \langle r, \Theta\rangle + \frac{b}{2\pi q_1} r_1 r_2 g(r_1) g(r_2) \sin(2\pi \langle q, \Theta\rangle).$$

[2] important to put g in the statement and then insist that g is positive and equal to 1 on $R(A)$, no? this is used in the proof of lemma 8

[3] always small and not possibly small, no?

It satisfies

$$\begin{aligned} R_1 &= \frac{\partial S(r,\Theta)}{\partial \Theta_1} = r_1(1 + br_2 g(r_1)g(r_2)\cos(2\pi\langle q,\Theta\rangle)), \\ R_2 &= \frac{\partial S(r,\Theta)}{\partial \Theta_2} = r_2(1 + b\frac{q_2}{q_1} r_1 g(r_1)g(r_2)\cos(2\pi\langle q,\Theta\rangle)), \\ \theta_1 &= \frac{\partial S(r,\Theta)}{\partial r_1} = \Theta_1 + \frac{b}{2\pi q_1}\left(r_1 g(r_1)\right)'_{r_1} r_2 g(r_2)\sin(2\pi\langle q,\Theta\rangle) \\ \theta_2 &= \frac{\partial S(r,\Theta)}{\partial r_2} = \Theta_2 + \frac{b}{2\pi q_1} r_1 g(r_1)\left(r_2 g(r_2)\right)'_{r_2}\sin(2\pi\langle q,\Theta\rangle). \end{aligned}$$

To see that S defines a diffeomorphism it is enough to verify that the following determinant does not vanish:

$$\begin{aligned} &\det\left(\frac{\partial R}{\partial r}\right) = \det\left(\frac{\partial \theta}{\partial \Theta}\right) = \det\left(\frac{\partial^2 S(r,\Theta)}{\partial r\partial\Theta}\right) = \\ &1 + b\left(\left(r_1 g(r_1)\right)'_{r_1} r_2 g(r_2) + \frac{q_2}{q_1} r_1 g(r_1)\left(r_2 g(r_2)\right)'_{r_2}\right)\cos(2\pi\langle q,\Theta\rangle) \\ &\geq 1 - 40bA^3 > 0 \end{aligned}$$

by the choice of $b = (10A)^{-3}$.

By a local inverse function theorem, Θ can be expressed as a function of (r,θ). We get $\Theta_i = \theta_i + O(|q|^{-1})$. Plugging in this expression into the first two lines, we get a formula for u in terms of (r,θ). The inverse u^{-1} exists by the same argument.

Since u is a diffeomorphism and equals identity in a neighborhood of $\{r_1 = 0\}$ and $\{r_2 = 0\}$, we get that the image of u satisfies $R_1 > 0$ and $R_2 > 0$.

Here we estimate $\|H_0^2 \circ u^{-1}(r,\theta) - H_0^2\|_s$. For the above coordinate change we have

$$\begin{aligned} H_0^2 \circ u(r,\theta) - H_0^2(r,\theta) = \langle R,\omega\rangle - \langle r,\omega\rangle = \\ b\frac{1}{q_1} r_1 g(r_1) r_2 g(r_2)\langle \omega, q\rangle\cos(2\pi\langle q,\Theta\rangle), \end{aligned}$$

where $\Theta = \Theta(r,\theta)$. We can estimate

$$\|H_0^2 \circ u^{-1} - H_0^2\|_s = \|(H_0^2 - H_0^2\circ u)\circ u^{-1}\|_s \leq F(q,A,s)\cdot\langle\omega,q\rangle\,,$$

where $F(q,A,s)$ is a polynomial of q_1, q_2 whose degree depends only on s, and the coefficients are bounded by functions of A and s. Since vector ω is Liouvillean, there exists (an infinite number of) q such that $\langle\omega,q\rangle < \varepsilon/F(q,A,s)$, and the desired estimate follows. □

Proof of Lemma 7. Fix V, s, A_0 and $\varepsilon > 0$. Since $(V - \mathrm{Id})$ is compactly supported inside $\widetilde{R}(A_0)$, the same holds for any $A > A_0$. Let b be as in Sublemma 8, $\Omega = |\omega_1/\omega_2|$ assumed WLOG to lie in $(0, 1]$, and let N be such that

$$(1 + b\Omega/(4A))^N > A^2, \quad (1 - b\Omega/(4A))^N < 1/A^2.$$

We shall define u_j, $j = 1, \ldots, N$, by Sublemma 8 inductively in j. We choose q_j in the construction of u_1 so that

$$|q_j| > |q_{j-1}|^3.$$

Moreover, for $j = 1, \ldots, N$, q_j are such that u_j satisfies a (much stronger) condition

$$\| \left(H_0^2 \circ u_j^{-1} - H_0^2\right) \circ u_{j-1}^{-1} \circ \cdots \circ u_1^{-1} \circ V^{-1} \|_s < 2^{-j}\varepsilon. \tag{3.7}$$

