A dichotomy between uniform hyperbolicity and zero Lyapunov epxonents for $SL(2,\mathbb{R})$ cocycles.

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Abstract

1 Introduction

Throughout this paper let $\mathbb{G} = SL(2,\mathbb{R})$.

We will be interested in $\mathbb{G}$-valued cocycles $A : X \to \mathbb{G}$ over dynamics $T : X \to X$ which preserve a finite measure μ on the space X. More concretely, we shall consider the following two general situations:

- *Measurable case:* (X,μ) is non-atomic Lebesgue space, $T : X \to X$ is an automorphism (i.e., a bi-measurable μ-preserving bijection), and $A : X \to \mathbb{G}$ is measurable such that $\log \|A\| \in L^1(\mu)$. The *discrete situation* where A takes finitely many values, is treated separately.

- *Continuous case:* X is a compact manifold of dimension at least 2, μ is a normalized volume measure, $T : X \to X$ is a μ-preserving homeomorphism, and $A : X \to \mathbb{G}$ is continuous.

In either case, the pair (A,T) is called a cocycle. It induces a skew-product map (also called a cocycle) $F_{T,A}\colon X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$ defined by $F_{T,A}(x,v) = (T(x), A(x)v)$. We will use the same notation (A,T) to denote the quotient map at the projective level from $X \times \mathbb{P}^1(\mathbb{R})$ to itself.

Denote

$$A_T^n(x) = A(T^{n-1}(x)) \cdots A(T(x))A(x).$$

The upper Lyapunov exponent of the cocycle (A,T) at $x \in X$, given by

$$\lambda(T,A,x) = \lim_{n\to\infty} \frac{1}{n} \log \|A_T^n(x)\|,$$

exists for μ-almost every $x \in X$. Denote also

$$LE(A,T) = \int_X \lambda(T,A,x)\, d\mu(x).$$

In this paper we address the question, given a cocycle $A : X \to \mathbb{G}$, of the dependence of $LE(A,T)$ on the dynamics T with respect to the following topologies:

- *Measurable case:* Let $Aut(X,\mu)$ be the set of automorphisms $T : X \to X$, endowed with the *weak topology*, according to which $T_n \to T$ iff $\mu(T_n(B) \vartriangle T(B)) \to 0$ for every measurable set $B \subset X$.

- *Continuous case:* Let $Homeo(X,\mu)$ be the set of μ-preserving homeomorphisms $T : X \to X$, endowed with the uniform (C^0) topology.

Remark 1.1. The generic maps $T \in Aut(X,\mu)$ and $T \in Homeo(X,\mu)$ are ergodic; these are classical theorems of Halmos and Rokhlin [H] and Oxtoby and Ulam [OU], respectively. Although we shall not use these results, we use some related ideas from [AP].

1.1 Duality between hyperbolicity and zero exponent

1.1.1 *The measurable case.*

We say that $A : X \to \mathbb{G}$ is *uniformly hyperbolic* if there exists $\lambda_0 > 0$ such that $LE(A,T) \geq \lambda_0$ for all $T \in Aut(X,\mu)$. This will be the case of course if A is a constant hyperbolic matrix, and more generally if the support of the measure $\nu = A_*\mu$ in $\mathbb{G}$ is a *hyperbolic set* in the following sense:

Definition 1.2. *A set $\Sigma \subset \mathbb{G}$ is called* hyperbolic *if there exists $\lambda > 1$ such that*

$$\|A_n \cdots A_1\| > \lambda^n \quad \textit{for all } A_1, \ldots, A_n \in \Sigma.$$

Let us say that A *satisfies duality* if

1. either the support of the measure $\nu = A_*\mu$ in $\mathbb{G}$ is a hyperbolic set and therefore A is uniformly hyperbolic[1]

2. or the set of $T \in Aut(X,\mu)$ for which $LE(A,T) = 0$ is residual in the weak topology.

[1] When the support of $A_*\mu$ is a finite set Σ, we will conjecture later that A can be uniformly hyperbolic without Σ being hyperbolic, cf. Problem 4.8.

Our goal is to express that "most" A's satisfy duality. For technical reasons that will be explained below, it is useful to ask differentiability of A (although this is not very natural since we are working in the measurable category):

Theorem A (Measurable case). *Let X be a C^r-manifold with finitly many connected components and let $r \in \mathbb{N}^*$. Then the set of $A \in C^r(X, \mathbb{G})$ that satisfy duality is open and dense.*

Remark 1.3. In opposition to uniform hyperbolicity, we shall define a condition over A, called *richness*, that guarantees the existence of maps T such that $LE(A,T) = 0$. In fact, we will prove that if A is rich then the generic $T \in Aut(X, \mu)$ satisfies $LE(A,T) = 0$. The richness condition explicit (it doesn't appeal to Baire's theorem, for instance). It only invovles the measure $\nu = A_*\mu$ and provides some "abundance" of matrices in the support of ν that makes it possible to find elliptic products, "mix directions", and make the exponents vanish after a perturbation of the dynamics. It is in showing that in general $A_*\mu$ is either rich or has a hyperbolic support that the differentiability condition on A is actually useful.

Indeed, in the case of differentiable A and compact connected base space X with volume measure μ, duality is actually satisfied by virtually every cocycle A (Cf. §1.3).

Remark 1.4. If the regularity of the dynamics on the base is more restrictive, a cocycle A can switch dramatically from alternative 2 above to alternative 1. This is the case for example for Schrödinger cocycles with polynomial potentials above the torus

$$S_{\lambda,V}(\theta) = \begin{pmatrix} \lambda V(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta \in \mathbb{T}^d, \lambda \in \mathbb{R}$$

and $V(\theta)$ is a trigonometrical polynomial. For these cocycles our theorems will imply alternative 2, while alternative 1 holds if the dynamics in the base is restricted to real analytic maps in some given band, and if λ is greater to some λ_0, by Herman's lower bound on the exponent [He1].

1.1.2 *The continuous case.*

We recall that a cocycle (T, A), $T \in Homeo(X, \mu)$ and $A \in C(X, \mathbb{G})$ is said to be *uniformly hyperbolic* if there exists $C > 0$, $\lambda > 1$ and a continuous section

$$\begin{array}{rccc} E^u: & X & \to & \mathbb{S}^1 \\ & x & \mapsto & E^u(x) \end{array}$$

such that for all $x \in X$ and $n \in \mathbb{N}$ we have $\|A_T^n(x)E^u(x)\| \geq C\lambda^n$. To recover the language of the precedent section we will use the following equivalent definition

Proposition 1.5 (Proposition 2. [Yo]). *A cocycle (T, A), $T \in Homeo(X, \mu)$ and $A \in C(X, \mathbb{G})$ is uniformly hyperbolic iff there exists $C > 0$, $\lambda > 1$ such that*

$$\|A_T^n(x)\| \geq C\lambda^n,$$

for all $n \geq 0$, $x \in X$.

In this setting, let us say that A *satisfies duality* if for the generic $T \in Homeo(X, \mu)$,

1. either the cocycle (A, T) is uniformly hyperbolic;
2. or $LE(A, T) = 0$.

We have

Theorem B (Continuous case). *Let X be a C^r-manifold, $1 \leq r \leq \infty$. Then the C^r-generic $A : X \to \mathbb{G}$ satisfies duality.*

Remark 1.6. Our problem is dual to the one addressed in [B], [BV], [ArB] where the map T on the base is fixed and the cocycle A is perturbed. Our result in the continuous case can in this way be related to the result from [B] stating that for any $T \in Homeo(X, \mu)$, there is a residual set $\mathcal{R}_T \subset C(X, \mathbb{G})$ ($C(X, \mathbb{G})$ being the set of continuous mappings $A : X \to \mathbb{G}$ endowed with the uniform topology) such that if $A \in \mathcal{R}_T$ then either the cocycle (A, T) is uniformly hyperbolic or $LE(A, T) = 0$.

The results of [B] are extended for higher-dimensional matrices in [BV]. The papers [B] and [BV] also deal with the special (and more difficult) situation where T is a volume-preserving (or symplectic) diffeomorphism and A is the derivative of T. There are related results for the measurable case: see [B] and [ArB].

1.1.3 The smooth case.

The phenomena of [B] and [BV] where in the absence of uniform hyperbolicity, the map A can be perturbed in the C^0 topology so as the exponent drops down, may disappear if higher regularity is requested of the perturbation. This is the case for example when the base map T is an Anosov diffeomorphism of the torus $\mathbb{T}^d$, $d \geq 1$, since the exponent is then positif and continuous on an open and dense set of $C^1(\mathbb{T}^d, \mathbb{G})$ [BnV].

Herman's lower bound on the exponent in the analytic setting gives also an example where the dichotomy between uniform hyperbolicity and zero exponents does not hold (Cf. remark 1.4 above). Above Diophantine translations there are open sets of C^r cocycles for which the expoenent is positive and continuous.

In our setting, given a smooth manifol X with a smooth measure μ and a map $A : X \to \mathbb{G}$ of class C^1 such that A is elliptic on some open set of X, we even do not know if there exists a single C^1 diffeomorphism T preserving μ (and ergodic) such that $LE(A, T) = 0$.

Remark 1.7 *(The smooth case. Non conservative dynamics).* We mention here two results obtained by Herman proving abundance of zero exponents in the absence of uniform hyperbolicity for smooth cocycles above uniquely ergodic diffeomorphisms of the circle; here the exponents are computed above the unique invariant measure. The results are based on Baire category arguments and the method used is to approximate the base dynamics by periodic maps and concentrate the measure on orbits above which the product of matrices are elliptic.

Define

$$F_I^\infty = \{f \in \mathrm{Diff}_+^\infty(\mathbb{T}^1), \rho(f) \in \mathbb{R} \setminus \mathbb{Q}\}$$

where $\rho(f) \in \mathbb{R}$ denotes the unique rotation number of f. We consider maps $A \in C^\infty_{\neq 0}(\mathbb{T}^1, \mathbb{G})$, that is, smooth maps that are not homotopic to a constant matrix. The set $\overline{F_I^\infty} \times C^\infty_{\neq 0}(\mathbb{T}^1, \mathbb{G})$ is a Baire space with the C^∞-topology. Then

Proposition 1.8. *[He2] There is a dense G_δ set $G \subset \overline{F_I^\infty} \times C^\infty_{\neq 0}(\mathbb{T}^1, \mathbb{G})$ of cocycles (f, A) such that f is uniquely ergodic and $LE(f, \mu_f, A) = 0$ with μ_f the unique invariant measure of f.*

Here the absence of uniform hyperbolicity is granted by the fact that the cocycle is not homotopic to identity.

The set of smooth maps that are homotopic to a constant matrix is denoted by $C^\infty_0(\mathbb{T}^1, \mathbb{G})$. Then

Proposition 1.9. *[He2] There exists a set $F \in F_I^\infty \times C_0^\infty(\mathbb{T}^1, \mathbb{G})$, such that the C^∞ closure of F, $\overline{F}$ is C^0 dense in the subset of non uniformly hyperbolic cocycles in $F_I^\infty \times C_0^\infty(\mathbb{T}^1, \mathbb{G})$, and such that there is a C^∞ dense G_δ set $G \subset \overline{F}$ of cocycles (f, A) such that f is uniquely ergodic and $LE(f, \mu_f, A) = 0$ with μ_f the unique invariant measure of f.*

1.1.4 *The discrete case.*

We return to the measurable case and consider this time the discrete situation where $A : X \to \mathbb{G}$ assumes a *finite* number of values. Such A cannot satisfy the richness condition, so the previous results do not apply. Nevertheless we can prove a generic duality result:

Theorem C. *Let $N \geq 2$ be an integer. There exists a residual set $\mathcal{R} \subset \mathbb{G}^N$ such that for every $\Sigma \in \mathcal{R}$:*

- *either Σ is uniformly hyperbolic;*
- *or for every measurable map $A\colon X \to \Sigma$ which assumes every value in Σ on a set of positive measure, there is a residual set $\mathcal{R}_A \subset Aut(x, \mu)$ such that for every $T \in \mathcal{R}_A$, $LE(A, T) = 0$.*

Given N-uple of matrices $\Sigma = (A_1, \ldots, A_N)$, we also write Σ for the set $\{A_1, \ldots, A_N\}$.

