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AN EFFECTIVE VERSION OF KATOK'S HORSESHOE THEOREM FOR CONSERVATIVE C^2 SURFACE DIFFEOMORPHISMS

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ABSTRACT. For area preserving C^2 surface diffeomorphisms, we give an explicit finite information condition on the exponential growth of the number of Bowen's (n,δ)-balls needed to cover a positive proportion of the space, that is sufficient to guarantee positive topological entropy. This can be seen as an effective version of Katok's horseshoe theorem in the conservative setting. We also show that the analogous result is false in dimension larger than 3.

1. INTRODUCTION

Let X be a compact smooth surface with a Riemannian metric. Denote by $\mathrm{Diff}^r_{\mathrm{vol}}(X)$ the group of C^r diffeomorphisms which preserve the volume form m induced by the Riemannian metric. Without loss of generality, we assume that $m(X) = 1$.

A well-known result of Katok, based on Pesin theory, says that if $f \in \mathrm{Diff}^{1+\epsilon}(X)$ has non-zero Lyapunov exponent for some f-invariant non-atomic ergodic measure, then the topological entropy of f is positive and that f actually has invariant horseshoes that carry most of the topological entropy (see, for example, [5] or [6]). In particular, this is the case for any $f \in \mathrm{Diff}^{1+\epsilon}_{\mathrm{vol}}(X)$ having positive Lyapunov exponents on a positive measure set, or, in other words, when f has positive metric entropy by Pesin's formula.

Besides the positivity of Lyapunov exponents, another manifestation of positive metric entropy is the exponential rate of growth of the Bowen (n,δ)-balls (see Definition 1) that are needed to cover a definite proportion of X (see, for example, [6]).

DEFINITION 1. Given a continuous map $f : X \to X$. For any $\delta > 0$, integer $n \geq 1$, any $x \in X$, we define Bowen's (n,δ)-ball centered at x by

$$B_f(x,n,\delta) = \left\{ y \,\middle|\, d(f^i(x), f^i(y)) < \delta,\ \forall\ 0 \leq i \leq n-1 \right\}.$$

Given an f-invariant measure μ. For any $\varepsilon \in (0,1)$, let $N_f(n,\delta,\varepsilon) = \inf_{\mathcal{U}} \#\mathcal{U}$, where $\mathcal{U}$ is taken over all the subsets of $\left\{B_f(x,n,\delta)\right\}_{x\in X}$ such that the union

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of the (n,δ)-balls in $\mathcal{U}$ has μ-measure not less than $1-\varepsilon$. For a finite set I, we use $\#I$ to denote the cardinality of I.

By the sub-multiplicative growth of the number of Bowen balls and Katok's horseshoe theorem, the following statement is true by compactness.

Fact: *Given $f \in \mathrm{Diff}^2(X)$ so that the C^2 norm of f is bounded by $D > 0$, and $h,\delta,\varepsilon > 0$. Then there exists $n_0 = n_0(D,h,\delta,\varepsilon) > 0$ such that if $N_f(n,\delta,\varepsilon) \geq e^{nh}$ for some integer $n > n_0$, then f has positive topological entropy.*

Sketch of proof. Assume by contradiction that there exists $h,\delta,\varepsilon > 0$ and a sequence f_n with a uniform bound on its C^2 norm for which $N_{f_n}(n,\delta,\varepsilon) \geq e^{nh}$ and $h_{\mathrm{top}}(f_n) = 0$. By compactness we can, up to passing to a subsequence, assume that f_n has a limit f that is $C^{1+\mathrm{Lip}}$. Since for any g, the minimal number $N_g(n,\delta)$ needed to cover all of X is essentially sub-multiplicative in n, we have that for a fixed $k \in \mathbb{N}$, and for any n sufficiently large $N_{f_n}(k,\delta) \geq e^{kh/2}$. Therefore $N_f(k,\delta) \geq e^{kh/2}$ for any $k \in \mathbb{N}$ and hence f has positive topological entropy. By Katok's horseshoe theorem, this contradicts the assumption $h_{\mathrm{top}}(f_n) = 0$ for all n. □

In this paper, we will give a direct proof of the above fact for area preserving f, that also provides an explicit upper bound for $n_0(D,h,\delta,\varepsilon)$. Our bound will essentially be a tower-exponential of height $K \sim \log(\frac{\log A}{h})$ where $A = \|f\|_{C^1}$. The norm of the second derivative of f enters into the argument of the tower-exponential bound. We will not use in our proof *any* ergodic theory.

Our main tool is a finite information closing lemma for a map $g \in \mathrm{Diff}^2_{\mathrm{vol}}(X)$ that generalizes the one obtained in [2, Theorem 4]. [2, Theorem 4] asserts that if x is such that $|Dg^q(x)|$ is comparable to $|Dg|^{\theta q}$ where θ is close to 1 and q is sufficiently large compared to powers of the C^2 norm of g, then there exists a hyperbolic periodic point that shadows a piece of a length q orbit of x. A similar effective closing lemma was previously obtained by Climenhaga and Pesin in [4] for $C^{1+\epsilon}$-diffeomorphisms in any dimension, assuming however the existence of a splitting of the tangent spaces along a long orbit with some additional estimates of effective hyperbolicity. For an interesting application of the latter effective approach, we refer the reader to [3]. In this note we will need a generalized version of the effective closing lemma in [2] that gives a shadowing of x by a hyperbolic periodic orbit, even when $|Dg^q(x)|$ is much smaller than $|Dg|^{\theta q}$, provided that $|Dg^q(x)| \geq |Dg(g^i(x))|^{\theta q}$, for most of the $i \in [0,q]$. An inductive use of this closing lemma allows one to obtain, under the growth condition of the (n,δ)-balls, sufficiently many hyperbolic periodic points with a good control on their local stable and unstable manifolds to insure the existence of a horseshoe. Note that, in order to exploit the growth condition of the Bowen balls, we need sufficiently precise informations from the shadowing property, which are not covered by the direct bootstrapping of [2, Theorem 4].

With the same approach, we are also able to conclude positive topological entropy from derivative growth at an explicit time scale along a single, yet not too concentrated, orbit.

1.1. **Statements of the main results.** Throughout this note, X is a compact surface with a volume form m. Without loss of generality, we assume that $m(X) = 1$. We will denote by $f : X \to X$ a C^2 diffeomorphism that preserves m such that for constants $A, D > 0$,

$$(*) \qquad \begin{cases} |Df| \le A, \\ |D^2 f| \le D. \end{cases}$$

Here $|Df|$, $|D^2 f|$ denote respectively the supremum of the first and second derivatives of f.

All the constants that appear in the text will implicitly depend on the surface X.

To simplify notations, we define the following.

DEFINITION 2. For $R_0, R_1 > 0$, $K \in \mathbb{Z}_+$, we define function Tower : $\mathbb{R}_+^2 \times \mathbb{Z}_+ \to \mathbb{R}$ by the following recurrence relation,

$$\mathrm{Tower}(R_0, R_1, K) = \begin{cases} R_0, & K = 1, \\ R_1^{\mathrm{Tower}(R_0, R_1, K-1)}, & K \ge 2. \end{cases} \tag{1.1}$$

Our main result is the following.

THEOREM A. *There exists a constant $C_0 = C_0(X) > 0$ such that the following is true. For any $A, D > 1$, $h \in (0, \log A]$, $\varepsilon \in (0,1)$, $\delta > 0$, denote by*

$$P_0 = \max(\varepsilon^{-1} e^{C_0 (\log(\frac{\log A}{h}))^2 + C_0}, C_0 h^{-1} \log \delta^{-1}), \tag{1.2}$$

$$P_1 = e^{C_0 h^{-1} \log D \log A}. \tag{1.3}$$

If $f : X \to X$ is a C^2 diffeomorphism preserving m that satisfies $(*)$, and $N_f(n, \delta, \varepsilon) > e^{nh}$ for some $n \ge \mathrm{Tower}(P_0, P_1, K_0)$, where $K_0 = \lceil C_0 \log(\frac{\log A}{h}) + C_0 \rceil$, then f has positive topological entropy.

Theorem A gives positive topological entropy from complexity growth at an explicit large time scale. Some adaptation of the proof also allows us to conclude positive topological entropy from derivative growth at an explicit time scale along a single, yet not too concentrated, orbit. To precisely formulate such a result, we introduce the following notation.

DEFINITION 3. Given a continuous map $f : X \to X$, for any subset $I \subset \mathbb{Z}$, any $x \in X$, we set $\mathrm{Orb}(f, x, I) = \{ f^i(x) \mid i \in I \}$. For constants $c, \delta > 0, \varepsilon \in (0,1)$, we say that x is $(n, c, \delta, \varepsilon)$*-sparse* if for any subset $I \subset \{0, \ldots, n-1\}$ satisfying $|I| > cn$ we have $m(B(\mathrm{Orb}(f, x, I), \delta)) > \varepsilon$.

THEOREM B. *There exists a constant $C_0 = C_0(X) > 0$ such that the following is true. For any $A, D > 1$, $h \in (0, \log A]$, $\varepsilon \in (0,1)$, let*

$$P_0 = \varepsilon^{-1} e^{C_0 (\log(\frac{\log A}{h}))^2 + C_0}, \quad P_1 = e^{C_0 h^{-1} \log D \log A}.$$

If $f : X \to X$ is a C^2 diffeomorphism preserving m that satisfies $(*)$, and there exists $x \in X$ such that for some $n \ge \mathrm{Tower}(P_0, P_1, K_0)$, where $K_0 = \lceil C_0 \log(\frac{\log A}{h}) + C_0 \rceil$, we have

- $|Df^n(x)| > e^{nh}$,
- *x is $(n, \mathrm{Tower}(P_0, P_1, K_0-1)^{-1}, D^{-\mathrm{Tower}(P_0,P_1,K_0-1)}, \varepsilon)$-sparse,*

then f has positive topological entropy.

