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RIGIDITY RESULTS FOR QUASIPERIODIC SL(2, ℝ)-COCYCLES

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ABSTRACT. In this paper we introduce a new technique that allows us to investigate reducibility properties of smooth SL(2, ℝ)-cocycles over irrational rotations of the circle beyond the usual Diophantine conditions on these rotations.

For any given irrational angle on the base, we show that if the cocycle has bounded fibered products and if its fibered rotation number belongs to a set of full measure $\Sigma(\alpha)$, then the matrix map can be perturbed in the C^∞ topology to yield a C^∞-reducible cocycle. Moreover, the cocycle itself is *almost rotations-reducible* in the sense that it can be conjugated arbitrarily close to a cocycle of rotations. If the rotation on the circle is of super-Liouville type, the same results hold if instead of having bounded products we only assume that the cocycle is L^2-conjugate to a cocycle of rotations.

When the base rotation is Diophantine, we show that if the cocycle is L^2-conjugate to a cocycle of rotations and if its fibered rotation number belongs to a set of full measure, then it is C^∞-reducible. This extends a result proven in [5].

As an application, given any smooth SL(2, ℝ)-cocycle over a irrational rotation of the circle, we show that it is possible to perturb the matrix map in the C^∞ topology in such a way that the upper Lyapunov exponent becomes strictly positive. The latter result is generalized, based on different techniques, by Avila in [1] to quasiperiodic $SL(2,\mathbb{R})$-cocycles over higher-dimensional tori.

Also, in the course of the paper we give a quantitative version of a theorem by L. H. Eliasson, a proof of which is given in the Appendix. This motivates the introduction of a quite general KAM scheme allowing to treat bigger losses of derivatives for which we prove convergence.

1. INTRODUCTION

We will be interested in smooth quasiperiodic cocycles on $\mathbb{T}\times SL(2,\mathbb{R})$, where $\mathbb{T}$ denotes the circle $\mathbb{R}/\mathbb{Z}$ (the base) and $SL(2,\mathbb{R})$ the set of 2 by 2 real matrices with determinant 1 (the fiber). For $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ and $A\in C^r(\mathbb{T},SL(2,\mathbb{R}))$, $r\in\mathbb{N}\cup\{\infty,\omega\}$, we define the cocycle (α, A) as a diffeomorphism on the product $\mathbb{T}\times SL(2,\mathbb{R})$:

$$\begin{aligned}(\alpha,A)\colon\ \mathbb{T}\times SL(2,\mathbb{R}) &\to \mathbb{T}\times SL(2,\mathbb{R}),\\ (\theta,y)&\mapsto(\theta+\alpha,A(\theta)y).\end{aligned}$$

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We denote by $C_0^r(\mathbb{T}, SL(2,\mathbb{R}))$ the set of maps $A \in C^r(\mathbb{T}, SL(2,\mathbb{R}))$ that are homotopic to the identity. We say that a C^r-cocycle (α, A) is C^r*-reducible* if there exist $B \in C^r(\mathbb{R}/2\mathbb{Z}, \mathbb{P}SL(2,\mathbb{R}))$ and $A_* \in SL(2,\mathbb{R})$ such that

$$B(\theta+\alpha)^{-1}A(\theta)B(\theta) = A_* \qquad \forall \theta \in \mathbb{R}/2\mathbb{Z}.$$

Reducibility is an important issue in the study of quasiperiodic cocycles and their applications (*e.g.,* to the spectral theory of Schrödinger operators). An obvious obstruction to reducibility is nonuniformly hyperbolic behavior of the cocycle, that is, a nonuniform exponential increase almost everywhere of the fibered products of matrices over the base rotation. On the other hand, when certain parametrized families of cocycles are considered, Kotani's theory asserts essentially that there is an almost sure dichotomy between nonuniform hyperbolicity and L^2-rotations-reducibility (conjugacy to a cocycle with values in $SO(2,\mathbb{R})$, see the exact definition below). More precisely, to any cocycle (α, A) one can associate a *fibered Lyapunov exponent* $LE(\alpha, A)$ (we refer to Section 2.1 for the definitions and basic results). Now let $Q_\varphi(\cdot) = R_\varphi A(\cdot)$, where $R_\varphi \in SO(2,\mathbb{R})$ is the rotation matrix $\begin{pmatrix} \cos 2\pi\varphi & -\sin 2\pi\varphi \\ \sin 2\pi\varphi & \cos 2\pi\varphi \end{pmatrix}$. Then, for Lebesgue almost every $\varphi \in [0,1]$, either $LE(\alpha, A) > 0$ or $(\alpha, R_\varphi A)$ is L^2-rotations-reducible (see [5] for references). This dichotomy also holds for Lebesgue almost every value of $E \in \mathbb{R}$ in the setting of Schrödinger cocycles

$$S_{V,E} = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix},$$

$(V \in C^r(\mathbb{T}, \mathbb{R}))$, which, combined with rigidity results that we now describe, gives information on the spectrum of the corresponding Schrödinger operator.

In [5], the latter dichotomy was pushed further to a global analytic rigidity result (also true in the smooth category). By analytic (resp. smooth) rigidity, we mean the fact by which weak regularity information (L^2) on the conjugating map is enough to guarantee its analyticity (resp. smoothness) under some additional assumptions. An important role is played by the fibered rotation number of the cocycle (see Section 2.2 for its definition).

THEOREM A. [5] *If $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ is recurrent-Diophantine and if $A \in C_0^\omega(\mathbb{T}, SL(2,\mathbb{R}))$ satisfies:*

(i) $\rho_f(\alpha, A)$ is Diophantine with respect to α and
(ii) (α, A) is L^2−conjugate *to a cocycle of rotations,*

then (α, A) is C^ω-reducible. As a consequence, for Lebesgue a.e. $\varphi \in [0,1]$) (resp. if $V \in C^\omega(\mathbb{T})$, for Lebesgue a.e. $E \in \mathbb{R}$), either $(\alpha, R_\varphi A)$ (resp. $(\alpha, S_{V,E})$) is C^ω-reducible or it is nonuniformly hyperbolic.

The set of recurrent-Diophantine numbers is by definition the full Lebesgue-measure set of irrationals with the property that the iteration of α by the Gauss map $G\colon (0,1) \to (0,1)$, $G(x) = \{x^{-1}\}$, falls infinitely many times in some Diophantine set with fixed constant and exponent. Also, for any fixed α, numbers that are Diophantine with respect to α form a set of full Lebesgue measure. See Section

2.3 for precise definitions. One aim of this note is to extend in a smooth setting the result of [5] to all Diophantine frequencies as follows.

1.1. **Almost-sure dichotomy for every Diophantine α.**

THEOREM 1. *If $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ is Diophantine, there exists a set $\Sigma(\alpha) \subset \mathbb{T}$ of measure 1 such that if $A \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ satisfies*

(i) $\rho_f(\alpha, A) \in \Sigma(\alpha)$ and

*(ii) (α, A) is L^2-*conjugate *to a cocycle of rotations,*

then (α, A) is C^∞-reducible.

In the analytic case, this theorem is also true, by different techniques, as a consequence of [4, 2], and [6] where, in addition, the Diophantine condition on α can be relaxed to be $\lim_{n\to\infty} \frac{\log q_{n+1}}{q_n} = 0$, where q_n denotes the denominator of the n-th continued-fraction expansion of α (see Section 2.3).

As in [5], Theorem 1 yields a global dichotomy that generalizes Theorem A to all Diophantine frequencies (in the C^∞ category).

COROLLARY 1. *Let $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ be Diophantine and let $A \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ (resp. $V \in C^\infty(\mathbb{T},\mathbb{R})$). Then, for Lebesgue almost every $\varphi \in [0,1]$ (resp. $E \in \mathbb{R}$), the cocycle $(\alpha, R_\varphi A)$ (resp. $(\alpha, S_{V,E})$) is either nonuniformly hyperbolic or C^∞-reducible.*

1.2. **Almost rigidity.**

It is clear that the notion of reducibility for cocycles defined over Liouvillean translations is too restrictive since in that case even an $\mathbb{S}^1$ or $\mathbb{R}$-valued cocycle is in general not reducible. The appropriate notion in this case is rotations-reducibility. We say that (α, A) is L^2-conjugate, (resp. C^r-conjugate), to a *cocycle of rotations* (for short, we shall say that $(\alpha.A)$ is L^2 (resp. C^r) rotations-reducible) if there exists a measurable $B\colon \mathbb{T} \to SL(2,\mathbb{R})$ such that $\|B(\cdot)\| \in L^2(\mathbb{T},\mathbb{R})$, (resp. $B \in C^r(\mathbb{T}, SL(2,\mathbb{R}))$, and

$$B(\theta+\alpha)^{-1} A(\theta) B(\theta) \in SO(2,\mathbb{R}), \qquad \forall \theta \in \mathbb{T}.$$

How to extend Theorem A or Corollary 1 to any irrational number when reducibility is replaced by rotations-reducibility is an interesting and important problem.

Here we introduce and study the following notions of almost-reducibility and almost-rotations reducibility: a cocycle (α, A) is *almost reducible* (resp. *almost rotations-reducible*) if there exist sequences $B_{(n)} \in C^r(\mathbb{R}/2\mathbb{Z}, SL(2,\mathbb{R}))$ and $A_{(n)} \in SL(2,\mathbb{R})$ (resp. $A_{(n)} \in C^r(\mathbb{T}, SO(2,\mathbb{R}))$) such that $B_{(n)}^{-1}(\cdot+\alpha)A(\cdot)B_{(n)}(\cdot) - A_{(n)} \to 0$ in the C^r-topology as $n \to \infty$.

We prove the following "almost rigidity" results

THEOREM 2. *Let $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$. There exists a set $\Sigma(\alpha) \subset \mathbb{T}$ of measure 1 such that if $A(\cdot) \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ satisfies*

(i) *$\rho_f(\alpha, A) \in \Sigma(\alpha)$ and*

(ii) *(α, A) is C^0-rotations-reducible,*

then A is C^∞-almost rotations-reducible and is C^∞-accumulated by functions $\tilde{A}(\cdot) \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ such that $(\alpha, \tilde{A})$ is C^∞-reducible.

REMARK 1. Note that from [16], it follows that (ii) is equivalent to the fact that the fibered products of $A(\cdot)$ be C^0-bounded; if we use the notation $(\alpha, A(\cdot))^n = (n\alpha, A_n(\cdot))$ $(n \in \mathbb{Z})$, this means $\sup_{n\in\mathbb{Z}} \|A_n(\cdot)\|_{C^0(\mathbb{T})} < \infty$.

This theorem can be strengthened in two cases. The first is that the frequency in the base is Diophantine, in which case, as stated in Theorem 1, (ii) can be replaced by L^2-rotations-reducible and the conclusion is reducibility. The other case is that α is very well approximated by rational numbers: we shall say that $\alpha \in \mathbb{R}\smallsetminus\mathbb{Q}$ is *super Liouville* if

$$\limsup_{n\to\infty} \frac{\log\log q_{n+1}}{\log q_n} = \infty,$$

where q_n denotes the denominator of the n-th continued-fraction expansion of α (see Section 2.3).

