A REMARK ON CONSERVATIVE DIFFEOMORPHISMS

JAIRO BOCHI, BASSAM R. FAYAD, AND ENRIQUE PUJALS

Abstract. We show that a stably ergodic diffeomorphism can be C^1 approximated by a diffeomorphism having stably non-zero Lyapunov exponents. The proof is a simple application of several recent results, by Bonatti-Díaz-Pujals, Arbieto-Matheus, Bonatti-Baraviera and Bochi-Viana.

Two central notions in Dynamical Systems are ergodicity and hyperbolicity. In many works showing that certain systems are ergodic, some kind of hyperbolicity (e.g. uniform, non-uniform or partial) is a main ingredient in the proof. In this note we do something in the converse direction.

Let M be a compact manifold of dimension $d \geq 2$, and let μ be a volume measure in M. Take $\alpha > 0$ and let $\mathrm{Diff}_\mu^{1+\alpha}(M)$ be the set of μ-preserving $C^{1+\alpha}$ diffeomorphisms, endowed with the C^1 topology. Let $\mathcal{SE} \subset \mathrm{Diff}_\mu^{1+\alpha}(M)$ be the set of stably ergodic diffeomorphisms (i.e., the set of diffeomorphisms such that every sufficiently C^1-close $C^{1+\alpha}$ conservative diffeomorphism is ergodic).

Our result answers positively a question of Burns, Dolgopyat, and Pesin [BuDP]:

Theorem 1. *There is open and dense set $\mathcal{R} \subset \mathcal{SE}$ such that if $f \in \mathcal{R}$ then f is non-uniformly hyperbolic, that is, all Lyapunov exponents of f are non-zero. Moreover, every $f \in \mathcal{R}$ admits a dominated splitting.*

Remark. It is not true that every stably ergodic diffeomorphism can be approximated by a partially hyperbolic system (in the weaker sense), by the examples of Tahzibi [T].

Remark. Let $\mathcal{SE}'$ be the set of diffeomorphisms $f \in \mathcal{SE}$ such that every power f^k, $k \geq 2$, is ergodic. Then every f in an open dense subset of $\mathcal{SE}'$ is Bernoulli. This follows from theorem 1 and Pesin theory.

The proof of theorem 1 has three steps.

1. A stably ergodic (or stably transitive) diffeomorphism f must have a dominated splitting. This is true because if it doesn't, Bonatti, Díaz, and Pujals [BDP] permits us to perturb f and create a periodic point whose derivative is the identity. Then, using the Pasting Lemma of Arbieto and Matheus [AM] (for which $C^{1+\alpha}$ regularity is an essential hypothesis), one breaks transitivity.
2. A result of Bonatti and Baraviera [BB] – which is a refinement of a technique developed by Shub and Wilkinson [SW] – gives a perturbation of f

Date: November 9, 2003.

J. B. is supported by CNPq-Profix. J.B. wishes to thank the hospitality of the LAGA – Univertité Paris 13.

such that the sum of the Lyapunov exponents "inside" each of the bundles of the (finest) dominated splitting is non-zero.

3. Using a result of Bochi and Viana [BV], we find another perturbation such that the Lyapunov exponents in each of the bundles become almost equal. (If we attempted to make the exponents exactly equal, we couldn't guarantee that the perturbation is $C^{1+\alpha}$.) Since the sum of the exponents in each bundle varies continuously, we conclude there are no zero exponents.

Remark. The perturbation techniques of [BB] and [BV] in fact don't assume ergodicity, but are only able to control the integrated Lyapunov exponents. That's why we have to assume stable ergodicity (in place of stable transitivity) in theorem 1.

Remark. The ideas of the proof were already present in [DP].

Let us recall briefly the definition and some properties of dominated splittings, see [BDP] for details. Let $f \in \mathrm{Diff}^1_\mu(M)$.

A Df-invariant splitting $TM = E^1 \oplus \cdots \oplus E^k$, with $k \geq 2$, is called a *dominated splitting* (over M) if there are constants c, $\tau > 0$ such that

$$\frac{\|Df^n(x) \cdot v_j\|}{\|Df^n(x) \cdot v_i\|} < ce^{-\tau^n} \tag{1}$$

for all $x \in M$, all $n \geq 1$, and all unit vectors $v_i \in E_i(x)$ and $v_j \in E_j(x)$, provided $i < j$. Dominated splittings are always continuous and stable under C^1-perturbations.