Recall from the proof of Sublemma 8 that this is done by choosing vector q_j at each step so that $\langle \omega, q_j \rangle$ is sufficiently small depending on $q_1, \ldots, q_j, s, A$ and ϵ. Define

$$U = u_1 \circ \cdots \circ u_N.$$

Then the first statement of the lemma holds since each u_j is identity outside $\widetilde{R}(2A)$ by construction. The second one follows from (3.7) and the triangle inequality:

$$\| \left(H_0^2 \circ U^{-1} - H_0^2\right) \circ V^{-1} \|_s \leq$$
$$\sum_{j=1}^{N} \| \left(H_0^2 \circ u_j^{-1} - H_0^2\right) \circ u_{j-1}^{-1} \circ \cdots \circ u_0^{-1} \circ V^{-1} \|_s < \varepsilon \sum_{j=1}^{N} 2^{-j} < \varepsilon,$$

where $u_0 = id$ for the uniformity if notations.

To prove (3), fix $p^0 = (r^0, \theta^0) \in \widetilde{R}(A)$, and consider the flat torus $\mathcal{T}_{p^0}$ passing through p^0. Then the invariant torus of $H_0 \circ U^{-1}$ passing through $P^0 = U(p^0)$ has the form

$$U(\mathcal{T}_{p^0}) = u_1 \circ \cdots \circ u_N(\mathcal{T}_{p^0}).$$

It lies on the invariant surface $\{H_0^2 \circ U^{-1}(p) = const\}$, which is ε-close to $\widetilde{E}_{p^0}$, due to (2).

Since V is Identity on the margin sets, we just have to show that $U(\mathcal{T}_{p^0})$ intersects the two margin sets. We assume without loss of generality that $\theta_1(p^0) = 0$. Given $\theta \in \{0\} \times \mathbb{T}$ we define $(r^1, 0, \theta^1) = u_N(r^0, \theta^0)$, $(r^2, \theta^2) = u_{N-1}(r^1, \theta^1)$, and in general

$$(r^{j+1}, \theta^{j+1}) = u_{N-j}(r^j, \theta^j) = u_{N-j} \circ \ldots u_N(r^0, \theta^0), \quad j = 0, \ldots N-1.$$

The lower index indicates the component: $(r^j, \theta^j) = (r_1^j, r_2^j, \theta_1^j, \theta_2^j)$.

So, (3) follows if we prove the following two claims.

Claim 1. There exists $\hat\theta \in \{0\} \times \mathbb{T}$ and $\check\theta \in \{0\} \times \mathbb{T}$ such that

$$\cos(2\pi \left\langle \hat\theta^{j+1}, q_{N-j} \right\rangle) > 1/2 \quad \text{for all} \quad j = 0, \dots N-1 \tag{3.8}$$

$$\cos(2\pi \left\langle \hat\theta^{j+1}, q_{N-j} \right\rangle) < -1/2 \quad \text{for all} \quad j = 0, \dots N-1 \tag{3.9}$$

Claim 2. $U(r, \hat\theta)$ and $U(r, \check\theta)$ lie close to the two different ends of $\widetilde{E}_{p^0} \cap R(A)$

Proof of Claim 1. We[4] will actually restrict ourselves to an interval $K(p^0)$ in $\mathcal{T}_{p^0}$ and show that even $U(K(p^0))$ intersects two margin sets. We do so in order to use one dimensional calculus.

So, assume without loss of generality that $\theta_1(p^0) = 0$ and let

$$K(p^0) := \{r^0\} \times \{0\} \times \mathbb{T}$$

be an *interval in θ_2-direction* passing through p^0.

We shall present two subsets of $K(p^0)$, that we denote by $p^0 \times \hat L_0$ and $p^0 \times \check L_0$, such that $U(p^0 \times \hat L_0)$ intersects $\{r_1 \geq A\} \cup \{r_2 \geq A\}$, and $U(p^0 \times \check L_0)$ intersects $\{r_1 \leq A^{-1}\} \cup \{r_2 \geq A\}$.