THEOREM 3. *Let $\alpha \in \mathbb{R}\smallsetminus\mathbb{Q}$ be super Liouville. There exists a set $\Sigma(\alpha) \subset \mathbb{T}$ of measure 1 such that if $A(\cdot) \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ satisfies*

(i) *$\rho_f(\alpha, A) \in \Sigma(\alpha)$ and*
(ii) *(α, A) is L^2-rotations-reducible,*

then A is C^∞-almost rotations-reducible and is C^∞-accumulated by functions $\tilde{A}(\cdot) \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ such that $(\alpha, \tilde{A})$ is C^∞-reducible.

1.3. Density of cocycles with positive Lyapunov exponent.

As a corollary of Theorems 1 and 2 we have the following result.

THEOREM 4. *For fixed irrational $\alpha \in \mathbb{T}$, the set of $A \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ such that (α, A) has positive Lyapunov exponent is dense in $C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ for the C^∞-topology.*

Proof. The proof relies on Kotani's theory and proceeds along the lines of [14].

Let $\alpha \in \mathbb{R}\smallsetminus\mathbb{Q}$ and $A(\cdot) \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$. Then, by a theorem of De Concini and Johnson [7], either for any $\delta > 0$ there exists $\epsilon_0 \in (-\delta, \delta)$ for which the Lyapunov exponent of $(\alpha, A(\cdot)R_{\epsilon_0})$ is positive, or $(\alpha, A(\cdot)R_\epsilon)$ has C^0-bounded fibered products for any $\epsilon \in (-\delta, \delta)$, in which case we also have that the continuous function $(-\delta, \delta) \ni \epsilon \to \rho_f(\alpha, A(\cdot)R_\epsilon)$ is not constant.

As mentioned earlier, we know from [16] that C^0-boundedness of the fibered products is equivalent to the fact that the cocycle is C^0-conjugate to rotations. Since the fibered rotation number of this cocycle is continuous and not constant as ϵ varies in $(-\delta, \delta)$, we can choose ε_0 such that, depending on α, hypothesis (i) of Theorem 2 or of Theorem 1 is satisfied. We conclude that the cocycle $(\alpha, A(\cdot)R_{\epsilon_0})$ is accumulated by reducible cocycles.

Theorem 4 then follows if we make the following observation (*cf.* [14])

LEMMA 1. *Any constant A_0 in $SL(2,\mathbb{R})$ is C^∞-accumulated by functions $A(\cdot) \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ such that $(\alpha, A(\cdot))$ is hyperbolic (in the fiber).* □

REMARK 2. In fact, as proven in [1], a stronger result holds: Theorem 4 is true for $SL(2,\mathbb{R})$-cocycles over translations on tori of any dimension (and even more general dynamics).

Let us describe briefly the novelty of this paper. There are usually two techniques to attack the reducibility problem for quasiperiodic systems: KAM theory and renormalization. In the case of $SL(2,\mathbb{R})$-cocycles, a third approach to tackle reducibility, based on localization and Aubry duality, proved to be fruitful in [15, 3, 2, 4]. Renormalization is usually used to reduce to a local (perturbative) situation where KAM techniques are applicable. This naturally requires Diophantine conditions on both the rotation number on the base and on the fibered rotation number of some renormalized system. The crucial observation in the present paper is that if the Diophantine condition on the base fails, it is indeed possible to take advantage of this fact and conjugate the cocycle closer to rotations (depending on the base point) with an error that will be even smaller as the involved small divisor is small (*cf.* Theorem 2 and 3). This remark is also useful to treat the Diophantine case in full generality (*cf.* Theorem 1). We refer to Sections 3.2 and 4.1 for a more detailed discussion concerning each case.

1.4. **A quantitative version of the Eliasson Theorem and a generalized KAM scheme.** Given an analytic cocycle (α, A), α Diophantine and A sufficiently close to a constant (as a function of α), Eliasson proved in [9] a theorem that guarantees reducibility of (α, A), provided the fibered rotation number satisfies some (weak) Diophantine condition.

We will prove a precise version of Theorem 1 (see Theorem 8 below) that uses an extension of Eliasson's reducibility theorem to the smooth case and also provides estimates, involving the Diophantine constants, on the required closeness to constants. This is our Theorem 5.

As pointed out to us by the referee (and as will be explained in Section 4.1), a usual KAM scheme with estimates would be sufficient for the purpose of proving Theorem 1. However, proving the quantitative version of Eliasson's theorem is itself interesting and allows for a slightly more general result on reducibility (*cf.* §4.1).

Thus, we present a proof of Theorem 5 in the Appendix together with a quite general KAM scheme that we think may be of broader utility, especially in problems where small divisors cause losses of derivatives that are proportional to the order of differentiability.

2. DEFINITIONS AND PRELIMINARIES.

2.1. **The fibered Lyapunov exponent.** Given a cocycle (α, A), for $n \in \mathbb{Z}$, we denote the iterates of (α, A) by $(\alpha, A)^n = (n\alpha, A_n(\cdot))$, where for $n \geq 1$,

$$A_n(\cdot) = A(\cdot + (n-1)\alpha) \cdots A(\cdot)$$
$$A_{-n}(\cdot) = A(\cdot - n\alpha)^{-1} \cdots A(\cdot - \alpha)^{-1}.$$

We call the matrices $A_n(\cdot)$ for $n \in \mathbb{N}$ fibered products of (α, A).

The fibered Lyapunov exponent is defined as the limit

$$L(\alpha, A) := \lim_{n\to\infty} \frac{1}{n} \int_{\theta\in\mathbb{T}} \log \|A_n(\theta)\| \, d\theta,$$

which by the subadditive theorem always exists (similarly, the limit when n goes to $-\infty$ exists and is equal to $L(\alpha, A)$).

2.2. **The fibered rotation number.** Assume that $A(\cdot)\colon \mathbb{T} \to SL(2,\mathbb{R})$ is continuous and homotopic to the identity; then the same is true for the map

$$\begin{aligned} F\colon \mathbb{T}\times\mathbb{S}^1 &\to \mathbb{T}\times\mathbb{S}^1 \\ (\theta, v) &\mapsto \left(\theta+\alpha, \frac{A(\theta)v}{\|A(\theta)v\|}\right); \end{aligned}$$

thus F admits a continuous lift $F\colon \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ of the form $\tilde{F}(\theta, x) = (\theta+\alpha, x + f(\theta, x))$ such that $f(\theta, x+1) = f(\theta, x)$ and $\pi(x + f(\theta, x)) = A(\theta)\pi(x)/\|A(\theta)\pi(x)\|$, where $\pi\colon \mathbb{R} \to \mathbb{S}^1$, $\pi(x) = e^{i2\pi x} := (\cos(2\pi x), \sin(2\pi x))$. In order to simplify the terminology, we shall say that $\tilde{F}$ is a lift for (α, A). The map f is independent of the choice of the lift up to the addition of a constant integer $p \in \mathbb{Z}$. Following [10] and [11], we define the limit

$$\lim_{n\to\pm\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tilde{F}^k(\theta, x)),$$

which is independent of (θ, x) and where the convergence is uniform in (θ, x). The class of this number in $\mathbb{R}/\mathbb{Z}$, which is independent of the chosen lift, is called the *fibered rotation number* of (α, A) and denoted by $\rho_f(\alpha, A)$. Moreover $\rho_f(\alpha, A)$ is continuous as a function of A (with respect to the uniform topology on $C_0^0\big(\mathbb{T}, SL(2,\mathbb{R})\big)$).

2.3. **Continued fraction expansion. Diophantine conditions.** Define as usual for $0 < \alpha < 1$,

$$a_0 = 0, \qquad \alpha_0 = \alpha,$$

and inductively for $k \geq 1$,

$$a_k = [\alpha_{k-1}^{-1}], \qquad \alpha_k = \alpha_{k-1}^{-1} - a_k = G(\alpha_{k-1}) = \left\{\frac{1}{\alpha_{k-1}}\right\},$$

where $[\ ]$ denotes the integer part and $G(\cdot)$ the fractional part (the Gauss map). We also set

$$\beta_k = \prod_{j=0}^{k} \alpha_j.$$

For all $k \geq 0$,

$$\beta_k = (-1)^k (q_k \alpha - p_k), \tag{2.1}$$

$$\frac{1}{q_{k+1} + q_k} < \beta_k < \frac{1}{q_{k+1}}, \tag{2.2}$$

$$\beta_k = \frac{1}{q_{k+1} + \alpha_{k+1} q_k}. \tag{2.3}$$

We use the notation

$$|||x||| := \inf_{p \in \mathbf{Z}} |x - p|.$$

Recall that

$$\forall 1 \le k < q_n, \qquad |||k\alpha||| \ge |||q_{n-1}\alpha|||. \tag{2.4}$$

We say that $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ satisfies a *Diophantine condition* $\mathrm{DC}(\gamma, \tau)$, where $\gamma > 0$, $\tau > 0$, if for every $p \in \mathbb{Z}, q \in \mathbb{N}^*$,

$$|q\alpha - p| \ge \frac{\gamma^{-1}}{q^{1+\tau}}.$$

Let $\alpha \in \mathbb{R}$ and $\theta > 0$. We say that $\rho \in \mathbb{R}/\mathbb{Z}$ is *θ-Diophantine with respect to α* if there exists C such that

$$|2\rho - k\alpha - l| > C(1 + |k| + |l|)^{-\theta}, \qquad (k, l) \in \mathbb{Z}^2.$$

For short, we shall say that ρ is *Diophantine with respect to α* (no mention to θ is made) when it is θ-Diophantine with respect to α with $\theta = 2$. Note that the set $\bigcup_{\gamma>0} \mathrm{DC}(\gamma, \tau)$ of Diophantine numbers with given exponent $\tau > 0$ is a set of full Lebesgue measure. Note also that given any $\alpha \in \mathbb{R}$, the set of $\rho \in \mathbb{T}$ that are Diophantine with respect to α is a set of full Haar measure on the circle.

2.4. **Eliasson's local theorem on reducibility.**

We will need the following *quantitative* extension of a result by Eliasson [9] (where the case of analytic continuous quasiperiodic Schrödinger cocycles is considered).

THEOREM 5. *There exist two constants $C_1, C_2 > 0$ such that the following holds. Let $\hat{A} \in \mathrm{SL}(2, \mathbb{R})$ and $\tau > 0$ be fixed. Then there exists $\epsilon(\tau, \hat{A}) > 0$ such that if a cocycle $(\alpha, A) \in \mathbb{R} \times C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$ satisfies*

(i) $\alpha \in \mathrm{DC}(\gamma, \tau)$,
(ii) $\rho_f(\alpha, A)$ *is Diophantine with respect to* α, *and*
(iii) $\|A - \hat{A}\|_{C^0} \le \gamma^{-d_0}\epsilon$ *and* $\|A - \hat{A}\|_{C^{s_0}} \le 1$, *where* $d_0 = C_1(\tau + 1)$, $s_0 = [C_2(\tau + 1)]$,

then (α, A) *is* C^∞ *reducible.*

REMARK 3. Assume that only τ is fixed and $\hat{A}$ varies in some compact subset K of $\mathrm{SL}(2, \mathbb{R})$. Then ϵ can be taken uniform with respect to $\hat{A}$.

