# Herman's Last Geometric Theorem

B. Fayad and R. Krikorian

**Abstract**

We present a proof of Herman's Last Geometric Theorem asserting that if $F$ is a smooth diffeomorphism of the annulus having the intersection property, then any given $F$-invariant smooth curve on which the rotation number of $F$ is Diophantine is accumulated by a positive measure set of smooth invariant curves on which $F$ is smoothly conjugated to rotation maps. This implies in particular that a Diophantine elliptic fixed point of an area preserving diffeomorphism of the plane is stable. The remarkable feature of this theorem is that it does not require any twist assumption.

## Le dernier théorème géométrique d'Herman

**Résumé**

Nous présentons une preuve du dernier théorème géométrique d'Herman qui affirme que si un difféomorphisme $F$ de l'anneau possède la propriété d'intersection, alors toute courbe $C^\infty$, $F$-invariante sur laquelle le nombre de rotation de $F$ est diophantien est accumulée par un ensemble de mesure positive de courbes invariantes $C^\infty$ sur lesquelles $F$ est $C^\infty$-conjuguée à une rotation. Ceci implique en particulier la stabilité des points fixes elliptiques diophantiens des difféomorphismes du plan qui préservent l'aire. Le caractère remarquable de ce théorème est qu'il ne requiert aucune condition diophantienne.

# 1 Introduction

In his 1998 ICM address [7], M. Herman asked the following question: " *Let $f$ be a $C^\infty$- diffeomorphism preserving the Lebesgue measure of $\mathbb{T}^1 \times [-1, 1]$, homotopic to the identity, that has a finite number of periodic points (...) and is such that the rotation number $\rho(f_{|\mathbb{T}^1 \times [-1,1]}) = \alpha$ satisfies a diophantine condition. Is $f$ $C^\infty$-conjugated to $R_\alpha(\theta, r) = (\theta + \alpha, r)$?*

*I would expect a counter-example, but there is some evidence in the opposite direction.*

*We will show elsewhere this is the case if $f$ is $C^\infty$-close to $R_\alpha$ and* [in this case] *$f$ is always $C^\infty$-conjugated to $R_\alpha$ near $\mathbb{T}^1 \times \{\pm 1\}$."*

By "Herman's Last Geometric Theorem"[1], we shall refer to the latter local rigidity result (see Corollary 1 for an exact statement), together with Herman's discovery that an invariant diophantine circle of an area preserving planar diffeomorphism is always accumulated by a positive measure set of invariant circles (see exact statement in Theorem 1).

It is possible to trace back Herman's first statement of the theorem no later than 1995 in his "Séminaire de Systèmes Dynamiques" at the Université Paris VII, and later on in the same seminar at various occasions. To our knowledge, Herman never wrote a complete proof of the theorem and the only available material was a set of notes (given to the participants of the aforementioned seminar) where he explained the strategy of the proof. It is based on this strategy that we give here a complete proof of "Herman's last geometric Theorem" . Of course, the content of the paper is under the responsibility of the authors.

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## 1.1 Stability and Ergodicity

Probably the best way to introduce Herman's last geometric theorem is in its relation to the stability question of elliptic fixed points. Indeed, the study of the (Lyapunov) stability of fixed points is a fundamental problem in the theory of dynamical systems and its applications.

In the case of an area-preserving plane diffeomorphism $f$, the fixed points are classified accroding to the eigenvalues of the Jacobian *df* at these fixed points in the following way. If the eigenvalues of *df* are distinct, then the fixed point is said to be *hyperbolic* if they are real, and the point is said to be *elliptic* if they lie on the unit circle. In the exceptional case of two equal eigenvalues $\pm 1$, the point is called *parabolic*.

While it has been known since very long that hyperbolic fixed points are unstable, the question of stability of elliptic fixed points remained essentially unsolved until the discovery of KAM theory (named after Kolmogorov, Arnold and Moser).

Prior to that, Birkhoff had introduced an important tool for the study of stability, the so called *normal forms*. They give a simple description, up

[1] This denomination was suggested to us by A. Katok

to canonical change of coordinates, of the map near an elliptic fixed point, in the spirit of Taylor series for real functions. For a smooth map $F$ fixing the origin, a normal form expression of order $N$ is given in polar coordinates $(\theta, r)$ by

$$(\theta, r) \mapsto \left(\theta + \sum_{i=0}^{N-1} a_i r^i + \varphi_1(\theta, r), r + \varphi_2(\theta, r)\right)$$

where $\varphi_1$ and $\varphi_2$ vanish with their derivatives up to order $N-1$ at $r = 0$.

Birkhoff proved that if a $C^\infty$ map $F$ has an *irrational* elliptic fixed point, i.e. with eigenvalues that are not roots of unity, then it admits, after canonical coordinate changes, normal forms at any order. He further showed that there exists a formal power series that conjugates $F$ to a *complete* normal form $(\theta + \sum_{i=0}^{\infty} a_i r^i, r)$ . Clearly, a map that is *exactly* a normal form $(\theta + \sum_{i=0}^{\infty} a_i r^i, r)$ is completely integrable and thus stable at the origin. Not surprisingly, complete integrability turns out to be too much to ask (it was known to Poincaré that resonant tori usually break up under small perturbations of a completely integrable system) [2] and it was shown by Siegel that the formal power series that conjugate $F$ to a complete normal form are in general divergent.

Nevertheless, Birkhoff normal forms proved to be very useful in the result of stability discovered by Moser [10] in line with Kolmogorov's seminal approach asserting the persistence of a positive measure set of invariant circles when a completely integrable system is perturbed, provided a non-degeneracy condition is imposed on the initial system (here the Birkhoff normal form). One invariant circle being sufficient for Lyapunov stability, it indeed follows from usual KAM theory that if the series $a_i$ contains nonzero terms (torsion) then an irrational elliptic fixed point is stable. Actually, Moser proved the stability of an elliptic fixed points in finite regularity ($C^4$), provided that the eigenvalues merely avoid the six roots of unity of order $1, 2, 3, 4$, and that $a_1 \neq 0$ in the Birkhoff normal form of order 2. The latter is of course a generic transversality condition.

On the other hand, Anosov and Katok constructed in [1] smooth area preserving diffeomorphisms of the unit disc in $\mathbb{R}^2$, with an irrational elliptic fixed point at the origin, that are ergodic. These examples showed that the existence of torsion was necessary in establishing stability in the KAM setting, at least when no arithmetical conditions, besides avoiding the first six roots of unity or even having irrational arguments, are imposed on the eigenvalues.

[2] Note however that, in the holomorphic case, complete integrability is equivalent to stability; see Siegel's theorem in the next section.

In fact, besides being infinitely tangent to the rotation at the origin, the Anosov-Katok examples were obtained only for a family of rotation numbers (arguments of the eigenvalues) at the origin that contained a dense $G_\delta$-set of the circle but that avoided all Diophantine numbers.

While the strength of Moser's result lies in the fact that stability is insured by the finite number of conditions stated above, its non-zero torsion condition involves the behavior of the map in the neighborhood of the fixed point. A tantalizing question naturally arose, to decide whether as it is the case with instability for hyperbolic fixed points, a sole information on the Jacobian at a fixed elliptic point could be enough to insure stability.

This is precisely what was established in the real analytic category by H. Rüssmann who proved in [11] the following dichotomy, that implies stability, if the rotation number of the fixed elliptic point satisfies a Brjuno condition: *either the Birkhoff normal form has some non zero term, in which case Moser's Theorem applies, or the Birkhoff form completely vanishes and the map is analytically linearizable in the neighborhood of the fixed point.*

This dichotomy clearly fails in the smooth category, as is shown by the following example (in cylindrical coordinates) : $(r, \theta) \mapsto (r, \theta + \alpha + e^{-1/r})$.

Thus, the question of whether an elliptic fixed point with a Diophantine rotation number (satisfying no *a priori* twist condition) is always stable remained unsolved for smooth maps until Herman gave it an affirmative answer as a corollary of his last geometric theorem

**Theorem 1.** *Let $F$ be a smooth diffeomorphism of the annulus having the intersection property. Then given a smooth curve $\Gamma$ invariant by $F$ on which the rotation number $\alpha$ of $F$ is Diophantine, it holds that $\Gamma$ is accumulated by a positive measure set of smooth invariant curves on which $F$ is smoothly conjugated to rotation maps.*

The result stems actually from the following alternative: either there is an open neighborhood of $\Gamma$ on which $F$ is conjugated to a rigid rotation of the annulus of rotation number equal to that of $F$ on $\Gamma$ or $\Gamma$ is accumulated by smooth invariant curves on which $F$ is smoothly conjugated to rotation maps with frequencies covering a positive measure set inside a Diophantine class obtained by slightly relaxing the Diophantine condition on $\alpha$.

Besides the elegance and conciseness of the result, its importance lies in the fact that in many of the physical situations where quasi-periodic stability is involved, the non degeneracy of torsion is either hard to prove or at least untrue at the first orders.

The technique used to prove the theorem is based on a general approach to KAM theory where useful dynamical informations are obtained from Whitney dependent normal forms (which are derived from a systematic use of

Hamilton's Implicit function theorem in judicious Fréchet spaces). This approach proved to be very helpful in dealing with delicate KAM problems such as, for example, Herman's rigorous approach to a proof of Arnol'd's results on the stability of the solar system (a proof of which was nicely written by Jacques Féjoz [5]).

Before stating more precisely the main results of this paper, let us mention that the ergodic examples of Anosov and Katok on the unit disc were extended in [4] to cover all Liouville rotation numbers at the origin (and the boundary) which gives, together with Herman's last geometric theorem, an additional example of the complete dichotomy between Diophantine stable and Liouville unstable paradigms.

**Theorem 2** ([4])**.** *For any given Liouville number $\alpha$, there exists a smooth area-preserving diffeomorphism of the unit disk, preserving the boundary and having rotation number $\alpha$ on the boundary, which is weakly mixing with respect to Lebesgue measure.*

In fact, the method of the proof of Theorem 1 shows that given a Diophantine class to which $\alpha$ belongs, there exists a class of differentiability of the map $F$ that insures the validity of Theorem 1, with however invariant curves that will have less regularity then the map $F$ itself. On the other hand, given a Diophantine class, it is also possible to construct by quantitative Anosov-Katok methods, as the one used in [4], weakly mixing examples as in Theorem 2 but with finite regularity.

## 1.2 Results

We now pass to a more detailed description and precise statement of Herman's results.

We denote the circle by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We denote by $\mathrm{Diff}^r_+(\mathbb{T})$, $r \in \mathbb{N} \cup \{\infty\}$ the group of orientation preserving diffeomorphisms of the circle of class $C^r$. We represent the lifts of these diffeomorphisms as elements of $D^r(\mathbb{T})$, the group of $C^r$-diffeomorphisms $\tilde{f}$ of the real line such that $\tilde{f} - \mathrm{Id}_{\mathbb{R}}$ is $\mathbb{Z}$-periodic.

Following Poincaré, one can define the rotation number of a circle homeomorphism $f$ as the uniform limit $\rho(f) = \lim_{|j|\to\infty}(\tilde{f}^j(x) - x)/j \mathrm{mod}[1]$, where $\tilde{f}^j$ ($j \in \mathbb{Z}$) denotes the $j$-th iterate of a lift $\tilde{f}$ to $\mathbb{R}$ of $f$. A rotation map of the circle with angle $\alpha$, that we denote by $R_\alpha : x \mapsto x + \alpha$, has clearly a rotation number equal to $\alpha$.

Denote the infinite annulus by $\mathbb{A} = \mathbb{T} \times \mathbb{R}$. We shall use coordinates $(\theta, r)$ on $\mathbb{A}$. We denote by $\mathrm{Diff}^\infty_0(\mathbb{A})$ the set of diffeomorphisms of the annulus that are homotopic to the identity (see Section 2). Denote by $C^\infty(\mathbb{R})$ the set of smooth real maps $f : \mathbb{R} \to \mathbb{R}$ and by $C^\infty(\mathbb{T}, \mathbb{R})$ the set of smooth real maps $f \in C^\infty(\mathbb{R})$ that are 1-periodic.

We denote by $\Gamma_0$ the circle $\mathbb{T} \times \{0\}$ in $\mathbb{A}$. More generally, we shall call *circle* in $\mathbb{A}$ any closed curve $\Gamma = \{(\theta, \gamma(\theta))\}_{\theta\in\mathbb{T}}$, where $\gamma$ belongs to $C^\infty(\mathbb{T}, \mathbb{R})$. For $c \in \mathbb{R}$, we denote by $\mathcal{G}^c$ the set of circles $\Gamma = \{(\theta, \gamma(\theta)),\ \theta \in \mathbb{T}\}$ such that $\int_{\mathbb{T}} \gamma(\theta)d\theta = c$.

We say that a diffeomorphism $F \in \mathrm{Diff}^\infty_0(\mathbb{A})$ has the *intersection property* if for any non homotopically trivial continuous curve $\Gamma \subset \mathbb{A}$, $F(\Gamma) \cap \Gamma \neq \emptyset$. A circle is said to be *$F$-invariant* if $F(\Gamma) = \Gamma$, that is, if there exists $f \in \mathrm{Diff}^\infty_+(\mathbb{T})$ such that $F(\theta, \gamma(\theta)) = (f(\theta), \gamma(f(\theta)))$. The restriction of $F$ on $\Gamma$ is then said to be smoothly *conjugate to a rotation $R_\beta$* on $\Gamma$ if there exists $h \in \mathrm{Diff}^\infty_+(\mathbb{T})$ such that $f = h \circ R_\beta \circ h^{-1}$. In this case, we will say that $F$ is *linearizable* on $\Gamma$.