Given (r^0, θ^0), denote $(r^1, \theta^1) = u_N(r^0, \theta^0)$, $(r^2, \theta^2) = u_{N-1}(r^1, \theta^1)$, and in general

$$(r^{j+1}, \theta^{j+1}) = u_{N-j}(r^j, \theta^j) = u_{N-j} \circ \dots u_N(r^0, \theta^0), \quad j = 0, \dots N-1.$$

The lower index indicates the component: $(r^j, \theta^j) = (r_1^j, r_2^j, \theta_1^j, \theta_2^j)$. By Sublemma 8, if $r^j \in R(A)$, we have: $r_1^{j+1} = r_1^j(1 + b r_2^j \cos(2\pi \left\langle \theta^{j+1}, q_{N-j} \right\rangle))$ for all $j = 0, \dots N-1$.

The simplified idea is the following. Imagine that $\theta^{j+1} = \theta^0$ for all j. Since; by construction, the frequences $|q_j|$ grow very fast with j, there are many points where $\cos(2\pi \left\langle \theta^0, q_j \right\rangle)) > 1/2$ for all $j \leq N$. This implies that $r_1^{j+1} = r_1^j + b r_1^j r_2^j/2$ for all $j = 0, \dots N-1$, and we can hope that $r_1^N > A$.

In reality though, θ^{j+1} can be different from θ^0 (but close), and should be more careful. We want to describe the set of points $\hat\theta^0 \in \{0\} \times \mathbb{T}$ such that for the images of $(r^0, \hat\theta^0)$ we have:

$$\cos(2\pi \left\langle \hat\theta^{j+1}, q_{N-j} \right\rangle) > 1/2 \quad \text{for all} \quad j = 0, \dots N-1.$$

We claim that the set

$$\hat J_{j+1} = \{\theta^0 \in \{0\} \times \mathbb{T} \mid \cos(2\pi \left\langle \theta^{j+1}, q_{N-j} \right\rangle) > 1/2\}$$

consists of $|q_{N-j}|$ disjoint intervals of size at least $C/(10|q_{N-j}|)$ whose midpoints are at most $C/|q_{N-j}|$ distant from the nearest neighbour, where C is a constant only depending on $\Omega = -\omega_1/\omega_2$. Indeed, this statement becomes evident if we replace θ^{j+1} in the definition of $\hat J_{j+1}$ by θ^0. In this simplified case, the desired intervals are of size $C/(3|q_{N-j} <)$, and the midpoints are $C/|q_{N-j}|$ distant. By Sublemma 8, $|\theta^1 - \theta^0|_0 < 10^{-3}|q_N|^{-1}$, $|\theta^2 - \theta^1|_0 < 10^{-3}|q_{N-1}|^{-1}$, and in general $|\theta^{l+1} - \theta^l|_0 < 10^{-3}|q_{N-l}|^{-1}$. Then

$$|\theta^{j+1} - \theta^0|_0 \leq 10^{-3} \left(\frac{1}{|q_N|} + \dots + \frac{1}{|q_{N-j}|} \right), \tag{3.10}$$

and $| \left\langle \theta^{j+1}, q_{N-j} \right\rangle - \left\langle \theta^0, q_{N-j} \right\rangle | \leq 10^{-3}(\frac{1}{|q_N|} + \dots + \frac{1}{|q_{N-j}|})|q_{N-j}| \leq 10^{-2}$. Hence, $\hat L_0 := \cap_{j=1}^N \hat J_j$ is nonempty.

By the same argument, there exists a nonempty subset $\check L_0$ of $\{0\} \times \mathbb{T}^1$ such that for $(r^0, \check\theta^0)$ with $\check\theta^0 \in \check L_0$ we have: $\cos(2\pi \left\langle \check\theta^{j+1}, q_{N-j} \right\rangle) < -1/2$ for all $j = 0, \dots N-1$.

Proof of Claim 2.

[4]The proof of Claim 1 should be contained in what follows in orange but it would be nice to rewrite it cleanly

[5] Now we verify that $U(r,\hat\theta)$ and $U(r,\check\theta)$ lie close to the two different margin sets. Consider the case $\omega_1\omega_2 < 0$. In this case $\widetilde{E}_p$ intersects $\{r_1 \ge A\}$ or $\{r_2 \ge A\}$ (or both). Suppose first $\widetilde{E}_p \cap \{r_1 \ge A\} \neq \emptyset$. For every $j = 1,\dots N$, we have the following recursive estimate:

$$\hat r_1^j = \hat r_1^{j-1}\left(1 + b\hat r_2^{j-1} g(\hat r_1^{j-1}) g(\hat r_2^{j-1}) \cos\left(2\pi\left\langle q_j, \hat\theta^j\right\rangle\right)\right) \ge \hat r_1^{j-1}.$$

In the same way, $\hat r_2^j \ge \hat r_2^{j-1}$. Hence, if for some $j < N$, $\hat r_1^j \ge A$, then $\hat r_1^N \ge A$.