A sketch of the proof of Theorem 5, based on Eliasson's original proof, is given in Appendix B.

2.5. $\mathbb{Z}^2$**-actions** [14, 5]**.** Let $\Omega^r = \mathbb{R} \times C^r(\mathbb{R}, SL(2,\mathbb{R}))$ be the subgroup of Diff$(\mathbb{R} \times \mathbb{R}^2)$ made of skew-product diffeomorphisms $(\alpha, A)\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}\times\mathbb{R}^2$,

$$(\alpha, A)(x, w) = (x+\alpha, A(x)w).$$

A C^r fibered $\mathbb{Z}^2$-action is a homomorphism $\Phi\colon \mathbb{Z}^2 \to \Omega^r$. We denote by Λ^r the space of such actions. When, in the above definitions, the group $SL(2,\mathbb{R})$ is replaced with $SO(2,\mathbb{R})$, we denote by $\Lambda^r(SO(2,\mathbb{R}))$ the subset of Λ^r thus obtained. As $\Phi(1,0)$ and $\Phi(0,1)$ determine Φ, we often write for short $\Phi = ((\Phi(1,0), \Phi(0,1))$. Let $\pi_1\colon \mathbb{R}\times C^r(\mathbb{R}, SL(2,\mathbb{R})) \to \mathbb{R}$, $\pi_2\colon \mathbb{R}\times C^r(\mathbb{R}, SL(2,\mathbb{R})) \to C^r\big(\mathbb{R}, SL(2,\mathbb{R})\big)$ be the coordinate projections. Let also $\gamma^{\Phi}_{n,m} = \pi_1\circ\Phi(n,m)$ and $A^{\Phi}_{n,m} = \pi_2\circ\Phi(n,m)$. We let Λ^r_0 be the set of $\Phi\in\Lambda^r$ such that $\gamma^{\Phi}_{1,0} = 1$ and $\gamma^{\Phi}_{0,1}\in[0,1]$.

We say that an action is:

– *constant* if for all $(n,m)\in\mathbb{Z}^2$, the maps $A^{\Phi}_{n,m}$ are constants;

– *normalized* if $\Phi(1,0) = (1,\mathrm{Id})$, and in that case, if $\Phi(0,1) = (\alpha, A)$ the map $A\in C^r\big(\mathbb{R}, SL(2,\mathbb{R})\big)$ is clearly $\mathbb{Z}$-periodic.

If $\Phi\in\Lambda^\infty$, $r\in\mathbb{N}$, and $I\subset\mathbb{R}$ is an interval or $I=\mathbb{T}$, we denote by $\|\Phi\|_{r,I}$ the quantities

$$\|\Phi\|_{r,I} := \max(\|\partial^r A^{\Phi}_{1,0}\|_{C^0(I)}, \|\partial^r A^{\Phi}_{0,1}\|_{C^0(I)}), \qquad \|\Phi\|^{\max}_{r,I} := \max_{0\le s\le r}\|\Phi\|_{s,I}.$$

We define

$$d_{r,I}(\Phi_1,\Phi_2) := \max\big(\|\partial^r(A^{\Phi_1}_{1,0} - A^{\Phi_2}_{1,0})\|_{C^0(I)}, \|\partial^r(A^{\Phi_1}_{0,1} - A^{\Phi_2}_{0,1})\|_{C^0(I)}\big).$$

and a distance on Λ^r: if $\Phi_1,\Phi_2\in\Lambda^r$ we set

$$d^{\max}_{r,I}(\Phi_1,\Phi_2) := \max_{0\le s\le r} d_{s,I}(\Phi_1,\Phi_2).$$

When I is the interval $[0,T]$ $(T\in\mathbb{R})$, we denote $\|\cdot\|_{r,I}$ and $d_{r,I}(\cdot,\cdot)$ by $\|\cdot\|_{r,T}$ and $d_{r,T}(\cdot,\cdot)$.

DEFINITION 1. Two fibered $\mathbb{Z}^2$ actions Φ,Φ' are said to be *conjugate* if there exists a smooth map $B\colon\mathbb{R}\to SL(2,\mathbb{R})$ such that

$$\forall (n,m)\in\mathbb{Z}^2 \quad \Phi'(n,m) = (0,B)\circ\Phi(n,m)\circ(0,B)^{-1},$$

that is, if

$$A^{\Phi'}_{n,m}(t) = B(t+\gamma^{\Phi}_{n,m})A^{\Phi}_{n,m}(t)B(t)^{-1} \quad \text{and} \quad \gamma^{\Phi'}_{n,m} = \gamma^{\Phi}_{n,m}.$$

We write $\Phi' := \mathrm{Conj}_B(\Phi)$ and get from the Hadamard–Kolmogorov convexity estimates (see Appendix A, (5.3)) the following estimates:

$$\|\Phi'\|_{r,I} \le K_r(I)(1+\|B\|_{0,I})^3(\|\Phi\|^{\max}_{r,I}\|B\|_{0,I} + \|\Phi\|_{0,I}\|B\|_{r,I}) \tag{2.5}$$

and, similarly, if $\Phi'_j = \mathrm{Conj}_B(\Phi_j)$, $j=1,2$,

$$d_{r,I}(\Phi'_1,\Phi'_2) \le K_r(I)(1+\|B\|_{0,I})^3(d^{\max}_{r,I}(\Phi_1,\Phi_2)\|B\|_{0,I} + d_{0,I}(\Phi_1,\Phi_2)\|B\|_{r,I}), \tag{2.6}$$

where the constant $K_r(I)$ is, for any r, a decreasing function of the length of I (and can be chosen equal to C^r, where $C>0$, when $I=\mathbb{T}$). We shall sometime denote $K_r(I)$ by $K_{r,I}$; when $I=[0,T]$, we write $K_{r,T}$ in place of $K_{r,[0,T]}$, and when $I=\mathbb{T}$ we simply write K_r.

A fibered action is said to be *reducible* if it is conjugate to a constant action. A fibered $\mathbb{Z}^2$-action can always be conjugated to a normalized one.

LEMMA 2. *If $\Phi \in \Lambda^r$ with $\gamma_{1,0}^{\Phi} = 1$, then there exist a conjugation $B \in C^r\big(\mathbb{R}, SL(2,\mathbb{R})\big)$ and a normalized action $\tilde{\Phi}$ such that $\tilde{\Phi} = \mathrm{Conj}_B(\Phi)$; then, letting $s, T \in \mathbb{N}$ with $0 \le s \le r$, there exists K_s such that*

$$\|B\|_{C^s([0,T])} \le K_s T^s (\|\Phi(1,0)\|_{0,T})^T \|\Phi(1,0)\|_{s,T}^{\max}.$$

If furthermore, $P := \max_{T\in\mathbb{Z}} \|\Phi(T,0)\|_{C^r(\mathbb{R})} < \infty$, then

$$\|B\|_{C^s([0,T])} \le K_s P^{4s} T^s \|\Phi(1,0)\|_{0,1}^2 \|\Phi(1,0)\|_{s,T}^{\max}.$$

The same results are true if $\Phi \in \Lambda^r(SO(2,\mathbb{R}))$.

We give the proof of Lemma 2 in Appendix A.

Finally, we observe that a normalized action $\Phi = ((1,\mathrm{Id}),(\alpha, A))$ is reducible if and only if the cocycle (α, A) is reducible (*cf.* [14, 5]).

2.6. **Renormalization of actions.** A fundamental tool in this note will be the results on convergence of renormalizations of bounded cocycles obtained in [14, 5, 6]. We recall here, following [5], the scheme of renormalization of $\mathbb{Z}^2$ actions introduced in [14].

Let $\lambda \neq 0$. Define $M_\lambda : \Lambda^r \to \Lambda^r$ by

$$M_\lambda(\Phi)(n,m) := (\lambda^{-1}\gamma_{n,m}^{\Phi}, A_{n,m}^{\Phi}(\lambda\cdot)).$$

Let $\theta_* \in \mathbb{R}$. Define $T_{\theta_*} : \Lambda^r \to \Lambda^r$ by

$$T_{\theta_*}(\Phi)(n,m) := (\gamma_{n,m}^{\Phi}, A_{n,m}^{\Phi}(\cdot + \theta_*)).$$

Let $U \in GL(2,\mathbb{Z})$. Define $N_U : \Lambda^r \to \Lambda^r$ by

$$N_U(\Phi)(n,m) := \Phi(n', m'), \quad \text{where} \quad \begin{pmatrix} n' \\ m' \end{pmatrix} = U^{-1}\begin{pmatrix} n \\ m \end{pmatrix}.$$

Notice that these three operations commute. Also, T_{θ_*} and U_N commute with Conj_B while $M_\lambda \circ \mathrm{Conj}_B = \mathrm{Conj}_{B(\lambda\cdot)} \circ M_\lambda$. Further, if $\Phi_1, \Phi_2 \in \Lambda^\infty$, then

$$\|M_\lambda \Phi_1\|_{r,\lambda^{-1}T} = \lambda^r \|\Phi_1\|_{r,T}, \qquad d_{r,\lambda^{-1}T}(M_\lambda\Phi_1, M_\lambda\Phi_2) = \lambda^r d_{r,T}(\Phi_1,\Phi_2). \tag{2.7}$$

Let

$$Q_n = \begin{pmatrix} q_n & p_n \\ q_{n-1} & p_{n-1} \end{pmatrix}, \tag{2.8}$$

and define for $n \in \mathbb{N}$ and $\theta_* \in \mathbb{R}$ the *renormalized* actions

$$\mathcal{R}^n(\Phi) := M_{\beta_{n-1}} \circ N_{Q_n}(\Phi)$$

$$\mathcal{R}^n_{\theta_*}(\Phi) := T_{\theta_*}^{-1}[\mathcal{R}^n(T_{\theta_*}(\Phi))].$$

Renormalization is a powerful tool with which to study reducibility due to the following elementary fact:

LEMMA 3 ([14, 5]). *If there exist n and θ_* such that $\mathcal{R}^n_{\theta_*}(\Phi)$ is C^r-reducible, then Φ is C^r-reducible.*