Observe that a non-concentration condition, such as the second condition of Theorem B, is necessary to conclude positive entropy, for otherwise x could just belong to a hyperbolic periodic orbit with a small period.

We remark that Theorem A does not hold in general in dimension at least 4 as the following example shows.

EXAMPLE 4. Denote by $\{g_t\}_{t\in\mathbb{R}}$ a geodesic flow on $Y := S_1 M$, the unit tangent space of a hyperbolic surface M, preserving the Liouville measure μ. We set $h_0 := h_\mu(g_1) > 0$. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle and let $\varphi \in C^\infty(\mathbb{T})$ be a function such that $\int_{\mathbb{T}} \varphi d\theta = 0$ and $\varphi|_{[0,\frac{1}{2}]} \equiv 1$. For any $\alpha \in \mathbb{R}$, denote by $R_\alpha : \mathbb{T} \to \mathbb{T}$ the rotation $\theta \mapsto \theta + \alpha[1]$, and consider the C^2 map $f_\alpha : \mathbb{T}\times Y \to \mathbb{T}\times Y$ defined as follows:

$$f_\alpha(\theta, x) = (\theta+\alpha, g_{\varphi(\theta)}(x)), \quad \forall\ (\theta, x) \in \mathbb{T}\times Y.$$

Observe that for any $\alpha \in \mathbb{R}$, f_α preserves the smooth measure $\nu := \mathrm{Leb}_{\mathbb{T}} \times \mu$. It is clear that $\sup_{\alpha\in\mathbb{T}} |f_\alpha|_{C^2} < \infty$. Moreover, we have the following which shows that Theorem A does not hold in general in dimension at least 4.

PROPOSITION 5. *We have that*

(1) For any $\alpha \in \mathbb{R}-\mathbb{Q}$, the topological entropy $h_{\mathrm{top}}(f_\alpha) = 0$.

(2) There exists $\delta > 0$ such that for any $\varepsilon \in (0,1)$, any integer $n_0 > 0$, there exists $n > n_0$, $\bar{\alpha} \in \mathbb{T}$, such that for any $\alpha \in [0, \bar{\alpha}]$ it holds that $N_{f_\alpha}(n, \delta, \varepsilon) > e^{\frac{nh_0}{2}}$.

Proof. Abramov-Rohlin formula [1] for the entropy of a skew product yields (1). To see (1) directly, let $(q_n)_{n\in\mathbb{N}}$ be the sequence of denominators of the best rational approximations of α. Then by Denjoy-Koksma theorem, the partial sums $S_{q_n}\varphi$ defined as $S_{q_n}\varphi(\theta) := \sum_{i=0}^{q_n-1} \varphi(\theta + i\alpha)$, $\forall\ \theta \in \mathbb{T}$, converge uniformly in the C^∞ topology to 0, as n tends to infinity. By direct computations, we see that

$$f_\alpha^{q_n}(\theta, x) = (\theta + q_n\alpha, g_{S_{q_n}(\theta)}(x)), \quad \forall\ (\theta, x) \in \mathbb{T}\times Y.$$

This implies that $f_\alpha^{q_n}$ converge to Id in the C^∞ topology, as n tends to infinty. By Ruelle's entropy inequality, such convergence can happen only if $h_{\mathrm{top}}(f_\alpha) = 0$.

To see (2), we notice that by $h_\mu(g_1) = h_0 > 0$, there exists $\delta > 0$, such that for any $\varepsilon \in (0,1)$, any $n_0 > 0$, there exists $n > n_0$ such that $N_{g_1}(n, \delta, \varepsilon) > e^{\frac{nh_0}{2}}$. Then by choosing α to be sufficiently close to 0, so that $i\alpha \in [0, \frac{1}{2}]$ for all $0 \le i \le n$, we have $f_\alpha^i(\theta, x) = (\theta + i\alpha, g_i(x))$ for any $(\theta, x) \in \mathbb{T}\times Y$, any $0 \le i \le n$. Then it is direct to see that $N_{f_\alpha}(n, \delta, \varepsilon) \ge N_{g_1}(n, \delta, \varepsilon) > e^{\frac{nh_0}{2}}$. This concludes the proof. □

NOTATION 6. For any $n \ge 1$, any $x \in X$, we will denote $\mu_{x,n} = \frac{1}{n}\sum_{m=0}^{n-1} \delta_{f^m(x)}$. For any $x \in X$, any linear subspace $E \subset T_x X$ and any $r > 0$, we denote $B_E(r) = \{v \in E \mid |v| < r\}$. For any subset $A \subset X$ and any $r > 0$, we denote $B(A, r) = \{x \mid d(x, A) < r\}$. For any measurable subset $K \subset X$, we denote the measure of K by $|K| := m(K)$. For a finite set A, we use $\#A$ to denote the cardinality of A.

We will use $c, c_1, \dots$ to denote generic positive constants which are allowed to vary from line to line, and may or may not depend on X, but independent of everything else. Under our notations, expressions like $cA \le B \le cA$ are legitimate. For two variables $A, B > 0$, we denote $A \gg B$ (resp. $A \ll B$) if we have $A \ge cB$ (resp. $cA \le B$) for some constant c as above.

2. FROM HYPERBOLIC POINTS TO POSITIVE ENTROPY

DEFINITION 7. Let $g : X \to X$ be a C^1 diffeomorphism. For $\alpha \in (0, \pi), r \in (0, 1)$, a hyperbolic periodic point of g, denoted by $y \in X$, is said to be (α, r)-hyperbolic if the following is true. Let $E^s(y), E^u(y)$ be respectively the stable and unstable direction at y. Then

(1) The angle between $E^s(y)$ and $E^u(y)$ is at least α;
(2) The local stable (resp. local unstable) manifold of g at x contains the set $\exp_y(\mathrm{graph}(\gamma_s))$ (resp. $\exp_y(\mathrm{graph}(\gamma_u))$), where $\gamma_s : B_{E^s(y)}(r) \to E^u(y)$ (resp. $\gamma_u : B_{E^u(y)}(r) \to E^s(y)$) is a Lipschitz function such that $\gamma_s(0) = 0$ and $\mathrm{Lip}(\gamma_s) < \frac{1}{100}$ (resp. $\gamma_u(0) = 0$ and $\mathrm{Lip}(\gamma_u) < \frac{1}{100}$).

Moreover, we denote $\exp_y(\mathrm{graph}(\gamma_s))$ (resp. $\exp_y(\mathrm{graph}(\gamma_u))$) by $\mathcal{W}^s_r(y)$ (resp. $\mathcal{W}^u_r(y)$).

For any $\alpha \in (0, \pi), r > 0$, the set of all (α, r)-hyperbolic points of g is denoted by $\mathcal{H}(g, \alpha, r)$. To simplify notations, for any $\lambda \in (0, 1)$, a (λ^2, λ^3)-hyperbolic point of g is said to be λ-hyperbolic. The set of all λ-hyperbolic points of g is denoted by $\mathcal{H}(g, \lambda)$.

DEFINITION 8 (Heteroclinic intersection)**.** For any C^1 diffeomorphism $g : X \to X$, for any two distinct hyperbolic periodic points of g denoted by p, q, we say that p, q has a heteroclinic intersection, if the global stable submanifold of p intersects transversely the global unstable manifold of q, and the global unstable submanifold of p intersects transversely the global stable manifold of q.

The following proposition shows that for any given α, r, there cannot be too many (α, r)-points unless there is a heteroclinic intersection.

PROPOSITION 9. *There exist $C_1, C_2 > 1$ depending only on X such that, for any $\alpha \in (0, \pi)$, any $0 < r < C_1^{-1}$, if a C^1 diffeomorphism $g : X \to X$ satisfy $\#\mathcal{H}(g, \alpha, r) > C_2 r^{-2} \alpha^{-4}$, then there exists a heteroclinic intersection for g. In particular, g has positive topological entropy. In particular, if $\lambda \ll 1$ and $\#\mathcal{H}(g, \lambda) \gg \lambda^{-14}$, then there exists a heteroclinic intersection for g.*

Proof. In order to be able to measure the angles between vectors in nearby tangent spaces, we cover the surface X by finitely many C^∞ local charts $\{\psi : [-1, 2]^2 \to X\}_{\psi \in \mathcal{B}}$ indexed by $\mathcal{B}$. For any three distinct points $x, y, z \in \mathbb{R}^2$, let $\angle(x, y, z)$ denote $\angle(x - y, z - y)$. For any $\beta > 0$, any $v \in \mathbb{R}^2 \setminus \{0\}$, let $C(v, \beta) := \{u \mid \angle(u, v) < \beta\} \cup \{0\}$.