If $\hat r_1^j \le A$ for all $j < N$, then

$$\begin{aligned}\hat r_1^N &\ge r_1^0 \prod_{j=0}^{N-1}(1 + b\hat r_2^j \cos(2\pi\left\langle q_j, \hat\theta^j\right\rangle))\\ &\ge A^{-1}(1 + b(4A)^{-1})^N \ge A^{-1}(1 + b\Omega(2A)^{-1})^N > A^{-1}\cdot A^2 = A.\end{aligned}$$

We used the fact that $R(2A)$ is invariant, so $\hat r_i^j \ge 1/(2A)$ for $i = 1, 2$ and $j = 1,\dots N$.

If for the given r we have $\widetilde{E}_p \cap \{r_2 = A\} \neq \emptyset$, then the same argument (applied to the r_2-coordinate) shows that $\hat r_2^N \ge A$.

At the same time, $\widetilde{E}_p$ has to intersect one of the two remaining margin sets. We prove that if for some $i \in \{1, 2\}$ we have $\widetilde{E}_p \cap \{r_i = 1/A\} \neq \emptyset$, then $\check r_i^N \le 1/A$. This is done in an analogous way. Namely, suppose first that $\widetilde{E}_p \cap \{r_1 = 1/A\} \neq \emptyset$. If for some $j < N$, $\check r_1^j \le 1/A$, then $\check r_1^N \le 1/A$. Indeed, for every $j = 1,\dots N$, the condition $\cos(2\pi\langle q_j, \check\theta^j\rangle)) \le -1/2$ implies

$$\check r_1^j = \check r_1^{j-1}\left(1 + b\check r_2^{j-1} g(\check r_1^{j-1}) g(\check r_2^{j-1}) \cos\left(2\pi\langle q_j, \check\theta^j\rangle\right)\right) \le \check r_1^{j-1}.$$

If $\check r_1^j \ge 1/A$ for all $j < N$, then

$$\begin{aligned}\check r_1^N &\le r_1^0 \prod_{j=0}^{N-1}\left(1 + b\check r_2^j \cos\left(2\pi\langle q_j, \check\theta^j\rangle\right)\right) < A(1 - b(4A)^{-1})^N\\ &\le A(1 - b\Omega(4A)^{-1})^N < A\cdot A^{-2} = A^{-1}.\end{aligned}$$

The case $\widetilde{E}_p \cap \{r_2 = 1/A\} \neq \emptyset$ is similar.

We will treat separately the cases $\omega_1\omega_2 < 0$ and $\omega_1\omega_2 > 0$.

Case 1 : $\omega_1\omega_2 < 0$**.** We will show that $U(r,\hat\theta)$ lies in $\{r_1 \ge A\}\cup\{r_2 \ge A\}$, while $U(r,\check\theta)$ lies in $\{r_1 \le 1/A\}\cup\{r_2 \le 1/A\}$.

For every $j = 1,\dots N$, we have the following recursive estimate:

$$\hat r_1^j = \hat r_1^{j-1}\left(1 + b\hat r_2^{j-1} g(\hat r_1^{j-1}) g(\hat r_2^{j-1}) \cos\left(2\pi\left\langle q_j, \hat\theta^j\right\rangle\right)\right) \ge \hat r_1^{j-1}.$$

In the same way, $\hat r_2^j \ge \hat r_2^{j-1}$ (here we use that $q_2/q_1 > 0$).

Hence, if for some $i \in \{1, 2\}$ and some $j < N$, $\hat r_i^j \ge A$, then $\hat r_i^N \ge A$.

Assume to the contrary that for all $j < N$, $\hat r_i^j < A$ for $i = 1, 2$. Then, since $g \equiv 1$ inside $R(A)$ we have that no mention of Ω in the

[5]what follows in blue was supposed to contain the proof of claim 2 but I could not follow the proof correctly. In particular I could not understand how the $\omega_1\omega_2 > 0$ is used, and when g is used and not, and how $\widetilde{E}_p \cap \{r_1 \ge A\} \neq \emptyset$ is really used in the proof, etc... so I put it in blue and include a modified proof that looks easy to follow but that I did not carefully check, so you can check it and switch back to the old blue version (to be revised) if you see that the new version is not ok... otherwise just wave the blue part.

next inequality

$$\hat{r}_1^N \geq r_1^0 \prod_{j=0}^{N-1} (1 + b\hat{r}_2^j \cos(2\pi \left\langle q_j, \hat{\theta}^j \right\rangle)) \geq A^{-1}(1 + b(4A)^{-1})^N \geq A.$$

We used the fact that $R(2A)$ is invariant, so $\hat{r}_i^j \geq 1/(2A)$ for $i = 1, 2$ and $j = 1, \dots N$.