We denote by $S_\alpha$ the rotation of angle $\alpha$ on the annulus, that is the map

$$S_\alpha : \begin{array}{ccc} \mathbb{A} & \to & \mathbb{A} \\ (\theta, r) & \mapsto & (\theta + \alpha, r). \end{array}$$

Finally, for a pair of constants $(\sigma, \tau)$ such that $\sigma > 0$ and $\tau > 1$, we denote by $\mathrm{DC}(\sigma, \tau)$ the set of real numbers $\alpha$ satisfying the *Diophantine condition*:

$$\forall (k, l) \in \mathbb{N}^* \times \mathbb{Z}, |k.\alpha - l| \geqslant \frac{1}{\sigma\, |k|^\tau}.$$

If $\tau > 1$ and $\sigma$ is big enough then $\mathrm{DC}(\sigma, \tau)$ has positive Lebesgue measure. The set $\mathrm{DC}(\tau) := \bigcup_{\sigma>0} \mathrm{DC}(\sigma, \tau)$ is by definition the set of diophantine number of *exponent* $\tau$ and is a set of full Lebesgue measure provided $\tau > 1$. Without any further specification, a diophantine number is a point in $\bigcup_{\tau>0} \mathrm{DC}(\tau)$.

**Theorem 3.** *Assume that $F \in \mathrm{Diff}^\infty_0(\mathbb{A})$ satisfies the following assumptions:*

- *$\Gamma_0$ is $F$-invariant and the rotation number $\alpha = \rho\left(F_{|\Gamma_0}\right)$ of the circle diffeomorphism induced by $F$ on $\Gamma_0$ is Diophantine;*
- *$F$ has the intersection property.*

*Then, $\Gamma_0$ is accumulated by $F$-invariant circles on which $F$ is linearizable.*

*More precisely, given any pair of constants $(\sigma, \tau)$ such that $\sigma > 0$ and $\tau > 1$, we will obtain $\varepsilon > 0$ and a $C^1$ map $\beta : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that whenever $\beta(c) \in \mathrm{DC}(\sigma, \tau)$, there exists an $F$-invariant circle $\Gamma(c) \in \mathcal{G}^c$ on which the restriction of the diffeomorphism $F$ is $C^\infty$-conjugate to the rotation $R_{\beta(c)}$.*

*Also, if we consider $\sigma' < \sigma$ and $\tau' > \tau + 1$, there exists $\varepsilon_1$ for which the following alternative holds:*

1. *either the application $\beta$ is locally constant at $0 \in (-\varepsilon_1, \varepsilon_1)$ , in which case there exists an $F$-invariant neighborhood $\mathcal{O}$ of the circle $\Gamma_0$ in $\mathbb{A}$ such that the diffeomorphism $F$ restricted to $\mathcal{O}$ is $C^\infty$-conjugate to the rotation $S_\alpha$ on the annulus $\mathbb{A}$.*

2. *or the application $\beta$ is not constant at 0 in which case for any $0 < \varepsilon' < \varepsilon_1$, we have $\mathrm{Leb}_1\left(\beta(-\varepsilon', \varepsilon') \cap \mathrm{DC}(\sigma', \tau')\right) > 0$, that is, the frequencies on the invariant circles accumulating $\Gamma_0$ cover a set of positive Lebesgue measure. Moreover, denoting by $\mathcal{G}_F(\varepsilon_1)$ the set of $F$-invariant circles contained in $\mathbb{T} \times (-\varepsilon_1, \varepsilon_1)$, we have $\mathrm{Leb}_2(\mathcal{G}_F(\varepsilon_1)) > 0$.*

**Remark 1.** A diffeomorphism $F \in \mathrm{Diff}_0^\infty(\mathbb{A})$ preserving the area and fixing some circle $\Gamma$ has the intersection property, hence the consequences of the theorem hold for an area preserving diffeomorphism of the annulus. On the other hand, if it is not assumed that the diffeomorphism $F$ has the intersection property the proof of Theorem 3 would provide *translated* curves ($F(\Gamma) = \Gamma + \mu$, $\mu \in \mathbb{R}$) instead of invariant ones.

**Remark 2.** The following alternative also holds (”locally constant” in the last Theorem being replaced by ”constant”)

1. *either the application $\beta$ is constant on $(-\varepsilon_1, \varepsilon_1)$, in which case there exists an invariant neighborhood $\mathcal{O}$ of the circle $\Gamma_0$ in $\mathbb{A}$ foliated by the circles $\Gamma(c)$, $c \in (-\varepsilon_1, \varepsilon_1)$ such that the diffeomorphism $F$ restricted to $\mathcal{O}$ is $C^\infty$-conjugate to the rotation $S_\alpha$ on the annulus $\mathbb{A}$.*

2. *or the application $\beta$ is not constant on $(-\varepsilon_1, \varepsilon_1)$ in which case*

$$\mathrm{Leb}_1\left(\beta(-\varepsilon_1, \varepsilon_1) \cap \mathrm{DC}(\sigma', \tau')\right) > 0 \text{ and } \mathrm{Leb}_2(\mathcal{G}_F(\varepsilon_1)) > 0$$

**Remark 3.** If the diffeomorphism $F$ is only defined on $\mathbb{A}_+ = \mathbb{T} \times [0, \infty)$, the results of the theorem remain true with $\beta$ defined on $[0, \varepsilon)$ instead of $(-\varepsilon, \varepsilon)$. The reason is that $F$ can be extended to a smooth diffeomorphism of $\mathbb{A}$. It is not necessary however to require the intersection property for the extended map beyond $\mathbb{A}_+$, since this property is only used in the proof of Theorem 3 to insure that a translated curve is actually invariant. We refer the reader to the Appendix for further details.

**Remark 4.** There exists an integer $k$ (resp. $k_1$) depending only on $\sigma, \tau$ for which the constants $\varepsilon$ (resp. $\varepsilon_1$) in the preceding theorem can be chosen uniformly in $F$ as long as[3] $\|F - S_\alpha\|_k \leqslant 1$ (resp. $\|F - S_\alpha\|_{k_1} \leqslant 1$).

[3]If $U$ is an open set of $\mathbb{R}^d$, $f : U \to \mathbb{R}$ is a smooth map, we define the $C^k$-norm $\|f\|_{C^k(U)}$ of $f$ (or for short $\|f\|_k$) by $\|f\|_k := \max_{|j| \leqslant k} \sup_{x \in \mathbb{R}^d} |\partial^j f(x)|$, where we use Whitney's notations : if $j = (j_1, \dots, j_d) \in \mathbb{N}^d$ we define $|j| = j_1 + \cdots + j_d$ and $\partial^j f = \partial_1^{j_i} \cdots \partial_d^{j_d} f$

A consequence of the alternative described in Theorem 3 is the following *local rigidity* result for diffeomorphisms of the closed annulus $\mathbb{T} \times [0,1]$ that are free of periodic points[4] and that satisfy a Diophantine condition on the boundary. We denote by $\mathrm{Diff}^\infty_0(\mathbb{T} \times [0,1])$ the subset of $\mathrm{Diff}^\infty_0(\mathbb{A})$ of diffeomorphisms fixing the circles $\Gamma_0 = \mathbb{T} \times \{0\}$and $\Gamma_1 = \mathbb{T} \times \{1\}$.

**Corollary 1.** *For any pair of positive constants $(\sigma, \tau)$ such that $\tau > 1$, there exist $\eta > 0$ and $s \in \mathbb{N}$ such that given any $F \in \mathrm{Diff}^\infty_0(\mathbb{T} \times [0,1])$ satisfying the following conditions:*

- $\rho\left(F_{|\Gamma_0}\right) \in \mathrm{DC}(\sigma, \tau)$,
- *$F$ has the intersection property,*
- *$F$ has only finitely many periodic point in $\mathbb{T} \times (0,1)$,*
- $\|F - S_\alpha\|_{C^s(\mathbb{T}\times[0,1])} < \eta$

*is $C^\infty$-conjugate to $S_\alpha$ on $\mathbb{T} \times [0,1]$.*

M. Herman asked whether the rigidity result of Corollary 1 remains true in a global setting, i.e. without the assumption that $F$ is close to $S_\alpha$.

**Question 1.** *Can one find a $C^\infty$-diffeomorphism $F$ on $\mathbb{T} \times [0,1]$ with the intersection property, having a Diophantine rotation number $\alpha$ on one of the boundary circles and no periodic point in $\mathbb{T} \times [0,1]$, that is not $C^\infty$-conjugate on $\mathbb{T} \times [0,1]$, to the map $S_\alpha : (\theta, r) \mapsto (\theta + \alpha, r)$?*

## 1.3 Examples of application: Elliptic fixed points and Siegel Theorem

Let $F : \mathbb{D} \to \mathbb{D}$, $F(x,y) = (f(x,y), g(x,y))$ be a smooth diffeomorphism of the disk such that $F(0) = 0$. We say that 0 is an irrational elliptic fixed point if $DF(0)$ (the derivative of $F$ at 0) has eigenvalues of the form $e^{\pm 2\pi i\alpha}$, $\alpha \in \mathbb{R} - \mathbb{Q}$. As is well known, one can reduce the study of such a diffeomorphism to that of a map of the annulus in the following way. First, one can assume (after conjugation) that $DF(0)$ is a rotation matrix of angle $\alpha$. If we introduce the diffeomorphism $H : \mathbb{T} \times (\mathbb{R} - \{0\}) \to \mathbb{C} - \{0\} \simeq \mathbb{R}^2 - \{(0,0)\}$ defined by $H(\theta, r) = re^{2\pi i\theta} \simeq (r\cos(2\pi\theta), r\sin(2\pi\theta))$ one has $F \circ H(\theta, r) = e^{2\pi i\alpha} re^{2\pi i\theta} U(\theta, r)$. The function

$$U(\theta, r) = \frac{f(r\cos(2\pi\theta), r\sin(2\pi\theta)) + ig(r\cos(2\pi\theta), r\sin(2\pi\theta)))}{re^{2\pi i(\theta+\alpha)}}$$

[4]or equivalently have only a finite number of periodic points

is clearly $C^\infty$ and equals 1 on $\mathbb{R}/\mathbb{Z}\times\{0\}$; the function $\log U(\theta,r)$ is then also smooth on a neighborhood of $\mathbb{R}/\mathbb{Z}\times\{0\}$, where log is a branch of logarithm such that $\log 1 = 0$. Consequently, there exist smooth functions $\Theta(\theta,r)$, $R(\theta,r)$ ($R(\theta,r)>0$) such that on a neighborhood of $\mathbb{R}/\mathbb{Z}\times\{0\}$

$$R(\theta,r)e^{2\pi i\Theta(\theta,r)} = U(\theta,r), \qquad F\circ H(\theta,r) = rR(\theta,r)e^{2\pi i(\theta+\alpha+\Theta(\theta,r))}.$$

This proves that the function $\bar{F} := H^{-1}\circ F\circ H$ can be extended as a smooth function which is clearly a diffeomorphism of $\mathbb{T}\times\mathbb{R}$.

Geometric properties of $F$ such as the intersection property translate to $\bar{F}$. A nice application of this fact in the holomorphic setting was given by M. Herman to provide a new proof of Siegel Theorem : *if $f(z) = e^{2\pi i\alpha}z + O(z^2)$ is a holomorphic germ and if $\alpha$ is diophantine, then it is linearizable at 0.* Indeed, if we denote by $\bar{f}:\mathbb{A}_\delta^+\to\mathbb{C}$ the smooth map of the annulus provided by the previous construction we see that $\bar{f}$ restricted to $\mathbb{T}\times\{0\}$ is the diophantine rotation by angle $\alpha$. This map has the intersection property because otherwise this would mean that $f$ sends a neighborhood of $0\in\mathbb{D}$ strictly inside itself; but, this is clearly impossible by Schwarz Lemma. Hence, there is some $f$-invariant circle around 0, which means the existence of an invariant domain around 0. Conformal representation and Scharwz Lemma give the conclusion.

Theorem 3 has also an immediate consequence for surface diffeomorphisms. We recall that a fixed point $p$ for a surface diffeomorphism $f$ is said to be elliptic if the Jacobian $Df(p)$ of $f$ at the point $p$ is an elliptic matrix. We then say that $p$ is Diophantine if $Df(p)$ is conjugate to a rotation matrix with a Diophantine angle $\alpha$. Diophantine elliptic periodic points are defined similarly.

**Theorem 4.** *Let $f$ be a surface diffeomorphism that has the intersection property. If $p$ is a Diophantine elliptic periodic point for $f$ with period $q$, then $p$ is accumulated by a positive measure set of $f^q$-invariant circles. In particular, an area preserving surface diffeomorphism with a Diophantine elliptic periodic point is not ergodic.*

# 2 Notations and Preliminaries

Define the set $\mathrm{Diff}^\infty(\mathbb{A})$ of smooth diffeomorphisms $F$ of the annulus as follows:

$$F:\begin{array}{ccc}\mathbb{A} & \to & \mathbb{A}\\ (\theta,r) & \mapsto & (f(\theta,r),g(\theta,r)),\end{array}$$

where $f$ and $g$ are maps in $C^\infty(\mathbb{R}^2,\mathbb{R})$ that are 1-periodic with respect to the first variable.