We will choose $\{\psi : [-1, 2]^2 \to X\}_{\psi \in \mathcal{B}}$ and a constant $c_0 > 0$, depending only on X, such that for any $x \in X$, any $\psi \in \mathcal{B}$ such that $x \in \psi([0, 1]^2)$, for any $v_1, v_2 \in T_x X \setminus \{0\}$, set $\hat{x} := \psi^{-1}(x)$, $\hat{v}_1 := D\psi^{-1}(x, v_1)$, $\hat{v}_2 := D\psi^{-1}(x, v_2)$, then:

1. $2^{-1}\angle(v_1,v_2) \le \angle(\hat v_1,\hat v_2) \le 2\angle(v_1,v_2)$;
2. If $|v_1|,|v_2| < 2c_0^{-1}$, then $\psi^{-1}exp_x(v_i)$ is defined and

$$2^{-1}\angle(v_1,v_2) \le \angle(\psi^{-1}\exp_x(v_1),\hat x,\psi^{-1}\exp_x(v_2)) \le 2\angle(v_1,v_2).$$

We fix an arbitrary smooth measure $\hat m$ on compact manifold

$$\widehat X = \left\{(x,v_1,v_2) \;\middle|\; x\in X, v_1,v_2\in T_xX, |v_1|=|v_2|=c_0^{-1}\right\}.$$

Let $c_1>0$ be a large constant to determined later, and for any $(x,v_1,v_2)\in\widehat X$, any $\psi\in\mathcal B$ so that $x\in\psi((0,1)^2)$ and set

$$Q_\psi(x,v_1,v_2) = \left\{(y,u_1,u_2)\in\widehat X \;\middle|\; |\hat x-\hat y| < \frac{r\alpha}{c_1}, \angle(\hat v_1,\hat u_1), \angle(\hat v_2,\hat u_2) < \frac{\alpha}{40}\right\}.$$

Then there exists $c_2>0$ depending only on X, c_1, such that for all $(x,v_1,v_2)\in\widehat X$, any $\psi\in\mathcal B$ so that $x\in\psi((0,1)^2)$, we have

$$\hat m(\psi(Q_\psi(x,v_1,v_2))) > c_2^{-1}r^2\alpha^4.$$

By pigeonhole principle, there exists a constant $c_3>0$ depending only on X, c_2, such that whenever $\#\mathcal H(g,\alpha,r) > c_3r^{-2}\alpha^{-4}$, there exists a chart $\psi\in\mathcal B$, $(y_i,v_i^s,v_i^u)\in\widehat X$, $i=1,2$ so that

(1) $y_1,y_2\in\mathcal H(g,\alpha,r)\cap\psi((0,1)^2)$ are two distinct points;
(2) For $i=1,2$, $\angle(v_i^s,v_i^u)\le\frac{\pi}{2}$, and v_i^s (resp. v_i^u) lies in the stable (resp. unstable) direction of y_i;
(3) $Q_\psi(y_1,v_1^s,v_1^u)\cap Q_\psi(y_2,v_2^s,v_2^u)\ne\emptyset$.

This implies that $|\hat y_1-\hat y_2| < \frac{2r\alpha}{c_1}$, $\angle(\hat v_1^s,\hat v_2^s) < \frac{\alpha}{20}$ and $\angle(\hat v_1^u,\hat v_2^u) < \frac{\alpha}{20}$.

For $i=1,2$, let us denote $\alpha_i = \angle(v_i^u,v_i^s)$. By the definition of $\mathcal H(g,\alpha,r)$ we have $\alpha_1,\alpha_2\ge\alpha$. Then $\angle(\hat v_i^u,\hat v_i^s)\ge 2^{-1}\alpha_i$ for $i=1,2$. Moreover for $r\ll 1$, we have $\psi^{-1}(\mathcal W_r^u(y_i))\subset\hat y_i + C(\hat v_i^u,\frac{1}{20}\alpha_i)$ since there exists $\gamma_u : B_{E^u(y_i)}(r)\to E^s(y_i)$ with $\mathrm{Lip}(\gamma_u) < \frac{1}{100}$, such that $\mathcal W_r^u(y_i) = exp_{y_i}\mathrm{graph}(\gamma_u)$ and $\mathrm{graph}(\gamma_u)\subset C(v_i^u,\frac{1}{40}\alpha_i)$. Similarly, we have $\psi^{-1}(\mathcal W_r^s(y_i))\subset\hat y_i + C(\hat v_i^s,\frac{1}{20}\alpha_i)$.

By straightforward calculations, when c_1 is chosen to be sufficiently large, y_1,y_2 above have a heteroclinic intersection. Thus for any $r\ll 1$, any C^1 diffeomorphism $g: X\to X$ so that $\#\mathcal H(g,\alpha,r)\gg r^{-2}\alpha^{-4}$, there exists a heteroclinic intersection for g. It is a standard fact that for C^1 surface diffeomorphism, the existence of a heteroclinic intersection implies positive topological entropy. This concludes the proof. □

3. A CLOSING LEMMA

DEFINITION 10. For any $\eta>0$, any integer $l>0$, any C^0 map $g: X\to X$, any subset $Y\subset X$, a point $x\in X$ is said to be (η,l,g)-recurrent for Y if we have

$$\frac{1}{l}\#\left\{0\le j\le l-1 \;\middle|\; g^j(x)\in Y\right\} > \eta.$$

For any subset $Y\subset X$, we denote by

$$\mathcal R(Y,\eta,l,g) := \left\{(\eta,l,g)\text{-recurrent points for } Y\right\}.$$

If in addition $g : X \to X$ is a C^1 diffeomorphism, we set for any $\lambda, \xi > 0$ that

$$\mathcal{G}(\lambda,\xi,g) := \bigcup_{y\in\mathcal{H}(g,\lambda)} B(\mathcal{W}^u_{\lambda^3}(y),\xi). \tag{3.1}$$

By our definition, we clearly have $\mathcal{G}(\lambda,\xi,g) = \mathcal{G}(\lambda,\xi,g^k)$ for any $k \geq 1$, since $\mathcal{H}(g,\lambda) = \mathcal{H}(g^k,\lambda)$ for any $k \geq 1$.

The theorem [2, Theorem 4] can be strengthened to prove the following proposition.

PROPOSITION 11. *There exist absolute constants $\bar{C} > 0$, $\theta_0 \in (\frac{1}{2}, 1)$, and $C = C(X) > 1 + \bar{C}$ such that the following is true. For each $\Delta \geq 1$, we set*

$$\eta = \eta(\Delta) := C^{-1}\Delta^{-2} \in (0,1). \tag{3.2}$$

Let $g : X \to X$ be a C^2 diffeomorphism preserving m. If for $A_1 \geq C$, $D_1 \geq A_1$, an integer $q \geq D_1^{C\Delta}$ and $x \in X$, we have the following:

(1) $|Dg| \leq A_1^{\Delta}$,
(2) $|D^2 g| \leq D_1$,
(3) $x \notin \mathcal{R}\big(\{y \,\big|\, |Dg(y)| > A_1^{\theta_0^{-1}}\}, \eta, q, g\big)$, or equivalently,

$$\frac{1}{q}\#\left\{0 \leq j \leq q-1 \,\middle|\, |Dg(g^j(x))| \leq A_1^{\theta_0^{-1}}\right\} \geq 1-\eta,$$

(4) $|Dg^q(x)| > A_1^q$,

then

$$x \in \mathcal{F}(A_1, D_1, \Delta, q, g) := \bigcup_{1\leq j\leq q} g^{-j}\Big(\mathcal{G}\big(D_1^{-\bar{C}\Delta}, A_1^{-\frac{q}{2D_1^{C\Delta}}}, g\big)\Big).$$

The proof of Proposition 11 follows closely that of [2, Theorem 4]. In our case we need to get more precise informations on the regularity of local invariant manifolds, as well as the location of the hyperbolic point. We defer its proof to Appendix A relying on many estimates from [2].

4. ESTIMATES ALONG A TOWER EXPONENTIAL SEQUENCE

Without loss of generality, we will always assume that D, A in Theorem A, B satisfy

$$D > A \gg 1. \tag{4.1}$$

Then we can assume that for any C^2 map $g : X \to X$ such that $|Dg|, |D^2 g| \leq D$, we have

$$|D^2 g^k| < D^{2k}, \quad \forall\ k \geq 1.$$

Let $C, \bar{C}, \theta_0$ be defined in Proposition 11. For D, A, h given in Theorem A or B, set C' to be a large positive constant depending only on X to be determined later. We set

$$\Delta = \frac{16\log A}{h}, \quad K = \left\lceil \frac{\log(\frac{\Delta}{4})}{-\log\theta_0} \right\rceil \geq 2, \quad \eta = \eta(\Delta) \text{ (see (3.2))}. \tag{4.2}$$

Define

$$H = H(X, A, h) := C'\Delta. \tag{4.3}$$

In the following, let $n \geq 1$ be given by Theorem A or B. Moreover, we will assume that $C_0 = C_0(X) \gg C'$ where C_0 is given by Theorem A or B.

Given $\varepsilon \in (0,1)$, we inductively define $\{q_k\}_{k=0}^{K+1}$ and $\{l_k\}_{k=0}^{K}$ as follows:

$$q_0 = \lceil \varepsilon^{-1} e^{C'(\log\Delta)^2} \rceil, \tag{4.4}$$

$$l_k = \begin{cases} \lceil D^{q_k H} \rceil, & 0 \leq k \leq K-1 \\ \lceil \frac{n}{q_K} \rceil, & k = K \end{cases}, \quad q_{k+1} = q_k l_k. \tag{4.5}$$

For $0 \leq k \leq K$, we set

$$\lambda_k = D^{-2\bar{C}\Delta q_k}, \quad \xi_k = A^{-\frac{q_{k+1}\theta_0^{k+1}}{2D^{2C\Delta q_k}}}, \tag{4.6}$$

and set

$$Q_0 = \varepsilon^{-1} e^{C'(\log\Delta)^2}, \quad Q_1 = e^{20C'h^{-1}\log D \log A}.$$

We have the following lemma.

LEMMA 12.