In conclusion, $U(r, \hat{\theta})$ lies in $\{r_1 \geq A\} \cup \{r_2 \geq A\}$.

Now, for $\check{\theta}$ the argument is similar : For every $j = 1, \dots N$

$$\check{r}_1^j = \check{r}_1^{j-1} \left(1 + b\check{r}_2^{j-1} g(\check{r}_1^{j-1}) g(\check{r}_2^{j-1}) \cos\left(2\pi \left\langle q_j, \check{\theta}^j \right\rangle\right)\right) \leq \check{r}_1^{j-1}.$$

In the same way, $\check{r}_2^j \leq \hat{r}_2^{j-1}$ (here we use that $q_2/q_1 > 0$).

Hence, if for some $i \in \{1, 2\}$ and some $j < N$, $\hat{r}_i^j \leq 1/A$, then $\hat{r}_i^N \leq 1/A$.

Assume to the contrary that for all $j < N$, $\hat{r}_i^j > 1/A$ for $i = 1, 2$. Then, since $g \equiv 1$ inside $R(A)$ we have that no mention of Ω in the next inequality

$$\check{r}_1^N \leq r_1^0 \prod_{j=0}^{N-1} \left(1 + b\check{r}_2^j \cos\left(2\pi \left\langle q_j, \check{\theta}^j \right\rangle\right)\right) < A(1 - b(4A)^{-1})^N < A^{-1}.$$

In conclusion, $U(r, \check{\theta})$ lies in $\{r_1 \leq 1/A\} \cup \{r_2 \leq 1/A\}$.

Case 2 : $\omega_1\omega_2 > 0$. We will show that $U(r, \hat{\theta})$ lies in $\{r_1 \geq A\} \cup \{r_2 \leq 1/A\}$ while $U(r, \check{\theta})$ lies in $\{r_1 \leq 1/A\} \cup \{r_2 \geq A\}$.

Contrary to **case 1**, here we have that $\check{r}_1^j$ is increasing while $\check{r}_2^j$ is decreasing (this because $q_1/q_2 < 0$). If for some $j < N$, $\check{r}_2^j \leq 1/A$ we finish. If not, and if for some $j < N$, $\check{r}_1^j \geq A$ we also finish. Hence we assume that for all $j < N$, $\check{r}_1^j < A$ and $\check{r}_2^j > 1/A$. In such a situation and due to the fact that $g \equiv 1$ in $R(A)$ we have that

$$\hat{r}_1^N \geq r_1^0 \prod_{j=0}^{N-1} (1 + b\hat{r}_2^j \cos(2\pi \left\langle q_j, \hat{\theta}^j \right\rangle)) \geq A^{-1}(1 + b(4A)^{-1})^N \geq A.$$

To treat $U(r, \check{\theta})$ we just exchange the roles of the coordinates r_1 and r_2. here the Ωb appears in the condition on N instead of b, I think for the first time

We proceed now to the proof of (4). Fix $p \notin \widetilde{R}(A)$. Consider the case $r_1(p) < A^{-1}$, the other cases being similar. Let $\bar{p}$ be such that $E_p = E_{\bar{p}}$ and $\bar{p} \in \widetilde{R}(A)$. The torus $\mathcal{T}_{\bar{p}}$ separates the space $\widetilde{E}_p$, and the torus $\mathcal{T}_p$ lies on one side of it, namely the one intersecting the set $\{r_1 < A^{-1}\}$. Therefore, the torus $U(\mathcal{T}_{\bar{p}})$ separates the space $U(\widetilde{E}_p)$, having $U(\mathcal{T}_p)$ on one side of it. Now, $U(\mathcal{T}_{\bar{p}})$ intersects two margin sets, so $U(\mathcal{T}_p)$ has to intersect at least one. Since $\widetilde{E}_p$ is close to $U(\widetilde{E}_p)$, it has to intersect $\{r_1 < A^{-1}\}$.

We have shown that $U(\mathcal{T}(p))$ is ε–close to (one or two components of) $\widetilde{E}_p \cap M(A)$. Finally, notice that $V \circ U(\mathcal{T}(p))$ has the same property since $V =$Id outside $\widetilde{R}(A)$.