More generally, for any neighborhood $U$ of the circle $\Gamma_0 = \{(\theta_0, 0), \theta \in \mathbb{T}\}$ in the annulus $\mathbb{A}$, we define the set $\mathrm{Diff}^\infty(U, \mathbb{A})$ of maps $F$ such that:

$$F: \begin{array}{ccc} U & \to & \mathbb{A} \\ (\theta, r) & \mapsto & (f(\theta, r), g(\theta, r)), \end{array}$$

where $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are smooth maps that are 1-periodic with respect to $\theta$.

We denote

$$C_0^\infty(\mathbb{T}, \mathbb{R}) = \{v \in C^\infty(\mathbb{T}, \mathbb{R}) \ / \ \int_{\mathbb{T}} v(t)dt = 0\}.$$

For $\gamma_0 \in C_0^\infty(\mathbb{T}, \mathbb{R})$ we denote $\mathcal{B}(\gamma_0, \varepsilon)_s = \{\gamma \in C_0^\infty(\mathbb{T}, \mathbb{R}) \ / \ \|\gamma - \gamma_0\|_s < \varepsilon\}$.

We identify any circle $\Gamma = \{(\theta, \gamma(\theta))\}_{\theta \in \mathbb{T}}$ with the associated smooth application $\gamma \in C^\infty(\mathbb{T}, \mathbb{R})$.

## 2.1 Tame Fréchet spaces/maps

For this section, we refer the reader to [6], [2], [9].

A topological vector space $E$ is said to be *locally convex* if its topology derives from a family of seminorms $(\| \cdot \|_n)$ $(n \in \mathbb{N})$ (a seminorm satisfies all the properties of a norm except for "$\|x\| = 0$ implies $x = 0$"), that is if the family $U_{i,j} = \{(\|x\|_i < j^{-1}\}, (i, j) \in \mathbb{N} \times \mathbb{N}^*$, constitutes a basis of neighborhoods for the topology of $E$. The space $E$ is *Hausdorff* if $x \in E$ vanishes if and only if for all $n \in \mathbb{N}$, $(\|x\|_n) = 0$.

A *Fréchet space* is a locally convex topological vector space that is Hausdorff and complete for the metric given by $d(x, y) = \sum_{i \geqslant 0} 2^{-i} \|x - y\|_i$.

**Example.** The space $C_0^\infty(\mathbb{R}/\mathbb{Z})$ with the topology given by the $C^r$ seminorms $(\| \cdot \|_j)_{j \in \mathbb{N}}$ $(\|f\|_j = \sup_{x \in \mathbb{T}} |\partial^j f(x)|)$ is a Fréchet space. More trivially, every Banach space is a Fréchet space. The collection of norms reduces to a single one.

A *graded* family of semi-norms on a Fréchet space satisfies $\|x\|_{i+1} \geqslant \|x\|_i$ for every $x \in E$ and $i \in \mathbb{N}$. Any family of semi-nroms can be transformed into a graded family by simply summing up for every $i \in \mathbb{N}$ the first $i$ semi-norms.

**Definition 1.** A family of *smoothing operators* on a graded Fréchet space $(E, (\| \cdot \|_i)_{i \in \mathbb{N}})$ is a real 1-parameter family $(S_t)_{t \geqslant 1}$ of continuous linear applications from $E$ to itself, such that there exist an integer $r$ and real constants

$C_{j,k},\ (j,k) \in \mathbb{N}^2$ such that, for any vector $x \in E$, for any $t > 1$ and any $j \leqslant k$ both following inequalities hold:

$$\begin{cases} \|S_t x\|_k \leqslant C_{k,j} t^{k-j+r} \|x\|_j \\ \|(\mathrm{Id} - S_t)x\|_j \leqslant C_{j,k}.t^{j-k+r} \|x\|_k \end{cases}$$

A *tame* Fréchet space is a graded Fréchet space endowed with a smoothing operators family.

**Example.** It is not difficult to see, using Fourier series, that the space $C_0^\infty(\mathbb{T}, \mathbb{R})$ is a tame Fréchet space. A simple choice for the family $S_t$ is given by the truncation operators:

$$(S_t f)(x) = \sum_{k \in \mathbb{Z}, |k| \leqslant t} \hat{f}(k) e^{2\pi i k x}, \qquad \hat{f}(k) = \int_{\mathbb{T}} f(x) e^{-2\pi i k x} dx$$

for $f \in C_0^\infty(\mathbb{R}/\mathbb{Z})$ ($\hat{f}(k)$ is the $k$-th Fourier coefficient of $f$); with this choice one can choose $r = 2$. Using Fourier integrals instead of Fourier series, it is possible to prove the existence of a family of smoothing operators on $C_0^\infty(\mathbb{T}, \mathbb{R})$ for which $r = 0$ (this is useful when one wants to prove accurate Hadamard inequalities); see [8].

**Definition 2.** Let $E$ and $F$ be two Fréchet spaces and consider $\Phi : U \to F$ a continuous map from an open subset $U$ in $E$ to $F$. The map $\Phi$ is said to be *Gâteaux differentiable*, if there exists an application

$$\begin{array}{rccc} D\Phi : & U \times E & \to & F \\ & (x, \Delta x) & \mapsto & D\Phi(x) \cdot \Delta x, \end{array}$$

continuous in $(x, \Delta x)$ and linear in the second variable, such that for every $(x, \Delta x) \in U \times E$, the following limit exists and satisfies

$$\lim_{t \to 0} \frac{\Phi(x + t\Delta x) - \Phi(x)}{t} = D\Phi(x) \cdot \Delta x.$$

By induction, it is possible to define $C^k$ differentiability of $\Phi$ for $k \geqslant 2$. The map $\Phi$ is said to be of class $C^\infty$ if it is of class $C^k$ for every integer $k$.

**Definition 3.** An application $\Phi : U \subset E \to F$ is *tame* if for any point $x_0$ in $U$ there exists a neighborhood $V$ of $x_0$ in $U$, an integer $p \in \mathbb{N}$ and a sequence of strictly positive constants $\{c_j\}_{j \in \mathbb{N}}$ such that for any $x \in E$, for any $j \in \mathbb{N}$

$$\|\Phi(x)\|_j \leqslant c_j (1 + \|x\|_{j+p})$$

The application $\Phi$ is a $C^k$*-tame application* ($k \in \mathbb{N} \cup \{\infty\}$) if $\Phi$ is of class $C^k$ and if all its differentials of order $j \leqslant k$ are tame. We use the notation $\Phi \in C^k(U, F)$. The map $\Phi$ is a $C^k$*-tame diffeomorphism* if it is invertible and if $\Phi$ and its inverse $\Phi^{-1}$ are $C^k$-tame applications.

**Remark 5.** The integer $p$ that appears in the previous definition is called the *differentiability loss* of the application $\Phi$, with reference to the particular case where $E$ and $F$ are graded function spaces endowed with the $C^k$-topologies.

**Example.** If $\Phi$ is a linear map its tameness is equivalent to the existence of $r, p \in \mathbb{N}$ such that for any $j \in \mathbb{N}$

$$\|\Phi x\|_j \leqslant C_j(\|x\|_r + \|x\|_{j+p}).$$

**Proposition 1** ([6]). *Le $M$ be a compact smooth finite dimensional manifold, and let $E$ and $F$ be two real vector spaces of finite dimension. Then,*

*(i) the space $C^\infty(X, E)$ is a tame Fréchet space*

*(ii) the composition map*

$$\begin{array}{ccc} C^\infty(X,E) \times C^\infty(E,F) & \to & C^\infty(X,F) \\ (f,g) & \mapsto & g \circ f, \end{array}$$

*is well defined and is a $C^\infty$ tame map.*

*(iii) if $f \in C^\infty(E, E)$ is invertible, and if $U$ is a sufficiently small neighborhood of $f$, then*

$$\begin{array}{ccc} U & \to & C^\infty(E,E) \\ g & \mapsto & g^{-1}, \end{array}$$

*is a $C^\infty$ tame map, where $g^{-1}$ denotes the inverse map of $g$.*

## 2.2 Hamilton Inversion Theorem

**Theorem 5** ([6]). *Consider two tame Fréchet spaces $E$ and $F$, an open set $U$ in $E$, $f$ a tame $C^r$ ($r \geqslant 2$, $r \in \mathbb{N} \cup \{\infty\}$) map from $U$ to $F$, $x_0$ a point in $U$ and $y_0 = f(x_0)$. Suppose there exists an open neighborhood $V_0$ of $x_0$ in $U$ and a tame continuous map which is linear in the second variable : $J : V_0 \times F \to E$ and such that if $x \in V_0$ then $Df(x)$ is invertible and its inverse is $J$. Then, there exist open neighborhoods $V \subset V_0$ and $W \subset F$ of $x_0$ and $y_0$ respectively such that $f : V \to W$ is a tame $C^r$ diffeomorphism.*

A corollary of the preceding theorem is the implicit function theorem in tame Fréchet spaces.

**Corollary 2** (Implicit function theorem [6]). *Let $E, F$ be two tame Fréchet spaces, $U \subset E$, $V \subset F$ open sets such that $(x_0, y_0) \in U \times V$ and $G : U \times V \to F$ a tame map of class $C^r$ ($r \geqslant 2$) such that $G(x_0, y_0) = 0$. Assume that there exists a tame continuous map $J : U \times V \times F \to F$ linear in the third variable* *and such that for any $(x,y) \in U \times V$ the linear map $D_yG(x,y)$ is invertible and its inverse is $J(x,y)$. Then there exist a neighborhoods $U_0$ of $x_0 \in U$, $V_0$ of $y_0 \in V$ and a map $g : U_0 \to F$ of class $C^r$ such that for any $(x,y) \in U_0 \times V_0$ the equality $G(x,y) = 0$ holds if and only if $y = g(x)$.*

## 2.3 The Diophantine condition and the linearized equation.

The following elementary fact known as the triviality of linear cohomology above Diophantine rotations lies nevertheless at the heart of the proof of Theorem 3 and more generally underlies all the stability results related to the Diophantine paradigm.

**Proposition 2.** *Let $\sigma > 0$, $\tau > 0$ and $\alpha \in \mathrm{DC}(\sigma,\tau)$. Then, for any smooth map $f \in C^\infty(\mathbb{T},\mathbb{R})$ and any real number $a$ there exists a unique map $g \in C_0^\infty(\mathbb{T},\mathbb{R})$ such that:*

$$\forall \theta \in \mathbb{T}, \quad f(\theta) = \eta + g(\theta+\alpha) - ag(\theta)$$

*with $\eta = \int_{\mathbb{T}} f(\theta)d\theta$. Moreover, the application*

$$\begin{array}{rcl} \tilde{\mathcal{L}}_{\alpha,a} : C^\infty(\mathbb{T},\mathbb{R}) & \to & \mathbb{R} \times C_0^\infty(\mathbb{T},\mathbb{R}) \\ f & \mapsto & (\eta, g), \end{array}$$

*is a $C^\infty$-tame (linear) map. In the following we shall denote $g$ by $\mathcal{L}_{\alpha,a} f$.*

*Proof.* The proof is obtained by a simple Fourier series computation and is based on the fact that for any pair of constants $(\sigma,\tau)$ such that $\sigma > 0$ and $\tau > 0$, there exists a positive constant $C(\sigma)$ such that:

$$\forall \alpha \in \mathbb{R},\ \forall \alpha \in \mathrm{DC}(\sigma,\tau),\ \forall n \in \mathbb{Z}\ , \left| \frac{1}{e^{2i\pi n\alpha} - a} \right| \leqslant C\,|n|^\tau .$$

□

**Remark 6.** *Actually, the proof of proposition 2 gives that for any pair of constants $(\sigma,\tau)$ such that $\sigma > 0$ and $\tau > 0$, there exist $p(\tau) \in \mathbb{N}$ and $C(\sigma)$ such that for $\alpha \in \mathrm{DC}(\sigma,\tau)$, $k \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{T},\mathbb{R})$*

$$\|\mathcal{L}_{\alpha,a}(f)\|_k \leqslant C\|f\|_{k+p}$$

*($p = \tau + 1 + 1$).*

**Remark 7.** *If the application $f$ is strictly positive, then there exist a positive constant $\nu = \exp\left(\int_{\mathbb{T}} \ln f(\theta) d\theta\right)$ and a unique strictly positive $g \in C^\infty(\mathbb{T}, \mathbb{R})$ such that for every $\theta \in \mathbb{T}$,*

$$f(\theta) = \nu \frac{g(\theta + \alpha)}{g(\theta)}.$$

*Here also, the application that associates the map $g$ to the map $f$ is a $C^\infty$-tame application (with a uniform derivative loss as $\alpha$ satisifes a given Diophantine condition).*

In the proof of Theorem 3, we need to insure that $\alpha$ is well accumulated (from both sides and with positive measure) by numbers satisfying a single Diophantine condition. For this, it is sufficient to *relax* the Diophantine condition satisfied by $\alpha$, as shown by the following.