(1) $e^{\frac{h}{16}} < A^{\theta_0^{K+1}} \leq e^{\frac{h}{4}}$;

(2) For any $C' \gg 1$, *for all* $0 \leq k \leq K-1$, *we have*

$$D^{q_k H} \leq l_k \leq \mathrm{Tower}(Q_0, Q_1, k+2).$$

If $n > \mathrm{Tower}(Q_0, Q_1, K+3)$, *then* $l_K \geq D^{q_K H}$;

(3) For any $C' \gg 1$, *set* $\delta_0 = D^{-\mathrm{Tower}(Q_0,Q_1,K+1)}$, *we have*

$$\xi_i \leq \xi_0, \quad \delta_0 < \min(\lambda_K^3, \xi_0),$$
$$C'\lambda_i^{-11}\max(\delta_0, \xi_i) < \varepsilon, \quad \forall\ 0 \leq i \leq K.$$

Proof. We omit the proof since this lemma is derived from the definitions of $K, H, q_k, l_k, \lambda_k, \xi_k, Q_0, Q_1$ by straightforward computations. □

We define for $0 \leq k \leq K$,

$$\mathcal{G}_k := \mathcal{G}(\lambda_k, \xi_k, f), \tag{4.7}$$

$$\mathcal{F}_k := \mathcal{F}(A^{q_k\theta_0^{k+1}}, D^{2q_k}, \Delta, l_k, f^{q_k}). \tag{4.8}$$

The following is a corollary of Proposition 11.

COROLLARY 13. *If* $n > \mathrm{Tower}(Q_0, Q_1, K+3)$, *then for any* $0 \leq k \leq K$ *we have*

$$x \notin \mathcal{R}\left(\{y \mid |Df^{q_k}(y)| > A^{q_k\theta_0^k}\}, \eta(\Delta), l_k, f^{q_k}\right) \cup \mathcal{F}_k \implies |Df^{q_{k+1}}(x)| \leq A^{q_{k+1}\theta_0^{k+1}}.$$

Proof. By Lemma 12(2), if $n > \mathrm{Tower}(Q_0, Q_1, K+3)$ then for any $0 \leq k \leq K$, we have $l_K \geq D^{q_k H}$. By our choice of A, D, we have

$$|Df^{q_k}| \leq A^{q_k}, \quad |D^2 f^{q_k}| < D^{2q_k}, \quad \forall\ 0 \leq k \leq K.$$

We take any $0 \le k \le K$ and an arbitrary $x \in X$ such that $|Df^{q_{k+1}}(x)| > A^{q_{k+1}\theta_0^{k+1}}$. It suffices to show that $x \in \mathcal{R}\left(\left\{y \,\middle|\, |Df^{q_k}(y)| > A^{q_k\theta_0^k}\right\}, \eta(\Delta), l_k, f^{q_k}\right) \cup \mathcal{F}_k$. By Lemma 12(1) we have

$$|Df^{q_k}(x)| \le A^{q_k} \le (A^{q_k\theta_0^{k+1}})^{\frac{16\log A}{h}}.$$

If $x \in \mathcal{R}\left(\left\{y \,\middle|\, |Df^{q_k}(y)| > A^{q_k\theta_0^k}\right\}, \eta(\Delta), l_k, f^{q_k}\right)$, we are done. Otherwise, by (4.5), (4.3) and by letting $C' \gg C$, we can verify conditions (1)-(4) in Proposition 11 for $(f^{q_k}, A^{q_k\theta_0^{k+1}}, D^{2q_k}, \frac{16\log A}{h}, l_k)$ in place of (g, A_1, D_1, Δ, q). We can apply Proposition 11 for map $g = f^{q_k}$ to show that $x \in \mathcal{F}_k$. This completes the proof. □

The following is a straightforward consequence of Proposition 9.

COROLLARY 14. *For all $C' \gg 1$ the following is true. If we have at least one of following:*

(1) there exists $0 \le i \le K$ such that $|\mathcal{G}_i| \ge \frac{\eta^{K-i}\varepsilon}{(K+1)l_i}$,
(2) there exists $0 \le i \le K-1$ such that $|B(\mathcal{G}_i, D^{-\mathrm{Tower}(Q_0,Q_1,K+3)})| > \varepsilon$,

then f has a heteroclinic intersection, in which case f has positive topological entropy.

We include the proof of Corollary 14 in Appendix B.

REMARK 15. Given A, D, h as in Theorem A or B, we will choose C' to be sufficiently large so that the conclusions of both Lemma 12 and Corollary 14 hold.

5. AN ITERATIVE DECOMPOSITION

Now we say a few words about the general strategy behind the proof of Theorem A and Theorem B. We will inductively define a sequence of decompositions of the surface X, denoted by $X = M_i \sqcup E_i$. To start the induction, we define $M_0 = X$ and $E_0 = \emptyset$. Assume that for $k \ge 0$, we have defined M_k, E_k satisfying the following condition:

$$\textit{For each } x \in M_k, \textit{ we have } |Df^{q_k}(x)| \le A^{q_k\theta_0^k}.$$

Then E_{k+1} is defined as the set of the points that up till some finite time scale, either run into E_k with frequency $\ge \eta$, or is shadowed by hyperbolic orbits (of course the first case does not happen if E_k is empty). We will use Proposition 11 to show that the complement of E_{k+1}, defined as M_{k+1}, again satisfies the induction hypothesis. We then argue that after roughly $K = O(\log(\frac{\log A}{h}))$ steps, E_{K+1} has to be large. This will show that at some previous time scale, there are enough different hyperbolic hyperbolic points to create a homoclinic intersection.

The formal construction is the following. For all $0 \le k \le K+1$, we define M_k, E_k through the following inductive formula. Let

$$E_0 = \emptyset, \quad M_0 = X \tag{5.1}$$

and for all $0 \le k \le K$, we define

$$E_{k+1} = \mathcal{R}(E_k, \eta, l_k, f^{q_k}) \cup \mathcal{F}_k, \tag{5.2}$$

$$M_{k+1} = X \smallsetminus E_{k+1} \tag{5.3}$$

LEMMA 16. *If $n > \mathrm{Tower}(Q_0, Q_1, K+3)$, then for any $0 \le k \le K+1$ we have*

$$x \in M_k \implies |Df^{q_k}(x)| \le A^{q_k \theta_0^k}.$$

Proof. This is clear when $k = 0$ by $|Df| \le A$ and sub-multiplicativity. Assume that the lemma is valid for some integer $k \in \{0, \dots, K\}$, then $\{x \,|\, |Df^{q_k}(x)| > A^{q_k\theta_0^k}\} \subset E_k$ (we consider the inclusion valid if both sides are empty). By Corollary 13 and (5.2), we see that any $x \in X$ such that $|Df^{q_{k+1}}(x)| > A^{q_{k+1}\theta_0^{k+1}}$ is contained in E_{k+1}. This completes the induction, thus finishes the proof. □

We will give the proof of Theorem A and B in the next two subsections. In the following, we let C, θ_0 be defined in Proposition 11, let $A, D, h > 0$ be given by Theorem A or B, and let C' be sufficiently large depending only on X, satisfying Remark 15.

5.1. **Proof of Theorem A.**

PROPOSITION 17. *Let C_0 in Theorem A be sufficiently large. Then under the conditions of Theorem A, we have*

$$|E_{K+1}| \ge \varepsilon.$$

Proof. We first show the following lemma.

LEMMA 18. *Let C_0 in Theorem A be sufficiently large, and let n be given as in Theorem A. Then for each $y \in M_{K+1}$, we have*

$$B(y, e^{-2nh/5}\delta) \subset B_f(y, n, \delta).$$

Proof. It is clear from (4.5), Lemma 12(2) and $D \gg 1$ that

$$\frac{n}{q_K} \le l_K \le \frac{n}{q_K} + 1 < \frac{21}{20}\frac{n}{q_K}.$$

Let $y \in M_{K+1}$. For each $0 \le i \le l_K - 1$, we denote by

$$a_i = \log|Df^{q_K}(f^{iq_K}(y))|, \quad \delta_i = e^{-2nh/5 + iq_Kh/24 + \sum_{j=0}^{i-1} a_i}\delta, \quad B_i = B(f^{iq_K}(y), \delta_i).$$

By letting C_0 in Theorem A be sufficiently large, we can ensure that $n > \mathrm{Tower}(P_0, P_1, K_0) > \mathrm{Tower}(Q_0, Q_1, K+3)$. Then by Lemma 12(1) and Lemma 16, we have for each $z \in M_K$, $\log|Df^{q_K}(z)| \le q_K\theta_0^K \log A \le hq_K/4$. Then by $y \in M_{K+1}$, (5.2) and Lemma 16, we have $y \notin \mathcal{R}\left(\left\{z \,\middle|\, \log|Df^{q_K}(z)| > hq_K/4\right\}, \eta, l_k, f^{q_k}\right)$, thus

$$\#\left\{0 \le i \le l_K - 1 \,\middle|\, a_i > \tfrac{hq_K}{4}\right\} \le \eta l_K.$$

Since $0 \le a_i \le q_K \log A$ for any $0 \le i \le l_K - 1$, we have

$$\sum_{j=0}^{i-1} a_j \le \sum_{j=0}^{l_K-1} a_j \le \eta l_K q_K \log A + \frac{l_K q_K h}{4} \le \frac{7 l_K q_K h}{24}, \quad \forall\ 0 \le i \le l_K - 1.$$

The last inequality follows from $\eta \le \frac{h}{24\log A}$ which is a consequence of (4.2), (3.2) and $h \in (0, \log A]$. Then for any $0 \le i \le l_K - 1$, we have

$$\delta_i \le e^{-2nh/5 + l_K q_K h/3}\delta \le e^{-\frac{1}{20}nh}\delta. \tag{5.4}$$

We claim that for any integer $0 \le i \le l_K - 1$,

$$f^{iq_K}(B_0) \subset B_i. \tag{5.5}$$

We first show that the above claim concludes the proof of our lemma. Indeed, for any $0 \le l \le n$, there exist $0 \le i \le l_K - 1, 0 \le j \le q_K - 1$ such that $l = iq_K + j$. Then we have

$$f^l(B_0) = f^j(f^{iq_K}(B_0)) \subset f^j(B_i) \subset B(f^l(y), \delta)$$

The last inclusion follows from $A^{q_K}\delta_i \le A^{q_K}e^{-nh/20}\delta \le \delta$, by $|Df^j| \le A^{q_K}$, (5.4) and $\frac{n}{q_K} \ge \frac{40\log D}{h}$.

