SKEW PRODUCTS OVER TRANSLATIONS ON $\mathbf{T}^d$, $d \geq 2$.

BASSAM R. FAYAD

ABSTRACT. We give an example on $\mathbf{T}^4$ of a minimal translation R_α and a real analytic function φ, such that the circle-valued skew product extension of R_α by φ has a Lebesgue spectrum in the orthocomplement of the space of eigenfunctions.

1. INTRODUCTION.

Let $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$, $d \geq 1$ and denote by μ the Haar measure on $\mathbf{T}^d$. Given a minimal translation R_α on $\mathbf{T}^d$ and a real function φ on $\mathbf{R}^d$, smooth and $\mathbf{Z}^d$-periodic, we will consider the map (skew product) $S_{\alpha,\varphi}$:

$$\begin{array}{ccc} \mathbf{T}^d \times \mathbf{T}^1 & \longrightarrow & \mathbf{T}^d \times \mathbf{T}^1 \\ (z,s) & \longrightarrow & (z+\alpha, s+\varphi(z) \bmod 1) \end{array}$$

When $d=1$, it was proved in [1] that if φ is absolutely continuous, the spectrum of the skew product is singular with respect to the Lebesgue measure[1]. The proof in [1] is based on an improved Denjoy-Koksma inequality implying that $S_{\alpha,\varphi}^{q_n}$ tends uniformly to the identity map on $\mathbf{T}^2$ as q_n runs through the sequence of denominators of the convergents of α. Hence, $S_{\alpha,\varphi}$ is said to be *rigid* and one easily deduces that its spectrum is purely singular. Here, we want to prove that this is not anymore true when $d \geq 2$; namely, we give an example of a skew product over $\mathbf{T}^2$ with a real analytic-function that is nonrigid (it actually displays "mixing in the fibers") and we derive from it an example of a skew product over $\mathbf{T}^4$ that has countable Lebesgue spectrum in the orthocomplement of the space of eigenfunctions. Our argument is essentially based on the construction by J-C. Yoccoz [4] of a minimal translation on $\mathbf{T}^2$ and a real-analytic complex function φ of $\mathbf{T}^2$ that give a counterexample, in dimension 2, to the Denjoy-Koksma inequality valid for functions over the circle.

[1] In this paper, the authors, P. Gabriel, M. Lemanczyk and P. Liardet, consider skew products over irrational rotations with *circle-valued* functions. They prove that when φ is absolutely continuous with degree 0, the corresponding skew product is rigid. In contrast, if the degree of φ is not zero, and φ' is of bounded variation, the skew product has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of R_α (see [3]), the most known example being $(x,y) \to (x+\alpha, x+y)$.

2. Mixing in the fibers.

Take α and α' rationaly independent such that the denominators of their convergents, q_n and q_n', satisfy the following, for $n \geq n_0$

$$q_n \geq e^{3nq'_{n-1}}, \tag{1}$$

$$q'_n \geq e^{3nq_n}, \tag{2}$$

and let $S_{\alpha,\alpha',\varphi}$ be the skew product constructed from $R_{\alpha,\alpha'}$ and the real analytic function

$$\varphi(x,y) = 1 + \mathrm{Re}\left(\sum_{j=0}^{\infty} \frac{e^{i2\pi q_j x}}{e^{q_j}}\right) + \mathrm{Re}\left(\sum_{j=0}^{\infty} \frac{e^{i2\pi q'_j y}}{e^{q'_j}}\right). \tag{3}$$

For $l \in \mathbf{Z}$ and $k \in \mathbf{Z}^2$, we denote by $\psi_{k,l} \in L^2(\mathbf{T}^3, \mathbf{C})$ the character $e^{i2\pi<k,z>}e^{i2\pi ls}$, where $< .,. >$ is the scalar product on $\mathbf{R}^2$. In this note we will prove the following estimate

Proposition 1 (Mixing in the fibers)**.** *Assume $l \neq 0$. Then given any $\epsilon > 0$, we have*

$$\begin{aligned}(\psi_{k,l} o S^m_{\alpha,\alpha',\varphi}/\psi_{k,l}) &:= \int_{\mathbf{T}^3} \psi_{k,l} o S^m_{\alpha,\alpha',\varphi}(x,y,s)\overline{\psi_{k,l}(x,y,s)}dxdyds \\ &= O(\frac{1}{m^{\frac{1}{3}-\epsilon}})\end{aligned}$$

when m goes to infinity.