2.7. **Fibered rotation number of nondegenerate $\mathbb{Z}^2$-actions.** It is possible to associate to any (nondegenerate) fibered $\mathbb{Z}^2$-action Φ a notion of *fibered degree*. Since our initial quasiperiodic cocycle (α, A) is homotopic to the identity, the degree of the associated $\mathbb{Z}^2$-action Φ^A is zero (*cf.* [14, 5]) and in that case, once a lift is chosen for the corresponding quasiperiodic projective cocycle, we can define a *fibered rotation number* $\mathrm{rot}(\Phi)$ for the $\mathbb{Z}^2$-action Φ^A. A change in the choice of the lift results in adding to the fibered rotation number an element of the frequency module $\{k\gamma^{\Phi}_{1,0} + l\gamma^{\Phi}_{0,1} : (k,l) \in \mathbb{Z}^2\}$. We refer the reader to [14, 5] for the definition of this rotation number and its behavior under renormalization. It is proven in the aforementioned references that

$$\mathrm{rot}(\mathcal{R}^n\Phi) = (-1)^n \mathrm{rot}(\Phi)/\beta_{n-1}. \tag{2.9}$$

2.8. **Convergence of the renormalized actions** [14, 5, 6]**.** Let Φ be the action $((1,\mathrm{Id}),(\alpha,A))$. Write $((1,C_n^{(1)}),(\alpha_n,C_n^{(2)})) = \mathcal{R}^n_{\theta_*}(\Phi)$. By definition

$$C_n^{(1)} = A_{(-1)^{n-1}q_{n-1}}(\beta_{n-1}\cdot), \qquad C_n^{(2)} = A_{(-1)^n q_n}(\beta_{n-1}\cdot). \tag{2.10}$$

[6] established the following result:

THEOREM 6 ([6]). *If (α, A) is of degree 0 and is L^2-conjugate to a cocycle of rotations, then for almost every $\theta_* \in \mathbb{T}$, there is a constant matrix B such that if $\mathrm{Conj}_B(\Phi)$ denotes the conjugate action of Φ by B, then one has $\mathcal{R}^n_{\theta_*}(\mathrm{Conj}_B(\Phi)) = ((1,\tilde{C}_n^{(1)}),(\alpha_n,\tilde{C}_n^{(2)}))$ with*

$$\tilde{C}_n^{(j)} = e^{U_n^{(j)}} R_{\rho_n^{(j)}}, \quad j = 1,2,$$

such that for any $r \in \mathbb{N}$, $\|U_n^{(j)}\|_{C^r([0,1],sl(2,\mathbb{R}))} \to 0$, and $\rho_n^{(j)} \in \mathbb{R}$.

3. THE WELL APPROXIMATED CASE

3.1. Statement of the result.

DEFINITION 2. A number $\alpha \in \mathbb{R}\smallsetminus\mathbb{Q}$ is said to be of Roth type if for every $\epsilon > 0$ there exist at most finitely many integers q such that $|\!|\!|q\alpha|\!|\!| \le \frac{1}{q^{1+\epsilon}}$.

We will say that a number $\alpha \in \mathbb{R}\smallsetminus\mathbb{Q}$ is *well approximated* if it is not of Roth type. In this case, there exist $\epsilon > 0$ and an infinite set $\mathcal{N}(\epsilon) \subset \mathbb{N}$ such that for any $n \in \mathcal{N}(\epsilon)$, we have

$$|\!|\!|q_n\alpha|\!|\!| \le \frac{1}{q_n^{1+\epsilon}}.$$

Notice that if α is well-approximated, then there exists $M \in \mathbb{N}$ such that if $n \in \mathcal{N}(\epsilon)$ is sufficiently large, then

$$\alpha_n^M \le \frac{1}{q_n}. \tag{3.1}$$

We say that $\alpha \in \mathbb{T}\smallsetminus\mathbb{Q}$ is *super-Liouville* if $\limsup_{n\to\infty} \frac{\log\log q_{n+1}}{\log q_n} = \infty$. Equivalently, this means that for any $A > 0$, $\limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n^A} = \infty$. In that case, we denote by $\mathcal{L} \subset \mathbb{N}$ an infinite set for which $\lim_{n\in\mathcal{L},n\to\infty} \frac{\log\log q_{n+1}}{\log q_n} = \infty$.

In this section, we prove the following theorem that encompasses both Theorem 2 and Theorem 3.

THEOREM 7. *Suppose $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ is well approximated and $A(\cdot) \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ satisfies one of the two following sets of assumptions:*
Case (I)

(H1) $|||\dfrac{2\rho_f(\alpha,A)}{\beta_{n_i-1}}||| \geq \rho_0 > 0$ *for some sequence* $n_i \to \infty,\ n_i \in \mathcal{N}(\epsilon)$*;*

(H2) (α, A) *is C^0-conjugate to a cocycle of rotations (or equivalently has uniformly bounded products);*

Case (II)

(SL) α *is super-Liouville;*

(H1) $|||\dfrac{2\rho_f(\alpha,A)}{\beta_{n_i-1}}||| \geq \rho_0 > 0,$ *for some sequence* $n_i \to \infty,\ n_i \in \mathcal{L}$*;*

(H2') (α, A) *is L^2-conjugate to a cocycle of rotations.*

Then $A(\cdot)$ is C^∞-accumulated by functions $\tilde{A}(\cdot) \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ such that $(\alpha, \tilde{A})$ is C^∞-reducible. Moreover, (α, A) is almost rotation-reducible.

REMARK 4. Given any sequence of numbers $\beta_n \to 0$, the set of numbers ρ for which there exist $\rho_0 > 0$ and an infinite sequence $n_i \in \mathcal{N}(\epsilon)$ (resp. $n_i \in \mathcal{L}$) such that for all i, we have $|||\rho/\beta_{n_i}||| \geq \rho_0$, is of full Lebesgue measure.

3.2. **Plan of the proof of Theorem 7.** We concentrate on the description of case (I). The basic observation for the proof of Theorem 7 is the following: let (α, A) be a cocycle (homotopic to the identity), where $A(\cdot)$ is close to some cocycle of rotations $R_\varphi(\cdot) \in C^\infty(\mathbb{T}, SO(2,\mathbb{R}))$, and assume that φ is close to $\varphi(0)$ that satisfies $\min_{l\in\mathbb{Z}} \|\varphi(0) - (l/2)\| \geq \delta$ where δ is not too small. On the other hand, let us assume that α is very well approximated by rational numbers and consider the even worse case where α is very small. The assumptions on A allow us to find a conjugation $B \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ that "diagonalizes" $A(\cdot)$ in the usual algebraic sense: $B(\cdot)^{-1}A(\cdot)B(\cdot) = R_{\varphi_1(\cdot)} \in C^\infty(\mathbb{T}, SO(2,\mathbb{R}))$, the C^k-norms of B being under control. Though this conjugation relation is only algebraic, it can be put in a "dynamical" form by writing

$$B(\cdot+\alpha)^{-1}A(\cdot)B(\cdot) = \Big(B(\cdot+\alpha)^{-1}B(\cdot)\Big)R_{\varphi_1(\cdot)}.$$

But $B(\cdot+\alpha)^{-1}B(\cdot) = I + O(\alpha\|\partial B\|)$ and since we have assumed that α is very small, this quantity will also be very small (as we said before, we have good estimates on the norms of B). The virtue of this remark is to reduce by conjugation the situation to a perturbative case where the size of the perturbation is now related to the "badness" of α. This is the content of Lemma 4 below. Notice that this step can be iterated a finite number of times and this will allow us to prove Proposition 2, which is the main ingredient of the paper.

We now illustrate how this argument coupled with renormalization gives the proof of Theorem 7. Consider now the case where α is a Liouville number. If we assume that (α, A) is C^0-rotations-reducible, renormalization will converge to constants. This means (after normalizing the renormalized action) that there exists a cocycle $(\alpha_n, \tilde{A}^{(n)})$, with $\alpha_n = G^n(\alpha)$ (G is the Gauss map $G(x) = \{1/x\}$) and with $\tilde{A}^{(n)}$ close to a constant rotation in C^k-topology for some fixed k, from which one can retrieve many of the dynamical properties of (α, A). If we make some assumption on the fibered rotation number of (α, A), we can expect that $\tilde{A}^{(n)}(0)$ will be not too close to I. Also, if α is a Liouville number, there are renormalization times n_k where α_{n_k} goes to zero much faster than any power of $q_{n_k}^{-1}$. If we apply the basic observation we have described in the previous paragraph, we can conjugate $(\alpha_{n_k}, \tilde{A}^{(n_k)})$ to a cocycle $(\alpha_{n_k}, \hat{A}^{(n_k)})$ that will be closer to an $SO(2,\mathbb{R})$-valued cocycles $(\alpha_{n_k}, R_{\varphi_{n_k}})$ with α_{n_k} small. Now invert the renormalization both for $(\alpha_{n_k}, \hat{A}^{(n_k)})$ and $(\alpha_{n_k}, R_{\varphi_{n_k}})$: this basically means that we iterate each of these cocycles about q_{n_k} times. But these two cocycles are much closer than $q_{n_k}^{-1}$; this has as a consequence that the two cocycles obtained after applying inverse renormalization will remain close. Since conjugation and renormalization commute (up to dilation that is of the order of q_{n_k} at the k^{th} step), this will give the desired result. Some technicalities about actions complicate the argument a little bit, which essentially remains the same for case (II).

The same set of ideas and Proposition 2, coupled with the quantitative version of Eliasson's reducibility theorem, will also be useful in the proof of Theorem 1 (*cf.* Subsection 4.1).

3.3. **The renormalized actions.** Theorem 6 (about the convergence of renormalization) and Lemma 2 (about normalization) imply the following proposition.

PROPOSITION 1. *Let Φ be the action $((1, Id), (\alpha, A))$, and assume it is L^2-reducible. Then there exist a subsequence n_i and a sequence of matrices B_i such that for any $r \in \mathbb{N}$, $T \in \mathbb{R}$,*

$$\|B_i\|_{r,T} \le u_i(r, T). \tag{3.2}$$

Here

$$u_i(r, T) = K_r T^r \|A\|_0^{q_{n_i-1}(T+1)} \|A\|_r, \tag{3.3}$$

and the action $\mathrm{Conj}_{B_i}(\mathcal{R}_{\theta_*}^{n_i}(\Phi))$ is of the form $((1, Id), (\alpha_{n_i}, G_i))$ and satisfies

$$G_i(\cdot) = e^{U_i(\cdot)} R_{\rho_i}$$

with, for any $r \in \mathbb{N}$,

$$U_i \in C^r\big(\mathbb{T}, sl(2,\mathbb{R})\big), \qquad \|U_i\|_{C^r(\mathbb{T})} \to 0,$$

and (cf. 2.9))

$$\rho_i = (-1)^{n_i} \frac{\rho_f(\alpha, A)}{\beta_{n_i-1}}.$$

Furthermore, if the fibered products $A_n(\cdot)$ are uniformly C^0-bounded, namely if $\max_{n\in\mathbf{Z}} \|A_n\|_{C^0(\mathbf{T})} < \infty$, one can take $u_i(r, T)$ to be equal to $u(r, T)$:

$$u(r, T) = \tilde{K}_r(A) T^r, \tag{3.4}$$

where $\tilde{K}_r(A)$ depends on A.