This finishes the proof of Lemma 7. □

4. Proofs for the case $d = 4$

The proof of Theorem 2 follows from the proposition below that encloses one step of the successive conjugation scheme. Fix $I_n = [a_n, b_n]$ for some n. Given a (large) A, we define

$$I_n(A) := [a_n + A^{-1}, b_n - A^{-1}], \quad \widetilde{I}^4_n(A) := [A^{-1}, A]^3 \times I_n(A) \times \mathbb{T}^4.$$

Define the margins set

$$M_4(A) = \bigcup_{i=1}^{3} \Big(\{r_i > A\} \bigcup \{r_i < A^{-1}\} \Big).$$

Proposition 3. *Let $n \in \mathbb{N}$, $\epsilon > 0$, $s \in \mathbb{N}$ and $V \in \mathcal{U}_4$ that is identity outside $\widetilde{I}^4_n(A_0)$ for some $A_0 > 0$, be given. Then for any $A > A_0$ there exist $U \in \mathcal{U}_4$, $T > 0$ and $i \in \{1, 2, 3\}$ with $H = H_0 \circ U^{-1} \circ V^{-1}$ satisfying the following properties:*

1. $U = \mathrm{Id}$ *in the complement of $\widetilde{I}^4_n(2A)$,*
2. $\|H - H_0 \circ V^{-1}\|_s < \epsilon$,
3. *For any $P \in \widetilde{I}^4_n(A)$ we have that $\mathcal{O}^T_H(P)$ intersects the set $M_4(A)$*
4. *Moreover, if $\omega_{0,1}\omega_{0,2} < 0$ and if $n = 3m$ for some $m \in \mathbb{Z}$, then there is a point $p' \in \widetilde{I}^4_n(A)$ such that $\mathcal{O}^T_H(p')$ intersects both $\cap_{i=1}^3 \{r_i < 2A^{-1}\}$ and $\{r_1 > A\} \cup \{r_2 > A\}$.*

Proof of Theorem 2. For $n, A, T \in \mathbb{N}^*$, let

$$\mathcal{D}(n, A, T) := \Big\{ H \in \bar{\mathcal{H}}_4 \mid \forall P \in \widetilde{I}^4_n(A), \mathcal{O}^T_H(P) \text{ intersects } M_4(A) \Big\}$$

It is clear that $\mathcal{D}(n, A, T)$ are open subsets of $\bar{\mathcal{H}}_4$ in any C^s topology. Proposition 3 implies that $\bigcup_{T \in \mathbb{N}^*} \mathcal{D}(n, A, T)$ is dense in $\bar{\mathcal{H}}_4$ in any C^s topology. Hence the following set $\bar{\mathcal{D}} \subset \mathcal{D}$ is a dense G^δ set (in any C^s topology)

$$\bar{\mathcal{D}} = \bigcap_{A \in \mathbb{N}^*} \bigcap_{n \in \mathbb{N}^*} \bigcup_{T \in \mathbb{N}^*} \mathcal{D}(n, A, T).$$

In case $\omega_{0,1}\omega_{0,2} < 0$ we add to the definition of $\mathcal{D}(n, A, T)$, when $n = 3m$ for some $m \in \mathbb{Z}$, the existence of a point $p' \in \widetilde{I}_n^4(A)$ that satisfies $\mathcal{O}_H^T(p')$ intersects both $\cap_{i=1}^3 \{r_i < 2A^{-1}\}$ and $\{r_1 > A\} \cup \{r_2 > A\}$. The second part of Theorem 2 then follows from the fact that if for some fixed $A > 0$, we consider $H \in \mathcal{D}(n, A, T)$ for n sufficiently large, then the orbit of the corresponding point $p' \in \widetilde{I}_n^4(A)$ has its r_4 coordinate always smaller than A^{-1}. The orbit of p' hence intersects both $\cap_{i=1}^4 \{r_i < 2A^{-1}\}$ and $\{r_1 > A\} \cup \{r_2 > A\}$. □

Proof of Proposition 3. Since V equals identity near the axes, we can, by increasing A, assume without loss of generality that $V = \mathrm{Id}$.

Assume that $I = [a, b] = I_{3n}$, the other cases being exactly similar. In this case $f_1(r_4) \equiv \bar{f}_1$ and $f_2(r_4) \equiv \bar{f}_2$ for $r_4 \in I$. Moreover, for $\omega_1 = \bar{f}_1 + \omega_{0,1}$ and $\omega_2 = \bar{f}_2 + \omega_{0,2}$, the vector (ω_1, ω_2) is Liouville. We will hence be able to use the two-dimensional Liouville construction of Proposition 2.