Now we obviously have (5.5) for $i = 0$. Assume that we have (5.5) for some $0 \le i \le l_K - 1$, we will show that we have (5.5) for $i+1$. It suffices to show that $f^{q_K}(B_i) \subset B_{i+1}$. Using the C^2 bound $|D^2 f^{q_K}| \le D^{2q_K}$ and $\frac{n}{q_K} \ge \frac{40\log D}{h}$, we see that for any $z \in B_i$,

$$\begin{aligned}|Df^{q_K}(z)| &\le e^{a_i} + \delta_i D^{2q_K}\\ &\le e^{a_i} + D^{2q_K}e^{-nh/20}\delta \le e^{a_i + hq_K/24}.\end{aligned}$$

Since $\delta_{i+1} = e^{a_i + hq_K/24}\delta_i$, we obtain $f^{q_{K+1}}(B_i) \subset B_{i+1}$. This proves (5.5) and concludes the proof of Lemma 18. □

To proceed with the proof of Proposition 17, observe that by Lemma 18, $M_{K+1} = X \setminus E_{K+1}$ can be covered by $ce^{4nh/5}\delta^{-2}$ many Bowen's (n,δ)-balls. By (1.2), $n > P_0$ and by letting C_0 be large, we have $c\delta^{-2} < e^{P_0 h/5} < e^{nh/5}$. Thus M_{K+1} can be covered by less than e^{nh} many Bowen's (n,δ)-balls. By our hypothesis that $N_f(n,\delta,\varepsilon) > e^{nh}$, we obtain $|M_{K+1}| < 1 - \varepsilon$. This implies that $|E_{K+1}| \ge \varepsilon$. □

Proof of Theorem A. Since f is area preserving, by Markov's inequality we have

$$|\mathcal{R}(E_k, \eta, l_k, f^{q_k})| \le \eta^{-1}|E_k|.$$

Again by the fact that f is area preserving, we obtain the following inequality by (5.2), (4.8)

$$|E_{k+1}| \le \eta^{-1}|E_k| + |\mathcal{F}_k| \le \eta^{-1}|E_k| + l_k|\mathcal{G}_k|. \tag{5.6}$$

By (5.6) and (5.1), we have

$$|E_{K+1}| \le \sum_{i=0}^{K}\eta^{i-K}l_i|\mathcal{G}_i|.$$

Thus $|E_{K+1}| \ge \varepsilon$ implies that $|\mathcal{G}_i| \ge \eta^{K-i}\frac{\varepsilon}{(K+1)l_i}$ for some $0 \le i \le K$, which by Corollary 14 (1) implies that f has positive entropy. □

5.2. **Proof of Theorem B.** The proof of Theorem B is parallel to that of Theorem A. The following proposition is an analogue of Proposition 17.

PROPOSITION 19. *Let C_0 in Theorem B be sufficiently large, and let n be as in Theorem B. Then under the condition of Theorem B, we have*

$$\mu_{x,n}(E_K) \geq \frac{h}{2\log A}.$$

We recall that $\mu_{x,n} = \frac{1}{n}\sum_{m=0}^{n-1}\delta_{f^m(x)}$.

Proof. By letting C_0 in Theorem A be sufficiently large, we can ensure that $n > \mathrm{Tower}(P_0, P_1, K_0) > \mathrm{Tower}(Q_0, Q_1, K+3)$. Then by Lemma 16, for each $y \in M_K$, we have $|Df^{q_K}(y)| \leq A^{q_K\theta_0^K} \leq e^{\frac{q_K h}{2}}$.

We let p_1 be the smallest integer $p \in \{0, \dots, n-q_K\}$ so that $f^p(x) \in M_K$. Then for each integer $j \geq 1$ so that $q_1, \dots, q_j$ are defined and belong to $\{0, \dots, n-q_K\}$, we let q_{j+1} be the first entry to M_K after $p_j + q_K - 1$ if such entry exists, otherwise we terminate the construction. We thus obtain $\{p_1, \dots, p_l\} \subset \{0, \dots, n-q_K\}$ so that $I_j := \{p_j, \dots, p_j + q_K - 1\}, 1 \leq j \leq l$ are disjoint subsets of $\{0, \dots, n-1\}$ with $f^{p_j}(x) \in M_K$ for all j; and for any $k \in \{0, \dots, n-1\} \setminus \bigcup_{j=1}^{l} I_j$, we have $f^k(x) \in E_K$.

Then by sub-multiplicativity, we have

$$\begin{aligned}\log|Df^n(x)| &\leq \sum_{i=1}^{l}\log|Df^{q_K}(f^{p_i}(x))| + (n - lq_K)\log A \\ &\leq \frac{1}{2}lhq_K + (n - lq_K)\log A.\end{aligned}$$

By the condition in Theorem B, we have $\log|Df^n(x)| > nh$. Thus $n(\log A - h) > lq_K(\log A - \frac{h}{2})$. Then we see that $\mu_{x,n}(E_K) \geq \frac{n - lq_K}{n} \geq \frac{h}{2\log A}$. □

Proof of Theorem B. For any measurable set $B \subset X$, any integers $n, l \geq 1$, any $x \in X$, we have

$$\mu_{x,n}(f^{-l}(B)) \leq \frac{l}{n} + \mu_{x,n}(B).$$

Then for any $k = 0, \cdots, K-1$, by Markov's inequality we have

$$\begin{aligned}\mu_{x,n}(\mathcal{R}(E_k, \eta, l_k, f^{q_k})) &\leq (\eta l_k)^{-1}\int \sum_{i=0}^{l_K - 1} 1_{f^{-iq_K}(E_k)}\, d\mu_{x,n} \\ &\leq (\eta l_k)^{-1}\sum_{i=0}^{l_k - 1}\mu_{x,n}(f^{-iq_k}(E_k)) \\ &\leq (\eta l_k)^{-1}\sum_{i=0}^{l_k - 1}(\mu_{x,n}(E_k) + \frac{iq_k}{n}) \\ &\leq \eta^{-1}\mu_{x,n}(E_k) + \frac{q_{k+1}}{2n\eta}.\end{aligned}$$

Similarly, we have

$$\mu_{x,n}(\mathcal{F}_k) \le \sum_{i=0}^{l_k-1} \mu_{x,n}(f^{-iq_k}(\mathcal{G}_k))$$
$$\le l_k \mu_{x,n}(\mathcal{G}_k) + \frac{l_k q_{k+1}}{n}.$$

Then we have an inequality analogous to (5.6), as follows:

$$\mu_{x,n}(E_{k+1}) \le \mu_{x,n}(\mathcal{R}(E_k, \eta, l_k, f^{q_k})) + \mu_{x,n}(\mathcal{F}_k)$$
$$\le \eta^{-1}\mu_{x,n}(E_k) + \frac{q_{k+1}}{2n\eta} + l_k\mu_{x,n}(\mathcal{G}_k) + \frac{l_k q_{k+1}}{n}.$$

By the simple observation that $l_k \ge l_0 \ge \eta^{-1}$ for all $0 \le k \le K$, we have

$$\mu_{x,n}(E_K) \le \sum_{i=0}^{K-1} \eta^{-K+i+1}(l_i \mu_{x,n}(\mathcal{G}_i) + \frac{2l_i q_{i+1}}{n}).$$

By (4.5) and Proposition 19, we see that there exists $0 \le i \le K-1$ such that

$$\mu_{x,n}(\mathcal{G}_i) \ge l_i^{-1}(\frac{\eta^{K-i-1}}{K}\frac{h}{2\log A} - \frac{2q_K l_{K-1}}{n}) \ge l_i^{-2}.$$

The last inequality follows from

$$\frac{2q_K l_{K-1}}{n} < \frac{2l_{K-1}}{l_K} < q_0^{-1} \le e^{-C'(\log\Delta)^2} < \frac{\eta^K h}{4K\log A},$$
$$\frac{\eta^K h}{4K\log A} > l_0^{-1} \ge l_i^{-1}, \quad \forall\ 0 \le i \le K-1,$$

by letting C' be larger than some absolute constant. In particular, by Lemma 12(2), (4.2), (4.4), (4.5) and by letting C_0 in Theorem B be sufficiently large, we have

$$\mu_{x,n}(\mathcal{G}_i) > \mathrm{Tower}(Q_0, Q_1, K+1)^{-2} > \mathrm{Tower}(Q_0, Q_1, K+2)^{-1},$$
$$K_0 \ge K+4, \quad P_i > Q_i, i = 0, 1.$$

By the condition that x is $(n, \mathrm{Tower}(P_0, P_1, K_0-1)^{-1}, D^{-\mathrm{Tower}(P_0,P_1,K_0-1)}, \varepsilon)$-sparse in Theorem B, we see that

$$|B(\mathcal{G}_i, D^{-\mathrm{Tower}(Q_0,Q_1,K+3)})| \ge |B(\mathcal{G}_i, D^{-\mathrm{Tower}(P_0,P_1,K_0-1)})| > \varepsilon.$$

This concludes the proof by Corollary 14(2). □

APPENDIX A.

In this section we prove the main technical result Proposition 11. We start with a slight generalization of Pliss lemma [7].