**Proposition 3.** *Let $\alpha \in \mathrm{DC}(\sigma, \tau)$. Then, for any $0 < \sigma' < \sigma$ and $\tau' > \tau + 1$, and for any $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ such that $\varphi(0) = \alpha$, the set $\{x \in [-\delta, \delta] \ / \ \varphi(x) \in \mathrm{DC}(\sigma', \tau')\}$ has a strictly positive Lebesgue measure for every $\delta > 0$.*

*Proof.* We just have to show that for any $\varepsilon > 0$, we have $\lambda[\mathrm{DC}(\sigma', \tau') \cap (\alpha - \varepsilon, \alpha)] > 0$, as well as $\lambda[\mathrm{DC}(\sigma', \tau') \cap (\alpha, \alpha + \varepsilon)] > 0$ (where $\lambda(\cdot)$ is the one-dimensional Lebesgue measure). The proof of the two inequalities being identical, we will only consider the latter. For $q \geqslant 1$, define

$$L_q = \{x \in (\alpha, \alpha + \varepsilon) \ : \ \exists p \in \mathbb{Z} \ / \ |x - \frac{p}{q}| < \frac{\sigma'}{q^{\tau'+1}}\}$$

We want to show that $\lambda[\cup_{q \geqslant 1} L_q] < \varepsilon$. We actually claim that for $\varepsilon > 0$ sufficiently small $\lambda[\cup_{q \geqslant 1} L_q] < \varepsilon - \varepsilon(\sigma - \sigma')/2$. Observe first that

$$\lambda(L_q) \leqslant \frac{2\sigma' \varepsilon}{q^{\tau'}} + \frac{2\sigma'}{q^{\tau'+1}}$$

The fact that $\tau' > \tau + 1$ directly implies that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{q \geqslant \varepsilon^{-\frac{1}{1+\tau}}} \lambda(L_q) = 0.$$

Hence, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ we have

$$\sum_{q \geqslant \varepsilon^{-\frac{1}{1+\tau}}} \lambda(L_q) < \frac{1}{2} \varepsilon (\sigma - \sigma').$$

On the other hand, for any $1 \leqslant q \leqslant \varepsilon^{-\frac{1}{1+\tau}}$, $x \in L_q$ and since $\alpha \in \mathrm{DC}(\sigma, \tau)$ we get

$$\begin{aligned} x - \alpha &> \frac{\sigma}{q^{\tau+1}} - \frac{\sigma'}{q^{\tau'+1}} \\ &> \frac{\sigma - \sigma'}{q^{\tau+1}} \\ &> \varepsilon(\sigma - \sigma') \end{aligned}$$

and the claim follows. □

## 2.4 Plan of the proof of the main result Theorem 3.

First, the application of the Herman-Yoccoz theorem on smooth linearizability of diffeomorphisms of the circle with diophantine number allows to reduce the problem to that of a diffeomorphism of the annulus $F$ fixing $\Gamma_0$ on which its restriction is a diophantine rotation $R_\alpha$. Using the Diophantine property of $\alpha$ and the intersection property of $F$ we can perform a change of coordinates that allows to write $F$, in the neighborhood of $\Gamma_0$, in a Birkhoff Normal Form given by $F(\theta, r) = \left(\theta + \alpha + \sum_{i=1}^{N-1} a_i r^i, r\right) + O\left(r^N\right)$. In this perturbative context, it is then possible to look for invariant circles using the Hamilton implicit function theorem. To insure the solvability of the linearized equations we have to introduce parameters as it is the case often in KAM theory. Namely, given a diophantine frequency $\beta$, then to any sufficiently close to 0 height $c$, it is possible to find a curve $\Gamma$ of which the average height on the annulus is $c$ and two parameters $\lambda$ and $\mu$ such that $F(\theta, c+\gamma(\theta)) = (\lambda + h \circ R_\beta \circ h^{-1}, \mu + c + \gamma(h \circ R_\beta \circ h^{-1}))$ where $h$ is a smooth conjugacy that depends on $c$ and $\beta$. This is the content of Section 5.

It is crucial to note that whenever $\lambda(\beta, c) = 0$, this means that the curve $\Gamma = (\theta, c + \gamma(\theta))$ is a translated curve by $F$ ($F(\Gamma) = \Gamma + \mu$); and since $F$ is supposed to have the intersection property, $\mu$ is then bound to be null and we end up with an invariant curve $c + \gamma$ on which the restricted dynamics of $F$ is $C^\infty$ conjugated to $R_\beta$.

The object of Sections 7 and 8 is to let $\beta$ vary and solve implicitly $\lambda(\beta(c), c) = 0$. For this, the dependence on $\beta$ of $\lambda$ is studied in Section 7 and to insure its regularity $\beta$ is restricted to a single Diophantine class $K$ (to fix the loss of differentiability in the linearized equations). The Whitney dependance of $\lambda$ on $\beta$ allows then to extend $\lambda$ to a $C^1$ function $\bar{\lambda}$ defined on a neighborhood of $(0, \alpha)$ in $\mathbb{R}^2$. Since $\bar{\lambda}(\beta, 0) = \lambda(\beta, 0) = \alpha - \beta$, it will be possible to apply the (usual) implicit function theorem to find a function $c \mapsto \beta(c)$

such that $\bar{\lambda}(\beta(c), c) = 0$. Now, if $\beta(c) \in K$, then $\lambda(\beta(c), c) = \bar{\lambda}(\beta(c), c) = 0$ and the curve $c + \gamma(\beta(c), c)$ is indeed invariant by $F$ with rotation number $\beta(c)$. The alternative of the Main Theorem follows hence from the fact that either $\beta$ happens to be locally constant in the neighborhood of 0 equal to $\beta(0) = \alpha \in K$, or $\beta$ varies and takes on a positive measure set of heights $c$ values in $K$ since the Diophantine set $K$ is chosen as in Proposition 3 so that $\alpha$ is well accumulated (from both sides and with positive measure) by numbers in $K$. This is explained in Section 8.1 while the proof of Corollary 1 is given in Section 8.2.

# 3 Herman-Yoccoz theorem on the boundary

By the Theorem of Herman and Yoccoz ([8], [14]), since the restriction of the smooth diffeomorphism $F$ to $\Gamma_0$ has a Diophantine rotation number $\alpha$, it is possible to conjugate $F$, via a $C^\infty$-diffeomorphism of $\mathbb{A}$, to a diffeomorphism fixing $\Gamma_0$ and equal to the circle rotation $R_\alpha$ on $\Gamma_0$. More generally one can prove:

**Proposition 4.** *If the diffeomorphism $F$ of $\mathbb{A}$ has a smooth invariant graph $\Gamma := \{(\theta, \gamma(\theta)), \theta \in \mathbb{T}\}$ on which the dynamics has a diophantine rotation number $\alpha$, then, there exists a diffeomorphism $G$ of $\mathbb{A}$ which sends $\Gamma$ to $\Gamma_0$ and such that $\tilde{F} := G \circ F \circ G^{-1}$ has $\Gamma_0$ as an invariant curve and $\tilde{F}$ restricted to $\Gamma_0$ is the rotation of angle $\alpha$.*

*Proof.* Assume that $F(\theta, r) = (\theta + \alpha + \phi(\theta, r), \psi(\theta, r))$ and that $F(\theta, \gamma(\theta)) = (f(\theta), \gamma(f(\theta)))$ where $f$ is a diffeomorphism of the circle of rotation number $\alpha$. By the Theorem of Herman and Yoccoz, there exists a smooth diffeomorphism $h$ of the circle such that $f = h \circ R_\alpha \circ h^{-1}$. If we define $K : (\theta, r) \mapsto (\theta, r - \gamma(\theta))$ and $H : (\theta, r) \mapsto (h^{-1}(\theta), r)$, we can take $G = H \circ K$. $\square$

Therefore, we will assume hereafter that $F_{|\Gamma_0} = R_\alpha$.

# 4 Birkhoff Normal Form reduction

Using the Diophantine property of $\alpha$ and the intersection property of $F$ we get the following Birkhoff Normal Forms for $F$ in the neighborhood of $\Gamma_0$ [5]:

[5] In the case of an elliptic fixed point of an area preserving surface map, the Diophantine property would not be necessary and the same Birkhoff normal form can be obtained by a symplectic change of coordinate for any irrational rotation number [10].

**Proposition 5.** *For any $N \geqslant 2$, there exist a neighborhood $U$ of $\Gamma_0$ in the annulus $\mathbb{A}$ and a smooth diffeomorphism $G \in \mathrm{Diff}^\infty(\mathbb{A})$ leaving the circle $\Gamma_0$ invariant such that,*

- *the smooth diffeomorphism $\tilde{F} = G \circ F \circ G^{-1}$ leaves the circle $\Gamma_0$ invariant and has the intersection property;*
- *there exist $(N-1)$ constants $a_i \in \mathbb{R}, i = 1, \ldots, N-1$, and two smooth maps $\varphi_j \in C^\infty(\mathbb{A}, \mathbb{R})$, $j = 1, 2$ such that, for any $(\theta, r) \in U$:*

$$\tilde{F}(\theta, r) = \left( \theta + \alpha + \sum_{i=1}^{N-1} a_i r^i + r^N \varphi_1(\theta, r), r + r^N \varphi_2(\theta, r) \right).$$

We shall use the short hand notation

$$\tilde{F}(\theta, r) = \left( \theta + \alpha + \sum_{i=1}^{N-1} a_i r^i, r \right) + O\left(r^N\right).$$

*Proof.* Since we assumed that the restriction of $F$ to the circle $\Gamma_0$ is the rotation map $R_\alpha$, we have

$$F(\theta, r) = (\theta + \alpha + \phi_1(\theta) r, \phi_2(\theta) r) + O\left(r^2\right),$$

with $\phi_i \in C^\infty(\mathbb{T}, \mathbb{R})$, $i = 1, 2$. Since $F$ is a smooth diffeomorphism, $\phi_2$ never vanishes (notice that the Jacobian of $F$ at the points $(\theta, 0)$ is equal to $\phi_2(\theta)$). Without any loss of generality, we can assume that $\phi_2 > 0$. Since $\alpha$ is Diophantine, there exists (see Remark 7) $g_2 \in C^\infty(\mathbb{T}, \mathbb{R})$, $g_2 > 0$ and a constant $C_2 > 0$ such that: for any $\theta$ in $\mathbb{T}$,

$$\phi_2(\theta) = C_2 \frac{g_2(\theta)}{g_2(\theta + \alpha)}.$$

Define $G_2 \in \mathrm{Diff}_0^\infty(\mathbb{A})$ as follows: for any $(\theta, r)$ in $\mathbb{A}$,

$$G_2(\theta, r) = (\theta, g_2(\theta) r).$$

Then, conjugating the diffeomorphism $F$ by $G_2$, we get:

$$\tilde{F}(\theta, r) = G_2 \circ F \circ G_2^{-1}(\theta, r) = (\theta + \alpha + \tilde{\phi}_1(\theta) r, C_2 r) + O(r^2),$$

where $\tilde{\phi}_1 \in C^\infty(\mathbb{T}, \mathbb{R})$ is defined by $\tilde{\phi}_1(\theta) = \phi_1(\theta)/g_2(\theta)$, for any $\theta \in \mathbb{T}$. The conjugate diffeomorphism $\tilde{F}$ has the intersection property, because $F$ has it, hence, *a posteriori*, $C_2 = 1$.

Using the Diophantine property again (see Proposition 2), we get another smooth application $g_1 \in C^\infty(\mathbb{T}, \mathbb{R})$ and a constant $a_1$ such that: for any $\theta$ in $\mathbb{T}$,

$$\tilde{\phi}_1(\theta) = a_1 + g_1(\theta) - g_1(\theta + \alpha).$$

Define a smooth application $G_1 \in C^\infty(\mathbb{A})$ as follows: for any $(\theta, r)$ in $\mathbb{A}$,

$$G_1(\theta, r) = (\theta + r g_1(\theta), r).$$

On a neighborhood of $\Gamma_0$, the application $G_1$ induces a smooth diffeomorphism on its image, and we can assume that $G_1$ is extended to a diffeomorphism of $\mathbb{A}$ without altering it in a neighborhood of $\Gamma_0$. Also, in the neighborhood of $\Gamma_0$, the inverse of $G$ has the form

$$G_1^{-1}(\theta, r) = (\theta - r g_1(\theta) + O(r^2), r).$$

Conjugating the diffeomorphism $\tilde{F}$ by $G_1$, we get: for any $(\theta, r)$ in a small neighborhood of $\Gamma_0$ in $\mathbb{A}$,

$$F_2(\theta, r) = G_1 \circ \tilde{F} \circ G_1^{-1}(\theta, r) = (\theta + \alpha + a_1 r, r) + O(r^2).$$

Developing further, we can locally write: for any $(\theta, r)$ in a small neighborhood of $\Gamma_0$ in $\mathbb{A}$,

$$F_2(\theta, r) = (\theta + \alpha + a_1 r + \phi_1^{(2)}(\theta) r^2, r + \phi_2^{(2)}(\theta) r^2) + O(r^3),$$

with $\phi_i^{(2)} \in C^\infty(\mathbb{T}, \mathbb{R})$, $i = 1, 2$.