A dominated splitting $E^1 \oplus \cdots \oplus E^k$ is called the *finest dominated splitting* if there is no dominated splitting defined over all M with more than k bundles. If some dominated splitting exists, then the finest dominated splitting exists, is unique, and refines every dominated splitting. The continuation of the finest dominated splitting is the finest dominated splitting of the perturbed diffeomorphism.

Let $\lambda_1(f,x) \geq \cdots \geq \lambda_d(f,x)$ be the Lyapunov exponents of f (counted with multiplicity), defined for almost all x. We write also

$$\lambda_i(f) = \int \lambda_i(f,x)\, d\mu(x). \tag{2}$$

Assume f has a dominated splitting $E^1 \oplus \cdots \oplus E^k$. Then the Oseledets splitting is a measurable refinement of it. For simplicity of writing, we will say the exponent λ_p *belongs* to the bundle E^i if

$$d_1 + \cdots + d_{i-1} < p \leq d_1 + \cdots + d_i,$$

By (1), there is an uniform gap between Lyapunov exponents that belong to different bundles.

We now give the proof of theorem 1 in detail. Take $f \in \mathrm{Diff}^{1+\alpha}_\mu(M)$ a stably ergodic diffeomorphism. As mentioned, this implies that f has a dominated splitting, see [AM].

Let $E^1_f \oplus \cdots \oplus E^k_f$ be the finest dominated splitting of f. If g is a perturbation of f, let $E^1_g \oplus \cdots \oplus E^k_g$ be the continuation of the splitting. Let us indicate by

$J_i(g)$ the sum of all Lyapunov exponents $\lambda_p(g)$ that belong to E^i_g. Then we can also write

$$J_i(g) = \int \log \left| \det Dg|_{E^i_g} \right| d\mu. \tag{3}$$

In particular, $J_i(\cdot)$ is a continuous function in the neighborhood of f.

By the theorem of Bonatti and Baraviera [BB], up to C^1-perturbing f, we may assume $J_i(f) \neq 0$ for all i. (It is important to notice that the perturbation can be taken $C^{1+\alpha}$ since so is the original f.)

In the last step we need the following proposition:

Proposition 2. *Let $f \in \mathcal{SE}$ and let $E^1_f \oplus \cdots \oplus E^k_f$ be the finest dominated splitting. Then for all $\varepsilon > 0$ and $i = 1, \ldots, k$ there exists a perturbation $g \in \mathrm{Diff}^{1+\alpha}_\mu(M)$ of f such that if the Lyapunov exponents $\lambda_p(g)$, $\lambda_q(g)$ belong to E^i_g then*

$$|\lambda_p(g) - \lambda_q(g)| < \varepsilon.$$

Applying the proposition, we find g close to f such that all $\lambda_p(g)$ in E^i_g are close to $J_i(g)/\dim E^i$ and therefore are non-zero. This finishes the proof of theorem 1, modulo giving the:

Proof of proposition 2. Take $f \in \mathcal{SE}$ and p, q such that both λ_p and λ_q belong to E^i. It is enough to consider the case $q = p + 1$.

Let us write $\Lambda_p(f) = \lambda_1(f) + \cdots + \lambda_p(f)$. Then $\Lambda_p(\cdot)$ is an upper semicontinuous function (see e.g. [BV]). Since $\mathrm{Diff}^{1+\alpha}_\mu(M)$ is not a complete metric space, we can't deduce that the set of continuity points of $\Lambda_p(\cdot)$ is dense. Nevertheless, for every $\varepsilon > 0$, the set

$$\mathcal{D}_\varepsilon = \{f \in \mathrm{Diff}^{1+\alpha}_\mu(M);\ \exists\, \mathcal{U} \ni f \text{ open s.t. } |\Lambda_p(g) - \Lambda_p(f)| < \varepsilon\ \forall g \in \mathcal{U}\}$$

is dense in $\mathrm{Diff}^{1+\alpha}_\mu(M)$, simply because $\Lambda_p \geq 0$.