Proof. From Section 2.8, consider a subsequence of n_i (that we still call n_i to simplify notation) such that $\mathcal{R}^{n_i}_{\theta_*}(\mathrm{Conj}_B(\Phi)) = ((1,\tilde{C}^{(1)}_i),(\alpha_{n_i},\tilde{C}^{(2)}_i))$, with

$$\tilde{C}^{(j)}_i = e^{U^{(j)}_i} R_{\rho^{(j)}_i}, \quad j=1,2,$$

and for any $r\in\mathbb{N}$, $\varepsilon_{i,r} := \max_{j=1,2}\|U^{(j)}_i\|_{C^r([0,1])} \to 0$ as i goes to infinity, and $\rho^{(j)}_i \in \mathbb{R}$.

Since $\tilde{C}^{(1)}_i = \mathrm{Ad}_B \cdot A_{(-1)^{n_i-1}q_{n_i-1}}(\beta_{n_i-1}\cdot)$, from Lemma 8 of Appendix A and (2.10) we deduce that for any $T\in\mathbb{R}$,

$$\begin{aligned}\|\partial^r \tilde{C}^{(1)}_i\|_{r,T} &\le K_r \beta^r_{n_i-1} q^r_{n_i-1}\|A\|_r \|A\|_0^{q_{n_i-1}}\\ &\le K_r\|A\|_r\|A\|_0^{q_{n_i-1}}.\end{aligned}$$

Using Lemma 2, this provides us with a normalizing conjugation $\tilde{B}_i$ such that $B_i := \tilde{B}_i B$ satisfies the required bound (3.2) and the conclusion of the theorem.

If we now assume that $\sup_{n\in\mathbb{Z}}\|A_n\|_0<\infty$, then the fibered products of $\tilde{C}^{(1)}_i$ are C^0-bounded and we have

$$\|(1,\tilde{C}^{(1)}_i)\|_{C^r([0,T])} \le \beta^r_{n_i-1} q^r_{n_i-1} K_r \|A\|_{C^r(\mathbb{T})}.$$

Using Lemma 2 concludes the proof. □

3.4. **Further reduction.** We now come to the crucial basic observation that allows us to conjugate the renormalized system, which is close to a constant system, to a cocycle which is a perturbation of a cocycle of rotations, the size of the perturbation now being explicit in terms of the new rotation number on the base (α_{n_i}).

We use the notation of Proposition 1.

PROPOSITION 2. *There exists a sequence of conjugacies $D_i \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ that is bounded (in $C^r(\mathbb{T}, SL(2,\mathbb{R}))$-norms, for each $r\in\mathbb{N}$) and such that the sequence $\tilde{\Phi}^{(1)}_i = \mathrm{Conj}_{D_i}((1,Id),(\alpha_{n_i},G_i)) = ((1,Id),(\alpha_{n_i},\tilde{G}_i))$ satisfies*

$$\tilde{G}_i(\cdot) = e^{\tilde{U}_i(\cdot)} R_{\tilde{\rho}_i(\cdot)} \tag{3.5}$$

with, for any $l,l'\in\mathbb{N}$,

$$\lim_{i\to\infty}\frac{\|\tilde{U}_i\|_{C^l}}{\alpha^{l'}_{n_i}} = 0 \tag{3.6}$$

$$\lim_{i\to\infty}\|\tilde{\rho}_i(\cdot)-\rho_i\|_{C^l} = 0. \tag{3.7}$$

Moreover, there exists constants $\mu_s(\rho_0)$ depending only on s and ρ_0 such that

$$\|D_i\|_{C^s(\mathbb{T})} \le \mu_s(\rho_0).$$

Combining this proposition with Proposition 1, we immediately get the following corollary.

COROLLARY 2. *If we define* $Z_i := D_i B_i$, *we have*

$$\tilde{\Phi}_i = \mathrm{Conj}_{Z_i}(\mathrm{R}_{\theta_*}^{n_i}(\Phi)) = ((1, Id), (\alpha_{n_i}, \tilde{G}_i)),$$

and Z_i *satisfies the following estimate for* $n_i \in \mathcal{N}(\varepsilon)$ *(Case (I)) or* $n_i \in \mathcal{L}$ *(Case(II)):*

$$\|Z_i\|_{C^r([0,T])} \le \mu_r(\rho_0) u_i(r, T). \tag{3.8}$$

Proposition 2 follows inductively from the following.

LEMMA 4. *Let* $\rho_0 \in (0, 1/2)$ *and* $M > 0$. *There exist* $\varepsilon_0 > 0$ *and, for any* $s \in \mathbb{N}$, *constants* $C_s > 0$ *such that for any* $U \in C^r(\mathbb{T}, sl(2,\mathbb{R}))$, $\rho \in C^r(\mathbb{T}, \mathbb{R})$ *satisfying*

$$\|U\|_{C^0(\mathbb{T})} \le \varepsilon, \quad \inf_{l \in \mathbb{Z}, x \in \mathbb{T}} |2\rho(x) - l| \ge 2\rho_0 \quad \text{and} \quad \|\rho\|_{C^s} \le M,$$

there exist $Y, \tilde{U} \in C^r(\mathbb{T}, sl(2,\mathbb{R}))$, *and* $\tilde{\rho} \in C^r(\mathbb{T}, \mathbb{R})$ *such that*

$$(0, e^{Y(\cdot)}) \circ (\alpha, e^{U(\cdot)} R_{\rho(\cdot)}) \circ (0, e^{Y(\cdot)})^{-1} = (\alpha, e^{\tilde{U}(\cdot)} R_{\tilde{\rho}(\cdot)}),$$

with

$$\|\tilde{U}\|_{C^{s-1}} \le C_s(\rho_0) \alpha \|U\|_{C^s} \tag{3.9}$$

$$\|\tilde{\rho} - \rho\|_{C^s} \le C_s(\rho_0) \|U\|_{C^s}, \tag{3.10}$$

with a conjugacy $e^{Y(\cdot)}$ *satisfying*

$$\|Y\|_{C^s} \le C_s(\rho_0) \|U\|_{C^s}. \tag{3.11}$$

Proof. By simple algebra, there exists $\varepsilon_0 > 0$ such that for any $u \in sl(2,\mathbb{R})$, $\rho \in \mathbb{R}$ satisfying $\|u\| \le \varepsilon_0$, $\rho \in (\rho_0, 1/2 - \rho_0) M$, there exist $y \in sl(2,\mathbb{R})$ and $\tilde{\rho} \in R$ such that

$$e^u R_\rho = e^{-y} R_{\tilde{\rho}} e^y.$$

Also, $y = H_1(u, \rho)$, $\tilde{\rho} = H_2(u, \rho)$ for some real analytic functions H_1, H_2 defined on $E_{\varepsilon_0, \rho_0, M} := \{u \in sl(2,\mathbb{R}) : \|u\| \le \varepsilon_0\} \times (\rho_0, 1/2 - \rho_0)$, and we clearly have $H_1(0, \rho) = 0$ and $H_2(0, \rho) = \rho$ for any $\rho \in (\rho_0, 1/2 - \rho_0)$. We define $Y(\cdot) = H_1(U(\cdot), \rho(\cdot))$ and $\tilde{\rho}(\cdot) = H_2(U(\cdot), \rho(\cdot))$ so that

$$e^{U(\theta)} R_\rho(\theta) = e^{-Y(\theta)} R_{\tilde{\rho}(\theta)} e^{Y(\theta)}.$$

It is then a standard fact that $\|Y\|_{C^s} \le C_s \|U\|_{C^s}$, $\|\tilde{\rho} - \rho\|_{C^s} \le C_s \|U\|_{C^s}$.

Next, we write

$$e^{U(\theta)} R_\rho(\theta) = e^{-Y(\theta+\alpha)} e^{\tilde{U}(\theta)} R_{\tilde{\rho}(\theta)} e^{Y(\theta)},$$

where $e^{\tilde{U}(\theta)} := e^{Y(\theta+\alpha)} e^{-Y(\theta)}$. Now it can be proven (cf [13], Prop. A.2.3) that if ϕ, f, u are smooth functions, for any $s \in \mathbb{N}$ there exists a constant C_s such that

$$\|\phi \circ (f + u) - \phi \circ u\|_s \le C_s \|\phi\|_s (1 + \|f\|_0)^s (1 + \|f\|_s) \|u\|_s.$$

If we choose $\phi = \exp$, $f = Y(\cdot)$, $u = Y(\cdot + \alpha) - Y(\cdot)$, then (3.9) follows from (3.11) if $\varepsilon_0 > 0$ is chosen sufficiently small. □

3.5. **End of proof of Theorem 7.** We introduce the notation $\Psi := \mathcal{IR}^n_{\theta_*}(\Psi')$ if $\Psi' = \mathcal{R}^n_{\theta_*}(\Psi)$. It is easy to check that

$$\mathcal{IR}^n_{\theta_*}(\Phi) = T^{-1}_{\theta_*} \circ N_{Q_n^{-1}} \circ M_{\beta^{-1}_{n-1}} \circ T_{\theta_*}(\Phi). \tag{3.12}$$

The proof of Theorem 7 from Corollary 2 will be a consequence of the following lemma, the proof of which is given in the Appendix.

LEMMA 5. *Let* $\Phi_1, \Phi_2 \in \Lambda^r$. *Then, provided* $d_{0,T}(\Phi_1,\Phi_2) \le (1/q_n^2)$, *we have*

$$d_{r,\beta^2_{n-1}T}(\mathcal{IR}^n(\Phi_1), \mathcal{IR}^n(\Phi_2)) \le K_{r,\beta_{n-1}T}\, q_n^{2r+1} \|\Phi_2\|_{0,T}^{q_n} \|\Phi_2\|_{r,T}\, d_{r,T}(\Phi_1,\Phi_2).$$