Once again, using the Diophantine condition on $\alpha$, there exists a constant $C_2^{(2)}$ and a smooth application $g_2^{(2)} \in C^\infty(\mathbb{T}, \mathbb{R})$ such that $\phi_2^{(2)}(\theta) = C_2^{(2)} + g_2^{(2)}(\theta) - g_2^{(2)}(\theta + \alpha)$. Consider the smooth application $G_2^{(2)} \in C^\infty(\mathbb{A})$ defined, for any $(\theta, r)$ in $\mathbb{A}$ by $G_2^{(2)}(\theta, r) = (\theta, r + r^2 g(\theta))$. This application induces a local diffeomorphism on some neighborhood of $\Gamma_0$ in $\mathbb{A}$. Locally conjugating $F_2$, we thus get: for any $(\theta, r)$ in some neighborhood of $\Gamma_0$,

$$\tilde{F}_2(\theta, r) = G_2^{(2)} \circ F_2 \circ \left(G_2^{(2)}\right)^{-1}(\theta, r) = (\theta + \alpha + a_1 r + \tilde{\phi}_1^{(2)}(\theta) r^2, r + C_2^{(2)} r^2) + O(r^3),$$

where $\tilde{\phi}_1^{(2)} \in C^\infty(\mathbb{T}, \mathbb{R})$.

Again, the intersection property yields $C_2^{(2)} = 0$ *a posteriori*. In this way, alternating local coordinate changes of the form $(\theta, r) \mapsto (\theta, r + r^i g(\theta))$ and $(\theta, r) \mapsto (\theta + r^i g(\theta), r)$ we obtain the normal form writing announced in the proposition. $\square$

In conclusion, and as far as will be necessary for us in the sequel, thanks to Proposition 5 where we take $N = 2$, we can assume that there exist a constant $a_1 \in \mathbb{R}$, a neighborhood $V$ of $\Gamma_0$ in $\mathbb{A}$ and two maps $\varphi_j \in C^\infty(V, \mathbb{R}),\ j = 1, 2$ such that for $(\theta, r) \in V$

$$F(\theta, r) = \big(\theta + \alpha + a_1 r + r^2 \varphi_1(\theta, r), r + r^2 \varphi_2(\theta, r)\big) \tag{1}$$

with possibly $a_1 = 0$.

# 5 Introducing a rotation parameter $\beta$

The results contained in this section are valid for a diffeomorphism $F \in \mathrm{Diff}^\infty(\mathbb{A})$ that can be expressed in a neighborhood $V$ of $\Gamma_0$ as

$$F(\theta, r) = (\theta + \phi(\theta, r), r + r^2 \psi(\theta, r)) \tag{2}$$

with $\phi, \psi \in C_0^\infty(\mathbb{T} \times \mathbb{R})$ such that

$$\phi(\theta, 0) = \alpha \tag{3}$$

for every $\theta \in \mathbb{T}$. No arithmetic condition on $\alpha$ will be needed.

Clearly, a diffeomorphisms as in (1) satisfies the latter conditions.

We denote $\mathcal{U}_0 = \{u \in C_0^\infty(\mathbb{R}/\mathbb{Z}) \ / \ \|u\|_{C^1} < 1\}$, and introduce $E = \mathcal{U}_0 \times \mathcal{U}_0 \times \mathbb{T} \times \mathbb{R}$. The goal of this section is to prove the following.

**Theorem 6.** *Let $F \in \mathrm{Diff}_0^\infty(\mathbb{A})$ that satisfies (2)–(3), and let $\beta$ be a Diophantine number. Then, there exists $\varepsilon > 0$ and a $C^\infty$ map*

$$\begin{array}{rcl} \Psi_\beta : (-\varepsilon, \varepsilon) & \to & E \\ c & \mapsto & (\overline{h}, \gamma, \lambda, \mu) \end{array}$$

*such that the diffeomorphism of the circle $h = \mathrm{id} + \overline{h}$ satisfies*

$$F(\theta, c + \gamma(\theta)) = \big(\lambda + h \circ R_\beta \circ h^{-1}, \mu + c + \gamma(h \circ R_\beta \circ h^{-1})\big)\,.$$

It is crucial to note that whenever $\lambda(c, \beta) = 0$, this means that $(\theta, c+\gamma(\theta))$ is a translated curve by $F$ ($F(\gamma) = \gamma + \mu$); and since $F$ is supposed to have the intersection property, $\mu$ is then bound to be null and we end up with an invariant curve $c+\gamma$ on which the restricted dynamics of $F$ is $C^\infty$ conjugated to $R_\beta$.

It will be the object of the next section to let $\beta$ vary and solve implicitly $\lambda(c, \beta(c)) = 0$. For this, the dependence on $\beta$ of $\Psi_\beta$ will have to be studied and to insure its regularity $\beta$ will be restricted to a single Diophantine class.

In the current section however, $\beta$ will be fixed.

*Proof.* We have

$$F(\theta, c+\gamma(\theta)) = \big(\theta + \phi(\theta, c+\gamma(\theta)), c+\gamma(\theta) + (c+\gamma(\theta))^2\psi(\theta, c+\gamma(\theta))\big)$$

so that the equations we need to solve with a good choice of $\overline{h}$, $\gamma$, $\lambda$ and $\mu$ are $\Phi_1(c,(\overline{h},\gamma,\lambda,\mu))(\theta) = 0$ and $\Phi_2(c,(\overline{h},\gamma,\lambda,\mu))(\theta) = 0$ where

$$\begin{aligned}
\Phi_1(c,(\overline{h},\gamma,\lambda,\mu))(\theta) &= \lambda + h\circ R_\beta \circ h^{-1}(\theta) - (\theta + \phi(\theta, c+\gamma(\theta))) \\
\Phi_2(c,(\overline{h},\gamma,\lambda,\mu))(\theta) &= \mu + \gamma(h\circ R_\beta\circ h^{-1}(\theta)) - \gamma(\theta) - (c+\gamma(\theta))^2\psi(\theta, c+\gamma(\theta))
\end{aligned}$$

Let

$$F = C^\infty(\mathbf{R}/\mathbf{Z}) \times C^\infty(\mathbf{R}/\mathbf{Z})$$

(so $E$ and $F$ are isomorphic tame Frechet spaces) and define

$$\begin{aligned}
\Phi : \mathbb{R}\times E &\to F \\
(c,(\overline{h},\gamma,\lambda,\mu)) &\mapsto \Big(\Phi_1(c,(\overline{h},\gamma,\lambda,\mu)), \Phi_2(c,(\overline{h},\gamma,\lambda,\mu))\Big).
\end{aligned}$$

First of all, observe that

$$\Phi(0,0,0,\alpha-\beta,0) = 0.$$

Next we want to apply Hamilton's Implicit Function Theorem (Corollary 2) in the neighborhood of $(0,0,0,\alpha-\beta,0)$.

Indeed, it is clear that the map $\Phi$ is $C^\infty$-tame and to prove the existence of the map $\Psi_\beta$ as in the statement of theorem 6, it is enough to prove that for *any* $(c,\overline{h},\gamma,\lambda,\mu)$ in some neighborhood of $(0,0,0,\alpha-\beta,0)$ in $\mathbb{R}\times E$, the partial derivative $D'\Phi(c,\overline{h},\gamma,\lambda,\mu)$ ($D'$ in all this section denotes the partial derivative with respect to $(\overline{h},\gamma,\lambda,\mu)$) is invertible with a tame inverse.

We start by computing $\Delta A = D'\Phi_1(c,\overline{h},\gamma,\lambda,\mu)\cdot(\Delta\overline{h},\Delta\gamma,\Delta\lambda,\Delta\mu)$ and $\Delta B = D'\Phi_2(c,\overline{h},\gamma,\lambda,\mu)\cdot(\Delta\overline{h},\Delta\gamma,\Delta\lambda,\Delta\mu)$:

$$\Delta A = \Delta\lambda + \left(\Delta\overline{h}\circ R_\beta - \frac{h'\circ R_\beta}{h'}\Delta\overline{h}\right)\circ h^{-1} - \partial_r\phi(\cdot, c+\gamma)\Delta\gamma, \tag{4}$$

$$\begin{aligned}
\Delta B = \Delta\mu + \Delta\gamma(h\circ R_\beta\circ h^{-1}) + \gamma'(h\circ R_\beta\circ h^{-1})\cdot\left(\Delta\overline{h}\circ R_\beta - \frac{h'\circ R_\beta}{h'}\Delta\overline{h}\right)\circ h^{-1} \\
- \Delta\gamma - \big((c+\gamma)^2\partial_r\psi(\cdot, c+\gamma) + 2(c+\gamma)\psi(\cdot,\gamma)\big)\Delta\gamma
\end{aligned} \tag{5}$$

We now prove that if $(c, \overline{h}, \gamma, \lambda, \mu)$ is in a small neighborhood of $(0, 0, 0, \alpha - \beta, 0)$ in $\mathbb{R} \times E$, then given $\Delta A, \Delta B$ one can find $(\Delta\overline{h}, \Delta\gamma, \Delta\lambda, \Delta\mu)$ solving these equations. The system of equations (4) and (5) is equivalent to (4) and

$$\begin{aligned}\Delta B &= \Delta\mu + \Delta\gamma(h \circ R_\beta \circ h^{-1}) \\ &\quad + \gamma'(h \circ R_\beta \circ h^{-1}) \cdot \Big(\Delta A \circ h + \partial_r\phi(h, c + \gamma \circ h)\Delta\gamma \circ h - \Delta\lambda\Big) \circ h^{-1} \\ &\qquad - \Delta\gamma - \big((c+\gamma)^2\partial_r\psi(\cdot, c+\gamma) + 2(c+\gamma)\psi(\cdot, \gamma)\big)\,\Delta\gamma\end{aligned}$$

which in its turn is equivalent to

$$\begin{aligned}\Delta B \circ h - \gamma'(h \circ R_\beta)\Delta A \circ h - \Delta\mu + \gamma'(h \circ R_\beta)\Delta\lambda \\ = (\Delta\gamma \circ h) \circ R_\beta - (1+\eta)(\Delta\gamma \circ h)\end{aligned} \quad (6)$$

where

$$\begin{aligned}\eta &= \gamma'(h \circ R_\beta)\partial_r\phi(h, c + \gamma \circ h) \\ &\quad - \big((c + \gamma \circ h)^2\partial_r\psi(h, c + \gamma \circ h) + 2(c + \gamma \circ h)\psi(h, \gamma \circ h)\big)\end{aligned}$$

is small in norm $C^\infty$ as $c$ is small and $\gamma$ and $\overline{h}$ are small in the $C^\infty$ topology. Since in addition $\beta$ is supposed to be Diophantine, if $\eta$ is small enough, we can write (cf. Remark 7) $1 + \eta = a g \circ R_\beta / g$ where the constant $a > 0$ is close to 1 and $g$ is some smooth function close to 1 in the $C^\infty$ topology and (6) finally becomes

$$\begin{aligned}\frac{1}{g \circ R_\beta}\left(\Delta B \circ h - \gamma'(h \circ R_\beta)\Delta A \circ h - \Delta\mu + \gamma'(h \circ R_\beta)\Delta\lambda\right) \\ = \left(\frac{\Delta\gamma \circ h}{g}\right) \circ R_\beta - a\frac{\Delta\gamma \circ h}{g}.\end{aligned} \quad (7)$$

Since $\beta$ is Diophantine this equation can be solved in $\Delta\gamma$ if the left hand side has zero mean. More precisely, given $\Delta\lambda, \Delta\mu$ such that

$$\Delta\mu \int_{\mathbb{T}} \frac{1}{g \circ R_\beta} - \Delta\lambda \int_{\mathbb{T}} \frac{\gamma'(h \circ R_\beta)}{g \circ R_\beta} = \int_{\mathbb{T}} \frac{\Delta B \circ h - \gamma'(h \circ R_\beta)\Delta A \circ h}{g \circ R_\beta}$$

there is a unique $\Delta\gamma$ of zero mean solving equation (7), namely

$$\begin{aligned}&\Delta\gamma = \xi g \circ h^{-1} \\ &+ g \circ h^{-1}\mathcal{L}_{\beta,a}\left(\frac{\Delta B \circ h - \gamma'(h \circ R_\beta)\Delta A \circ h}{g \circ R_\beta} - \frac{(\Delta\mu - \gamma'(h \circ R_\beta)\Delta\lambda)}{g \circ R_\beta}\right) \circ h^{-1},\end{aligned}$$

where the constant $\xi \in \mathbb{R}$ is chosen so that $\Delta\gamma$ has zero mean: this is possible since $g$ is close to 1. We write this solution as

$$\Delta\gamma = \xi g \circ h^{-1} + P - \Delta\mu Q + \Delta\lambda R,$$

with

$$Q = g \circ h^{-1} \mathcal{L}_{\beta,a}\left(\frac{1}{g \circ R_\beta} - \int_{\mathbb{T}} \frac{1}{g \circ R_\beta}\right) \circ h^{-1}$$

$$R = g \circ h^{-1} \mathcal{L}_{\beta,a}\left(\frac{\gamma'(h \circ R_\beta)}{g \circ R_\beta} - \int_{\mathbb{T}} \frac{\gamma'(h \circ R_\beta)}{g \circ R_\beta}\right) \circ h^{-1}$$

$$\xi g \circ h^{-1} = \left(-\int_{\mathbb{T}} P + \Delta\mu \int_{\mathbb{T}} Q - \Delta\lambda \int_{\mathbb{T}} R\right) \tilde{g}, \quad \tilde{g} = \frac{g \circ h^{-1}}{\int_{\mathbb{T}} g \circ h^{-1}}.$$

$$P = g \circ h^{-1} U$$

$$U = \mathcal{L}_{\beta,a}\left(\frac{\Delta B \circ h - \gamma'(h \circ R_\beta)\Delta A \circ h}{g \circ R_\beta} - \int_{\mathbb{T}} \frac{\Delta B \circ h - \gamma'(h \circ R_\beta)\Delta A \circ h}{g \circ R_\beta}\right) \circ h^{-1}$$

Notice that if $\gamma$ and $\overline{h}$ are sufficiently small in the $C^\infty$ topology it is possible to make $R$ and $Q$ also small.