Let $a \in C^\infty(\mathbb{R})$ be such that $a(\xi) = 0$ if $\xi \notin I(2A)$ and $a(\xi) = 1$ if $\xi \in I(A) = [a + A^{-1}, b - A^{-1}]$, and $\|a\|_s \leq C(s, A)$ where $C(s, A)$ is a constant that depends on s and A (recall that A^{-1} is assumed to be small compared to the size of $I = I_{3n}$).

We construct the map U as follows. First, U is independent of (r_3, θ_3) (i.e., $U(r, \theta) = U(r_1, r_2, r_4, \theta_1, \theta_2, \theta_4)$). Second, r_4 acts as a parameter, and for each $r_4 \in I(A)$ the map U equals the map provided by Lemma 2 (call the latter U_2, where 2 indicates the number of degrees of freedom). More precisely, in the proof of Proposition 2, U_2 is constructed as a composition of a certain number of symplectic maps $u_2^j : \mathbb{R}^2 \times \mathbb{T}^2$, $(r, \theta) \mapsto (R, \Theta)$, each one given by a generating function of the form

$$S_2^j(r_1, r_2, \Theta_1, \Theta_2) = \langle (r_1, r_2), (\Theta_1, \Theta_2) \rangle + g_2^j(r_1, r_2, \Theta_1, \Theta_2),$$

where g_2^j is some smooth function equal to zero in the neighborhood of the axes.

We extend S_2^j and u_2^j to S^j and u^j defined for $(r, \theta) \in \mathbb{R}^4 \times \mathbb{T}^4$ by letting

$$S^j(r, \Theta) = \langle r, \Theta \rangle + a(r_4) g_2^j(r_1, r_2, \Theta_1, \Theta_2).$$

Since $a \equiv 0$ on $I(2A)^c$, we get (1) of Proposition 3 from (1) of Proposition 2. To check (2), observe that $a(r_4)$ appears just like a parameter in the construction of S^j from that of S_2^j. Thus, since $\|a\|_s \leq C(s, A)$ we get (2) by just taking ϵ sufficiently small in Proposition 2.

Now for $P \in \widetilde{I}_n^4(A)$ we have that $a(r_4(P)) = 1$ and since r_4 is invariant under the flow we get that the dynamics in $(r_1, r_2, \theta_1, \theta_2)$ coordinates is exactly that of Proposition 2, hence (3) holds.

To prove (4) of the Proposition, choose $p' \in \widetilde{I}^4_n(A)$ with r_4, θ_4, θ_3 arbitrary, with $r_3 < 2A^{-1}$ and with the projection of p' on the $(r_1, r_2, \theta_1, \theta_2)$ coordinates being the point p'_2 that satisfies (4) of Proposition 2. Clearly, (4) of Proposition 3 holds for p'. □

5. Proofs for the case $d = 3$.

Fix $I_n = [a_n, a_{n+1}]$ for some n. Given a (large) A, we define

$$I_n(A) := [a_n + A^{-1}, a_{n+1} - A^{-1}], \quad \widetilde{I}^3_n(A) := R(A) \times I_n(A) \times \mathbb{T}^3.$$

Define the margins set

$$M_3(A) = \bigcup_{i=1}^{3} \{r_i > A\} \cup \{r_1 < A^{-1}\} \cup \{r_2 < A^{-1}\} \cup \bigcup_{n \in \mathbb{Z}} \{r_3 \in I_n \setminus I_n(A)\}.$$

The following proposition is an analog of Proposition 3.

Proposition 4. *Let $n \in \mathbb{N}$, $\epsilon > 0$, $s \in \mathbb{N}$ and $V \in \mathcal{U}_3$ that is identity outside $\widetilde{I}^3_n(A_0)$ for some $A_0 > 0$, be given. Then for any $A > A_0$ there exist $U \in \mathcal{U}_3$, $T > 0$ and $(i_1, i_2) \in \{1, 2, 3\}$ distinct, with the following properties:*

(1) $U = \mathrm{Id}$ *in the complement of* $\widetilde{I}^3_n(2A)$,
(2) $\|H^3_0 \circ U^{-1} \circ V^{-1} - H^3_0 \circ V^{-1}\|_s < \epsilon$,
(3) *For any $P \in \widetilde{I}^3_n(A)$ we have that $\mathcal{O}^T_H(P)$ intersects $M_3(A)$.*
(4) *Moreover, if $\omega_{0,1}\omega_{0,2} < 0$ and if $n = 3m$ for some $m \in \mathbb{Z}$, then there is a point $p' \in \widetilde{I}^3_n(A)$ such that $\mathcal{O}^T_H(p')$ intersects both $\cap_{i=1}^2 \{r_i < 2A^{-1}\}$ and $\{r_1 > A\} \cup \{r_2 > A\}$.*

Proof of Theorem 3. The proof follows exactly the same lines as the proof of Theorem 2.