We can now conclude the derivation of Theorem 7 from Corollary 2. We use the notation of Section 3.4. We define $\Phi_1 = \Phi := ((1,\mathrm{Id}),(\alpha,A))$, $\Phi_1^{(i)} = \mathcal{R}^{n_i}_{\theta_*}(\Phi_1)$, $\tilde\Phi_1^{(i)} = \mathrm{Conj}_{Z_i}(\Phi_1^{(i)}) = ((1,\mathrm{Id}),(\alpha_{n_i},\tilde G_i))$ and $\tilde\Phi_2^{(i)} = ((1,\mathrm{Id}),(\alpha_{n_i},R_{\tilde\rho_i}))$. We then set $\Phi_2^{(i)} = \mathrm{Conj}_{Z_i^{-1}}(\tilde\Phi_2^{(i)})$, and $\Phi_{2,i} = \mathcal{IR}^{n_i}(\Phi_2^{(i)})$. From Proposition 2, we know that there exists $K_{r,l'}$ $(r = l)$ such that

$$d_{r,T}(\tilde\Phi_1^{(i)},\tilde\Phi_2^{(i)}) \le K_{r,l'}\,\alpha_{n_i}^{l'}$$

for any r,T,l'. If we set $\tilde\Phi_{1,i} = \mathrm{Conj}_{Z_i^{-1}(\beta^{-1}_{n_i-1}\cdot)}(\Phi_1)$, $\tilde\Phi_{2,i} = \mathrm{Conj}_{Z_i^{-1}(\beta^{-1}_{n_i-1}\cdot)}(\Phi_{2,i})$, we have $\tilde\Phi_{j,i} = \mathcal{IR}^{n_i}(\tilde\Phi_j^{(i)})$ $(j=1,2)$. By Lemma 5, the fact that $\tilde\Phi_2^{(i)}$ is in $\Lambda^\infty(SO(2,\mathbb{R}))$, and, since for n_i big enough, $d_{r,T}(\tilde\Phi_1^{(i)},\tilde\Phi_2^{(i)}) \le (1/q_{n_i}^2)$ (see (3.1)), we can write

$$d_{r,\beta^2_{n_i-1}T}(\tilde\Phi_{1,i},\tilde\Phi_{2,i})) \le K_{r,l',\beta_{n_i-1}T}\, q_{n_i}^{2r+1} \|\tilde\Phi_2^{(i)}\|_{r,T}\,\alpha_{n_i}^{l'}. \tag{3.13}$$

Since $\Phi_{j,i} = \mathrm{Conj}_{Z_i(\beta^{-1}_{n_i-1}\cdot)}(\tilde\Phi_{j,i})$, using inequalities (2.6) and (3.8) we then get

$$d_{r,\beta^2_{n_i-1}T}(\Phi_1,\Phi_{2,i}) \le K_{r,l',\beta_{n_i-1}T}(\mu(\rho_0)u_i(r,\beta_{n_i-1}T))^4 q_{n_i}^{3r+1} \|\tilde\Phi_2^{(i)}\|_{r,T}\,\alpha_{n_i}^{l'}. \tag{3.14}$$

From Lemma 8 of the Appendix,

$$\|\tilde\Phi_2^{(i)}\|_{r,T} \le K_r\, q_{n_i}^{2r}, \tag{3.15}$$

and thus

$$d_{r,\beta^2_{n_i-1}T}(\Phi_1,\Phi_{2,i}) \le K_{r,l',\beta_{n_i-1}T}(\mu(\rho_0)u_i(r,\beta_{n_i-1}T))^4 q_{n_i}^{5r+1}\alpha_{n_i}^{l'}.$$

Denote by ε_{r,n_i} the quantity on the right-hand side of this last equation with $T = \beta^{-2}_{n_i-1}$. That is (notice that $K_{r,l',\beta^{-1}_{n_i-1}} = K_{r,l'}$ since $\beta^{-1}_{n_i} > 1$),

$$\varepsilon_{r,n_i} = K_{r,l'}(\mu(\rho_0)u_i(r,\beta^{-1}_{n_i-1}))^4 q_{n_i}^{5r+1}\alpha_{n_i}^{l'}. \tag{3.16}$$

In case (I), it is enough to assume that α is well-approximated to ensure that $\varepsilon_{r,n_i} \to 0$ as $n_i \to \infty$, $n_i \in \mathcal{N}(\varepsilon)$, since in that case $\alpha_{n_i} \le q_{n_i}^{-\frac{1}{M}}$ for some subsequence; notice that if this is so, then case (I) has the improved estimates on u_i (which is u), to work with (*cf.* Proposition 1). In case (II), the stronger assumption $\lim_{n\in\mathcal{L},n\to\infty}(\log q_n)^{-1}\log\log q_{n+1} = \infty$ is a sufficient condition to ensure this convergence to 0.

The action $\Phi_{2,i}$ is conjugate to an action in $\Lambda^r(SO(2,\mathbb{R}))$ and is ε_{r,n_i}-close to Φ_1 in the C^r-topology, but it is not necessarily normalized. Since

$$\|\mathrm{Id}-\Phi_{2,i}(1,0)\|_{C^r([0,1]} = \|\Phi_1(1,0)-\Phi_{2,i}(1,0)\|_{C^r([0,1]} \leq \varepsilon_{r,n_i},$$

we can construct from Lemma 2 a conjugacy $\tilde{B}_i \in C^r\big(\mathbb{R}, SL(2,\mathbb{R})\big)$ such that $\tilde{B}_i(\cdot+1)\Phi_{2,i}(1,0)\tilde{B}_i(\cdot)^{-1} = \mathrm{Id}$ and satisfying

$$\|\tilde{B}_i - \mathrm{Id}\|_{C^r([0,1])} \leq K_r\varepsilon_{r,n_i}. \tag{3.17}$$

As a consequence, from (2.6),

$$d_{r,1}(\Phi_{2,i},\bar{\Phi}_{2,i}) \leq K_r\varepsilon_{r,n_i}(1+\varepsilon_{r,n_i})^3\|\Phi_{2,i}\|_{r,1}$$

and from (3.15),

$$d_{r,1}(\Phi_{2,i},\bar{\Phi}_{2,i}) \leq 2K_r^2\varepsilon_{r,n_i}q_{n_i}^{2r}.$$

The *normalized* action $\bar{\Phi}_{2,i} := \mathrm{Conj}_{\tilde{B}_i}\Phi_{2,i}$ is conjugate to one in $\Lambda^r(SO(2,\mathbb{R}))$ satisfying

$$d_{r,1}(\Phi_1,\bar{\Phi}_{2,i}) \leq 3K_r^2\varepsilon_{r,n_i}q_{n_i}^{2r}. \tag{3.18}$$

With the preceding notation, observe that $\bar{\Phi}_{2,i} = \mathrm{Conj}_{\tilde{B}_i}\mathrm{Conj}_{Z_i(\beta_{n_i-1}^{-1}\cdot)}(\tilde{\Phi}_{2,i})$ and that $\tilde{\Phi}_{2,i} \in \Lambda^r_{SO(2,\mathbb{R})}$. This means that for any i, there exist $\bar{A}_i \in C^r(\mathbb{T}, SL(2,\mathbb{R}))$, $\varphi_i \in C^r(\mathbb{T},\mathbb{R})$, and $\bar{B}_i \in C^r(\mathbb{T}, SL(2,\mathbb{R}))$, where $\bar{B}_i(\cdot) = \tilde{B}_i(\cdot)Z_i(\beta_{n_i-1}^{-1}\cdot)$ satisfies (*cf.* Corollary 2 and (3.17))

$$\|\bar{B}_i\|_{C^r(\mathbb{T})} \leq K_r q_{n_i}^r \mu_r(\rho_0)u_i(r,\beta_{n_i-1}^{-1}), \tag{3.19}$$

such that

$$\|A(\cdot)-\bar{A}_i(\cdot)\|_{C^r(\mathbb{T})} \leq 3K_r^2\varepsilon_{r,n_i}q_{n_i}^{2r} \tag{3.20}$$

and

$$(\alpha,\bar{A}_i) = (0,\bar{B}_i)\circ(\alpha,R_{\varphi_i})\circ(0,\bar{B}_i)^{-1}. \tag{3.21}$$

Proof of the fact that (α,A) *is accumulated by reducible cocycles.* In both cases (I) and (II), $\varepsilon_{r,n_i}q_{n_i}^{2r}$ goes to zero as n_i goes to infinity (n_i being in $\mathcal{N}(\varepsilon)$ in case (I) and n_i being in $\mathcal{L}$ in case (II)). Thus (3.20) shows that (α,A) is C^r-accumulated by cocycles that are C^r-conjugate to the C^r-cocycles $(\alpha,R_{\varphi_i(\cdot)})$ with values in $SO(2,\mathbb{R})$. But such cocycles can be accumulated by reducible ones: just truncate the Fourier series of φ_i far enough to get a trigonometric polynomial $\bar{\varphi}_i(\cdot)$ and solve the usual cohomological equation $\psi(\cdot+\alpha)-\psi(\cdot) = \bar{\varphi}_i(\cdot)-\int_{\mathbb{T}}\bar{\varphi}_i(x)\,dx$.

Proof of the fact that (α,A) *is almost rotations-reducible.* From (3.21), (3.20), (3.19) and the convexity inequalities we have

$$\|(0,\bar{B}_i)^{-1}\circ(\alpha,A)\circ(0,\bar{B}_i)-R_{\varphi_i}\|_{C^r(\mathbb{T})} \leq 3K_r^4 q_{n_i}^{4r}(\mu_r(\rho_0)u_i(r,\beta_{n_i-1}^{-1}))^2\varepsilon_{r,n_i}. \tag{3.22}$$

But from (3.3) and (3.16) the quantity in the right-hand side of this inequality goes to zero as n_i goes to infinity (n_i being in $\mathcal{N}(\varepsilon)$ in case (I) and n_i being in $\mathcal{L}$ in case (II)). This proves the almost rotations-reducibility and completes the proof of Theorem 7.

3.6. **Proofs of Theorems 2, 3.** Theorems 2, 3 now clearly follow from Theorem 7.

4. THE DIOPHANTINE CASE.

In this section, we prove Theorem 1.

4.1. **Plan of the proof of Theorem 1.** We show in this subsection how the arguments given in Section 3.2 can be used to prove Theorem 1; we now assume that (α, A) is a smooth cocycle homotopic to the identity and that α is a Diophantine number. We observed in Section 3.2 that after renormalization and conjugation, we can associate to (α, A) cocycles $(\alpha_n, \hat{A}^{(n)})$, where $\alpha_n = G^n(\alpha)$ and $\hat{A}^{(n)}$ is α_n^C-close in the C^r-topology (r fixed) to some $SO(2,\mathbb{R})$-valued cocycle $(\alpha_n, R_{\varphi_n})$, where $C > 0$ is an arbitrarily large constant. Since α is Diophantine, α_n is also Diophantine with the same exponent, but its Diophantine constant can be small. The relevant observation is that there exist infinitely many times n_k such that the Diophantine constant of α_{n_k} is not less than $\mathrm{cst}\cdot\alpha_{n_k}$. This allows us, first to find a further conjugation that conjugates $(\alpha_{n_k}, \hat{A}^{(n_k)})$, $\alpha_{n_k}^{C'}$-close ($C' > 0$) in C^r-topology to a *constant* cocycle (α_{n_k}, R_{n_k}), and second to apply a quantitative version of Eliasson's Theorem (Theorem 5, the proof of which can be found in the Appendix) that guarantees reducibility of (α_{n_k}, R_{n_k}) provided the fibered rotation number satisfies some (weak) Diophantine condition. Since reducibility is preserved by renormalization, this proves that (α, A) is reducible.

In fact, it was pointed to us by the referee that a local KAM theorem on reducibility with estimates would suffice to prove Theorem 1. Indeed, in Theorem 1 it is supposed that the fibered rotation number should belong to a set $\Sigma(\alpha)$ of full measure, while in Theorem 8, that is, a precise version implying Theorem 1, the set $\Sigma(\alpha)$ is partly described by the fact that the fibered rotation number $\rho_f(\alpha, A)$ is supposed to be Diophantine with respect to α, a condition that is evidently of full measure. With this condition, a quantitative version of Eliasson's result is needed to obtain reducibility.