The $\Delta A$-equation (4) becomes

$$\widetilde{\Delta A} = (\frac{1}{h' \circ R_\beta} - \rho)\Delta\lambda + \omega\Delta\mu + \left(\frac{\Delta\overline{h}}{h'} \circ R_\beta - \frac{\Delta\overline{h}}{h'}\right), \tag{8}$$

where

$$\widetilde{\Delta A} = \frac{1}{h' \circ R_\beta}\left[\Delta A \circ h + \partial_r \phi(h, c + \gamma \circ h)\left(P \circ h - \tilde{g} \circ h \int_{\mathbb{T}} P\right)\right],$$

and

$$\begin{aligned} \rho &= \frac{\partial_r \phi(h, c + \gamma \circ h)}{h' \circ R_\beta}\left(R \circ h - \tilde{g} \circ h \int_{\mathbb{T}} R\right), \\ \omega &= \frac{\partial_r \phi(h, c + \gamma \circ h)}{h' \circ R_\beta}\left(Q \circ h - \tilde{g} \circ h \int_{\mathbb{T}} Q\right). \end{aligned}$$

We notice again that $\rho$ and $\omega$ are small if $c, \overline{h}$, and $\gamma$ are small.

Equation (8) has a unique solution $\Delta\overline{h}$ of mean zero provided

$$\Delta\lambda(\int_{\mathbb{T}} \frac{1}{h' \circ R_\beta} - \int_{\mathbb{T}} \rho) + \Delta\mu \int_{\mathbb{T}} \omega = \int_{\mathbb{T}} \widetilde{\Delta A},$$

which equals

$$\Delta\overline{h} = h'\mathcal{L}_{\beta,0}(V) + \zeta h'$$

with $\zeta \in \mathbb{R}$ and $V$ defined by

$$V = \widetilde{\Delta A} - (\frac{1}{h' \circ R_\beta} - \rho)\Delta\lambda - \omega\Delta\mu, \qquad \zeta = -\frac{\int_{\mathbb{T}} h'\mathcal{L}_{\beta,0}V}{\int_{\mathbb{T}} h'}.$$

The constants $\Delta\lambda, \Delta\mu$ are then determined as the unique solution of the system

$$\begin{cases} \Delta\lambda(\int_{\mathbb{T}} \frac{1}{h'\circ R_\beta} - \int_{\mathbb{T}} \rho) + \Delta\mu \int_{\mathbb{T}} \omega = \int_{\mathbb{T}} \widetilde{\Delta A} \\ -\Delta\lambda \int_{\mathbb{T}} \frac{\gamma'(h\circ R_\beta)}{g\circ R_\beta} + \Delta\mu \int_{\mathbb{T}} \frac{1}{g\circ R_\beta} = \int_{\mathbb{T}} \frac{\Delta B\circ h - \gamma'(h\circ R_\beta)\Delta A\circ h}{g\circ R_\beta} \end{cases}$$

which is invertible since it is almost in triangular form with diagonal close to 1. With this choice for $(\Delta\lambda, \Delta\mu)$ we get $\Delta\overline{h}$ and $\Delta\gamma$ that solve (8) and (7), or equivalently (4) and (5).

To summarize, we have a obtained a map $\Phi : \mathbb{R} \times E \to F$ (with $E$ and $F$ isomorphic tame Fréchet spaces) such that $\Phi$ is $C^\infty$-tame and $\Phi(0,0,0,\alpha-\beta,0) = 0$, and if we denote by $(c,u)$ $(u = (\overline{h},\gamma,\lambda,\mu))$ the variables in $\mathbb{R}\times E$, we have proved that for $(c,u)$ in a small neighborhood of $(0,(0,0,\alpha-\beta,0))$ then $D_u\Phi$ is invertible. Furthermore, it is not hard to see from the proof and from propositions 2 and 7 that the inverse of $D_u\Phi$ is in fact tame. The result of Theorem 6 then follows from Hamilton's Implicit Function Theorem (Corollary 2).

□

# 6 Hamilton's Theorem in Whitney spaces

## 6.1 Whitney spaces

We refer the reader to [9] and [5] for this section. Let $(E, (\|\cdot\|_i))$ and $(F, (\|\cdot\|_i))$ be tame Fréchet spaces, $K$ a compact set of $\mathbb{R}^d$ and $\nu \in \mathbf{R}$ and $p \in \mathbf{N}$ such that $p < \nu \leqslant p+1$. We say that an element $x(\cdot) \in E^K$ is in $\mathrm{Lip}^\nu(K,E)$ if for any $0 \leqslant j \leqslant p$ there exist elements $x^{(j)}(\cdot)$ with $x^{(0)} = x$ and $R_j \in E^{K\times K}$ such that for any $\alpha, \beta \in K$

$$x^{(j)}(\beta) = \sum_{|j+l|\leqslant p} \frac{x^{(j+l)}(\alpha)}{l!}(\beta-\alpha)^l + R_j(\beta,\alpha),$$

(we use here Whitney's multi-indices notations[6]) satisfying the following estimates (for any $j, \alpha, \beta$):

$$\|x^{(j)}\|_s \leqslant M_s, \qquad \|R_j(\beta,\alpha)\|_s \leqslant M_s|\beta-\alpha|^{\nu-j}.$$

The choice for the $x^{(j)}$ ($j \neq 0$) is in general not unique (unless $d = 1$ and $K$ does not contain isolated points). We denote by $\|x(\cdot)\|_s$ the infimum of $M_s$ for all the possible choices of $x^{(j)}$; it is not difficult to check that these are seminorms and that $(\mathrm{Lip}^\nu(K,E), (\|\cdot\|_s))$ is a Fréchet space. One can define smoothing operators by

$$\forall \beta \in K, \ (\mathcal{S}(t)x(\cdot))(\beta) = S(t)x(\beta),$$

which makes the Whitney space $(\mathrm{Lip}^\nu(K,E), (\|\cdot\|_s))$ a tame Fréchet space.
The notion of tame maps between Whitney spaces is then clear.

## 6.2 Whitney Extension

We mention a general result about extensions of Whitney regular functions, that we will only need in the simple case of a finite dimensional target space $E$.

**Theorem 7** (Whitney Extension Theorem (cf. [12])**.** *For any integer $d \geqslant 1$, there exists a positive constant $\kappa_d$, such that for any closed set $K \subset \mathbb{R}^d$ and any integer $\nu \geqslant 1$, there exists a linear extension operator $\mathrm{Ext}_\nu : \mathrm{Lip}^\nu(K,E) \to \mathrm{Lip}^\nu(\mathbb{R}^d,E)$, such that for any $x \in \mathrm{Lip}^\nu(K,E)$, $\mathrm{Ext}_\nu(x)_{|K} = x$, and for any $s \in \mathbb{N}$, the following holds:*

$$\|\mathrm{Ext}_\nu(x)\| \leqslant \kappa_d \, \|x\| \, .$$

**Remark:** It is possible to extend Whitney's Extension Theorem to the case where $E$ is a Tame Fréchet space.

# 7 Whitney dependence in $\beta$

In this section we improve Theorem 6 into the following central theorem of the paper.

[6] If $j = (j_1,\dots,j_d) \in \mathbb{N}^d$, $z = (z_1,\dots,z_d)$ we define $|j| = j_1 + \cdots + j_d$, $j! = j_1! \cdots j_d!$ and $z^j = z_1^{j_1} \cdots z_d^{j_d}$. If $i$ is also in $\mathbb{N}^d$ we write $i \leqslant j$ iff $i_1 \leqslant j_1, \dots, i_d \leqslant j_d$. Also $\binom{j}{i}$ is by definition $\binom{j_1}{i_1} \cdots \binom{j_d}{i_d}$

**Theorem 8.** *Let $F \in \mathrm{Diff}_0^\infty(\mathbb{A})$ that satisfies (2)–(3), and let $\tau$, $\sigma$ be positive numbers and denote $K = \mathrm{DC}(\sigma,\tau) \cap [0,1]$. Fix $\nu > 0$. Then, there exists $\varepsilon > 0$ and a $C^\infty$-tame map $\Psi : (-\varepsilon,\varepsilon) \to \mathrm{Lip}^\nu(K,E)$ such that if $c \in (-\varepsilon,\varepsilon)$, $\beta \in K$, and $(\overline{h},\gamma,\lambda,\mu) = (\Psi(c))(\beta)$, we have*

$$F(\theta, c+\gamma(\theta)) = \left(\lambda + h \circ R_\beta \circ h^{-1}(\theta), \mu + c + \gamma(h \circ R_\beta \circ h^{-1}(\theta))\right).$$

The above theorem states that the familly of maps $\Psi_\beta$ obtained by the implicit function theorem in Theorem 6 actually depends $C^\nu$-Whitney on $\beta$ as $\beta$ belongs to a compact set of numbers satisfying a Diophantine condition with fixed constant and exponent.

The rest of the current section is devoted to the proof of Theorem 8. We will see in the next section how this theorem easily implies Theorem 3. We refer the reader to the last section of the Appendix for further consequences of Theorem 8; in particular we explain how this normal form theorem can be used to give short proofs of Moser's twist theorem and of a theorem of Cheng and Sun [3] and Xia [13] (see also the survey of J.-C. Yoccoz [15]).

Rename $\Phi_\beta$ the map that was introduced in the proof of Theorem 6. Recall that the pair $(c,u) = (c,(\overline{h},\gamma,\lambda,\mu))$ denotes the variables in $\mathbb{R} \times E$. If we denote by $(u_\beta)_{\beta\in\mathbb{R}} = (0,0,\alpha-\beta,0)_{\beta\in\mathbb{R}}$ we have that $\Phi_\beta(0,u_\beta) = 0$.

From the proof of the invertibility of $D_u\Phi_\beta$ in Section 5, it is easy to observe that there exist $l \in \mathbb{N}$ and $\varepsilon > 0$ and $a \in \mathbb{N}$, such that for every $s \in \mathbb{N}$ there exists a constant $C_s$, such that if $|c| \leqslant \varepsilon$, $\|\overline{h}\|_{C^l} \leqslant \varepsilon$, $\|\gamma\|_{C^l} \leqslant \varepsilon$, $\lambda,\mu \in \mathbb{R}$ ($|\lambda| \leqslant 1$, $|\mu| \leqslant 1$) and $\beta \in K$, then $D_u\Phi_\beta(c,u)$ is invertible and if $J_\beta(c,u)$ denotes its inverse, we have

$$\|J_\beta(c,u)\cdot\Delta u\|_s \leqslant C_s((1+\|u\|_{s+a})\|\Delta u\|_a + \|\Delta u\|_{s+a}),$$

for every $|c|, \|\overline{h}\|_{C^l}, \|\gamma\|_{C^l} \leqslant \varepsilon$, for every $\lambda,\mu \in \mathbb{R}$ ($|\lambda| \leqslant 1$, $|\mu| \leqslant 1$) and for every $\Delta u \in E$.

This implies Theorem 8 due to the following Implicit Function Theorem that in its turn will be obtained by the application of Hamilton's Implicit Function Theorem in some adequate tame Whitney spaces.