1.2 Richness and zero exponents.

As we mentionned in a remark above, the property opposed to hyperbolicity of a cocycle is richness. To define this property we need first to introduce some notation. If ν is a measure in $\mathbb{G}$ and $v \in \mathbb{P}^1$, then the push-forwards of ν by the maps

$$M \in \mathbb{G} \mapsto M^{-1} \in \mathbb{G} \quad \text{and} \quad M \in \mathbb{G} \mapsto M \cdot v \in \mathbb{P}$$

are indicated by ν^{-1} and $\nu * v$, respectively. If $n \in \mathbb{N}$, the push-forward of ν^n by the map

$$(M_1, \ldots, M_n) \in \mathbb{G}^n \mapsto M_n \cdots M_1 \in \mathbb{G}$$

is indicated by ν^{*n}. Finally, assuming that ν is finite and has bounded support, we write

$$|\nu| = \nu(\mathbb{G}) \quad \text{and} \quad \|\nu\|_\infty = \inf\{C > 1;\ \|A\| < C \text{ for } \nu\text{-a.e. } A \in \mathbb{G}\},$$

where $\|\cdot\|$ is some fixed operator norm.

Definition 1.10. *Let ν be a finite measure on $\mathbb{G}$ of bounded support. ν is called* rich *if there are $N \in \mathbb{N}$ and $\kappa > 0$ such that for every $v \in \mathbb{P}^1$ we have*

$$\nu^{*N} * v \geq \kappa m \quad \text{and} \quad (\nu^{*N})^{-1} * v \geq \kappa m,$$

where m denotes Lebesgue measure in $\mathbb{P}^1$. If $N = 1$, we call ν spreading.

The richness property is studied in appendix 5.4 where the following criteria are obtained:

Proposition 1.11. *Let M be a compact manifold (maybe with dimension* 1*, maybe not connected, maybe with boundary), with a smooth volume measure μ, and let $A : M \to \mathbb{G}$ be a C^1 map. Assume there are points $p_1, \dots, p_k \in M$ such that the matrix $A(p_k) \cdots A(p_1)$ is elliptic and moreover A is not locally constant at at least one of the p_i's. Then $A_*\mu$ is rich.*

Proposition 1.12. *If ν is a measure in $\mathbb{G}$ such that there is an open set $U \subset \mathbb{G}$ of elliptic matrices and $\kappa > 0$ such that $\nu|_U \geq \kappa h|_U$, where h is Haar measure in $\mathbb{G}$, then ν is rich.*

Our main theorems, that imply theorems A and B, are:

Theorem D (Measurable case). *Assume $A : X \to \mathbb{G}$ is rich. Then there is a residual set $\mathcal{R}_A \subset Aut(X, \mu)$ such that $LE(A, T) = 0$ for all $T \in \mathcal{R}_A$.*

Remark 1.13. There is a residual subset $\mathcal{R} \subset C(X, \mathbb{G})$ such that for every $A \in \mathcal{R}$, the conclusion of theorem D is true. This follows from the above-mentioned result from [B] and proposition 5.4.

Theorem E (Continuous case). *Assume the pair (A, T) is rich. Then for every $\varepsilon > 0$ there is $\tilde{T} \in Homeo(X, \mu)$ arbitrarily close to T such that $LE(A, \tilde{T}) < \varepsilon$.*

In the proof of the above theorem we use theorem D and tools from [AP].

Corollary 1.14 (Continuous case). *Assume the pair (A, T) is stably rich, that is, (A, S) is rich for every S in a neighborhood $\mathcal{U} \subset Homeo(X, \mu)$ of T. Then there is a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that $LE(A, \tilde{T}) = 0$ for all $\tilde{T} \in \mathcal{R}$.*

1.3 Complete classification in the case of smooth cocycle A and connected space X.

Theorem F. *Let X be a compact connected manifold, and μ be a volume measure. Let $A : X \to \mathbb{G}$ be a C^1 map. Then*

(i) either there is a closed interval $I \subsetneq \mathbb{P}^1$ such that $A(x) \cdot I \subset I$ for every $x \in X$;

(ii) or $LE(A, T) = 0$ for the generic $T \in Aut(X, \mu)$.

We study now the Lyapunov exponent if the first alternative holds. There are three possibilities:

1. The interval I is a point;

2. the interval I is not a point but $A(x) \cdot I \cap \partial I \neq \emptyset$ for all $x \in X$;

3. the interval I is not a point and $A(x) \cdot I \subset I^{\circ}$ for some $x \in X$.

For an open and dense set of $A \in C^1(X, \mathbb{G})$, (i) occurs under alternative 3. In this case we have

Claim: Under alternative 3 *above, A is uniformly hyperbolic; that is, there exists $\lambda_0 > 0$ such that $LE(A,T) > \lambda_0$ for all $T \in Aut(X, \mu)$.*

With this claim, theorem A clearly follows from F.

Proof of the claim. Let us recall some facts about the Hilbert metric. If $I = (a, b) \subset \mathbb{P}^1$ is an open interval, then

$$d_I(x, y) = |\phi_I(x) - \phi_I(y)|,$$

where $\phi_I : I \to \mathbb{R}$ is given by

$$\phi_I(x) = \frac{1}{2} \log \frac{|x - a|}{|b - x|}.$$

Let I and J be open intervals, and $A \in \mathbb{G}$. Then:

1. $d_{A(I)}(Ax, Ay) = d_I(x, y)$.

2. If $\bar{J} \subset I$ then there is some $\eta < 1$ such that $d_I \leq \eta d_J$.

3. Assume $A(I) \subset I$ and let $0 < \eta \leq 1$ be the Lipschitz constant of $A : I \to I$. Then the spectral radius $\rho(A) \geq \eta^{-1/2}$.

Now we come back to the proof of the claim. By continuity, there is an open set of $x \in X$ such that $A(x)(I) \subset I^{\circ}$. Let $\eta(x)$ be the Lipschitz constant of $A(x) : I^{\circ} \to I^{\circ}$, and

$$\lambda_0 = -\frac{1}{2} \int_X \log \eta \, d\mu.$$

Then $\lambda_0 > 0$ and $LE(A,T) \geq \lambda_0$ for all $T \in Aut(X, \mu)$. □

We continue our study of the Lyapunov exponent under alternative (i). In the subcase 1, there is $v_0 \in \mathbb{P}^1$ such that $A(x)v_0 = v_0$ for all x then the Lyapunov exponent does not depend on the dynamics:

$$LE(A,T) = \left| \int \log \frac{\|A(x)v_0\|}{\|v_0\|}\, d\mu(x) \right|,$$

for all $T \in Aut(X,\mu)$.

In the second subcase, the Lyapunov exponent may depend on the dynamics, and can be non-zero or zero (notice that sometimes 2nd subcase can be reduced to the 1st).

For example, let $X = [0,1]$ and

$$A(x) = \begin{pmatrix} 1 & b(x) \\ c(x) & 1 \end{pmatrix},$$

with $b \geq 0$, $c \geq 0$, $b = 0$ in $X_2 = [1/2, 1]$, $c = 0$ in $X_1 = [0, 1/2]$. If $T \in Aut(X,\mu)$ preserves X_1 and X_2 then $LE(A,T) = 0$. Otherwise, $LE(A,T) > 0$.

In 2nd subcase, $LE(A,\cdot)$ seems to be continuous and > 0 in an (open and) dense set (that includes all ergodic T).

To prove theorem F, we will need the following:

Lemma 1.15. *Let $\Sigma \subset \mathbb{G}$ be a compact connected set. Assume that there is no closed interval $I \subsetneq \mathbb{P}^1$ such that $A \cdot I \subset I$ for every $A \in \Sigma$. Then there are $A_1, \ldots, A_n \in \Sigma$ such that $A_1 \cdots A_n$ is elliptic.*

Proof. We claim that there is $n_0 \in \mathbb{N}$ such that for all v, $w \in \mathbb{P}^1$, there exist $A_1, \ldots, A_{n_0} \in \Sigma$ such that $A_1 \cdots A_{n_0} v = w$.

Indeed, fix any matrix $A_0 \in \Sigma$, and let $v_0 \in \mathbb{P}^1$ be such that $A_0 v_0 = v_0$. Let $I_n \subset \mathbb{P}^1$ be the set of directions $A_1 \cdots A_n(v_0)$, with $A_i \in \Sigma$. Since Σ is connected, each I_n is an interval or the circle. Also, $I_n \subset I_{n+1}$, because v_0 is invariant by a matrix in Σ. Let us see that $I_{n_1} = \mathbb{P}^1$ for some n_1. Assume the contrary, and let $I = \bigcup_n I_n$. We have $A(\bar{I}) \subset \bar{I}$ for all A. Since we are assuming Σ has no invariant interval, we must have $\bar{I} = \mathbb{P}^1$. Therefore $I = \mathbb{P}^1 \setminus \{z\}$ for some z. By the same assumption, there must be $A \in \Sigma$ such that $A(z) \neq z$. Then $A^{-1}(z) \in I$ and so there must exist $A_1, \ldots, A_n$ such that $A_1 \cdots A_n(v_0) = A^{-1}(z)$. But this implies $z \in I$, a contradiction.

We have shown that there is n_1 such that for any w there is a product of length n_1 which sends v_0 to w. The same reasoning applied to the set Σ^{-1} (which does not have an invariant interval as well) gives that there is

n_2 such that for any v there is a product of length n_2 sending v to v_0. Let $n_0 = n_1 + n_2$. The claim is proved.

Next we construct an elliptic product. Fix any $A \in \Sigma$, $A \neq id$. If A is elliptic, we are done.

If A is hyperbolic, with eigenvalues e^+, e^-, then let B be a product such that $B(e^+) = e^-$. Then a calculation shows that $\operatorname{tr} A^n B \to 0$ as $n \to \infty$, so there exist an elliptic product.

If A is parabolic then, relative to some basis $\{e_1, e_2\}$,

$$A = \pm \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad \text{with } \beta \neq 0.$$

Let B be a product such that $Be_1 = \mathbb{R}e_2$. Write

$$B = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}, \quad \text{with } c \neq 0.$$

Then $|\operatorname{tr} A^n B| = |\beta c n + d| \to \infty$ as $n \to \infty$. This shows that Σ has an hyperbolic product. Then we can repeat the previous reasoning and find an elliptic product. □

We are ready now to give the

Proof of theorem F. If the function A is constant then the first case holds if A is hyperbolic or parabolic, and the second case holds if A is elliptic. So we can assume A is not constant.