This *mixing in the fibers* already eliminates the rigidity encountered when $d = 1$. Before we prove the proposition we show how the estimation of the rate of "mixing" enclosed in it, enables us to construct a skew product with a Lebesgue component in its spectrum:

Assume $R_{\hat{\alpha}} = R_{\alpha_1,\alpha'_1,\alpha_2,\alpha'_2}$ is a minimal translation on $\mathbf{T}^4$ and the couples (α_i, α'_i), $i = 1, 2$ both satisfy (1) and (2). We will denote by (x_1, x'_1, x_2, x'_2) the coordinates on $\mathbf{T}^4$. Let $\varphi_1(x_1, x'_1)$ and $\varphi_2(x_2, x'_2)$ be as in (3), and define on $\mathbf{T}^4$ the real analytic function $\hat{\varphi} = \varphi_1 + \varphi_2$. Define now on $\mathbf{T}^5$ the skew product $\hat{S} = S_{\hat{\alpha},\hat{\varphi}}$. Let $H = L^2(\mathbf{T}^5, \mathbf{C})$ be the set of complex functions on $\mathbf{T}^5$ that are L^2 with respect to the Haar measure. By $U_{\hat{S}}$ we refer to the unitary operator on H associated to $\hat{S}$:

$$\begin{aligned} H &\longrightarrow H, \\ f &\longrightarrow fo\hat{S}. \end{aligned}$$

(The operator $U_{\hat{S}}$ is unitary because $\hat{S}$ preserves the Haar measure on $\mathbf{T}^5$.)

The space H decomposes under the action of $U_{\hat{S}}$ into a countable sum of *orthogonal invariant subspaces* H_l, $l \in \mathbf{Z}$, where H_l designates the subspace containing the functions of the form $a(x_1, x'_1, x_2, x'_2)e^{i2\pi ls}$.

Finally, from Proposition 1, it follows that

Theorem 1. *For any $l \in \mathbf{Z}^*$, $U_{\hat{S}}$ has a Lebesgue spectrum on H_l.*

Proof. For $(k_1, k_1', k_2, k_2') \in \mathbf{Z}^4$, let $\xi_{k_1,k_1',k_2,k_2',l}$ be the character $e^{i2\pi(k_1x_1+k_1'x_1'+k_2x_2+k_2'x_2'+ls)}$. We have immediately

$$\begin{aligned} |c_m| &:= \left|\left(U_{\hat{S}}^m \xi_{k_1,k_1',k_2,k_2',l}/\xi_{k_1,k_1',k_2,k_2',l}\right)\right| \\ &= \left|\left(\psi_{k_1,k_1',l} o S^m_{\alpha_1,\alpha_1',\varphi_1}/\psi_{k_1,k_1',l}\right)\left(\psi_{k_2,k_2',l} o S^m_{\alpha_2,\alpha_2',\varphi_2}/\psi_{k_2,k_2',l}\right)\right| \\ &= O\left(\frac{1}{m^{\frac{2}{3}-2\epsilon}}\right) \end{aligned}$$

when $l \neq 0$ and m goes to infinity. The latter is satisfied for any $\epsilon > 0$, hence the serie $\sum |c_m|^2$ converges and the spectral measure of $\xi_{k_1,k_1',k_2,k_2',l}$ is absolutely continuous with respect to Lebesgue (by definition, the numbers c_m are the Fourier coefficients of this measure). Furthermore, by the so-called "purity law" on the unitary operators arising from cocycles, we conclude that $U_{\hat{S}}$ has a Lebesgue spectrum on H_l, when $l \neq 0$ (the purity law, see [2], states that the spectral type of $U_{\hat{S}}$ on H_l is pure, i.e. either discrete, or continuous and purely singular, or equivalent to Lebesgue). □

Proof of Proposition 1. Assume α, α' and φ satisfy (1), (2) and (3). From now on, we will denote the Birkhoff sums of φ with respect to $R_{\alpha,\alpha'}$ by

$$\varphi_m(x,y) := \sum_{k=0}^{m-1} \varphi\left(R^k_{\alpha,\alpha'}(x,y)\right).$$