**Theorem 9.** *Fix $\nu > 0$. Consider three Fréchet spaces $E$, $F$ and $G$, two open sets $U \subset E$ and $C \subset G$, a relatively compact open set $O \subset \mathbb{R}^d$, $d \in \mathbb{N}$, and a $C^\infty$-tame application*

$$\Phi : \begin{array}{ccc} O \times C \times U & \to & F \\ (\beta,c,u) & \mapsto & \Phi_\beta(c,u) := \Phi(\beta,c,u) \end{array}$$

*such that there exists $c_0 \in C$ and $u^{(0)} \in C^\infty(O,U)$ satisfying*

$$\forall \beta \in O,\ \Phi(\beta,c_0,u^{(0)}(\beta)) = 0.$$

*Assume, moreover, that there exists a closed set $K \subset O$ such that the following property holds : there exist open sets $\tilde{C} \subset C$ and $\tilde{U} \subset U$ (containing all $u^{(0)}(\beta), \beta \in K$) such that for each $\beta \in K$, there exists a continuous map $\mathcal{J}_\beta : \tilde{C} \times \tilde{U} \times F \to E$ linear with respect to the third variable, which is the inverse map of $D_u\Phi_\beta$; assume moreover that this map satisfies a uniform tameness condition: there exists $a \in \mathbb{N}$, and for every $s \in \mathbb{N}$ there exists a constant $C_s$, such that*

$$\|\mathcal{J}_\beta(c,u) \cdot \Delta u\|_s \leqslant C_s((1 + \|u\|_{s+a})\|\Delta u\|_a + \|\Delta u\|_{s+a}), \tag{9}$$

*for every $\beta \in K$, $(c,u) \in \tilde{C} \times \tilde{U}$ and $\Delta u \in F$.*

*Then there exist an open neighborhood $C_0$ of $c_0$ in $\tilde{C}$, a neighborhood $V_0$ of the function $u^{(0)}$ in $\mathrm{Lip}^\nu(K, \tilde{U})$ and a $C^\infty$-tame map $\Psi : C_0 \to V_0$ such that $\Phi(\beta, c, (\Psi(c))(\beta)) = 0$. Moreover, $(c, \Psi(c)(\cdot))$ is the unique element of $C_0 \times V_0$ such that for all $\beta \in K$ this identity is satisfied.*

Before giving the proof of Theorem 9 we give a useful specification of the result of Theorem 8.

**Remark 8.** Rescaling the variable $r$, equation (1) can be written as

$$F(\theta, r) = \left(\theta + \alpha + \delta a_1 r + \delta^2 r^2 \overline{\varphi}_1(\theta, \delta r), r + \delta r^2 \overline{\varphi}_2(\theta, \delta r)\right). \tag{10}$$

The same proof as that of Theorem 8 would then yield a smooth tame map $\overline{\Psi}$ : $(-\varepsilon, \varepsilon)^2 \to \mathrm{Lip}^\nu(K, E)$ such that if $(c, \delta) \in (-\varepsilon, \varepsilon)^2$, $\beta \in K$, and $(\overline{h}, \gamma, \lambda, \mu) = (\Psi(c, \delta))(\beta)$, then

$$F(\theta, c + \gamma(\theta)) = \left(\lambda + h \circ R_\beta \circ h^{-1}(\theta), \mu + c + \gamma(h \circ R_\beta \circ h^{-1}(\theta))\right).$$

But it is clear that for any $\beta$, $\overline{h}(c, 0, \beta) = 0$, $\gamma(c, 0, \beta) = 0$, $\mu(c, 0, \beta) = 0$, while $\lambda(c, 0, \beta) = \alpha + a_1 c - \beta$. Hence, letting $|\delta|$ be sufficiently small, we can consider that for any $\beta \in K$, the maps $(\theta, c) \mapsto (\theta, c + \gamma(c, \beta)(\theta))$, and $(\theta, c) \mapsto (h(c, \beta)(\theta), c)$ obtained in Theorem 8, are smooth diffeomorphisms from some open neighborhood of $\Gamma_0$ in $\mathbb{A}$ onto its images, which will be useful in the proof of Theorem 3.

*Proof of Theorem 9.*
**Step 1.** Notice that the map

$$\tilde{\Phi} : \begin{array}{ccc} C \times \mathrm{Lip}^\nu(K, U) & \to & \mathrm{Lip}^\nu(K, F) \\ (c, (u_\beta)_{\beta \in K}) & \mapsto & (\Phi(\beta, c, u_\beta))_{\beta \in K} \end{array}$$

is well defined and $C^\infty$-tame. This follows from Taylor formula with integral remainder.
**Step 2.** The following proposition shows that under the uniformity condition (9) there is a tame inverse to $D_2\tilde{\Phi}$ (where $D_2\tilde{\Phi}$ denotes the derivative with respect to the variable $u(\cdot)$):

**Proposition 6.** *Assume that $E$ and $F$ are two Fréchet spaces and $V$ is an open subset of $E$. Assume that*

$$\begin{aligned} L : \mathbf{R}^d \times V \times E &\to F \\ (\beta, u, \Delta u) &\mapsto L(\beta, u) \cdot \Delta u \end{aligned}$$

*is a $C^\infty$-tame map, linear in the third factor with the following property:*

1. *for each $\beta \in K$ there exists a continuous tame map linear in the second factor*

$$\begin{aligned} J_\beta : V \times F &\to E \\ (u, \Delta u) &\mapsto J_\beta(u) \cdot \Delta u, \end{aligned}$$

*such that*

$$J_\beta(u) \cdot L(\beta, u) = \mathrm{Id}_E, \;\; L(\beta, u) \cdot J_\beta(u) = \mathrm{Id}_F$$

2. *there are constants $a, C_s$ such that for any $\beta \in K$, any $s \in \mathbf{N}$ and any $(u, \Delta u) \in V \times F$*

$$\|J(\beta, u) \cdot \Delta u\|_s \leqslant C_s((1 + \|u\|_{s+a})\|\Delta u\|_a + \|\Delta u\|_{s+a}),$$

*$(J(\beta, u) = J_\beta(u))$.*

*Then, for any choice of $\gamma > 0$, the map*

$$\begin{aligned} \mathcal{J} : \mathrm{Lip}^\gamma(K, V) \times \mathrm{Lip}^\gamma(K, F) &\to \mathrm{Lip}^\gamma(K, E) \\ (u(\cdot), \Delta u(\cdot)) &\mapsto J(\cdot, u(\cdot)) \cdot \Delta u(\cdot) \end{aligned}$$

*is well defined and is a continuous tame map linear in the second factor.*

The proof of this proposition will be given in the Appendix.

**Step 3.** Observe that we have $\tilde{\Phi}(c_0, u^{(0)}(\cdot)) = 0_{\beta \in K}$. From steps 1 and 2, and the hypothesis on the inverse of $D_u \Phi_\beta$ we thus obtain by Hamilton's Implicit Function Theorem applied to $\tilde{\Phi}$: There exists $C_0$ a neighborhood of $c_0$ in $C$, a neighborhood $V_0$ of the function $u^{(0)}$ in $\mathrm{Lip}^\nu(K, \tilde{U})$ and a $C^\infty$-tame map $\Psi : C_0 \to V_0$ such that $\tilde{\Phi}(c, \Psi(c)(\cdot)) = 0_{\beta \in K}$ which satisfies $\Phi(\beta, c, (\Psi(c)(\beta)) = 0$ for all $c \in C_0$ and all $\beta \in K$. $\square$

# 8 Proof of Theorem 3 and Corollary 1

## 8.1 Proof of Theorem 3

If we take $\nu = 2$ in Theorem 8, we have that $\lambda(\cdot,\cdot)$ is in $\mathrm{Lip}^2(K\times(-\varepsilon,\varepsilon),\mathbb{R})$. Thanks to Theorem 7, we are thus allowed to consider the extended application $\bar\lambda \in C^1([0,1]\times(-\varepsilon,\varepsilon),\mathbb{R})$ such that $\bar\lambda(\beta,c) = \lambda(\beta,c)$ for each $(\beta,c)\in K\times(-\varepsilon,\varepsilon)$. Recall that $\bar\lambda(\alpha,0) = \lambda(\alpha,0) = 0$ and that more generally for $\beta\in K$ we have $\bar\lambda(\beta,0) = \lambda(\beta,0) = \alpha-\beta$. Since $\alpha$ is not isolated in $K$ (see Proposition 3) we have that $\partial_\beta\bar\lambda(\alpha,0) = -1$. We thus obtain by the implicit function theorem applied to $\bar\lambda$ that there exists $\varepsilon'>0$ and a $C^1$ map

$$\beta : \begin{array}{ccc} (-\varepsilon',\varepsilon') & \to & \mathbb{R} \\ c & \mapsto & \beta(c) \end{array}$$

such that $\bar\lambda(\beta(c),c) = 0$.

Assume now that $c$ is such that $\beta(c)\in K$, then $\lambda(\beta(c),c) = \bar\lambda(\beta(c),c) = 0$ and the curve $c+\gamma(\beta(c),c)$ obtained in Theorem 8 is $\mu$-translated by $F$ with $\mu = \mu(\beta(c),c)$. By the intersection property, this implies that $\mu = 0$ and that the curve is actually invariant by $F$.

There are hence two possible scenarios: in the first one, there exists $\varepsilon''>0$ such that $\beta(c) = \alpha$ for all $c\in[-\varepsilon'',\varepsilon'']$ in which case the curves $c+\gamma(c,\alpha)$ are invariant since $\alpha\in K$. The annulus $\mathcal{O}$ bounded by $-\varepsilon''+\gamma(-\varepsilon'',\alpha)$ and $\varepsilon''+\gamma(\varepsilon'',\alpha)$ is then completely foliated by the invariant curves $c+\gamma(c,\alpha)$, $c\in[-\varepsilon'',\varepsilon'']$. This is due to the continuity of the maps $c\mapsto c+\gamma(c,\alpha)(\theta)$ for any given $\theta\in\mathbb{T}$. The annulus $\mathcal{O}$ is a neighborhood of the circle $\Gamma_0$ since $\gamma(0,\alpha) = 0$. As pointed out in Remark 8, the maps $G_1 : \mathbb{A}''_\varepsilon = \mathbb{T}\times[-\varepsilon'',\varepsilon'']\to\mathcal{O}$, $(\theta,c)\mapsto(\theta,c+\gamma(c)(\theta))$, and $G_2 : \mathbb{A}''_\varepsilon\to\mathbb{A}''_\varepsilon$, $(\theta,c)\mapsto(h(c)(\theta),c)$, are $C^\infty$-diffeomorphisms (here $h(c) = h(c,\alpha)$ and $\gamma(c) = \gamma(c,\alpha)$ since $\beta(c) = \alpha$ on $[-\varepsilon'',\varepsilon'']$ ). The composition $G := G_1\circ G_2$ gives a $C^\infty$-conjugation between $F$ on $\mathcal{O}$ and the rotation $S_\alpha$ on the annulus $\mathbb{T}\times[-\varepsilon'',\varepsilon'']$.

The second part of the alternative derives from Proposition 3 and from the fact that $c+\gamma(c,\beta(c))\in C^\infty(\mathbb{T},\mathbb{R})$ converges to 0 in the $C^1$ topology as $c\to 0$.

## 8.2 Proof of Corollary 1

Let us denote by $k_1$ and $\varepsilon_1>0$ the constants given by Remark 4. Given $F\in\mathrm{Diff}^\infty_0(\mathbb{T}\times[0,1])$ as in Corollary 1, we first extend it to a diffeomorphism of $\mathbb{A}$ such that $\|F-S_\alpha\|_{\mathrm{Diff}^{k_1}(\mathbb{A})} < 2\eta$. Next, given the contraction $C_{\varepsilon_1}(\theta,r) = (\theta,r\varepsilon_1)$, consider $\overline{F} = C_{\varepsilon_1}\circ F\circ C_{\varepsilon_1}^{-1}$. Since $C_{\varepsilon_1}\circ S_\alpha\circ C_{\varepsilon_1}^{-1} = \mathrm{Id}$, it is plain that if $\eta>0$ is chosen sufficiently small we will have that the $C^{k_1}$

distance on $\mathbb{T} \times [-1,1]$ between $\overline{F}$ and $S_\alpha$ is less than 1. Hence, there exists $\beta : [-\varepsilon_1, \varepsilon_1] \to \mathbb{R}$ such that Theorem 3 applies to $\overline{F}$ but with "translated circles" instead of invariant ones (cf. Remark 1), namely : if $\beta(c) \in \mathrm{DC}(\sigma', \tau')$ there exists a translated curve $\Gamma_c$. Notice that the circles $\mathbb{T} \times \{0\}$, $\mathbb{T} \times \{\varepsilon_1\}$ are $\overline{F}$ -invariant and that the dynamics of $\overline{F}$ on these circles is $R_\alpha$. Moreover, the dynamics of $\overline{F}$ on $\mathbb{T} \times [0, \varepsilon_1]$ has the intersection property. From Section 9.1 of the Appendix, any translated curve having a part in $\mathbb{T} \times [0, \varepsilon_1]$ is in fact invariant. The fact that $\overline{F}$ does not have any periodic point in the annulus $\mathbb{T} \times [0, \varepsilon_1]$ implies by Poincaré's last geometric theorem that there are no two invariant curves with different rotation numbers included in this annulus. Hence, $\beta$ has to be constant on $[0, \varepsilon_1]$. Now, the arguments given in the previous Subsection 8.1 to conclude the proof of Theorem 3 apply in this situation also.