Assume the first case does not hold. Applying lemma 1.15 to $\Sigma = \{A(x);\ x \in X\}$, we conclude that there is $n \in \mathbb{N}$ such that the function

$$(x_1, \ldots, x_n) \in X^n \mapsto A(x_n) \cdots A(x_1)$$

assumes an elliptic value. This function is not constant and X^n is connected, so proposition 1.11 applies and the exponent vanishes generically by theorem D. □

2 Proof of theorem D

In all this section, (X, μ) denotes a non-atomic Lebesgue space. We begin introducing some notation: If $A : (X, \mu) \to \mathbb{G}$ is a bounded measurable map, $k \in \mathbb{N}$, and $T \in Aut(X, \mu)$, let

$$\Lambda_k(A, T) = \frac{1}{k} \int_X \log \|A_T^k\| \, d\mu \, .$$

Observe that by subadditivity of $\int_X \log \|A_T^k\| d\mu$ we have $LE(A,T) \leq \Lambda_k(A,T)$.

We define the *infimum exponent of order* k of A

$$\Lambda_k(A) = \inf_{T \in Aut(X,\mu)} \Lambda_k(A,T).$$

In §2.1 we will study some properties of the maps $A \mapsto \Lambda_k(A)$ and show that they enjoy useful properties of continuity and convexity and that they actually depend only on the push-forward measure $A_*\mu$. We will then show that if $A_*\mu$ is spreading then $\Lambda_k(A)$ tends to zero as k tends to infinity. Finally, for $A : X \to \mathbb{G}$ rich and $T \in Aut(X,\mu)$, we will consider induced cocycles from (A,T) (cf. §2.5 and apply the latter result to perturb T so that the Lyapunov exponent under A comes close to zero.

2.1 Infimum exponent of order k.

The continuity property of Λ_k states as follows:

Lemma 2.1. *Given $k \in \mathbb{N}$, $C > 1$ and $\delta > 0$, there exists $\eta > 0$ with the following properties: If A, $B : (X,\mu) \to \mathbb{G}$ are measurable maps with $\|A\|_\infty$, $\|B\|_\infty \leq C$ and*

$$\|A - B\|_1 = \int_X \|A - B\| \, d\mu < \eta$$

then

$$|\Lambda_k(A) - \Lambda_k(B)| < \delta.$$

Proof. We take $\eta = C^{-k+1}\delta$. The lemma follows from the facts below:

1. If $x \geq 1$ and $y > 0$ then $\log(x+y) \leq \log x + y$;
2. if A_i, $B_i \in \mathbb{G}$ satisfy $\|A_i\|$, $\|B_i\| \leq C$ then

$$\|B_k \cdots B_1 - A_k \cdots A_1\| \leq C^{k-1} \sum_{i=1}^{k} \|B_i - A_i\|.$$

□

The map Λ_k has a convexity property:

Lemma 2.2. *Let ν_1, ν_2 be probability measures in $\mathbb{G}$, with bounded supports. If $0 \leq t \leq 1$ then*

$$\Lambda_k(t\nu_1 + (1-t)\nu_2) \leq t\Lambda_k(\nu_1) + (1-t)\Lambda_k(\nu_2).$$

Proof. Let $A_i : \mathbb{I} \to \mathbb{G}$ be such that $(A_i)_* m = \nu_i$. Let $f_1 : [0,t] \to \mathbb{I}$ and $f_2 : [t,1] \to \mathbb{I}$ be affine bijections. Given $T_1, T_2 \in Aut(\mathbb{I}, m)$, let

$$A(x) = \begin{cases} A_1 \circ f_1 & \text{in } [0,t), \\ A_2 \circ f_2 & \text{in } [t,1], \end{cases} \quad T(x) = \begin{cases} f_1^{-1} \circ T_1 \circ f_1 & \text{in } [0,t), \\ f_2^{-1} \circ T_2 \circ f_2 & \text{in } [t,1]. \end{cases}$$

Then $A_* m = t\nu_1 + (1-t)\nu_2$, $T \in Aut(\mathbb{I}, m)$, and

$$\Lambda_k(A,T) = t\Lambda_k(A_1,T_1) + (1-t)\Lambda_k(A_2,T_2).$$

The result follows. □

An important property of $\Lambda_k(A)$ is that it depends only on the push-forward $A_*\mu$:

Lemma 2.3. *Let $A, B : (X,\mu) \to \mathbb{G}$ be such that $A_*\mu = B_*\mu$. Then $\Lambda_k(A) = \Lambda_k(B)$.*

Remark 2.4. If $A : X \to \mathbb{G}$ is measurable and bounded, and $S : (X,\mu) \to (X',\mu')$ is an isomorphism, it is clear that $\Lambda_k(A \circ S) = \Lambda_k(A)$, because $\Lambda_k(A \circ S, T) = \Lambda_k(A, S \circ T \circ S^{-1})$ for every $T \in Aut(X,\mu)$.

Based on the lemma and remark above, we can introduce the following notation: If ν is a finite measure in $\mathbb{G}$ with bounded support, and $k \in \mathbb{N}$, we write

$$\Lambda_k(\nu) = \Lambda_k(A),$$

where $A : (\mathbb{I}, m) \to \mathbb{G}$ is any map such that $A_* m = \nu$.

To prove lemma 2.3 we will need lemmas 2.5 and 2.6 below

Lemma 2.5. *Let $A : \mathbb{I}^2 \to \mathbb{G}$ be measurable and bounded. Let $\pi : \mathbb{I}^2 \to \mathbb{I}$ be the projection on the first coordinate, and consider the map*

$$A \circ \pi : (\mathbb{I}^2, m) \to \mathbb{G}.$$

Then $\Lambda_k(A \circ \pi) = \Lambda_k(A)$ for every $k \in \mathbb{N}$.

We use m to denote the Lebesgue measure on $\mathbb{I}$ as well as on $\mathbb{I}^2 = \mathbb{I} \times \mathbb{I}$.

The idea of the proof is to approximate π by something invertible and to use remark 2.4. and the continuity property in lemma 2.1.

Proof. It is clear that $\Lambda_k(A \circ \pi) \le \Lambda_k(A)$, because $\Lambda_k(A \circ \pi, T \times id) = \Lambda_k(A,T)$ for any $T \in Aut(\mathbb{I}, m)$. Fix $\delta > 0$. Let $T \in Aut(\mathbb{I}^2, m)$ be such that $\Lambda_k(A \circ \pi, T) < \Lambda_k(A \circ \pi) + \delta$.

For $n \in \mathbb{N}$, define an isomorphism $P_n : \mathbb{I}^2 \to \mathbb{I}$ such that if $I \subset \mathbb{I}$ is a dyadic interval with $|I| = 2^{-n}$ then $P_n(I \times \mathbb{I}) = I$. Then the functions π and $P_n : \mathbb{I}^2 \to \mathbb{I}$ are uniformly 2^{-n}-close. This implies L^1-convergence:

$$\lim_{n\to\infty} \|A \circ P_n - A \circ \pi\|_1 = 0.$$

Indeed, given $\varepsilon > 0$, Lusin's theorem gives a compact $K \subset \mathbb{I}$ such that $A|_K$ is continuous and $m(K^c) < \varepsilon$. If n is large enough then for every $x, y \in K$ that are 2^{-n}-close we have $\|A(x) - A(y)\| < \varepsilon$. Let $G_n = \pi^{-1}(K) \cap P_n^{-1}(K)$; then $m(G_n^c) < 2\varepsilon$. Thus

$$\int_{\mathbb{I}} \|A \circ P_n - A \circ \pi\| \, dm = \int_{G_n} (\cdots) + \int_{G_n^c} (\cdots) < \varepsilon + 2\varepsilon\|A\|_\infty.$$

By lemma 2.1, if n is sufficiently large then $\Lambda_k(A \circ P_n, T) < \delta + \Lambda_k(A \circ \pi)$. Let $T' = P_n \circ T \circ P_n^{-1}$; then $\Lambda_k(A, T') = \Lambda_k(A \circ P_n, T)$. This shows that $\Lambda_k(A) < \delta + \Lambda_k(A \circ \pi)$. Since $\delta > 0$ is arbitrary, the lemma follows. $\square$

Following Rokhlin we can state

Lemma 2.6. *Let $A, A' : \mathbb{I} \to \mathbb{G}$ be measurable functions such that $A_* m = A'_* m = \nu$. Then there exists $S \in Aut(\mathbb{I}^2, m)$ such that $A' \circ \pi \circ S = A \circ \pi$ m-a.e.*

Now we can give the:

Proof of lemma 2.3. ¿From remark 2.4 we can assume that A and B are defined over $(X, \mu) = (\mathbb{I}, m)$. Since $A_* m = B_* m$, by lemma 2.6 there is an automorphism S such that $A \circ \pi = B \circ \pi \circ S$. In particular, $\Lambda_k(A \circ \pi) = \Lambda_k(B \circ \pi)$. So, by lemma 2.5, $\Lambda_k(A) = \Lambda_k(B)$. $\square$

2.2 An existence result

Now we prove that if a measure ν is spreading then $\Lambda_k(\nu) \to 0$ as $k \to \infty$. This implies the existence for A such that $A_* m = \nu$ of dynamics in $Aut(\mathbb{I}, m)$ with arbitrarilly small exponent.

Proposition 2.7. *Let $C > 1$, $\delta > 0$, and σ be a spreading measure with $\|\sigma\|_\infty \le C$. Then there exists $k \in \mathbb{N}$ with the following properties: If ω is a measure in $\mathbb{G}$ such that $|\omega| \le 1$ and $\|\omega\|_\infty \le C$ then*

$$\Lambda_k(\omega + \sigma) < |\omega|\delta + |\sigma| \log C.$$

The fact that k depends uniformly on ω, provided $\|\omega\|_\infty \le C$ will be important in the proof of theorem D. It also allows to observe that $\Lambda_k(\nu) \to_{k\to\infty} 0$ if ν is a bounded spreading measure since $\tau\nu$ is spreading for any $\tau > 0$ (write $\nu = (1-\tau)\nu + \tau\nu$).

Proof. Definition of k: Given $C > 1$, there is $\ell_0 \in \mathbb{N}$ with the following properties: If $H, R \in \mathbb{G}$ are matrices such that

- $\|H\| \le C$, $\|R\| \le C^2$;
- H is a hyperbolic matrix with expanding and contracting eigendirections e^u and $e^s \in \mathbb{P}^1$;
- $R(e^u) = e^s$;

then the matrix RH^ℓ is elliptic for every $\ell \ge \ell_0$.

Given the spreading measure σ, let $\kappa > 0$ be as in definition 1.10. Fix an integer

$$\ell \ge \max\{\ell_0, 2/\kappa\}.$$

Given $\delta > 0$, we take $k' \in \mathbb{N}$ such that if $E \in \mathbb{G}$ is an elliptic or parabolic matrix with $\|E\| \le C^{\ell+2}$ then

$$\frac{1}{n}\log\|E^n\| < \frac{\delta}{4} \quad \forall n \ge k'.$$

Finally take $k \in \mathbb{N}$ such that

$$k \ge (\ell+2)(k'+2) \quad \text{and} \quad \frac{1}{k}\log C^{2(\ell+2)} < \frac{\delta}{4}.$$

First case: We will first prove the proposition in the case where ω is a Dirac measure δ_H on some $H \in \mathbb{G}$.

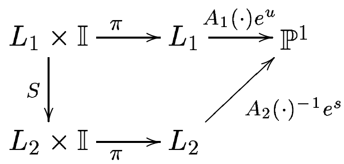
If H is elliptic or parabolic, we simply take $T = id$; then for any A such that $A_*m = \omega + \sigma$, we have $\Lambda_k(A,T) < |\omega|\delta + |\sigma|\log C$.