We have

$$\left|\left(\psi_{k,l} o S^m_{\alpha,\alpha',\varphi}/\psi_{k,l}\right)\right| = \left|\int_{\mathbf{T}^2} e^{i2\pi l\varphi_m(x,y)} dxdy\right|, \tag{4}$$

and we will derive the estimation of Proposition 1 from the large oscillations of $\varphi_m(x,y)$ for *all* integer m, large enough. We underlined the last quantifier because there lies the difference with the one-dimensional case, the oscillations of φ_m being large in one or in the other direction x and y depending on whether m is far from q_n or far from q_n'. Indeed, a direct computation on the Birkhoff sums of φ_m implies the following, where $\| \,.\, \|_{C^3}$ denotes a norm on the space of real functions on $\mathbf{R}^2$ of class C^3 and $\mathbf{Z}^2$-periodic

Lemma *For any $m \in [e^{nq_n}, e^{nq_n'}]$, we have*

$$\varphi_m(x,y) = \frac{m}{e^{q_n}}\cos(2\pi q_n x) + h^{(m)}(x) + \phi_m(y),$$

where $\phi_m(y)$ denotes the Birkhoff sums of the "y-part" of φ, and where $h^{(m)}$ satisfies, for any m in the above interval, $\| h^{(m)} \|_{C^3} = O(q_n)$.

There is of course an equivalent expression when $m \in [e^{nq_n'}, e^{(n+1)q_{n+1}}]$

in which we interchange x and q_n with y and q'_n respectively.

Proof of the lemma. For any $m \in \mathbf{N}$, we have

$$\varphi_m(x,y) = \mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{X(m,q_k)}{e^{q_k}} e^{i2\pi q_k x} + \sum_{k=1}^{\infty} \frac{Y(m,q'_k)}{e^{q'_k}} e^{i2\pi q'_k y}\right),$$

where

$$X(m,q_k) = \frac{1-e^{i2\pi m q_k \alpha}}{1-e^{i2\pi q_k \alpha}}, \qquad Y(m,q'_k) = \frac{1-e^{i2\pi m q'_k \alpha'}}{1-e^{i2\pi q'_k \alpha'}}.$$

We will need the following simple inequalities

$$\begin{array}{llrcl} (5) & \text{For all } k \in \mathbf{N}^*, \text{ and any } m \in \mathbf{N}, & |X(m,q_k)| & \leq & m; \\ (6) & \text{for } k < n, \text{ and any } m \in \mathbf{N}, & |X(m,q_k)| & \leq & q_n; \\ (7) & \text{for } m \leq e^{nq'_n}, & |X(m,q_n) - m| & = & o(1). \end{array}$$

Proof of (5), (6) and (7): First,

$$X(m,q_k) = \sum_{j=0}^{m-1} e^{i2\pi j q_k \alpha},$$

so the first inequality is trivial.
For the other inequalities remember that the denominators of the convergents of α satisfy

$$\| q_{n-1}\alpha \| \leq |||k\alpha|||, \quad \forall k < q_n, \tag{8}$$

and

$$\frac{1}{2q_n} \leq \frac{1}{q_{n-1}+q_n} \leq |||q_{n-1}\alpha||| < \frac{1}{q_n}, \tag{9}$$

where $|||.|||$ denotes the distance to the closest integer.
Next, notice that for any k and for any m

$$|X(m,q_k)| \leq \frac{2}{|1-e^{i2\pi q_k \alpha}|},$$

then using the inequality $\sin(\pi u) \geq 2u$, when $0 \leq u \leq \frac{1}{2}$, we have

$$\frac{2}{|1-e^{i2\pi q_k \alpha}|} = \frac{1}{\sin \pi |||q_k\alpha|||} \leq \frac{1}{2|||q_k\alpha|||},$$

so, if $k < n$, we have from (8) and the left hand side in (9), that this last term is bounded by q_n. Hence, (6) is proved.
For (7), we use again

$$X(m,q_n) = \sum_{j=0}^{m-1} e^{i2\pi j q_n \alpha};$$

since $|||q_n\alpha||| \leq \frac{1}{q_{n+1}} \leq \frac{1}{3e^{nq'_n}}$, one has for $j \leq e^{nq'_n}$

$$e^{i2\pi j q_n \alpha} = 1 + O(e^{-2nq'_n}),$$

which immediately leads to (7).