LEMMA 20 (a variant of Pliss). *For any real numbers $N \ge 1$, $1 > \theta_0 > \theta_1 > \theta_2 > 0$, $\eta \in (0, \frac{1}{2}\frac{1-\theta_0}{N-\theta_0}\frac{\theta_1-\theta_2}{N-\theta_2})$, for any integer $n \ge 1$, real number $l > 0$, the following is true. Given a sequence of n real numbers $a_1, ..., a_n$. Assume that*

(1) $a_i \le Nl$ for all $1 \le i \le n$,

(2) $\sum_{i=1}^{n} a_i > n\theta_1 l$,
(3) $\#\left\{1 \le i \le n \,\middle|\, a_i > \theta_0 l\right\} < \eta n$.

Then there exist at least $\frac{\theta_1-\theta_2}{1-\theta_2}n$ *indexes* i*'s such that* $\frac{1}{k}\sum_{j=i}^{i+k-1} a_j > \theta_2 l$ *for all* $1 \le k \le n+1-i$.

Proof. Denote by

$$A := \left\{ i \,\middle|\, \text{there exists } 1 \le k \le n+1-i \text{ such that } \tfrac{1}{k}\textstyle\sum_{j=i}^{i+k-1} a_j \le \theta_2 l \right\}.$$

Without loss of generality, we assume that $A \neq \emptyset$, for otherwise the conclusion of Lemma 20 is already true. We claim that A is contained in a non-empty set $I \subset \{1,\dots,n\}$ satisfying that $\frac{1}{\#I}\sum_{i\in I} a_i \le \theta_2 l$. Indeed, we can inductively construct I as follows. We set $I_0 = \emptyset$. Assume that I_n is constructed for some $n \in \mathbb{N}$ so that $A \not\subset I_n$, then we let i be the smallest element of $A \setminus I_n$, and let $1 \le k \le n+1-i$ be an integer such that $\frac{1}{k}\sum_{j=i}^{i+k-1} a_j \le \theta_2 l$. Then we set $I_{n+1} = I_n \cup \{i,\dots,i+k-1\}$. After finite steps, we obtain I_n satisfying $A \subset I_n$ for some $n \ge 1$. It is direct to see that $I = I_n$ satisfies the required property.

Then by (1),(2), we obtain that

$$lN\#(I^c) + l\theta_2\#I > ln\theta_1.$$

By $l > 0$, the above inequality implies that $\#(I^c) \ge \frac{\theta_1-\theta_2}{N-\theta_2}n$. We claim that

$$\frac{1}{\#(I^c)}\sum_{i\in I^c} a_i \le l. \tag{A.1}$$

Indeed, if (A.1) was false, by (1) we would have at least $\frac{1-\theta_0}{N-\theta_0}\#(I^c) \ge \frac{1-\theta_0}{N-\theta_0}\frac{\theta_1-\theta_2}{N-\theta_2}n > \eta n$ indexes $i \in I^c$ such that $a_i > \theta_0 l$, but this would contradict (3).

Now we use (2) again, with the improved estimate (A.1) in place of (1), and obtain

$$l\#(I^c) + \theta_2 l\#I \ge \sum_{i\in I^c} a_i + \sum_{i\in I} a_i > n\theta_1 l.$$

This implies $\#(I^c) \ge \frac{\theta_1-\theta_2}{1-\theta_2}n$. We conclude the proof by the definition of I. □

Let x be given by the condition of Proposition 11. We will define a collection of charts along a sub-orbit of x following the definitions and estimates in [2].

Let v_s be a unit vector in the most contracting direction of $Dg^q(x)$ in T_xX, and let v_u be a unit vector orthogonal to v_s. For each $0 \le i \le q$, we define

$$v_i^s := \frac{Dg^i(v_s)}{|Dg^i(v_s)|}, \qquad v_i^u := \frac{Dg^i(v_u)}{|Dg^i(v_u)|},$$
$$\lambda_i^s := \log\frac{|Dg(v_i^s)|}{|v_i^s|}, \qquad \lambda_i^u := \log\frac{|Dg(v_i^u)|}{|v_i^u|},$$
$$\overline{\lambda}_i^e := \min\{\lambda_i^u, -\lambda_i^s\}.$$

Given $r > 0, \tau > 0, \kappa > 0$, we define a (r,τ,κ)-*Box*, which we denote by $U(r,\tau,\kappa)$, to be

$$U(r,\tau,\kappa) = \left\{(v,w) \in \mathbb{R}^2 \,\middle|\, |v| \le r, |w| \le \tau + \kappa|v|\right\}.$$

For example, the narrow-lined contour in Figure 1 represents a (r,τ,κ)-Box after re-scaling.

For $\kappa > 0$, we denote by

$$C(\kappa) = \left\{ (v,w) \in \mathbb{R}^2 \,\middle|\, |w| < \kappa |v| \right\},$$
$$\tilde{C}(\kappa) = \left\{ (v,w) \in \mathbb{R}^2 \,\middle|\, |v| < \kappa |w| \right\}.$$

We will refer to these sets as cones.

We now recall some definitions in [2].

- A curve contained in $\mathbb{R}^2 = \mathbb{R}_x \oplus \mathbb{R}_y$ is called a *κ-horizontal graph* if it is the graph of a Lipschitz function from an closed interval $I \subset \mathbb{R}_x$ to $\mathbb{R}_y$ with Lipschitz constant equal to or less than κ. Similarly, we can define the *κ-vertical graphs.*
- The boundary of an (r,τ,κ)-Box U is the union of two 0-vertical graphs and two κ-horizontal graphs. We call these graphs respectively, the *left (resp. right) vertical boundary of U* and the *upper (resp. lower) horizontal boundary of U*. We call the union of the left and right vertical boundary of U the *vertical boundary of U*. Similarly we call the union of the upper and lower horizontal boundary of U the *horizontal boundary of U*.
- Horizontal and vertical graphs which connect the boundaries of U will be called full horizontal and full vertical graphs as in the following definition. Given $r,\tau,\kappa,\eta > 0$, for each (r,τ,κ)-Box U, an *η-full horizontal graph of U* is an η-horizontal graph L such that $L \subset U$ and the endpoints of L are contained in the vertical boundary of U. Similarly, we define the *η-full vertical graphs of U*.
- We define an *η-horizontal strip of U* to be a subset of U bounded by the vertical boundary of U and two disjoint η-full horizontal graphs of U which are both disjoint from the horizontal boundary of U. Similarly we can define *η-vertical strips of U*. Like Boxes, we define the horizontal, vertical boundary of a strip.
- Given a Box U, $\mathcal{R}'$ a vertical strip of U, and $\mathcal{R}$ a horizontal strip of U, a homeomorphism that maps $\mathcal{R}'$ to $\mathcal{R}$ is said to be *regular* if it maps the horizontal (resp. vertical) boundary of $\mathcal{R}'$ homeomorphically to the horizontal (resp. vertical) boundary of $\mathcal{R}$.

We recall the definition of hyperbolic map in [2].

DEFINITION 21. Given $r,\tau > 0$ and $0 < \kappa,\kappa',\kappa'' < 1$, denote $U = U(r,\tau,\kappa)$ and let $\mathcal{R}_1$ be a κ-vertical strip of U, $\mathcal{R}_2$ be a κ-horizontal strip of U. A C^1 diffeomorphism $G : \mathcal{R}_1 \to \mathbb{R}^2$ is called a hyperbolic map if it satisfies the following conditions:

$$G \text{ is a regular map from } \mathcal{R}_1 \text{ to } \mathcal{R}_2, \tag{A.2}$$
$$\forall\ x \in \mathcal{R}_1, DG_x(C(\kappa')) \subset C(\tfrac{1}{2}\kappa'), \tag{A.3}$$
$$\forall\ x \in \mathcal{R}_2, DG_x^{-1}(\tilde{C}(\kappa")) \subset \tilde{C}(\tfrac{1}{2}\kappa"). \tag{A.4}$$

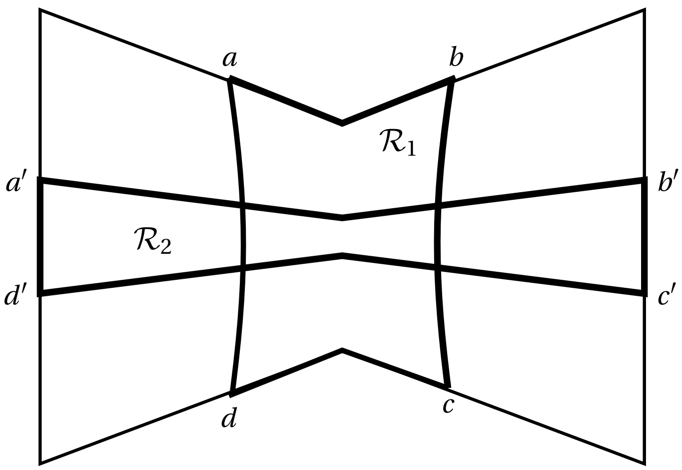

FIGURE 1. $\mathcal{R}_1$ is the topological rectangle $abcd$; $\mathcal{R}_2$ is the topological rectangle $a'b'c'd'$. Under a hyperbolic map G, ab is mapped to $a'b'$ and similarly bc, cd, da are mapped respectively to $b'c', c'd', d'a'$.

A sketch of a hyperbolic map is provided in Figure 1.