So we assume H is a hyperbolic matrix. Let e^u and $e^s \in \mathbb{P}^1$ be its expanding and contracting eigendirections, respectively. Since σ is spreading, we have

$$\sigma * e^u \ge \kappa m, \quad \sigma^{-1} * e^s \ge \kappa m.$$

There are measures $\sigma_1, \sigma_2 \le \sigma$ such that $\kappa m = \sigma_1 * e^u$ and $\kappa m = \sigma_2^{-1} * e^s$.[2] Let $L_1 \subset J_1$, $L_2 \subset J_2$ be intervals with $|L_i| = \frac{1}{2}\kappa = \frac{1}{2}|\sigma_i|$, $|J_i| = \frac{1}{2}|\sigma|$, $J_1 \cap J_2 = \emptyset$.

Consider two measurable maps $A_i : J_i \to \mathbb{G}$ ($i = 1, 2$) such that $(A_i)_*(m|_{L_i}) = \frac{1}{2}\sigma_i$ and $(A_i)_*(m|_{J_i \setminus L_i}) = \frac{1}{2}(\sigma - \sigma_i)$. By lemma 2.6, there exists an isomorphism S such that the following diagram commutes a.e:



That is, for every $z \in L_1 \times \mathbb{I}$,

$$(A_2 \circ \pi)(S(z)) \cdot (A_1 \circ \pi)(z) \cdot e^u = e^s.$$

Define a convenient Lebesgue space to work in:

$$X = \mathbb{I} \sqcup (J_1 \times \mathbb{I}) \sqcup (J_2 \times \mathbb{I}),$$

The measure μ in X restricted to $\mathbb{I}$, resp. $J_i \times \mathbb{I}$, is one, resp. two, dimensional Lebesgue measure. The map $A : X \to \mathbb{G}$ is defined as $A = H$ in $\mathbb{I}$, $A = A_1 \circ \pi$ in $J_1 \times \mathbb{I}$, and $A = A_2 \circ \pi$ in $J_2 \times \mathbb{I}$. Then $A_*\mu = \delta_H + \sigma$. At last, we define the measure-preserving dynamical system $T : X \to X$. Break $\mathbb{I}$ into disjoint intervals $I_1, \ldots, I_\ell$ of equal measure $m(I_1) = 1/\ell$. Since $1/\ell \le \kappa/2$, we can take a set $Z \subset L_1 \times \mathbb{I}$ with $m(Z) = m(I_1)$. Let $U_1 : I_\ell \to Z$ be an isomorphism. We define T as being the identity in

$$\mathcal{I} = (J_1 \times \mathbb{I} \setminus Z) \cup (J_2 \times \mathbb{I} \setminus S(Z)),$$

and in the rest as

$$I_1 \to I_2 \to \cdots \to I_\ell \xrightarrow{U_1} Z \xrightarrow{S} S(Z) \xrightarrow{U_2} I_1\,,$$

where the unspecified arrows are translations and the isomorphism U_2 is chosen so that $T^{\ell+2}|_{I_1}$ is identity. Note that $\int_{\mathcal{I}} \log \|A\| d\mu \le \log C|\sigma|$. On the other hand, given $p \in \mathbb{N}$, the product matrices along T of length $p(l+2)$ above I_1 are of the form $E(z)^p$ where

$$E(z) = A(S \circ T^{\ell-1}z)A(T^{\ell-1}z)H^\ell, \quad z \in I_1,$$

[2]Given $f : (X, \mu) \to (Y, \nu)$ homomorphism and $\nu_1 \le \nu$, define a measure μ_1 in X by $\frac{d\mu_1}{d\mu} = \frac{d\nu_1}{d\nu} \circ f$; then $f_*\mu_1 = \nu_1$.

with $A(S \circ T^{\ell-1}z)A(T^{\ell-1}z) \cdot e^u = e^s$. By our choice of ℓ, $E(z)$ is elliptic for every $z \in I_1$. Also, $\|E(z)\|_\infty \le C^{\ell+2}$. It then follows from our choice of k that $\Lambda_k(A,T) < \frac{\delta}{2} + |\sigma| \log C$.

General case: Now let ν be any measure satisfying the hypotheses of the lemma. Let $I_1 = [0, |\sigma|]$, $I_2 = [|\sigma|, |\sigma| + |\omega|]$ $A : I_1 \cup I_2 \to \mathbb{G}$ be such that

$$A_*(m|_{I_1}) = \sigma \quad \text{and} \quad A_*(m|_{I_2}) = \omega.$$

Let $\eta = \eta(k, C, |\omega|\delta/2)$ be given by lemma 2.1.

Let $B : I_2 \to \mathbb{G}$ be a simple function such that $\|A|_{I_2} - B\|_1 < \eta$. Extend B to $\mathbb{I}$ by taking $B = A$ in I_1. We can write

$$B_* m = \sigma + \sum_{i=1}^{n} t_i \delta_{H_i} = (1 - |\omega|)\sigma + \sum_{i=1}^{n} t_i (\delta_{H_i} + \sigma),$$

where $H_i \in \mathbb{G}$, $t_i \ge 0$ and $\sum_{i=1}^n t_i = |\omega| \le 1$.

By lemma 2.2 and the case already considered,

$$\begin{aligned}
\Lambda_k(B) &\le (1 - |\omega|)\Lambda_k(\sigma) + \sum t_i \Lambda_k(\delta_{H_i} + \sigma) \\
&\le (1 - |\omega|)|\sigma| \log \|\sigma\|_\infty + \left(\sum t_i\right)\left(\tfrac{\delta}{2} + |\sigma| \log \|\sigma\|_\infty\right) \\
&= \tfrac{1}{2}|\omega|\delta + |\sigma| \log \|\sigma\|_\infty.
\end{aligned}$$

Since $\|A - B\|_1 < \eta$, we obtain $\Lambda_k(A) < |\omega|\delta + |\sigma| \log \|\sigma\|_\infty$. This proves the proposition. □

2.3 Perturbing measures

We adopt a definition of closeness in the space of measures which is suitable to our purposes:

Definition 2.8. *Let ν_1, ν_2 be measures in $\mathbb{G}$ with bounded support and same mass $|\nu_1| = |\nu_2| = \nu$. Given $\eta > 0$, we say that ν_1 and ν_2 are η-*close *if there exists $A_1, A_2 : ([0, \nu], m) \to \mathbb{G}$ such that $(A_i)_*\mu = \nu_i$ and $\|A_1 - A_2\|_1 < \eta$.*

We can define a distance $d(\nu_1, \nu_2)$ as the infimum of the η such that ν_1 and ν_2 are η-close in the sense above. That this is indeed a metric follows from the lemma below:

Lemma 2.9. *Let $A : X \to \mathbb{G}$ and $\nu = A_*\mu$. If $\tilde{\nu}$ is η-close to ν then there exists $\tilde{A} : X \to \mathbb{G}$ such that $\|\tilde{A} - A\|_1 < \eta$ and $\tilde{\nu} = \tilde{A}_* m$.*

Proof. Without loss of generality we assume $|\nu| = 1$. By assumption, there are $A_1, A_2 : \mathbb{I} \to \mathbb{G}$ such that $(A_1)_* m = \nu$, $(A_2)_* m = \tilde{\nu}$, and $\eta' = \eta - \|A_1 - A_2\|_1 > 0$.

By lemma 2.6, there exists an $S \in Aut(\mathbb{I}^2, m)$ such that and $A_1 \circ \pi \circ S = A \circ \pi$ a.e. Let $P_n : \mathbb{I}^2 \to \mathbb{I}$ be as is the proof of lemma 2.5 and choose $n \in \mathbb{N}$ large enough so that $\|A \circ \pi - A \circ P_n\|_1 < \eta'$. Define $\tilde{A} = A_2 \circ \pi \circ S \circ P_n^{-1}$. Then $\tilde{A}_* m = \tilde{\nu}$ and

$$\|\tilde{A} - A\|_1 = \|\tilde{A} \circ P_n - A \circ P_n\|_1 = \|A_2 \circ \pi \circ S - A \circ P_n\|_1 \leq$$
$$\leq \|A_2 \circ \pi \circ S - A_1 \circ \pi \circ S\|_1 + \|A \circ \pi - A \circ P_n\|_1 < \|A_2 - A_1\|_1 + \eta' = \eta.$$

□

We will also need the following:

Lemma 2.10. *Let $A : (X, \mu) \to \mathbb{G}$, $\nu = A_* \mu$ and $\sigma \leq \nu$ Then for every $\eta > 0$ there exists a measurable set $Y \subset X$ such that $A_*(\mu|_Y)$ is η-close to σ.*

Proof. Let $f : \mathbb{G} \to \mathbb{I}$ be the Radon-Nikodym derivative $\frac{d\sigma}{d\nu}$. Define $Y_0 = \{(x, t) \in X \times \mathbb{I};\ 0 \leq t \leq f \circ A(x)\}$. Let $P_n : X \times \mathbb{I} \to X$ be as is the proof of lemma 2.5, with $n \in \mathbb{N}$ large enough so that $\|A \circ \pi - A \circ P_n\|_1 < \eta$. Let $Y = P_n(Y_0)$. Then $A_*(\mu|_Y) = (A \circ P_n)_*((\mu \times m)|_{Y_0})$ is η-close to $(A \circ \pi)_*((\mu \times m)|_{Y_0})$. The later measure equals σ. Indeed,

$$\big((A \circ \pi)_*((\mu \times m)|_{Y_0})\big)(Z) = (\mu \times m)\big(Y_0 \cap \pi^{-1}(A^{-1}(Z))\big)$$
$$= \int_{A^{-1}(Z)} f \circ A \, d\mu = \int_Z f \, d\nu = \sigma(Z),$$

for any measurable $Z \subset \mathbb{G}$. □

Remark 2.11. Since all Lebesgues with probability measures are isomorphic, lemmas 2.9 and 2.10 can be stated with (X, μ) instead of $(\mathbb{I}, m)$.

2.4 Towers and convolutions

In this section we show that convolution measures can be dynamically approximated in the following sense:

Lemma 2.12. *Let $A : \mathbb{I} \to \mathbb{G}$ be bounded and $\nu = A_* m$. Given $N \in \mathbb{N}$ and $\eta > 0$ there exists $F \in Aut(\mathbb{I}, m)$ and a set $Z \subset \mathbb{I}$ such that $F^N = id$, the sets Z, $F(Z)$, ..., $F^{N-1}(Z)$ are disjoint, and the measure $(A_F^N)_*(m|_Z)$ is η-close to $\frac{1}{N}\nu^{*N}$.*

Proof. If $N = 1$ there is nothing to prove; assume $N \geq 2$. Let us first consider the case where A has a special form, namely there is $M \in \mathbb{N}$ such that A restricted to each interval $I_j = [\frac{j-1}{M}, \frac{j}{M})$, $j = 1, \ldots, M$, is constant, say equal to A_j. Then

$$\nu = \frac{1}{M}\sum_{j=1}^{M}\delta_{A_j} \quad \text{and} \quad \nu^{*N} = \frac{1}{M^N}\sum_{\underline{j}\in\{1,\ldots,M\}^N}\delta_{A_{j_N}\cdots A_{j_1}}.$$

Break each interval I_j into NM^{N-1} disjoint intervals of equal length, $I_{j,k}$, $k = 1, \ldots, NM^{N-1}$. Take a bijection

$$\{1,\ldots,N\}\times\{1,\ldots,M\}^N \to \{1,\ldots,M\}\times\{1,\ldots,NM^{N-1}\}$$

of the form $(i,\underline{j}) \mapsto (j_i, k(i,\underline{j}))$, where $\underline{j} = (j_1,\ldots,j_M)$.