Thus we may assume, after taking a perturbation, that $f \in \mathcal{D}_\varepsilon$. Since f is ergodic and λ_p, λ_{p+1} belong to the same bundle of the finest dominated splitting, there is no dominated splitting of index p along the orbit of x, for a.e. $x \in M$. This shows that the set $\Gamma(f, \infty)$ (see [BV, §4]) has full measure. Therefore Proposition 4.17 in [BV] gives us a C^1-perturbation g of f such that

$$\Lambda_p(g) < \Lambda_p(f) - \frac{\lambda_p(f) - \lambda_{p+1}(f)}{2} + \varepsilon.$$

In fact, g can be taken $C^{1+\alpha}$ once f is $C^{1+\alpha}$.

Since $f \in \mathcal{D}_\varepsilon$ and g is close to f, we have $|\Lambda_p(g) - \Lambda_p(f)| < \varepsilon$ and accordingly $\lambda_p(f) - \lambda_{p+1}(f) < 4\varepsilon$. □

We close this note with some questions about what can be said in the absence of stable ergodicity. The following problem is likely to have a positive answer:

Problem 1. Is it true that for the generic $f \in \mathrm{Diff}^1_\mu(M)$, either all Lyapunov exponents are zero at almost every point, or f is non-uniformly hyperbolic (i.e., all Lyapunov exponents are non-zero almost everywhere)?

Notice this is true if $\dim M = 2$, by [B] (later extended in [BV]). We show now how this dichotomy can easily be derived from an eventual positive answer to the following well known conjecture of A. Katok:

Problem 2. Is it true that the generic map $f \in \mathrm{Diff}^1_\mu(M)$ is ergodic?

Remark. The theorem of Oxtoby-Ulam [OU] says that C^0-generic volume-preserving homeomorphisms are ergodic. Also, it was recently shown by Bonatti and Crovisier [BC] that the generic $f \in \mathrm{Diff}^1_\mu(M)$ is transitive.

Assume problem 2 has a positive answer. We define some subsets of $\mathrm{Diff}^1_\mu(M)$:

- Let $\mathcal{R}$ be the set of f such that f is ergodic and the Oseledets splitting is either trivial (all exponents zero) or dominated. Then $\mathcal{R}$ is a residual subset of $\mathrm{Diff}^1_\mu(M)$, by [BV].
- Let $\mathcal{Z}$ be the set of f such that $\lambda_i(f, x) = 0$ for all i and a.e. x. Then $\mathcal{Z}$ is a G_δ set (since $\lambda_1(\cdot)$ is semi-continuous).
- Let $\mathcal{S}$ be the set of f which have a dominated splitting $TM = E^+ \oplus E^-$ with (recall definition (2)) $\lambda_p(f) > 0 > \lambda_{p+1}(f)$, where $p = \dim E^+$. This is an open set. (Indeed, for g close to f, E^-_g depends continuously on g; since λ_{p+1} is the integrated top exponent in E^-, it defines an upper semicontinuous function in the neighborhood of f; analogously for λ_p.)

If $f \in \operatorname{int} \mathcal{Z}^c$ then we can take a perturbation $f_1 \in \mathcal{R}$. Since $f_1 \notin \mathcal{Z}$, the Oseledets splitting of f_1 is non-trivial. Consider the finest dominated splitting and let J_i be as in (3). By [BB] we can find another perturbation f_2 such that all $J_i(f_2)$ are non-zero. Finally, take $f_3 \in \mathcal{R}$ close enough to f_2 so that $J_i(f_3)$ are still non-zero. Then $f_3 \in \mathcal{S}$. This shows that $\mathcal{S}$ is an (open and) dense subset of int /cZ^c. Therefore $\mathcal{Z} \cup \mathcal{S}$ is a residual subset of $\mathrm{Diff}^1_\mu(M)$, answering positively problem 1.

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IMPA – Estr. D. Castorina 110 – 22460-320 Rio de Janeiro – Brazil.
E-mail address: bochi@impa.br

LAGA – Université Paris 13 – 99 Av. J-B. Clément – 93430 Villetaneuse – France.
E-mail address: fayadb@math.univ-paris13.fr

IMPA – Estr. D. Castorina 110 – 22460-320 Rio de Janeiro – Brazil.
E-mail address: enrique@impa.br