For $n, A, T \in \mathbb{N}^*$, we let

$$\mathcal{D}(n, A, T) := \left\{ H \in \bar{\mathcal{H}}_0 \mid \forall P \in \widetilde{I}^3_n(A), \mathcal{O}^T_H(P) \text{ intersects } M_3(A) \right\}.$$

and we see that $\bar{\mathcal{D}} \subset \mathcal{D}$ given by

$$\bar{\mathcal{D}} = \bigcap_{A \in \mathbb{N}^*} \bigcap_{n \in \mathbb{N}^*} \bigcup_{T \in \mathbb{N}^*} \mathcal{D}(n, A, T).$$

is a dense G^δ set (in any C^s topology) in $\bar{\mathcal{H}}_0$.

In case $\omega_{0,1}\omega_{0,2} < 0$ we add to the definition of $\mathcal{D}(n, A, T)$ the existence of a point p' satisfying conclusion (4) of Proposition 4. The second part of Theorem 2 then follows from the fact that if for some fixed $A > 0$, we consider $H \in \mathcal{D}(n, A, T)$ for n sufficiently large, then the orbit of the corresponding point $p' \in \widetilde{I}^3_n(A)$ has its r_3 coordinate always smaller than A^{-1} since it lies in I_n. The orbit of p' hence intersects both $\cap_{i=1}^3 \{r_i < 2A^{-1}\}$ and $\{r_1 > A\} \cup \{r_2 > A\}$. □

Proof of Proposition 4. The proof is similar to that of Proposition 3 (which relies on Lemma 2). We shall only describe the modifications that have to be done in order to get the conjugacy U for $d = 3$.

When $r_3 \in I_{3n}$, then (ω_1, ω_2) is a constant Liouville vector and the construction of Proposition 2 is carried out in the $(r_1, r_2, \theta_1, \theta_2)$-space, with r_3 acting as a parameter exactly as r_4 acted as a parameter in the proof of Proposition 3.

The situation for $r_3 \in I_n$ for $n = 3m + 1$ or $n = 3m + 2$ is slightly different since r_3 is not invariant anymore. Suppose that $n = 3m + 1$, the other case being similar. For $r_3 \in I_n$ we have that the vector (ω_1, ω_3) is constant and Liouville. The idea is that since r_3 will diffuse but remain inside I_n, one can still perform the construction of Proposition 2 with r_3 playing in the same time the role of a parameter and that of a diffusing action coordinate. The diffusion in the r_3 variable is thus limited to accumulating the set $\{r_3 \in I_n \setminus I_n(A)\}$. All the rest of the proof is similar to the proof of Proposition 3. □

References

[AK] D. V. Anosov and A. B. Katok, *New examples in smooth ergodic theory. Ergodic diffeomorphisms*, Transactions of the Moscow Mathematical Society 23, 1–35, 1970.

[D] R. Douady, *Stabilité ou instabilité des points fixes elliptiques*, Ann. Sci. Ec. Norm. Sup. 21, no. 1, 1–46, 1988.

[EFK1] L. H. Eliasson, B. Fayad, R. Krikorian *KAM-tori near an analytic elliptic fixed point*, Regular and Chaotic Dynamics 18, no. 6, 801–831, 2013.

[EFK2] L. H. Eliasson, B. Fayad, R. Krikorian *On the stability of KAM tori*, Analytic KAM-tori are never isolated Duke Math. J. 164, no. 9, 1733–1775, 2015.

[FH] A. Fathi and M. Herman, *Existence de difféomorphismes minimaux*, Asterisque 49, 39–59, 1977.

[FK] B. Fayad and R. Krikorian, *Herman's last geometric theorem*, Ann. Sci. Éc. Norm. Supér. 42, 193–219, 2009.

[H1] M. Herman, *Some open problems in dynamical systems, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, Doc. Math. 1998 Extra Vol. II, 797–808, 1998.

[H2] M. Herman, *Courbes invariantes sans propriété de torsion 1* Archives Herman, to appear.

[R] H. Rüssmann, *Über die Normalform analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung*, Math. Ann. 169, 55–72, 1967.

[S] M. B. Sevryuk, *KAM-stable Hamiltonians*, J. Dynam. Control Systems 1 no. 3, 351–366, 1995.

B.Fayad: IMJ-PRG CNRS, France; M. Saprykina: Inst. för Matematik, KTH, 10044 Stockholm, Sweden,

E-mail address: bassam.fayad@imj-prg.fr, masha@kth.se