Write $J_{i,\underline{j}} = I_{j_i,k(i,\underline{j})}$; then $\{J_{i,\underline{j}}\}_{i,\underline{j}}$ is a partition of $\mathbb{I}$. Define $F : \mathbb{I} \to \mathbb{I}$ by mapping each $J_{i,\underline{j}}$ to $J_{i+1,\underline{j}}$ (and $J_{N,\underline{j}}$ to $J_{1,\underline{j}}$, say) by a translation.

Let

$$Z = \bigsqcup_{\underline{j}\in\{1,\ldots,M\}^N} J_{1,\underline{j}}.$$

Then Z, $F(Z)$, $\ldots$, $F^{N-1}(Z)$ are disjoint and $(A_F^N)_*(m|_Z) = \frac{1}{N}\nu^{*N}$.

General case: Given any $A : \mathbb{I} \to \mathbb{G}$ bounded, define $C = \|A\|_\infty$ and assume that $N \in \mathbb{N}$ and $\eta > 0$ are aribtrarily chosen. Let $\tilde{A} : \mathbb{I} \to \mathbb{G}$ be a bounded simple function which is $\upsilon(C,N,\eta) - L^1$-close to A and such that $\tilde{A}$ has M level sets, all with the same measure $1/M$; where M is some integer and $\upsilon(C,N,\eta)$ will be defined later.

Take $S \in Aut(\mathbb{I},m)$ that map these level sets to intervals, so $\hat{A} = \tilde{A}\circ S$ falls in the later case. Accordingly there exist $\hat{F} \in Aut(\mathbb{I},m)$ and a set $\hat{Z} \subset I$ such that $\hat{Z}$, $\hat{F}(\hat{Z})$, $\ldots$, $\hat{F}^{N-1}(\hat{Z})$ are disjoint and $(\hat{A}_{\hat{F}}^N)_*(m|_{\hat{Z}}) = \frac{1}{N}(\hat{A}_*m)^{*N}$.

Let $F = S^{-1}\circ\hat{F}\circ S$ and $Z = S^{-1}(\hat{Z})$. Then $(\tilde{A}_F^N)_*(m|_Z) = \frac{1}{N}(\tilde{A}_*m)^{*N}$. From point 2 of the proof of lemma 2.1 we see that $\upsilon(C,N,\eta)$ can be chosen so that the $\upsilon(C,N,\eta) - L^1$ closeness of $\tilde{A}$ and A implies that $(A_*m)^{*N}$ and $(\tilde{A}_*m)^{*N}$ are $\eta/2$ close (in the sense of definiton 2.8 as well as $(A_F^N)_*(m|_Z)$ and $(\tilde{A}_F^N)_*(m|_Z)$. This concludes the proof.

□

2.5 End of the proof

Proof of theorem D. Assume $A : X \to \mathbb{G}$ is such that $\nu = A_*\mu$ is rich and let $T \in Aut(X,\mu)$. We have to show that for any $\delta, \in \mathbb{R}_+^*$, there exists $\tilde{T} \in$

$Aut(X, \mu)$ arbitrarily close to T in the weak topology such that $LE(A, \tilde{T}) < \delta$. Since we are working in the measurable category we can assume that X is the unit interval $\mathbb{I}$ and μ is the Lebesgue measure on it.

Let $\varepsilon > 0$ be fixed. By density of cyclic interval permutations, we can assume that T is a cyclic interval permutation and assume its rank M satisfies $M \geq 4/\varepsilon$.

By definition of richness, there exists σ spreading with $\sigma \leq \nu^{*N}$ for some N.

Before going into the details of the proof let us sketch how we will obtain the perturbation $\tilde{T}$ that will actually satisfy $m[\tilde{T} \neq T] < \varepsilon$ The perturbation is done in two steps. In the first one, we will use richness of the measure ν to produce a map T_1 that is close to T and that has two cyclic towers: a (big) cyclic tower of height M that fills most of the space and that comes from the original tower of T, a (small) cyclic tower of height N such that the push forward of the measure on its base by the product of A's along its N levels is a measure close to a spreading one, namely σ/N (from lemma 2.12). Then we consider the first return on the set W equal to the union of the basis of the big and the small tower: we obtain an induced cocycle over W with identity for dynamics and a matrix map $\hat{A}$ such that $\hat{A}_*(m_| W)$ contains a part that is close to σ/N.

Here we pass to the second step and perturb T_1 in $\tilde{T}$, keeping the two towers above W invariant but modifying the first return map in a map S so that the Lyapunov exponent $LE(\hat{A}, S)$ of the induced cocycle becomes small (this is done by taking $\tilde{T}$ equal to T_1 except on $T_1^{-1}W$ with nevertheless $\tilde{T}T_1^{-1}W = W$). Since $\cup_{n\in\mathbb{Z}}\tilde{T}^n W = \cup_{n\in\mathbb{Z}}T_1^n W$ has almost full measure, the latter implies smallness of $LE(A, \tilde{T})$. The map $\tilde{T}$ that we obtain is close to T_1 since we only modify the dynamics on $T_1^{-1}W$.

To understand how the map S is obtained, replace for a moment $\hat{A}_*(m_| W)$ by a map $\tilde{A}$ such that $\tilde{A}_*(m_| W)$ contains σ/N so that proposition 2.7 applies and identity on W can be replaced by a dynamics that reduces $\Lambda_k(\tilde{A}, S)$ close to zero (for some k that depends on σ/N). Now, the fact that k depends only on the spreading part of the measure and a careful choice of quantifiers allow to use the continuity of Λ_k and derive the same conclusion for $\hat{A}$ instead of $\tilde{A}$. Now we give the exact proof.

Since $\tau\sigma$ is spreading for any $\tau > 0$ we can assume $|\sigma| < \varepsilon'$ where

$$\varepsilon' = \min\{\varepsilon/(4M+4), \delta/(M\log C)\}, \quad C = \|A\|_\infty.$$

The measure σ/N is spreading as well and let $k = k(\sigma/N, \delta, C^M)$ be given by proposition 2.7.

Let $\eta = \eta(k, \delta, C^{\max\{M,N\}})$ be given by lemma 2.1.

Using lemma 2.12, we find $F \in Aut(\mathbb{I}, m)$ such that $F^N = id$, and a set $Z \subset \mathbb{I}$ such that $Z, \ldots, F^{N-1}(Z)$ are disjoint and $(A_F^N)_*(m|_Z)$ is η-close to $\frac{1}{N}\nu^{*N}$. By lemma 2.10, there exists a set $Y \subset Z$ such that $(A_F^N)_*(m|_Y)$ is η-close to $\frac{1}{N}\sigma$ (from the definition 2.8 this requires that $|Y| = |\sigma|/N$).

Let $\mathcal{T}_F = \bigsqcup_{i=0}^{N-1} F^i(Y)$; this set is an F-tower of height N, and has small measure:

$$m(\mathcal{T}_F) = Nm(Y) = |\sigma| < \varepsilon'.$$

Let $\mathcal{T}_T = \bigcap_{i=0}^{M-1} T^{-i}(\mathcal{T}_F^c)$. This set has almost full measure: $m(\mathcal{T}_T^c) \leq Mm(\mathcal{T}_F) < M\varepsilon' < \varepsilon/4$. It is also invariant by T (since $T^M = id$) and we can write it as a T-tower of height M over $I \cap \mathcal{T}_T$ where I is any interval of the cyclic permutation T.

Consider a first perturbation of T:

$$T_1(x) = \begin{cases} T(x) & \text{if } x \in \mathcal{T}_T, \\ F(x) & \text{if } x \in \mathcal{T}_F, \\ x & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}_T$ and $\mathcal{T}_F$ are two disjoint invariant towers for T_1 with heights M and N respectively, and basis $I \cap \mathcal{T}_T$ and Y respectively. We define $\mathcal{T} = \mathcal{T}_T \cup \mathcal{T}_F$, and $W = (I \cap \mathcal{T}_T) \sqcup Y$.

The first return map to W of T_1 is the identity. The return time function is $n_W(x) = M$ for $x \in I \cap \mathcal{T}_T$ and $n_W(x) = N$ for $x \in Y$. Hence we define on W the following map:

$$\hat{A}(x) = \begin{cases} A_T^M(x) & \text{if } x \in I \cap \mathcal{T}_T, \\ A_F^N(x) & \text{if } x \in Y. \end{cases}$$

Because $(A_F^N)_*(m|_Y)$ and $\frac{1}{N}\sigma$ are η-close, lemma 2.9 gives a map $\tilde{A} : Y \to \mathbb{G}$ such that $\tilde{A}_*(m|_Y) = \sigma/N$ and $\|\tilde{A} - A_F^N|_Y\|_1 \leq \eta$. Let $\hat{B} : W \to \mathbb{G}$ be such that

$$\hat{B}(x) = \begin{cases} A_T^M(x) & \text{if } x \in I \cap \mathcal{T}_T, \\ \tilde{A}(x) & \text{if } x \in Y. \end{cases}$$

By proposition 2.7, since we took $k = k(\sigma/N, \delta, C^M)$ we get

$$\Lambda_k(\hat{B}_*(m|_W)) = \Lambda_k\left((A_T^M)_*(m|_{I\cap\mathcal{T}_T}) + \frac{\sigma}{N}\right) < \frac{\delta}{M} + \frac{\varepsilon'}{N} \log C^M < 2\delta.$$

Since $\|\hat{A} - \hat{B}\|_1 \leq 2\eta$, with $\eta = \eta(k, \delta, C^{\max\{M,N\}})$ we get by lemma 2.1,

$$\Lambda_k(\hat{A}) < 3\delta.$$

This means that there exists an automorphism $S : W \to W$ such that $\Lambda_k(\hat{A}, S) < 3\delta$, and consequently $LE(\hat{A}, S) < 3\delta$.

Finally we define $\tilde{T}$ on $\mathbb{I}$:

$$\tilde{T}(x) = \begin{cases} S(T_1(x)) & \text{if } x \in T_1^{-1}(W), \\ T_1(x) & \text{otherwise.} \end{cases}$$

The set $\mathcal{T} = \mathcal{T}_T \cup \mathcal{T}_F$ is still invariant by $\tilde{T}$, the return time to the set W is still the function n_W as for T_1 and the products of matrices above W before the first return are still given by $\hat{A}$. But the first return map to W by $\tilde{T}$ is now S. Hence, by proposition 5.1, we have $LE(A|_{\mathcal{T}}, \tilde{T}|_{\mathcal{T}}) = LE(\hat{A}, S)$.

Recall that $m(\mathcal{T}^c) \le m(\mathcal{T}_T^c) \le M\varepsilon'$ and that we took $\varepsilon' \le \delta/(M \log C)$, therefore

$$LE(A, \tilde{T}) = LE(\hat{A}, S) + LE(A|_{\mathcal{T}^c}, \tilde{T}|_{\mathcal{T}^c}) < 4\delta.$$

We have

$$m[\tilde{T} \ne T] \le m[\tilde{T} \ne T_1] + m[T_1 \ne T] \le m(W) + m(\mathcal{T}_T^c) \le \frac{1}{M} + \frac{\varepsilon'}{N} + \frac{\varepsilon}{4} < \varepsilon$$

as required. □

3 The continuous case

From now on X will denote a compact connected manifold, possibly with boundary, of dimension $d \ge 2$. And μ will denote a smooth volume measure (or, more generally, an *OU measure*, see [AP, ch. 9]).