# 9 Appendix

## 9.1 Diffeomorphisms on the closed semi annulus

Let $F$ be a diffeomorphism of the closed semi annulus $\mathbb{A}_+ := \mathbb{T} \times [0, \infty)$, having the intersection property. We can write $F(\theta, r) = (F_1(\theta, r), F_2(\theta, r))$ with $F_2(\theta, 0) = 0$ for all $\theta \in \mathbb{T}$, $F_2(\theta, r) \geqslant 0$ if $r \geqslant 0$ and we can also assume that $F$ is a Birkhoff normal form: on $\mathbb{R}/\mathbb{Z} \times [0, \delta]$ it is close in $C^1$-topology to the diffeomorphism $G : (\theta, r) \mapsto (\theta + \alpha + P(r), r)$ which is defined on the annulus $\mathbb{A}_\delta := \mathbb{R}/\mathbb{Z} \times [-\delta, \delta]$, for $\delta$ sufficiently small. Since any real valued smooth function $f : [0; \infty) \to \mathbb{R}$ can be extended as a smooth function on $\mathbb{R}$, for any integer $k$ the Fourier coefficients $\hat{F}_{i,k}(r)$ $(i = 1, 2)$ of $F_i(\cdot, r)$, which define a smooth function, can be extended for any value of $r \in \mathbb{R}$. The same is then true for $F_i(\theta, r)$. If we denote by $\bar{F}{=}(\bar{F}_1, \bar{F}_2)$ the extended map, it will still be close in $C^1$-topology to $G$ on $\mathbb{A}_\delta := \mathbb{R}/\mathbb{Z} \times [-\delta, \delta]$ (maybe for a smaller $\delta$) and hence $\bar{F}$ is a diffeomorphism on that annulus. A simple argument shows that $F^{\pm 1}$ sends $\mathbb{A}_\delta \cap \{r \geqslant 0\}$ and $\mathbb{A}_\delta \cap \{r \leqslant 0\}$ into themselves. We claim that any graph $\Gamma := \{(\theta, \gamma(\theta)), \theta \in \mathbb{T}\}$ in $\mathbb{A}_\delta$, which has a part above the circle $\mathbb{T}$ and such that $\bar{F}^{\pm}(\Gamma)$ is in $\mathbb{A}_\delta$ has the intersection property. If the graph $\Gamma$ is strictly above the circle $\mathbb{T}$, this is clear since $F$ has the intersection property on that region. Otherwise, this means that there exists a graph $\Gamma$ which intersects the circle and which has no intersection with its image. This last property implies that either $\forall \theta \in \mathbb{T},\ \bar{F}_2(\theta, \gamma(\theta)) > \gamma(\theta)$ or $\forall \theta \in \mathbb{T},\ \bar{F}_2(\theta, \gamma(\theta)) < \gamma(\theta)$. Let us assume that the image of $\Gamma$ by $\bar{F}$ is above $\Gamma$ and let $J$ be the set of points $\theta \in \mathbb{T}$ where $\bar{F}_2(\theta, \gamma(\theta)) > 0$. It is an open set different from $\mathbb{T}$ such that $g^{-1}(J) \subset J$ where $g$ is the homeomorphism of the circle defined by $g(\theta) = \bar{F}_1(\theta, \gamma(\theta))$. Since $g^{-1}$ coincide with the rotation $R_{-\alpha}$ on the boundary of $J$ and since $g$ and $R_\alpha$ are homotopic, we also have $R_{-\alpha}J \subset J$, which is in contradiction with the fact that $J$ is an open set different from $\mathbb{T}$ . The case where the image of $\Gamma$ is below $\Gamma$, is dealt with in an analogous way.

## 9.2 Proof of Proposition 6

We sketch the proof in the case $\nu = p + 1$. In the proof we shall make use of Whitney's multi-indices notations. Since the map $L : \mathbf{R}^d \times V \times E \to F$ is $C^\infty$ tame, it holds for any $\alpha \in K$ that there exist tame maps $L_\alpha^{(j)} : V \times E \to E$ linear in the second factor such that

$$L_\beta^{(j)}(u, \Delta u) = \sum_{|j+l| \leqslant p} L_\alpha^{(j+l)}(u) \cdot \Delta u \frac{(\beta - \alpha)^l}{l!} + R_j(\beta, \alpha, u)\Delta u,$$

such that

$$\|L_\alpha^{(j)}(u) \cdot \Delta u\|_s \leqslant C_s(\|\Delta u\|_{s+a} + \|u\|_{s+a}\|\Delta u\|_a),$$
$$\|R_j(\beta, \alpha, u)\Delta u, \|_s \leqslant C_s(\|\Delta u\|_{s+a} + \|u\|_{s+a}\|\Delta u\|_a)|\beta - \alpha|^{p+1-j}.$$

Also, $L_\alpha^{(j)}(u) \cdot \Delta u, R_j(\beta, \alpha)(u) \cdot \Delta u$ are continuous in $(u, \Delta u)$. We now define by induction for any $\beta \in K$ and $r : |r| \leqslant p$,

$$\frac{J_\beta^{(r)}}{r!} = - \sum_{0 \leqslant k \leqslant r, k \neq 0} \frac{J_\beta^{(r-k)}}{(r-k)!} \frac{L_\beta^{(k)}}{k!} J_\beta^{(0)}. \tag{11}$$

By construction

$$\left( \sum_{|l| \leqslant p} J_\alpha^{(l)} \frac{(\beta - \alpha)^l}{l!} \right) \left( \sum_{|k| \leqslant p} L_\alpha^{(k)} \frac{(\beta - \alpha)^k}{k!} \right) = id + M_{\beta - \alpha} \tag{12}$$

where $(u, \Delta u) \mapsto M_{\beta-\alpha}(u) \cdot \Delta u$ is continuous, tame and is a polynomial in $\beta - \alpha$ with terms of total degree larger or equal to $p + 1$. We then have by construction

$$\left( \sum_{|l| \leqslant p} J_\alpha^{(l)} \frac{(\beta - \alpha)^l}{l!} \right) L_\beta = id + N_{\beta - \alpha}$$

where $N_{\beta-\alpha}$ is also continuous, tame and polynomial in $\beta - \alpha$ with monomials of total degree not smaller than $p + 1$. Since $J_\beta \circ L_\beta = id$ we get

$$\left( J_\beta - \sum_{|l| \leqslant p} J_\alpha^{(l)} \frac{(\beta - \alpha)^l}{l!} \right) L_\beta = N_{\beta - \alpha}$$

Since $L_\beta^{-1}$ is *uniformly* tame in the sense of Hypothesis 2 in Proposition 6, we get that

$$J_\beta = \sum_{|l|\leqslant p} J_\alpha^{(l)} \frac{(\beta-\alpha)^l}{l!} + R_0(\beta,\alpha)$$

where $|\beta-\alpha|^{-(p+1)} R_0(\beta,\alpha)$ is tame and continuous in $(u,\Delta u)$.

The consistency relations between each $J_\alpha^{(l)}$ and the corresponding higher order terms can be checked similarly using uniform tameness of $L_\beta^{-1}$ and formal identities relating Taylor expansions of $L$ with those of $J$ (in the special case where $K$ is a compact set without isolated point of the real line this is easier to prove). Let us be more specific. By definition we have the following formal identity

$$\left(\sum_{|l|\leqslant p} J_\alpha^{(l)} \frac{T^l}{l!}\right)\left(\sum_{|m|\leqslant p} L_\alpha^{(m)} \frac{T^m}{m!}\right) = id + Q_{p+1}(T)$$

(where $Q_{p+1}$ is a polynomial in the indeterminates $T_1,\dots,T_d$ all terms of which are of total degree larger or equal to $p+1$). Now, applying the differential operator $\partial_T^r := \partial_{T_1}^{r_1}\cdots\partial_{T_d}^{r_d}$ to the product in the left hand side and using Leibniz formula we get for any $n=(n_1,\dots,n_d) : |n| \leqslant p-|r|$

$$\sum_{k\leqslant r} \binom{r}{k} \sum_{l+m=n} \frac{J_\alpha^{(k+l)}}{l!} \frac{L^{(r-k+m)}}{m!} = 0 \tag{13}$$

We can now prove the consistency relations by induction on $r$: from (11) and the induction assumption

$$\begin{aligned}
\frac{J_\beta^{(r)}}{r!} L_\beta^{(0)} &= -\sum_{k\leqslant r:k\neq 0} \sum_n (\beta-\alpha)^n \sum_{l+m=n} \frac{J_\alpha^{(r-k+l)}}{(r-k)!l!} \frac{L_\alpha^{(k+m)}}{k!m!} \\
&= -\frac{1}{r!} \sum_{k\leqslant r:k\neq 0} \binom{r}{k} \sum_n (\beta-\alpha)^n \sum_{l+m=n} \frac{J_\alpha^{(r-k+l)}}{l!} \frac{L_\alpha^{(k+m)}}{m!} \\
&= -\frac{1}{r!} \sum_n (\beta-\alpha)^n \sum_{k\leqslant r:k\neq 0} \binom{r}{k} \sum_{l+m=n} \frac{J_\alpha^{(r-k+l)}}{l!} \frac{L_\alpha^{(k+m)}}{m!};
\end{aligned}$$

but, in view of (13)

$$\sum_{k\leqslant r:k\neq 0} \binom{r}{k} \sum_{l+m=n} \frac{J_\alpha^{(r-k+l)}}{l!} \frac{L_\alpha^{(k+m)}}{m!} = -\sum_{l+m=n} \frac{J_\alpha^{(r+l)}}{l!} \frac{L_\alpha^{(m)}}{m!}$$

so that

$$\frac{J_\beta^{(r)}}{r!}L_\beta^{(0)} = \frac{1}{r!}\sum_{|n|\leqslant p-|r|}(\beta-\alpha)^n\sum_{l+m=n}\frac{J_\alpha^{(r+l)}}{l!}\frac{L_\alpha^{(m)}}{m!} + U_{p+1-|r|}(\beta-\alpha)$$
$$= \frac{1}{r!}\Big(\sum_{|l|\leqslant p-|r|}\frac{J_\alpha^{(r+l)}}{l!}(\beta-\alpha)^l\Big)\Big(\sum_{|m|\leqslant p-|r|}\frac{L_\alpha^{(m)}}{m!}(\beta-\alpha)^m\Big) + V_{p+1-|r|}(\beta-\alpha)$$

where $U_{p+1-|r|}(\beta-\alpha)$, $V_{p+1-|r|}(\beta-\alpha)$ are polynomials the monomials of which are of total degree larger or equal to $p+1-|r|$ and are continuous and tame in $(u,\Delta u)$. Observing that

$$J_\beta^{(r)} = \Big(\sum_{|l|\leqslant p-|r|}\frac{J_\alpha^{(r+l)}}{l!}(\beta-\alpha)^l\Big)(id - R_{p+1-|r|}(\beta,\alpha)L_\beta^{-1}) + V_{p+1-|r|}(\beta-\alpha)L_\beta^{-1},$$

gives the proof of the consistency relations (here we use again the fact that $L_\beta^{-1}$ is uniformly tame). The proof of the proposition is complete.

## 9.3 Other applications of the normal form writing of Theorem 8.

**Nonzero twist. Moser's Twist Theorem (see [15]).** Consider the case where $F$ is a perturbation of a twist map $F_0(\theta,r) = (\theta+\phi(r),r)$, with $\phi'$ bounded away form zero (this corresponds to a twist coefficient $a_1 \neq 0$ in the Birkhoff normal form of $F$). The map $\lambda_0$ associated to $F_0$ as in Theorem 8 satisfies $\lambda_0(c,\beta) = \phi(c)-\beta$, and hence is such that $\partial_c\lambda_0(c,\beta)$ is bounded away from zero, while $\lambda_0(\phi^{-1}(\beta),\beta)\equiv 0$, for every $\beta$ in the range of $\phi$. By the (usual) Implicit Function Theorem, it is hence possible to find a map $\beta\mapsto c(\beta)$ such that $\lambda(c(\beta),\beta) = 0$, where $\lambda$ is the map corresponding to $F$. This yields an invariant curve of frequency $\beta$, whenever $\beta\in \mathrm{DC}(\sigma,\tau)$. Furthermore, to obtain the full strength of Moser's Twist Theorem it is suffient to observe as in Theorem 8 that the map associating an invariant curve to $\beta\in\mathrm{DC}(\sigma,\tau)$ is Whitney.

Note that Herman's Last Geometric Theorem cannot be generalized to symplectic maps of $T^*\mathbb{T}^n\simeq\mathbb{T}^n\times\mathbb{R}^n$, for arbitrary $n$, although the twisted normal form writing as in theorem 8 still holds (if we assume that the map admits a Birkhoff normal form writing after canonical coordinate change). The reason is that the map $c\mapsto\beta(c)$, even if it is not locally constant, does not have to pass through a Diophantine vector. Nevertheless, the same argumentation described in this remark does yield Moser's Twist Theorem in arbitrary dimension.

**Maps of the solid torus ([3],[13],[15]).** Consider the case where $F$ is a volume preserving perturbation of a completely integrable solid torus map $F_0(\theta, r) = (\theta + \phi(r), r)$, $\theta \in \mathbb{T}^n$. In the neighborhood of any invariant torus of $F_0$, it is possible to obtain a normal form writing for $F$ as in Theorem 8, as well as a map $c \mapsto \beta(c)$ such that $\lambda(c, \beta(c))$ vanishes. Assume that $\phi$ is nonplanar. In this case the map $c \mapsto \beta(c)$ is also nonplanar which forces it to pass by Diophantine vectors, thus yielding a positive measure set of invariant tori for $F$.

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Bassam Fayad, CNRS LAGA
Université Paris 13, 93430 Villetaneuse, France
et LPMA Université Paris 6, 75252–Paris Cedex 05, France
*email:* fayadb@math.univ-paris13.fr

Raphal Krikorian, Laboratoire de Probabilités et Modèles aléatoires
Université Pierre et Marie Curie, Boite courrier 188
75252–Paris Cedex 05, France
*email:* krikoria@ccr.jussieu.fr