However, one can further restrict the set $\Sigma(\alpha)$ and remain with a set of full measure ensuring nonetheless that, along a subsequence, the renormalized cocycle $(\alpha_{n_k}, \hat{A}^{(n_k)})$ has the vector $(\alpha_{n_k}, \rho_{n_k})$ (where ρ_{n_k} is the fibered rotation number of $(\alpha_{n_k}, \hat{A}^{(n_k)})$) satisfying a Diophantine condition with constant $\mathrm{cst}\cdot\alpha_{n_k}$. From there, a usual KAM theorem with estimates would allow to conclude, since $\hat{A}^{(n_k)}$ is $\alpha_{n_k}^{C'}$ close to a constant, where $C' > 0$ is arbitrarily large.

4.2. **Diophantine constants.** We will first make explicit the condition on the fibered rotation number that will be used in the proof of Theorem 1. We need the following.

LEMMA 6. *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ be a Diophantine number. Then there exist $\nu > 0$ and $C > 0$ that depend on α, and an infinite set $\mathcal{M} \subset \mathbb{N}$, such that for $n \in \mathcal{M}$, we have*

$$\forall k \geq 1, \qquad |\!|\!| k\alpha_n |\!|\!| \geq \frac{C\alpha_n}{k^{1+\nu}}, \tag{4.1}$$

that is, $\alpha_n \in \mathrm{DC}(C\alpha_n, \nu)$.

Proof. For $n \geq 1$ define $\nu_n > 0$ such that $q_{n+1} = q_n^{1+\nu_n}$. Since α is Diophantine, there exists $\nu_0 > 0$ such that for all $n \geq 1$ we have $\nu_n \leq \nu_0$. We define

$$\nu = 10\nu_0 + 10. \tag{4.2}$$

Let $\mathcal{M}$ be the subset of $n \in \mathbb{N}$ such that

$$\forall\, l \geq n, \qquad \nu_l \leq 2\nu_n. \tag{4.3}$$

The set $\mathcal{M}$ is infinite because ν_n is bounded, and we will show that the assertion of the lemma holds for every $n \in \mathcal{M}$ with $C = \frac{1}{2\cdot 4^{\nu+1}}$.

Recall that $\alpha_k = \beta_k/\beta_{k-1}$ and that $\beta_k = (-1)^k(q_k\alpha - p_k)$; hence for $k, l \in \mathbb{N}$, we have

$$\begin{aligned}|k\alpha_n - l| &= \frac{1}{\beta_{n-1}}|(kq_n + lq_{n-1})\alpha - kp_n - lp_{n-1}|\\ &\geq q_n|(kq_n + lq_{n-1})\alpha - kp_n - lp_{n-1}|.\end{aligned}$$

Since

$$\frac{q_n}{2q_{n+1}} \leq \alpha_n \leq 2\frac{q_n}{q_{n+1}},$$

we have

$$|||k\alpha_n||| = |k\alpha_n - l(k)| \quad \text{for some } l(k) \leq 3k\frac{q_n}{q_{n+1}}.$$

If $1 \leq k < \frac{q_{n+1}}{4q_n}$, we have

$$|||k\alpha_n||| = k\alpha_n \geq \alpha_n.$$

If $\frac{q_{n+1}}{4q_n} \leq k < \frac{q_{n+2}}{4q_n}$, we have $kq_n + l(k)q_{n-1} < q_{n+2}$; hence (2.2) and (2.4) imply

$$|||k\alpha_n||| = |k\alpha_n - l(k)| \geq q_n|(kq_n + lq_{n-1})\alpha - kp_n - lp_{n-1}| \geq \frac{q_n}{2q_{n+2}},$$

while we have

$$\frac{q_n}{2q_{n+2}} = \frac{1}{2q_n^{(1+\nu_n)(1+\nu_{n+1})-1}}.$$

Since $k \geq \frac{q_{n+1}}{4q_n}$, we have

$$\frac{1}{k^{\nu+1}} \leq \frac{4^{\nu+1}}{q_n^{\nu_n(\nu+1)}};$$

hence the fact that $\nu_n(\nu+1) \geq \nu_n\nu \geq (1+\nu_n)(1+\nu_{n+1})$ implies that

$$|||k\alpha_n||| \geq \frac{C}{k^{\nu+1}}.$$

More generally, for $\frac{q_{n+i}}{4q_n} \leq k < \frac{q_{n+i+1}}{4q_n}$, $i \geq 1$, we have $kq_n + l(k)q_{n-1} < q_{n+i+1}$; hence (2.2) and (2.4) imply

$$|||k\alpha_n||| \geq \frac{q_n}{2q_{n+i+1}},$$

or equivalently

$$|||k\alpha_n||| \geq \frac{1}{2q_n^{(1+\nu_n)\dots(1+\nu_{n+i+1})-1}};$$

having in mind that

$$\frac{1}{2\cdot 4^{\nu+1}}\left(\frac{4q_n}{q_{n+i}}\right)^{\nu+1} = \frac{1}{2q_n^{(\nu+1)[(1+\nu_n)\dots(1+\nu_{n+i})-1]}}.$$

Our choice of ν in (4.2) then implies

$$|||k\alpha_n||| \geq \frac{C}{k^{\nu+1}}.$$

Lemma 6 is proved. □

We can now state a precise version of Theorem 1.

THEOREM 8. *Let $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ be a Diophantine number and let $\mathcal{M} = \mathcal{M}(\alpha)$ be the set of integers given by Lemma 6. Then if $A \in C_0^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ satisfies*

(D1) *$\rho_f(\alpha, A)$ is Diophantine with respect to α,*

(D2) $|||\frac{2\rho_f(\alpha,A)}{\beta_{n_i-1}}||| \geq \rho_0 > 0$*, for some sequence $n_i \to \infty$, $n_i \in \mathcal{M}$, and*

(D3) *(α, A) is L^2-conjugate to a cocycle of rotations,*

then (α, A) is C^∞-reducible.

This theorem implies Theorem 1 since the arithmetic conditions imposed on $\rho_f(\alpha, A)$ are both clearly of full measure.

4.3. **Proof of Theorem 8.** We will need the following elementary fact.

LEMMA 7. *If ρ is Diophantine with respect to α, then for any $n \in \mathbb{N}$, $\frac{\rho}{\beta_{n-1}}$ is Diophantine with respect to α_n.*

Proof. As in the proof of Lemma 6, we have

$$\begin{aligned}
|2\frac{\rho}{\beta_{n-1}} - k\alpha_n - l| &= \frac{1}{\beta_{n-1}}|2\rho - (kq_n + lq_{n-1})\alpha + kp_n + lp_{n-1}| \\
&\geq \frac{1}{\beta_{n-1}}\frac{C}{(1+|kq_n+lq_{n-1}|+|kp_n+lp_{n-1}|)^2} \\
&\geq \frac{C_n}{(1+|k|+|l|)^2}
\end{aligned}$$

□

Let $\mathcal{M}$ be the set given by Lemma 6 and consider a sequence $n_i \to \infty$, $n_i \in \mathcal{M}$, such that (D2) holds for this sequence. Since reducibility is invariant under renormalization and conjugation, it is enough to show that the action $\Phi_i = ((1,\mathrm{Id}),(\alpha_{n_i}, G_i))$ of Section 2.8 is reducible.

By (D2), we can apply Proposition 2 and conjugate Φ_i to $((1,\mathrm{Id}),(\alpha_{n_i},\tilde{G}_i))$ such that $\tilde{G}_i = e^{\tilde{U}_i(\cdot)} R_{\tilde{\rho}_i(\cdot)}$, with, for any $l, l' \in \mathbb{N}$

$$\lim_{i\to\infty} \frac{\|\tilde{U}_i\|_{C^l}}{\alpha_{n_i}^{l'}} = 0 \tag{4.4}$$

$$\lim_{i\to\infty} \|\tilde{\rho}_i - \rho_i\|_{C^l} = 0, \tag{4.5}$$

where $\rho_i := (-1)^{n_i}\rho_f(\alpha,A)/\beta_{n_i-1}$.

From (4.5), we have that $\lim_{i\to\infty} \left\|\tilde{\rho}_i - \int_{\mathbb{T}} \tilde{\rho}_i\right\|_{C^l} = 0$. In addition, $\alpha_{n_i} \in \mathrm{DC}(C\alpha_{n_i},\nu)$, so there exists $h_i \in C^\infty(\mathbb{T},\mathbb{R})$ such that

$$\tilde{\rho}_i(\cdot) - \int_{\mathbb{T}} \tilde{\rho}_i = h_i(\cdot + \alpha_{n_i}) - h_i(\cdot)$$

and

$$\lim_{i\to\infty} \alpha_{n_i}\|h_i\|_{C^l} = 0.$$

Due to (4.4), conjugating $(1,\mathrm{Id}),(\alpha_{n_i},\tilde{G}_i))$ by $R_{h_i(\cdot)}$, we get $((1,\mathrm{Id}),(\alpha_{n_i},\overline{G}_i))$, where

$$\|\overline{G}_i - R_{\int_{\mathbb{T}}\tilde{\rho}_i}\|_{C^l} = o(\alpha_{n_i}^{l'}) \tag{4.6}$$

for any $l, l' \in \mathbb{N}$ as $i \to \infty$.

Finally, $\rho_f(\alpha_{n_i},\overline{G}_i) = \rho_f(\alpha_{n_i},G_i) = (-1)^{n_i}\rho_f(\alpha,A)/\beta_{n_i-1}$ is Diophantine with respect to α_{n_i}, as is asserted by Lemma 7. So, the fact that $\alpha_{n_i} \in \mathrm{DC}(C\alpha_{n_i},\nu)$ and (4.6) allow us to use Eliasson's local theorem on reducibility, Theorem 5 (*cf.* Remark 3), to conclude that for i sufficiently large, $(1,\mathrm{Id}),(\alpha_{n_i},\overline{G}_i))$ is reducible, and so is Φ_i and the original action Φ.

5. APPENDIX A

5.1. **Proof of Lemma 2.** One just has to prove that, given $C \in C^\infty\big(\mathbb{R}, SL(2,\mathbb{R})\big)$, one can always find $B \in C^\infty\big(\mathbb{R}, SL(2,\mathbb{R})\big)$ such that $C(t) = B(t+1)B(t)^{-1}$, which is not difficult in the C^∞-category (it is more difficult in the analytic category; see Lemma 4.1 of [5]). Indeed, one can find $B \in C^\infty([0,1], SL(2,\mathbb{R}))$ such that $B(0) = \mathrm{Id}$, $B(1) = C(0)$, and such that for any $k \geq 1$, $\partial^k B(0) = 0$ and $\partial^k B(1) = \partial^k(CB)(0)$, $\|B\|_{s,[0,1]} \leq K_s\|C\|_{s,[0,1]}$. Setting $B(t) = C(t-1)\cdots C(t-m)B(t-m)$ for $t \in [m, m+1]$ completes the definition of B. One then has, for any $T \in \mathbb{N}$,

$$\|B\|_{C^s([0,T])} \leq K_s \max_{0\leq l\leq T}\Big(\|C_l\|_{C^s([0,1])}\|C\|_{C^0([0,1])} + \|C_l\|_{C^0([0,1])}\|C\|_{C^s([0,1])}\Big), \tag{5.1}$$

where $(l, C_l(\cdot)) = (1, C(\cdot))^l$. Lemma 2 will follow from Lemma 8 and Corollary 3 below.