In this section we will use a restricted definition of richness.

Definition 3.1. *We will say that (T, A) is rich if for all $\varepsilon > 0$ there is a periodic (T, ε)-pseudo-orbit $\{x_0, x_1, \cdots, x_n = x_0\}$ such that $A(x_{n-1}) \cdots A(x_0)$ is elliptic.*

Lemma 3.2. *Let $A : X \to \mathbb{G}$ be C^1 smooth and $T \in Homeo(X, \mu)$. Assume that there exists a periodic point p such that $A_T^n(p)$ is elliptic and A_T^n is not locally constant at p. Then (A, T) is rich.*

3.1 Notations. Tools

For $y, \tilde{y} \in \mathbb{I}^d$ we denote the Euclidean distance between y and $\tilde{y}$ by $|y - \tilde{y}|$. We recall that the uniform topology on $Homeo(\mathbb{I}^d)$ is determined by the distance

$$\|T - \tilde{T}\| = \sup_{x \in \mathbb{I}^d} |T(x) - \tilde{T}(x)| + \sup_{x \in \mathbb{I}^d} |T^{-1}(x) - \tilde{T}^{-1}(x)|.$$

If T or $\tilde{T}$ (or both) are not in $Homeo(\mathbb{I}^d, m)$ but in $Aut(\mathbb{I}^d, m)$ then the distance above should be considered with ess sup instead of sup.

A map $P \in Aut(\mathbb{I}^d, m)$ is called a cube exchange map if there exists a partition of $\mathbb{I}^d$ in a finite number of collections of cubes such that P acts as a cyclic permutation of the atoms of each collection. Each collection is then called a tower for P.

Here we collect some tools from [AP] that we will use in the proof of theorem E.

The result below permits us to work in the cube $(\mathbb{I}^d, m)$ instead of the manifold (X, μ).

Theorem 3.3 (Theorem 9.6 from [AP]). *There exists a map $\Phi : \mathbb{I}^d \to X$ such that:*

1. *Φ is onto;*
2. *$\Phi|_{\mathrm{int}\,\mathbb{I}^d}$ is a homeomorphism of the interior of $\mathbb{I}^d$ onto its image;*
3. *$\Phi(\partial\mathbb{I}^d)$ is a closed nowhere dense set, disjoint from $\Phi(\mathrm{int}\,\mathbb{I}^d)$;*
4. *$\mu(\Phi(\partial\mathbb{I}^d)) = 0$;*
5. *$\Phi_* m = \mu$.*

In the proof of theorem E, we will make a non-continuous perturbation of the given homeomorphism, and then perturb again to get a homeomorphism. For that last step we will need the measure preserving Lusin theorem:

Theorem 3.4 (Theorem 6.2 from [AP]). *Let $T \in Homeo(\mathbb{I}^d, m)$ and $S \in Aut(\mathbb{I}^d, m)$, with $\|S - T\| < \varepsilon$. Then for any $\delta > 0$ there exists $\tilde{S} \in Homeo(\mathbb{I}^d, m)$, with $\|\tilde{S} - T\| < \varepsilon$ and equal to the identity in the boundary of $\mathbb{I}^d$, such that*

$$m\{x; |\tilde{S}(x) - S(x)| \geq \delta\} < \delta.$$

An important perturbation result due to Lax (with the cyclic property added by Alpern) is the following:

Theorem 3.5 (Theorem 3.3 from [AP]). *Let $T \in Homeo(\mathbb{I}^d, m)$. Then for any $\varepsilon > 0$ there is a cyclic cube exchange map P such that $\|P - T\| < \varepsilon$.*

3.2 Proof of theorem E

In light of theorem 3.3 we can assume that $T \in Homeo(\mathbb{I}^d, m)$. Let $\varepsilon > 0$. The perturbation of T is done in three steps:

1. By the Lax lemma (theorem 3.5), there exists $P \in Aut(\mathbb{I}^d, m)$ such that:
   - $\|P - T\| \leq \varepsilon/2$;
   - P is a cube exchange map with two towers $\mathcal{T}_1$ and $\mathcal{T}_2$ with heights n_1 and n_2 and with bases B_1 and B_2 satisfying $\mathrm{diam}(B_1 \sqcup B_2) \leq \varepsilon/2$;
   - There is $x_0 \in B_1$ such that $A_P^{n_1}(x_0)$ is elliptic and $A_P^{n_1}$ is not locally constant at x_1.
2. The measurable theorem applied to the induced cocycle on $B = B_1 \sqcup B_2$ yields a measurable dynamics $\tilde{P} \in Aut(\mathbb{I}^d, m)$ such that $\|\tilde{P} - P\| < \varepsilon/2$ and $\mathrm{LE}(A, \tilde{P}) \leq \varepsilon/2$;
3. By the semicontinuity of the Lyapunov exponent, let $\delta > 0$ be such that if $S \in Aut(\mathbb{I}^d, m)$ satisfies $m\{x; |S(x) - \tilde{P}(x)| \geq \delta\} < \delta$ then $\mathrm{LE}(A, S) \leq \varepsilon$. Theorem 3.4 then gives $\tilde{T} \in Homeo(\mathbb{I}^d, m)$ such that $\|\tilde{T} - P\| < \varepsilon/2$ such that $\mathrm{LE}(A, S) \leq \varepsilon$. From 1, we have $\|\tilde{T} - T\| \leq \varepsilon$. □

4 Discrete case, elementary open problem

4.1 Uniformly hyperbolic sets

For some results on the classification of uniformly hyperbolic sets, see [Yo].

Proposition 4.1. *The set of $\Sigma \in \mathbb{G}^N$ which are uniformly hyperbolic is open in $\mathbb{G}^N$.*

Proof. Let $\Sigma = (A_1, \dots, A_N)$ and denote by $\mathcal{N}$ the finite alphabet $\{1, 2, \dots, N\}$. Definte a map $A : \mathcal{N}^{\mathbb{Z}} \to \mathbb{G}$ that only depends on the letter in position 0 with $A(k) = A_k, k = 1, \dots, N$. Consider the full shift $\tau : \mathcal{N}^{\mathbb{Z}} \to \mathcal{N}^{\mathbb{Z}}$. Then our definition of Σ uniformly hyperbolic is equivalent to the uniform hyperbolicity of the cocycle (τ, A), which is an open condition [Yo]. □

4.2 Elliptic products

Lemma 4.2. *There is an open and dense subset $\mathcal{R}_0 \subset \mathbb{G}^N$ such that if $\Sigma \in \mathcal{R}_0$ then either Σ is uniformly hyperbolic or there is an elliptic matrix in the semigroup $\langle \Sigma \rangle$ generated by Σ.*

The lemma is an immediate consequence of the following result, which was pointed to us by Artur Avila.

Lemma 4.3. *For every $n \in \mathbb{N}$ and $A_1, \dots, A_n \in \mathbb{G}$, there is $\theta \in \mathbb{R}$ such that*

$$R_\theta A_n \cdots R_\theta A_2 R_\theta A_1 \text{ is elliptic and } \quad |\theta| \le \frac{C}{n} \log \rho(A_n \cdots A_1),$$

where R_θ is a rotation of angle θ, $\rho(\cdot)$ denotes spectral radius, and $C = \frac{\pi}{2}$.

The proof is given in the appendix (cf. [Yo]).

4.3 Liouville pairs

Recall that ρ denotes the spectral radius.

Definition 4.4. *Let $\psi : \mathbb{N} \to \mathbb{N}$, with $\lim_{n\to\infty} \psi(n) = \infty$. If R and H belong to $\mathbb{G}$, we say that the pair (R, H) is ψ-Liouville if R is elliptic and*

$$\liminf_{n\to+\infty} \frac{1}{\psi(n)} \log \rho\big(R^n H^{\psi(n)}\big) = 0.$$

We say that a pair (R, H) is Liouville if it is ψ-Liouville for some ψ such that $\psi(n) \ge n$.

Notice that if H is not hyperbolic, then (R, H) is Liouville for every elliptic R.

Lemma 4.5. *Given any $\psi : \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} \psi(n) = \infty$, let $\mathcal{R}$ be the set of $(R, H) \in \mathbb{G}^2$ such that R is not elliptic or (R, H) is ψ-Liouville. Then $\mathcal{R}$ is a residual subset of $\mathbb{G}^2$.*

Proof. Let $\mathbb{G}_{\text{ell}}$ be the subset of $\mathbb{G}$ formed by elliptic matrices, and let $\mathcal{L} \subset \mathbb{G}^2$ be the set of ψ-Liouville pairs. We have $\mathcal{L} = \bigcap_{m\ge1,\varepsilon>0} U_{m,\varepsilon}$, where

$$U_{m,\varepsilon} = \Big\{(R, H) \in \mathbb{G}_{\text{ell}} \times \mathbb{G};\ \exists n \ge m \text{ s.t. } \frac{1}{\psi(n)} \log \rho\big(R^n H^{\psi(n)}\big) < \varepsilon\Big\}.$$

Each $U_{m,\varepsilon}$ is open and we have to show it is dense in $\mathbb{G}_{\text{ell}} \times \mathbb{G}$. Given $(R, H) \in \mathbb{G}_{\text{ell}} \times \mathbb{G}$, with H hyperbolic, take a basis of $\mathbb{R}^2$ such that we can write

$$H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad |\lambda| > 1.$$

Arbitrarily close to R, there is an elliptic matrix $\tilde{R}$ such that $\tilde{R}^n(1, 0) \in \mathbb{R}(0, 1)$ for some $n \geq m$, that we can choose satisfying $1 < e^{\epsilon\psi(n)}$. Hence

$$\tilde{R}^n = \begin{pmatrix} 0 & c \\ b & d \end{pmatrix} \quad \text{and} \quad \tilde{R}^n H^{\psi(n)} = \begin{pmatrix} 0 & c\lambda^{-\psi(n)} \\ b\lambda^{\psi(n)} & d\lambda^{-\psi(n)}. \end{pmatrix}$$

Therefore $\left|\operatorname{tr} \tilde{R}^n H^{\psi(n)}\right| = |d||\lambda|^{-\psi(n)} < 2$. Hence $\rho\big(\tilde{R}^n H^{\psi(n)}\big) = 1 < e^{\varepsilon\psi(n)}$, that is, $(\tilde{R}, H) \in U_{m,\varepsilon}$. □

4.4 Monomials

Let $\mathcal{N} = \{1, 2, \ldots, N\}$. To every vector $k_1, \ldots, k_\ell \in \mathcal{N}^l$, $\ell \geq 1$, we can associate the *monomial* (map) $F : \mathbb{G}^N \to \mathbb{G}, (A_1, \ldots, A_N) \mapsto A_{k_1} \cdots A_{k_\ell}$. For each $i \in \mathcal{N}$, let us write

$$m_i(F) = \#\big\{j \in \{1, \ldots, \ell\};\ k_j = i\big\},$$

that is, the number of appearances of the letter A_i in the monomial F.

Let us call two monomials $F_1, F_2 : \mathbb{G}^N \to \mathbb{G}$ *independent* if the vectors $(m_1(F_1), \ldots, m_N(F_1))$ and $(m_1(F_2), \ldots, m_N(F_2)) \in \mathbb{R}^N$ are not collinear.