For each $0 \le n \le q$, we define $i_n : \mathbb{R}^2 \to T_{x_n}X$ as

$$i_n(a,b) = av_n^u + bv_n^s.$$

There exists a constant $R = R(X) > 0$ such that $\exp_{x_n}$ is a diffeomorphism restricted to $i_n(B(0, D_1^{-\Delta}R))$ and that $\exp_{x_{n+1}}^{-1}$ is a diffeomorphism restricted to $g \exp_{x_n} i_n(B(0, D_1^{-\Delta}R))$. Denote by g_n the C^2 diffeomorphism

$$g_n : B(0, D_1^{-\Delta}R) \to \mathbb{R}^2$$
$$g_n(v,w) = i_{n+1}^{-1} \exp_{x_{n+1}}^{-1} g \exp_{x_n} i_n(v,w).$$

We set $M := 1000$, and

$$\bar{r} = D_1^{-3\Delta M}, \quad \bar{\kappa} = D_1^{-\Delta M}, \quad \delta = \frac{\log A_1}{100}.$$

The main estimates in [2] are summarized in the following proposition.

PROPOSITION 22. *Under the conditions of Proposition 11 for some absolute constant $\theta_0 \in (0,1)$ sufficiently close to 1, and $C > 0$ sufficiently large depending only on X, there exist constant $C_1 = C_1(X)$ which can be made arbitrarily large by choosing C to be large, integers $0 \le i_1 \le i_2 \le q$, and sequences of positive numbers $\left\{(r_n, \tau_n, \kappa_n, \tilde{\kappa})_n\right\}_{i_1 \le n \le i_2}$ such that:*

(1) (Positive proportion)

$$i_2 - i_1 \ge D_1^{-C_1\Delta} q,$$

(2) *(Tameness at the starting and ending points)*

$$\cot\angle(v^u_{i_1}, v^s_{i_1}), \cot\angle(v^u_{i_2}, v^s_{i_2}) < \frac{D_1^{M\Delta}}{100},$$
$$10^6\bar{r} \geq r_i \geq \tau_i, \quad \forall\ i_1 \leq i \leq i_2$$
$$r_{i_1} = \tau_{i_1} = \bar{r}, \quad \kappa_{i_1} = \tilde{\kappa}_{i_1} = \bar{\kappa},$$
$$r_{i_2} = 10^6\bar{r}, \quad \tau_{i_2} \leq \frac{1}{10}\bar{r}, \quad \kappa_{i_2} = \frac{1}{100}\bar{\kappa}, \quad \tilde{\kappa}_{i_2} = 100\bar{\kappa},$$
$$\sum_{n=i_1}^{i_2-1}\lambda^u_n, \sum_{n=i_1}^{i_2-1} -\lambda^s_n \geq \frac{2}{3}(i_2 - i_1)a,$$

(3) *(Transversal mappings) Let r_n, τ_n, κ_n be as above, we let*

$$U_n = U(r_n, \tau_n, \kappa_n), \quad C_n = C(\kappa_n), \quad \tilde{C}_n = \tilde{C}(\tilde{\kappa}_n).$$

If Γ is a κ_n-full horizontal graph of U_n, then $g_n(\Gamma) \cap U_{n+1}$ is a κ_{n+1}-full horizontal graph of U_{n+1}. Moreover, the image of the horizontal boundary of U_n under g_n is disjoint from the horizontal boundary of U_{n+1}; the image of the vertical boundary of U_n under g_n is disjoint from the vertical boundary of U_{n+1}.

(4) *(Cone condition) Furthermore, for any $(v, w) \in U_n$, we have $(Dg_n)_{(v,w)}(C_n) \subset C_{n+1}$; for any $(v, w) \in g_n(U_n) \cap U_{n+1}$, we have $(Dg_n^{-1})_{(v,w)}(\tilde{C}_{n+1}) \subset \tilde{C}_n$. Moreover, for any $(v, w) \in U_n$, any $(V, W) \in C_n$, let $(\bar{V}, \bar{W}) = (Dg_n)_{(v,w)}(V, W)$, we have $|\bar{V}| \geq e^{\lambda^u_n - \delta}|V|$; for any $(v, w) \in g_n(U_n) \cap U_{n+1}$, any $(V, W) \in \tilde{C}_{n+1}$, let $(\bar{V}, \bar{W}) = (Dg_n^{-1})_{(v,w)}(V, W)$, we have $|\bar{W}| \geq e^{-\lambda^s_n - \delta}|W|$.*

(5) *(Hyperbolic map) Denote*

$$J = i_{i_1}^{-1} exp_{x_{i_1}}^{-1} exp_{x_{i_2}} i_{i_2},$$
$$G = i_{i_1}^{-1} exp_{x_{i_1}}^{-1} g^{i_2 - i_1} exp_{x_{i_1}} i_{i_1} = J g_{i_2 - 1} \cdots g_{i_1}.$$

There exist $\mathcal{R}_1$, a $100\bar{\kappa}$-vertical strip of U_{i_1}, and $\mathcal{R}_2$, a $100\bar{\kappa}$-horizontal strip of U_{i_1} such that G is a hyperbolic map from $\mathcal{R}_1$ to $\mathcal{R}_2$ with parameters $\kappa' = \bar{\kappa}, \kappa'' = 100\bar{\kappa}$. Moreover, for each $0 \leq j \leq i_2 - i_1$, we have $g_{i_1+j-1} \cdots g_{i_1}(\mathcal{R}_1) \subset U_{i_1+j}$.

REMARK 23. We stress that by (1), $i_2 - i_1$ is lower bounded by a definite proportion of the length of the orbit q, although their ratio could be extremely small, i.e., $D_1^{-C_1\Delta}$.

We will give a sketch of the proof and refer the detailed estimates to [2].

Proof. Set $a = \log A_1$. Condition (4) in Proposition 11 translates into

$$\frac{1}{q}\sum_{i=0}^{q-1}\lambda^s_i \leq -a, \quad \frac{1}{q}\sum_{i=0}^{q-1}\lambda^u_i \geq a.$$

Using condition (3) and Lemma 20 in place of the Pliss lemma, by setting $\theta_0 \in (0,1)$ to be an absolute constant sufficiently close to 1 and setting $C > 0$ to be sufficiently large depending only on X, we can show analogously to [2, Lemma 4.4] that there are more than $q/2$ points in $\left\{g^k(x)\,\middle|\,0 \le k \le q-1\right\}$ that are "good in the orbit of x." Here a point $g^n(x)$ is said to be *good in the orbit of x* if $n \in [1, q-1]$ satisfies the following conditions:

$$\frac{1}{k}\sum_{j=n}^{n+k-1}\overline{\lambda}_j^e > \left(1-\frac{1}{1000}\right)\theta_0^{-1}a,\ \forall\ 1 \le k \le q-n, \tag{A.5}$$

$$\frac{1}{k}\sum_{j=n-k}^{n-1}\overline{\lambda}_j^e > \left(1-\frac{1}{1000}\right)\theta_0^{-1}a,\ \forall\ 1 \le k \le n. \tag{A.6}$$

We can show in analogy to [2, Lemma 4.5] that $|\cot\angle(v_n^s, v_n^u)| \le A_1^{3\Delta}$ for all n such that $g^n(x)$ is good in the orbit. Let $L_0 = \lceil D_1^{-C_1\Delta}q \rceil$ and $J_0 := \left\lfloor \frac{q}{L_0} \right\rfloor$. Then the sequence $(x_k)_{k=0}^{L_0J_0-1}$ is the union of L_0 subsequences $(x_{i+jL_0})_{j=0}^{J_0-1}$ where $i = 0, \ldots, L_0-1$. Then there exists an integer $0 \le i < L_0$ such that the subsequence $(x_{i+jL_0})_{j=0}^{J_0-1}$ contains at least $\frac{1}{3}D_1^{C_1\Delta}$ many points which are good in the orbit of x. By letting C_1 to be sufficiently large depending only on X, we can apply the pigeonhole principle to the above subsequence as in the proof of [2, Proposition 4.1] and obtain $0 \le i_1 < i_2 \le q-1$ that satisfy the following conditions:

(1) $i_2 - i_1 \ge D_1^{-\Delta C_1}q$,
(2) $\sum_{j=i_1}^{i_1+k-1}\overline{\lambda}_j^e > (1-\frac{1}{1000})\theta_0^{-1}ak,\ \forall\ 1 \le k \le i_2 - i_1$,
(3) $\sum_{j=i_2-k}^{i_2-1}\overline{\lambda}_j^e > (1-\frac{1}{1000})\theta_0^{-1}ak,\ \forall\ 1 \le k \le i_2 - i_1$,
(4) The angles $\angle(v_{i_1}^s, v_{i_1}^u), \angle(v_{i_2}^s, v_{i_2}^u)$ satisfy
$$\log|\cot\angle(v_{i_1}^s, v_{i_2}^u)| \le 3\Delta a, \quad \log|\cot\angle(v_{i_2}^s, v_{i_2}^u)| \le 3\Delta a,$$
(5) Moreover, we have $d(g^{i_1}(x), g^{i_2}(x)) < D_1^{-\frac{C_1\Delta}{200}}$, and
$$d_{T^1X}(v_{i_1}^s, v_{i_2}^s) < D_1^{-\frac{C_1\Delta}{200}}, \quad d_{T^1X}(v_{i_1}^u, v_{i_2}^u) < D_1^{-\frac{C_1\Delta}{200}}.$$

We note the similarities between the above conditions and those of [2, Definition 4.3]. However here we have a large inverse power of D_1 in (5) instead of a small inverse power of q as in [2, Definition 4.3(4)]. This is sufficient for the rest of proof, since r_{i_1}, r_{i_2}, $\angle(v_{i_1}^s, v_{i_1}^u)$ and $\angle(v_{i_2}^s, v_{i_2}^u)$ are lower bounded by $D^{-O(\Delta)}$.