Coming back to the proof of the lemma, we want to find a bound, when $m \leq e^{nq'_n}$, to the C^2 norm of

$$h^{(m)}(x) = \mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{X(m,q_k)}{e^{q_k}} e^{i2\pi q_k x}\right) - \frac{m}{e^{q_n}} \cos(2\pi q_n x).$$

If we consider the second derivatives of the sum above we have, from (5)

$$\left|\mathrm{Re}\left(\sum_{k=n+1}^{\infty} \frac{X(m,q_k)}{e^{q_k}} (2\pi q_k)^2 e^{i2\pi q_k x}\right)\right| \leq m \sum_{k=n+1}^{\infty} \frac{(2\pi q_k)^2}{e^{q_k}},$$

which implies, since $m \leq e^{nq'_n}$, and $q_k \geq e^{3nq'_n}$ for $k \geq n+1$,

$$= o(1). \tag{10}$$

From (6), it follows that

$$\left|\mathrm{Re}\left(\sum_{k=1}^{n-1} \frac{X(m,q_k)}{e^{q_k}} (2\pi q_k)^2 e^{i2\pi q_k x}\right)\right| = O(q_n). \tag{11}$$

Finally, (7) implies that

$$\left|\frac{X(m,q_n) - m}{e^{q_n}} (2\pi q_n)^2\right| = o(1),$$

and we obtain the required bound on the second derivative of $h^{(m)}$, which is clearly also valid for $h^{(m)}$ and its first derivative. □

Assume now $m \in [e^{nq_n}, e^{nq'_n}]$. In light of the lemma we have just stated, our problem is reduced to majorizing, for $l \in \mathbf{Z}^*$, the absolute value of the integral

$$I_m(l) = \int_{\mathbf{T}} e^{i2\pi l\left[\frac{m}{e^{q_n}} \cos(2\pi q_n x) + h^{(m)}(x)\right]} dx,$$

with the hypothesis $\| h^{(m)} \|_{C^3} \leq q_n$.

Whenever $n \geq \frac{2}{\epsilon}$, one has for $m \in [e^{nq_n}, e^{nq'_n}]$

$$a_m := \frac{m}{e^{q_n}} \geq m^{1-\frac{\epsilon}{2}}. \tag{12}$$

With a slight abuse of notation we will write $\varphi_m(x)$ for the function $a_m \cos(2\pi q_n x) + h^{(m)}(x)$.

First, we break down the integral to avoid the zeros of $\sin(2\pi q_n x)$:

$$I_m(l) = \sum_{k=0}^{2q_n-1} \int_{\frac{k}{2q_n}}^{\frac{k+1}{2q_n}} e^{i2\pi l\varphi_m(x)}dx =$$

$$\sum_{k=0}^{2q_n-1} \int_{\frac{k}{2q_n}}^{u_k} e^{i2\pi l\varphi_m(x)}dx + \sum_{k=0}^{2q_n-1} \int_{u_k}^{v_k} e^{i2\pi l\varphi_m(x)}dx + \sum_{k=0}^{2q_n-1} \int_{v_k}^{\frac{k+1}{2q_n}} e^{i2\pi l\varphi_m(x)}dx,$$

where

$$u_k = \frac{k}{2q_n} + \frac{1}{4q_n}\left(\frac{1}{m}\right)^{\frac{1}{3}}, \qquad v_k = \frac{k+1}{2q_n} - \frac{1}{4q_n}\left(\frac{1}{m}\right)^{\frac{1}{3}}.$$

We have then

$$|I_m(l)| \leq m^{-\frac{1}{3}} + \sum_{k=0}^{2q_n-1} \left|I_m^k(l)\right| \tag{13}$$

where

$$I_m^k(l) := \int_{u_k}^{v_k} e^{i2\pi l\varphi_m(x)}dx, \quad k = 0, ..., 2q_n - 1.$$

Denote

$$\overline{\varphi}_m(x) := \frac{\varphi_m(x)}{a_m} = \cos(2\pi q_n x) + \frac{h^{(m)}(x)}{a_m}.$$