Lemma 4.6. *Let $F_1, F_2 : \mathbb{G}^N \to \mathbb{G}$ be independent monomials, and let $F = (F_1, F_2) : \mathbb{G}^N \to \mathbb{G}^2$. Then for every residual subset R of $\mathbb{G}^2$, the set $F^{-1}(R)$ is residual in $\mathbb{G}^N$.*

Proof. Let $C \subset \mathbb{G}^N$ be the set of critical points of F. We will show that C has empty interior. This will imply the lemma, because F restricted to the open dense set $\mathbb{G}^N \setminus C$ is an open map. Since C is a (real) algebraic submanifold of $\mathbb{G}^N$, if $C \neq \mathbb{G}^N$, then C has positive codimension and, in particular, has empty interior. So it is enough to prove that $C \neq \mathbb{G}$.

The derivative of F at $(id, \ldots, id)$ is easily computed; it is:

$$(a_1, \ldots, a_n) \in sl(2, \mathbb{R})^N \mapsto \left(\sum_{i=1}^{n} m_i(F_1)a_i, \sum_{i=1}^{n} m_i(F_2)a_i\right) \in sl(2, \mathbb{R})^2.$$

Due to the independence assumption, $DF(id, \ldots, id)$ is surjective, that is, $(id, \ldots, id) \notin C$ and $C \neq \mathbb{G}^N$. □

4.5 Proof of theorem C

In all the proof we fix the function $\psi = id_{\mathbb{N}}$.

First we define the residual set $\mathcal{R} \subset \mathbb{G}^N$ for which we will prove the conclusion of the theorem. Given two independent monomials F_1, F_2 : $\mathbb{G}^N \to \mathbb{G}$, let $\mathcal{R}(F_1, F_2)$ be the set of all $\Sigma \in \mathbb{G}^N$ such that

$$F_1(\Sigma) \text{ is not elliptic or } (F_1(\Sigma), F_2(\Sigma)) \text{ is } \psi\text{-Liouville.}$$

By lemmas 4.5 and 4.6, $\mathcal{R}(F_1, F_2)$ is a residual subset of $\mathbb{G}^N$. Take the intersection over all independent pairs F_1, F_2 and call it $\mathcal{R}_1$. Finally, let $\mathcal{R} = \mathcal{R}_0 \cap \mathcal{R}_1$, where $\mathcal{R}_0$ is the set from lemma 4.2.

Now take $\Sigma \in \mathcal{R}$. If Σ is uniformly hyperbolic, there is nothing to do. In the other case, since $\Sigma \in \mathcal{R}_0$, there is a monomial F_1 such that $R = F_1(\Sigma)$ is elliptic. F_1 will be fixed from now on. By construction, $(F_1(\Sigma), F_2(\Sigma))$ is Liouville for every monomial F_2 which is independent from F_1.

Let $A : X \to \Sigma$ be a measurable function such that every matrix in Σ is attained on a positive measure set of X. As usual, we assume X is the unit interval $\mathbb{I}$. We can also suppose there is a partition $\mathbb{I} = I_1 \sqcup \cdots \sqcup I_N$ into intervals such that $A|_{I_i} = A_i$, where $\Sigma = (A_1, \ldots, A_N)$.

Now let $T : \mathbb{I} \to \mathbb{I}$ be any given measure-preserving transformation. We will explain how to perturb it in the weak topology to make the exponent small.

1. A dyadic permutation of rank n on $\mathbb{I}$ is a map that permutes the intervals of the dyadic decomposition of $\mathbb{I}$ into intervals of size $1/2^n$. Following [H], we may assume T is a dyadic permutation of some arbitrarily high rank N.

2. Since the rank is high, most of the dyadic intervals will be completely contained in one of the intervals I_i (where A is constant). By perturbing T, we may assume the collection of "good" intervals is invariant by T. In fact, we don't need to worry about the "bad" intervals since their union has small measure, so, to simplify writing, we will assume that all the dyadic intervals of rank N are good.

3. A sequence of *disjoint* dyadic intervals $I_i = T^i(I_1), i = 1, \ldots, p$ is called a *tower of height* p. The tower is said to be *cyclic* if in addition $T^{p+1}(I_1) = I_1$. Since we assumed that the map $A : \mathbb{I} \to \Sigma$ is constant on each dyadic interval we can talk about the product of matrices along a tower, that we denote by $A(I_1) \cdots A(I_p)$. As a function of Σ the latter is by definiton a monomial of degree p.

Let p be the degree of F_1. Since the rank of T can be chosen arbitrarily high compared to p we can assume, after perturbation (see [AP]), that the dynamics of T decomposes into two cyclic towers, one of which is of height p and such that the product of matrices along its levels is precisely $F_1(\Sigma)$. The other tower occupies most of the space and we write it as

$$J_1 \to J_2 \to \cdots \to J_\ell \to J_1,$$

and we call $F_2(\Sigma)$ the product of matrices along its levels. Since we can take off any single level of this second tower via a small perturbation of T we may assume that F_1 and F_2 are independent.

Hence, by definition of $\mathcal{R}$, the pair $(R, H) = (F_1(\Sigma), F_2(\Sigma))$ is ψ-Liouville, that is, for any choice of ε there is an n such that

$$\frac{1}{n} \log \rho(H^n R^n) < \varepsilon. \tag{1}$$

4. Decompose each interval J_i into n intervals of equal length $J_i = J_{i,1} \sqcup J_{i,2} \sqcup \cdots \sqcup J_{i,n^2}$. Modify slightly T in order to form the following tower of height ℓn:

$$\begin{aligned} &J_{1,1} \to \cdots \to J_{\ell,1} \to \\ &J_{1,2} \to \cdots \to J_{\ell,2} \to \\ &\cdots \to \\ &J_{1,n^2} \to \cdots \to J_{\ell,n^2} \to J_{1,1}. \end{aligned}$$

The product along this new tower is H^n, and it covers almost all the space.

In the same way we decompose the R tower in n towers that we unfold as above into a signle tower along which the product of matrices will be R^n.

By our construction, the base of the H^n tower is of the same size as the base of the R^n tower so we can actually concatenate them one on top of the other to get a single cyclic tower along which the matrix product is $H^n R^n$(this is done by composing on the left the dynamics with a map that permutes $J_{1,1}$ and the base of the R^n tower). Since almost all the space is covered by this tower, we deduce from (1) that the integrated Lyapunov exponent corresponding to the perturbed dynamics is small.

4.6 Some open questions

Problem 4.7. *Does there exist a finite set* $\Sigma \subset \mathbb{G}$ *with the following properties:*

1. Σ *cannot be approximated by a uniformly hyperbolic set;*
2. *there exists a measurable map* $A\colon X \to \Sigma$ *which assumes every value in* Σ *on a set of positive measure such that* $LE(A,T) > 0$ *for* every $T \in \mathcal{M}$ *? such that* $LE(A,T) \geq \lambda_0 > 0$*?*

By theorem C, those Σ form a meager subset of $\mathbb{G}^N$.

A positive answer to the following more elementary question would (by Birkhoff's theorem) answer problem 4.7 positively:

Problem 4.8. *Does there exist a pair of matrices* A_1, $A_2 \in \mathbb{G}$, *with* A_1 *hyperbolic and* A_2 *elliptic, and constants* $0 < p < 1$ *and* $\lambda > 1$ *such that for every word* $A_{i_1} A_{i_2} \cdots A_{i_k}$ *satisfying the* frequency condition

$$\#\big\{j \in \{1, \dots, k\};\ i_j = 2\big\} < pk, \tag{2}$$

we have

$$\|A_{i_1} A_{i_2} \cdots A_{i_k}\| > \lambda^k \ ?$$

Assuming problem 4.8 has a positive answer, notice that the products $A_{i_1} A_{i_2} \cdots A_{i_k}$ that satisfy the frequency condition (2) form a semigroup of $\mathbb{G}$ which is not finitely generated and is formed only by hyperbolic matrices.

Fixing some integer $N \geq 2$, we can also ask whether the set of $\Sigma \in \mathbb{G}^N$ that have the properties as in problem 4.7 has positive, or even full measure in $\mathbb{G}^N$.

Remark 4.9. From the proof of theorem C we see that even if the right hand side in (2) is replaced by any function ϕ such that $\phi(k) \to \infty$ then the set of $\Sigma \in \mathbb{G}^2$ that satisfy the conclusion of problem 4.8 is meager.

5 Appendices

5.1 Derived cocycles and exponent

For $A : (X, \mu) \to \mathbb{G}$ and $T \in Aut(X, \mu)$, recall the definition of the *mean exponent*:

$$LE(A,T) = \lim_{n \to \infty} \frac{1}{n} \int_X \log \|A_T^n\| \, d\mu.$$

Attention: we can have $\mu(X) \neq 1$.

Now, given a set $Y \subset X$ of positive measure. Let $T_Y : Y \to Y$ be the *first return map*, i.e. $T_Y(x) = T^{n_Y(x)}$, where $n_Y(x) = \min\{n \geq 1;\ T^n(x) \in Y\}$. n_Y and T_Y are defined a.e. and we have $T_Y \in Aut(Y, \mu|_Y)$.

Define $A_Y : Y \to \mathbb{G}$ as $A_Y(x) = A_T^{n_Y(x)}(x)$; this is the *derived cocycle*

Let $Y_T = \bigcup_{n \in \mathbb{Z}} T^{-n}(Y)$; this is a T-invariant set.

Proposition 5.1. *We have* $LE(A_Y, T_Y) = LE(A|_{Y_T}, T|_{Y_T})$.

Proof. Adapt the proof of [K, lemma 2.2], using that

$$\int_Y n_Y \, d\mu = \mu(Y_T).$$

(even if T is not ergodic: adapt the first proof of Kac's lemma from Petersen). ☐

5.2 Semicontinuity

It is well-known that:

$$LE(A,T) = \inf_N \frac{1}{N} \int_X \log \|A_T^N\| \, d\mu.$$

Among the consequences, we have semicontinuity of LE.

Proposition 5.2. *Fix* $T \in Aut(X, \mu)$. *Let* $\mathcal{A}$ *be the space of measurable functions* $A : X \to \mathbb{G}$ *that satisfy* $\log \|A\| \in L^1(\mu)$. *Then the function* $A \in \mathcal{A} \mapsto LE(A,T)$ *is upper semicontinuous in the* L^1 *topology. That is, for every* $A \in \mathcal{A}$ *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $B \in \mathcal{A}$ *and* $\int_X \|B - A\| \, d\mu < \delta$ *then* $LE(B,T) < LE(A,T) + \varepsilon$.

For the proof, see [ArB].

Proposition 5.3. *Let* $A : X \to \mathbb{G}$ *be measurable and such that* $\log \|A\| \in L^1(\mu)$. *Then the function* $T \in Aut(X, \mu) \mapsto LE(A,T)$ *is upper semicontinuous.*

Proof. We may assume that X is the unit interval $[0,1]$ and μ is Lebesgue measure. The weak topology in $Aut(X, \mu)$ is then given by the *weak metric*

$$d(S,T) = \inf\{\rho > 0;\ \mu(\{|S - T| > \rho\}) < \rho\}.$$

Let $A : X \to \mathbb{G}$, $T \in Aut(X, \mu)$, and $\varepsilon > 0$ be fixed. There exists $N \in \mathbb{N}$ such that

$$LE(A,T) > -\varepsilon + \frac{1}{N} \int_X \log \|A_T^N\| \, d\mu.$$

Since $\log\|A\|$ is integrable, there is $\delta_1 > 0$ such that if $Z \subset X$ has measure $\mu(Z) < \delta_1$ then $\int_Z \log\|A\|\,d\mu < \varepsilon$. By Lusin's theorem, there exists a compact set $K \subset X$ such that the functions $A|_K$ and $T|_K$ are continuous, and $\mu(K^c) < \delta_1/(2N)$. Let $C = \sup_K \|A\|$. There is $\delta_2 > 0$ such that if $A_1, \dots, A_N, B_1, \dots, B_N \in \mathbb{G}$ are matrices with norm at most C and $\|A_i - B_i\| < \delta_2$ for each i then $\|\prod_N^1 A_i - \prod_N^1 B_i\| < \varepsilon$. Let $\delta_3 > 0$ be such that if $x, y \in K$, $|x - y| < \delta_3$ then $\|A(x) - A(y)\| < \delta_2$. Take numbers $\eta_1 > \dots > \eta_{N-1} > 0$ such that $\eta_1 = \delta_3/2$ and

$$x, y \in K,\ |x - y| < 2\eta_{i+1} \ \Rightarrow\ |T(x) - T(y)| < \eta_i.$$

Let $\rho = \min\{\eta_{N-1}, \delta_1/(2N)\}$.