At this point, we can invoke the proof of Proposition 4.2, and obtain (2) as a consequence of [2, Lemmas 4.6, 4.7, 4.8]; and obtain (3) and (4) as a consequence of [2, Proposition 4.5]. We obtain (5) following the proof of [2, Proposition 4.4]. □

Now we are ready to conclude the proof of Proposition 11.

Proof of Proposition 11. We apply Proposition 22 and obtain i_1, i_2, $\mathcal{R}_1$, $\mathcal{R}_2$, G, U_i, C_i, $\tilde{C}_i$ as in the proposition. We set $i = i_1$, $j = i_2$. By (5) in Proposition 22 and [2, Proposition 4.3], we obtain a hyperbolic periodic point in $\mathcal{R}_1 \cap \mathcal{R}_2$, denoted by y.

We note the following lemma whose proof follows from the standard construction of unstable / stable manifolds for uniformly hyperbolic maps using graph transform argument. For this reason, we omit its proof.

LEMMA 24. *Let $r,\tau > 0$, $L > 1$, $0 < \kappa,\kappa',\kappa'' < 1$, $U = U(r,\tau,\kappa)$, and let $G : \mathcal{R}_1 \to \mathcal{R}_2$ be a hyperbolic map, where $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) is the κ-vertical strip (resp. κ-horizontal strip) of U as in Definition 21, and κ',κ'' satisfy inclusion (A.3), (A.4) respectively. Assume that*

1. *For each $x \in \mathcal{R}_1$, each $(V,W) \in C(\kappa')$, set $(\bar{V},\bar{W}) = DG_x(V,W)$, then $|\bar{V}| \geq L|V|$,*
2. *For each $x \in \mathcal{R}_2$, each $(V,W) \in \tilde{C}(\kappa'')$, set $(\bar{V},\bar{W}) = DG_x^{-1}(V,W)$, then $|\bar{W}| \geq L|W|$.*

Then there exists a hyperbolic fixed point of G, $y \in \mathcal{R}_1 \cap \mathcal{R}_2$, whose local unstable manifold in $\mathcal{R}_2$, denoted by $\mathcal{W}_G^u(y)$, is a κ'-horizontal graph, and whose local stable manifold in $\mathcal{R}_1$, denoted by $\mathcal{W}_G^s(y)$, is a κ''-vertical graph. We also have

$$G(\mathcal{R}_1) \subset B(\mathcal{W}_G^u(y), 2L^{-1}diam(U)).$$

We set $L = A^{\frac{j-i}{2}}$. We now verify conditions (1),(2) of Lemma 24 for L, G, $U = U_{i_1}$, $\kappa = 100\bar{\kappa}, \kappa' = \bar{\kappa}, \kappa'' = 100\bar{\kappa}$. We only verify condition (2) in details since condition (1) can be verified in a similar fashion. By Proposition 22(5), for any $i \leq n \leq j-1$, we have $g_{n+1}^{-1}\cdots g_{j-1}^{-1}J^{-1}(\mathcal{R}_2) = g_n\cdots g_i(\mathcal{R}_1) \subset U_{n+1}\cap g_n(U_n)$. For any $i \leq n \leq j$, $(v,w) \in \mathcal{R}_2$, $(V,W) \in \tilde{C}_j$ (here $\tilde{C}_j$ is given by Proposition 22(3)), denote by $(v_n,w_n) = g_n^{-1}\cdots g_{j-1}^{-1}J^{-1}(v,w)$, $(V_n,W_n) = D(Jg_{j-1}\cdots g_n)^{-1}_{(v,w)}(V,W)$. Then we have $(v_n,w_n) \in U_n$ for all $i \leq n \leq j$. By Proposition 22(2,4), we have $|W_i| \geq e^{\sum_{n=i}^{j-1}(-\lambda_n^s-\delta)}|W_j| \geq A^{\frac{j-i}{2}}|W| = L|W|$.

By Lemma 24 and Proposition 22(2), we obtain

$$G(\mathcal{R}_1) \subset B(\mathcal{W}_G^u(y), 200A^{-\frac{j-i}{2}}\bar{r})$$

We denote by $z = \exp_{x_{i_1}} i_{i_1}(y)$. By Proposition 22(5) and the fact that y is a hyperbolic fixed point of G, we conclude that z is a g-hyperbolic periodic point. Then by Proposition 22 for sufficiently large C_1, we can ensure that $z \in \mathcal{H}(g, D_1^{-M\Delta})$, and

$$g^j(x) \in \exp_{x_{i_1}} i_{i_1} G(\mathcal{R}_1) \subset B(\mathcal{W}^u_{D_1^{-3M\Delta}}(z), A_1^{-\frac{q}{2D_1^{C_1\Delta}}})$$

We conclude the proof by letting $\bar{C} = M$, and C to be sufficiently large depending only on X. □

APPENDIX B.

Proof of Corollary 14. In the following, we briefly denote $\mathcal{H}(f,\alpha,r)$ by $\mathcal{H}(\alpha,r)$, and denote $\mathcal{H}(f,\lambda)$ by $\mathcal{H}(\lambda)$.

We first prove the corollary under condition (1). For any $\alpha,r,\xi > 0$, any $y \in \mathcal{H}(\alpha,r)$,

$$|B(\mathcal{W}_r^u(y),\xi)| \ll r\xi.$$

It is clear from the definition of $\mathcal{G}$ in (3.1) that for any $\lambda \in (0,1)$,

$$\begin{aligned}\#\mathcal{H}(\lambda) &\geq |\mathcal{G}(\lambda,\xi,f)|/|B(\mathcal{W}^u_{\lambda^3}(y),\xi)| \\ &\gg \lambda^{-3}\xi^{-1}|\mathcal{G}(\lambda,\xi,f)|\end{aligned}$$

By (4.6), (4.7) and Proposition 9, it suffices to check that

$$|\mathcal{G}_i| \gg \lambda_i^{-11}\xi_i = A^{-\frac{q_{i+1}\theta_0^{i+1}}{2D^{2C\Delta q_i}}} D^{22\bar{C}\Delta q_i}.$$

Since $\eta \in (0,1)$, $A \gg 1$ and $1 > xA^{-x}$ for $x \in (0,\infty)$, we have

$$\begin{aligned}|\mathcal{G}_i| &\geq \frac{\eta^{K-i}\varepsilon}{(K+1)l_i} \\ &> \frac{\eta^{K-i}\varepsilon}{(K+1)l_i}\frac{q_{i+1}\theta_0^{i+1}}{4D^{2C\Delta q_i}}A^{-\frac{q_{i+1}\theta_0^{i+1}}{4D^{2C\Delta q_i}}} \\ &> (10Kl_i)^{-1}\eta^K\varepsilon\frac{q_{i+1}\theta_0^{i+1}}{D^{2C\Delta q_i}}A^{-\frac{q_{i+1}\theta_0^{i+1}}{4D^{2C\Delta q_i}}} \\ &\gg A^{-\frac{q_{i+1}\theta_0^{i+1}}{2D^{2C\Delta q_i}}} D^{22C\Delta q_i}.\end{aligned}$$

The last inequality follows from by letting $C' \gg 1$, and

- $K^{-1}\eta^K\varepsilon q_{i+1}\theta_0^{i+1}l_i^{-1} \gg 1$, since $q_{i+1}l_i^{-1}\theta_0^{i+1} \geq \frac{1}{2}q_i\theta_0^i \geq \frac{1}{2}q_0$, and $q_0 \geq_{(4.4)} \varepsilon^{-1}e^{C'(\log(\frac{\log A}{h}))^2+C'} \gg_{(4.2)} \varepsilon^{-1}K\eta^{-K}$,
- $A^{\frac{q_{i+1}\theta_0^{i+1}}{4D^{2C\Delta q_i}}} \geq D^{24C\Delta q_i}$. Indeed by $A \gg 1$, we have

$$A^{\frac{q_{i+1}\theta_0^{i+1}}{4D^{2C\Delta q_i}}} \geq \frac{q_{i+1}\theta_0^{i+1}}{4D^{2C\Delta q_i}} \geq_{(4.2),\text{Lemma } 12(1)} \frac{q_{i+1}}{4\Delta D^{2C\Delta q_i}}$$

  and

$$\frac{q_{i+1}}{4\Delta D^{2C\Delta q_i}} \geq_{(4.3),(4.5),C'\gg C} D^{24C\Delta q_i}.$$

This concludes the proof since by our choice $C > \bar{C}$.

Now we consider condition (2). We set $\delta_0 = D^{-\mathrm{Tower}(Q_0,Q_1,K+3)}$.
By Lemma 12(3), we have

$$\delta_0 \leq \xi_0 \text{ and } \delta_0 \leq \lambda_i^3,\ \forall\ 0 \leq i \leq K.$$

For any $\lambda,\xi \in (0,1)$, any $y \in \mathcal{H}(\lambda)$, any $\delta \in (0,\lambda^3)$, we have

$$|B(B(\mathcal{W}^u_{\lambda^3}(y),\xi),\delta)| \ll \lambda^3\max(\xi,\delta).$$

By (3.1) and condition (2), we have for some $0 \leq i \leq K$ that,

$$\begin{aligned}\#\mathcal{H}(\lambda_i) &\geq |B(\mathcal{G}(\lambda_i,\xi_i,f),\delta_0)| \Big/ \sup_{y\in H(\lambda_i)} |B(B(\mathcal{W}^u_{\lambda_i^3}(y),\xi_i),\delta_0)| \\ &\gg \varepsilon\lambda_i^{-3}\min(\xi_i^{-1},\delta_0^{-1}).\end{aligned}$$

By Proposition 9, it suffices to observe from Lemma 12 that

$$\varepsilon \gg \lambda_i^{-11}\max(\xi_i,\delta_0), \quad \forall\ 0 \leq i \leq K. \qquad \square$$

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