(The function $\overline{\varphi}_m$ is not a Birkhoff sum.)
Notice that, for $x \in [u_k, v_k]$, $2\pi q_n x \in [k\pi + \frac{\pi}{2}m^{-\frac{1}{3}}, (k+1)\pi - \frac{\pi}{2}m^{-\frac{1}{3}}]$, which implies

$$|\sin(2\pi q_n x)| \geq m^{-\frac{1}{3}}. \tag{14}$$

Since $\| h^{(m)} \|_{C^3} \leq q_n \leq \log m$, we have for any $m \in [e^{nq_n}, e^{nq'_n}]$:
on one hand,

$$\| \overline{\varphi}_m(x) \|_{C^2} \leq 4\pi^2 q_n^2 + q_n \leq (\log m)^2, \tag{15}$$

and on the other hand, using (12) and (14) we have that $\overline{\varphi}'_m(x)$ is of constant sign when $x \in [u_k, v_k]$, and

$$|\overline{\varphi}'_m(x)| \geq 2\pi q_n m^{-\frac{1}{3}} - \frac{\log m}{a_m} \geq m^{-\frac{1}{3}}. \tag{16}$$

Hence, we can make in the expression of $I_m^k(l)$ the change of variable $s = \overline{\varphi}_m(x)$, and obtain

$$|I_m^k(l)| = \left| \int_{\overline{\varphi}_m(u_k)}^{\overline{\varphi}_m(v_k)} \frac{e^{i2\pi l a_m s}}{\overline{\varphi}'_m(\overline{\varphi}_m^{-1}(s))} ds \right|.$$

If now we integrate by parts, having (16) in mind we obtain

$$|I_m^k(l)| \leq \frac{m^{\frac{1}{3}}}{\pi l a_m} + \frac{1}{2\pi l a_m} \left| \int_{\overline{\varphi}_m(u_k)}^{\overline{\varphi}_m(v_k)} \frac{\overline{\varphi}''_m}{[\overline{\varphi}'_m]^3} \left(\overline{\varphi}_m^{-1}(s)\right) e^{i2\pi l a_m s} ds \right|. \tag{17}$$

Going back to the initial variable we have

$$\begin{aligned}\left|\int_{\overline{\varphi}_m(u_k)}^{\overline{\varphi}_m(v_k)} \frac{\overline{\varphi}''_m}{[\overline{\varphi}'_m]^3}\left(\overline{\varphi}_m^{-1}(s)\right) e^{i2\pi l a_m s} ds\right| &= \left|\int_{u_k}^{v_k} \frac{\overline{\varphi}''_m(x)}{[\overline{\varphi}'_m]^2(x)} e^{i2\pi l a_m \overline{\varphi}_m(x)} dx\right| \\ &\leq \frac{1}{2q_n} m^{\frac{2}{3}} \| \overline{\varphi}_m \|_{C^2} \\ &\leq \frac{1}{2q_n} m^{\frac{2}{3}} [\log m]^2.\end{aligned}$$

Hence, (17) becomes

$$|I_m^k(l)| \leq \frac{m^{\frac{1}{3}}}{\pi l a_m} + \frac{1}{4\pi l a_m q_n} m^{\frac{2}{3}} [\log m]^2.$$

In the right hand side of this inequality, the second term is clearly dominant and (12) implies

$$|I_m^k(l)| \leq \frac{1}{l q_n} m^{-\frac{1}{3}+\frac{\epsilon}{2}} [\log m]^2 \leq \frac{1}{q_n} m^{-\frac{1}{3}+\epsilon},$$

for any $m \in [e^{n q_n}, e^{n q'_n}]$, n large enough. Coming back to (13), we obtain

$$|I_m(l)| \leq m^{-\frac{1}{3}} + 2m^{-\frac{1}{3}+\epsilon} \leq 3m^{-\frac{1}{3}+\epsilon}.$$

When $m \in [e^{n q'_n}, e^{(n+1) q_{n+1}}]$, we proceed in the same way integrating along y in (4). Proposition 1 is hence proved. □

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References

[1] P. Gabriel, M. Lemańczyk, and P. Liardet. Ensemble d'invariants pour les produits croisés de Anzai. *Mémoire de la Société Mathématique de France. Nouvelle Série*, **47**, (1991), p. 102.

[2] Henry Helson. Cocycles on the circle. *J. Operator Theory*, **16**.

[3] A. Iwanik, M. Lemańczyk, and D. Rudolph. Absolutely continuous cocycles over irrational rotations. *Israel J. Math.*, **83**, (1993), p. 73–95.

[4] J-C. Yoccoz. *Petits diviseurs en dimension 1.* Astérisque (1982), Appendix 1.

Centre de Mathématiques, UMR 7640 du CNRS,
École Polytechnique, 91128 Palaiseau Cedex, France
fayadb@math.polytechnique.fr