Now assume $S \in Aut(X,\mu)$ is such that $d(S,T) < \rho$. Let $W = \{|S-T| \le \rho\}$; then $\mu(W^c) < \rho$. Define

$$G = \bigcap_{i=0}^{N-1} \left[T^{-i}(K \cap W) \cap S^{-i}(K \cap W)\right].$$

Then G^c has small measure: $\mu(G^c) \le N\mu(K^c + W^c) < \delta_1$. We are going to bound the expression $\frac{1}{N}\int_X \log\|A_S^N\|\,d\mu$. To do so, we are going to split the integral in two parts, $\int_X = \int_{G^c} + \int_G$. For the first part, we have

$$\frac{1}{N}\int_{G^c} \log\|A_S^N\|\,d\mu \le \frac{1}{N}\sum_{i=0}^{N-1}\int_{S^i(G^c)} \log\|A\|\,d\mu \le \varepsilon.$$

For the second part,

$$\begin{aligned}\frac{1}{N}\int_G \log\|A_S^N\|\,d\mu &\le \frac{1}{N}\int_G \log\|A_T^N\|\,d\mu + \frac{1}{N}\int_G \|A_S^N - A_T^N\|\,d\mu \\ &< LE(A,T) + \varepsilon + \frac{1}{N}\int_G \|A_S^N - A_T^N\|\,d\mu\,.\end{aligned}$$

(We used that $\log(a+b) \le \log a + b$ for $a \ge 1$, $b \ge 0$.)

Let $x \in G$. We claim that $|T^i(x) - S^i(x)| \le 2\eta_{N-i}$ for all $i = 0, 1, \dots, N-1$. This is easily shown by induction:

$$\begin{aligned}|T^{i+1}(x) - S^{i+1}(x)| &\le |T(T^i(x)) - T(S^i(x))| + |T(S^i(x)) - S(S^i(x))| \\ &\le \eta_{N-i-1} + \rho \le 2\eta_{N-i-1}.\end{aligned}$$

In particular, for all i we have $|T^i(x) - S^i(x)| < \delta_3$ and thus $\|A(T^i(x)) - A(S^i(x))\| < \delta_2$. Therefore $\|A_T^N(x) - A_S^N(x)\| < \varepsilon$.

Summing the two parts, we conclude that

$$LE(A,S) \leq \frac{1}{N}\int_G \log \|A_S^N\| \, d\mu \leq LE(A,T) + \varepsilon + \frac{\varepsilon}{N}.$$

This shows upper semicontinuity. □

5.3 Generic Fubini

Proposition 5.4. *If X, Y are separable Baire spaces and $\mathcal{R} \subset X \times Y$ is residual, then there is a residual subset $\mathcal{R}' \subset X$ such that for every $x \in \mathcal{R}'$, the set $\mathcal{R}_x = \{y \in Y;\ (x,y) \in \mathcal{R}\}$ is a residual subset of Y.*

Proof. First let $A \subset X \times Y$ be an open and dense set. For any open set $V \subset Y$, let $X_{A,V} = \{x \in X;$ there exists $y \in V$ such that $(x,y) \in A\}$. Then $X_{A,V}$ is open and dense in X. Let $\mathcal{V}$ be a countable base of open subsets of Y, and consider the residual set $X_A = \bigcap_{V \in \mathcal{V}} X_{A,V}$. If we define $A_x = \{y \in Y;\ (x,y) \in A\}$, then A_x is open and dense in Y for every $x \in X_A$.

Now, given a residual set $\mathcal{R} \subset X \times Y$, write $\mathcal{R} = \bigcap_{n \in \mathbb{N}} A_n$, where A_n are open and dense. Let $\mathcal{R}' = \bigcap_{n \in \mathbb{N}} X_{A_n}$. Then for every $x \in \mathcal{R}'$, the fiber $\mathcal{R}_x = \bigcap_{n \in \mathbb{N}} (A_n)_x$ is a residual subset of Y. □

5.4 Criterium for richness

We begin with a linear-algebraic lemma:

Lemma 5.5. *Let A, $B \in \mathbb{G}$ be elliptic matrices that are conjugated via a matrix in $\mathbb{G}$.*[3] *Then*

$$\operatorname{tr} AB \leq \operatorname{tr} A^2,$$

with equality if and only if $A = B$.

Proof. Write $B = CAC^{-1}$, with $C \in \mathbb{G}$. We can assume

$$A = R_\theta, \quad C = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},$$

with $a > 0$. Direct calculation gives

$$\operatorname{tr} AB = 2 - (2 + a^2 + a^{-2} + b^2)\sin^2\theta \leq 2 - 4\sin^2\theta = \operatorname{tr} A^2,$$

with equality if and only if $a = 1$ and $b = 0$. □

[3] Note that two matrices in $\mathbb{G}$ may be $\mathrm{GL}(2,\mathbb{R})$-conjugated and yet not $\mathbb{G}$-conjugated. E.g. R_θ and $R_{-\theta} = C^{-1} R_\theta C$, where $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Now we give the:

Proof of proposition 1.11. Write $\hat{x} = (x_1, \dots, x_k)$, and $\hat{A}(\hat{x}) = A(x_k) \cdots A(x_1)$. By assumption, there are open sets $V_i \subset M$, such that $\hat{A} : V_1 \times \cdots \times V_k \to \mathbb{G}$ is not constant and assumes only elliptic values. We can assume that V_i is connected and $V_i \cap \partial M = \emptyset$ for each i. Write $\hat{V} = V_1 \times \cdots \times V_k$. We can also assume that the function

$$\hat{x} \in \hat{V} \mapsto \operatorname{tr} \hat{A}(\hat{x}) \in (-2, 2)$$

is not constant. [Indeed, if the function is constant then by lemma 5.5 the function

$$(\hat{x}_1, \hat{x}_2) \in \hat{V} \times \hat{V} \mapsto \operatorname{tr} \hat{A}(\hat{x}_1)\hat{A}(\hat{x}_2) \in \mathbb{R}$$

is not constant. Therefore we can replace k with $2k$ and $\hat{V}$ by a subset of $\hat{V} \times \hat{V}$.] Finally, we can assume that $\operatorname{tr} \hat{A}$ has no critical points in $\hat{V}$.

We can find C^1 functions θ and C such that

$$\hat{A}(\hat{x}) = C(\hat{x}) R_{\theta(\hat{x})} C(\hat{x})^{-1}, \quad \text{where} \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Then $\theta = \arccos \frac{1}{2} \operatorname{tr} \hat{A}$ has no critical points in $\hat{V}$. An easy calculation shows that if n is sufficiently large then for any $v \in \mathbb{P}^1$ the map

$$\hat{x} \in \hat{V} \mapsto [\hat{A}(\hat{x})]^n \cdot v = C(\hat{x}) R_{n\theta(\hat{x})} C(\hat{x})^{-1} \cdot v \in \mathbb{P}^1$$

is surjective and has no critical points. Let $\delta > 0$ be small so that the map

$$(\hat{x}_1, \dots, \hat{x}_n) \in V_\delta(\Delta) \mapsto \hat{A}(\hat{x}_n) \cdots \hat{A}(\hat{x}_1) \in \mathbb{G},$$

defined in the δ-neighborhood of the diagonal $\Delta \subset \hat{V}^n$, assumes only elliptic values. Let σ be the push-forward of $\mu^{nk}|_{V_\delta(\Delta)}$. Taking smaller δ if necessary, we have that for each $v \in \mathbb{P}^1$, the map

$$(\hat{x}_1, \dots, \hat{x}_n) \in V_\delta(\Delta) \mapsto \hat{A}(\hat{x}_n) \cdots \hat{A}(\hat{x}_1) \cdot v \in \mathbb{P}^1$$

is surjective and has no critical points. Therefore the push-forward $\sigma * v$ has a density bounded from below. Accordingly, $\sigma \le (A_*\mu)^{nk}$ is spreading and so $A_*\mu$ is rich. $\square$

Proposition 1.12 follows from proposition 1.11.

5.5 Proof of lemma 4.3

Proof. Let, for $z \in \mathbb{C}$,

$$R_z = \begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}.$$

Let $C_z = R_z A_n \cdots R_z A_1$ and consider the function $h\colon \mathbb{C} \to \mathbb{R}$ defined by $h(z) = n^{-1} \log \rho(C_z)$. We will show that

$$h(z) \geq |\operatorname{Im} z| \text{ , for all } z \in \mathbb{C}. \tag{3}$$

In particular, $h^{-1}(0) \subset \mathbb{R}$. If z is such that $h(z) \neq 0$ then C_z has two different eigenvalues (notice $\det C_z = 1$). This shows that the restriction of h to $\mathbb{C} \setminus h^{-1}(0)$ is an harmonic function. To conclude the proof – assuming (3) – suppose that the matrix C_z is hyperbolic for $z \in [-\varepsilon, \varepsilon]$. Then disk $|z| \leq \varepsilon$ does not intersect $h^{-1}(0)$ and so

$$h(0) = \frac{1}{2\pi} \int_0^{2\pi} h(\varepsilon e^{i\theta})\, d\theta \geq \frac{\varepsilon}{2\pi} \int_0^{2\pi} |\sin \theta|\, d\theta = \frac{2\varepsilon}{\pi}.$$

The lemma follows.

It remains to prove (3). Let, for $w \in \mathbb{C}$,

$$T_w = \begin{pmatrix} \frac{1}{2}(w^2+1) & -\frac{1}{2i}(w^2-1) \\ \frac{1}{2i}(w^2-1) & \frac{1}{2}(w^2+1) \end{pmatrix}$$

and $D_w = T_w A_n \cdots T_w A_1$. Then, for $w = e^{iz}$, $T_w = w R_z$ and $D_w = w^{2n} C_z$. The eigenvalues of D_w define, for $|w| < 1$, two holomorphic functions $\lambda_1(w)$ and $\lambda_2(w)$ such that $|\lambda_1(w)| > |\lambda_2(w)|$ (see [AB, lemma 5]). Since $\lambda_1(w)\lambda_2(w) = \det D_w = w^{2n}$, we get $|\lambda_1(w)| > |w|^n$ and $|\lambda_2(w)| < |w|^n$, for $|w| < 1$. Given $z \in \mathbb{C}$ with $\operatorname{Im} z < 0$, let $w = e^{iz}$. Then

$$\rho(C_z) = |w|^{-2n} \rho(D_w) > |w|^{-n} = |e^{-inz}| = e^{-n \operatorname{Im} z}.$$

That is, $h(z) > -\operatorname{Im} z$ if $\operatorname{Im} z < 0$. So (3) follows from the reflection principle. ☐

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