LEMMA 8. *Let $(u_k)_{1\leq k\leq n}$ be a sequence of functions in $C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$, and let $U_n = u_1 \cdot \cdots \cdot u_n$. Denote by $M_s = \max_{1\leq k\leq n}(\|u_k\|_s)$. Then*

$$\|U_n\|_r \leq (nC)^r M_0^{n-1} M_r.$$

If the functions are defined on $[0,1]$ *one can replace* C^r *by some constant* K_r *depending only on* r *(and on the interval* $[0,1]$*).*

Proof.

$$\partial^r U_n(x) = \partial^r \left(\prod_{k=n-1}^{0} u_k \right)(x), \tag{5.2}$$

which by the Leibniz formula is a sum of n^r terms of the form ($s \le r$)

$$\begin{aligned} I_{(i^*)}(x) = \left(\prod_{l=n-1}^{i_1+1} u_l(x) \right) \cdot \partial^{m_1} u_{i_1}(x) \cdot \left(\prod_{l=i_1-1}^{i_2+1} u_l(x) \right) \cdot \\ \cdot \partial^{m_2} u_{i_2}(x) \cdot \left(\prod_{l=i_2-1}^{i_3+1} u_l(x) \right) \cdots \\ \cdot \partial^{m_s} u_{i_s}(x) \cdot \left(\prod_{l=i_s-1}^{0} u_l(x) \right), \end{aligned}$$

where i^* runs through $\mathcal{I} = \{0, \dots, n-1\}^{\{1,\dots,r\}}$, and where $\{i_1, \dots, i_s\} = i^*(\{1,\dots,r\})$ satisfy $n-1 \ge i_1 > i_2 > \cdots i_s \ge 0$ and $m_l = \#(i^*)^{-1}(i_l)$ (notice that $m_1 + \ldots + m_s = r$).

From this and the convexity (Hadamard–Kolmogorov) inequalities [12],

$$\|\partial^m u\|_{C^0} \le C \|u\|_0^{1-(m/r)} \|\partial^r u\|_{C^0}^{\frac{m}{r}}, \quad 0 \le m \le r, \tag{5.3}$$

we deduce (using $\sum_{p=1}^s m_p = r$)

$$\|I_{(i^*)}(x)\| \le M_0^{n-s} \prod_{p=1}^{s} \left(C M_0^{1-\frac{m_p}{r}} M_r^{\frac{m_p}{r}} \right) \le C^s M_0^{n-1} M_r,$$

so

$$\|U_n\| \le \sum_{i^* \in \mathcal{I}} \|I_{(i^*)}(x)\| \le n^r C^r M_0^{n-1} M_r. \qquad \square$$

Lemma 8 has an immediate corollary on the growth of the products A_n.

COROLLARY 3. *Let* $J \subset \mathbb{R}$ *be some interval and define* $J_n = \bigcup_{l=0}^{n-1}(J + l\alpha)$. *If* $M_k(x) > 1$ *is an upper bound of* $\|A_k(x)\|$ *for* $x \in J$, *then for any* $x \in J$

$$\|\partial^r A_n(x)\| \le K^r \|A_n(x)\|_{C^0(J)} M_0(x)^{2r} (M_0^2(x) + \cdots + M_{n-1}^2(x))^r \|\partial^r A(x)\|_{C^0(J_n)}.$$

Observe that when the fibered products $C_l(t)$ are uniformly bounded for $t \in \mathbb{R}$ ($P := \sup_{l \in \mathbb{R}} \|C_l\|_{C^0(\mathbb{R})} < \infty$), we have $\|C_l\|_{C^s(\mathbb{R})} \le K_s (M^4 l)^s \|C\|_{C^s([0,l])} \|C\|_{C^0([0,1])}$ for some constants K_s, $s \in \mathbb{N}$, and using (5.1) we get the estimate for B.

In the general case,

$$\|C_l\|_{C^s([0,1])} \le K_s l^s \|C\|_{0,l}^{l-1} \|C\|_{s,l}.$$

This and (5.1) conclude the proof of Lemma 2.

5.2. **Proof of Lemma 5.**

Proof. We use the following lemma.

LEMMA 9. *Let* (γ, A), $(\gamma, \tilde{A}) \in \Omega^r$, *and define* $(\gamma, A)^k := (k\gamma, A_k)$, $(\gamma, \tilde{A})^k := (k\gamma, \tilde{A}_k)$ *for* $k \in \mathbb{Z}$. *If we define* $I_n := \bigcup_{l=0}^{n-1}(I + l)$, $M_s := \|A\|_{C^s(I_n)}$, $H_s := \|A - \tilde{A}\|_{C^s(I_n)}$, *then*

$$\|\partial^s A_n - \partial^s \tilde{A}_n\|_{C^0(I)} \leq K_s n^{s+1}(M_0 + H_0)^{n-2}\Big((M_s + H_s)H_0 + (M_0 + H_0)H_s\Big).$$

Proof. Define $H = \tilde{A} - A$ and set $\Delta(\nu, t) = \prod_{l=n-1}^{0}(A(t + l\gamma) + \nu H(t + l\gamma))$. By the Mean-Value Theorem, we clearly have

$$\|\partial^s A_n - \partial^s \tilde{A}_n\|_{C^0(I)} \leq \max_{0 \leq \nu \leq 1} \|\partial_\nu(\partial^s \Delta(\nu, \cdot))\|_{C^0(I)}.$$

But $\partial_\nu(\partial^s \Delta(\nu, \cdot))$ is a sum of n terms of the form $(0 \leq m \leq n-1)$

$$\partial^s \Big(\Big(\prod_{l=n-1}^{m+1}(A(t + l\gamma) + \nu H(t + l\gamma))\Big) H(t + m\gamma)\Big(\prod_{l=m-1}^{0}(A(t + l\gamma) + \nu H(t + l\gamma))\Big)\Big).$$

By the convexity inequalities and Lemma 3, the C^s-norm of this expression is less than

$$C_s n^{s+1}(M_0 + H_0)^{n-2}\left((M_s + H_s)H_0 + (M_0 + H_0)H_s\right). \qquad \square$$

Denote $\Psi_i = M_{\beta_{n-1}^{-1}} \circ T_{\theta_*}(\Phi_i)$, $i = 1, 2$, and $\Gamma_n = T_{\theta_*}^{-1} \circ N_{Q_n^{-1}}$. We clearly have

$$d_{r, \beta_{n-1} T}(\Psi_1, \Psi_2) = \beta_{n-1}^{-r} d_{r,T}(\Phi_1, \Phi_2)$$

and

$$\|\Psi_1\|_{r, \beta_{n-1} T} = \beta_{n-1}^{-r} \|\Phi_1\|_{r,T}.$$

From (3.12), the previous lemma, and the definition of N_{Q_n}, we then get

$$d_{r, \beta_{n-1}^2 T}(\Gamma_n(\Psi_1), \Gamma_n(\Psi_2)) \leq K_{r, \beta_{n-1} T}\, q_n^{r+1} \|\Psi_1\|_{0, \beta_{n-1} T}^{q_n - 2}$$
$$\times \left(\|\Psi_1\|_{0, \beta_{n-1} T}\, d_{r, \beta_{n-1} T}(\Psi_1, \Psi_2) + \|\Psi_1\|_{r, \beta_{n-1} T}\, d_{0, \beta_{n-1} T}(\Psi_1, \Psi_2)\right),$$

which in view of the previous formulae is the conclusion of Lemma 5. $\square$

6. APPENDIX B
AN ABRIDGED PROOF OF THEOREM 5

The proof relies on a KAM scheme (named after Kolmogorov, Arnold, and Moser). Assume that $\alpha \in \mathrm{DC}(\gamma, \tau)$, that is,

$$|||q\alpha||| \geq \frac{\gamma^{-1}}{q^{1+\tau}},$$

and that $A(\cdot) = e^{F_0(\cdot)} A_0$; here $F_0 \in C^r\big(\mathbb{T}, sl(2, \mathbb{R})\big)$, where $sl(2, \mathbb{R})$ is the Lie algebra of $SL(2, \mathbb{R})$, whose elements are traceless 2×2 real matrices.

DEFINITION 3. For $N, K > 0$, define the set

$$DS(N, K) := \{\beta \in \mathbb{R} \mid \inf_{l \in \mathbb{Z}, 0 < |k| \leq N} |\beta - k\alpha - l| \geq K^{-1}\}.$$

It is easy to check the following.

LEMMA 10. *Let* $\alpha \in \mathrm{DC}(\gamma,\tau)$*,* $N > 0$*, and* $K \geq \gamma(2N)^{3(\tau+1)}$*. Then:*

(i) *If* $\beta \notin DS(N,K)$*, then there exists a unique* $k_0 \in \mathbb{Z} - \{0\}$*, such that* $|k_0| \leq N$ *and* $|||\beta - k_0\alpha||| < 1/K$*, and we have* $\beta - k_0\alpha \in DS(N^3,K)$*.*

(ii) *The following inclusion holds:* $\{|\beta| < K^{-1}\} \subset DS(N^3,K)$*.*

This lemma will be used with sequences N_n and K_n given by $N_n = L^{(1+\sigma)^n}$, $K_n = \gamma(2N_n)^{3(\tau+1)}$, where $\sigma > 0$ and $L > 1$ some constants that will be fixed later. The lemma shows that if β is almost *resonant* at step n, that is, $\beta \notin DS(N_n,K_n)$, then it is possible, by replacing β by $\tilde\beta = \beta - k_0\alpha$, to get a number $\tilde\beta$ that is nonresonant for a much longer period. The importance of the exponent in the Diophantine condition on $\tilde\beta$ will be explained later.

If g in $SL(2,\mathbb{R})$ is elliptic (that is, conjugate to a rotation matrix or, equivalently, if the absolute value of its trace is less than 2), it can be written e^M, where $M \in sl(2,\mathbb{R})$ is elliptic (conjugate to an infinitesimal rotation). In that case, $\det M$ is positive, and we write $\beta(g) := \pi^{-1}\sqrt{\det M}$. The eigenvalues of $\mathrm{Ad}(g)\colon sl(2,\mathbb{R}) \to sl(2,\mathbb{R})$ (which is defined by $\mathrm{Ad}(g)\cdot X = gXg^{-1}$) are 1 and $\pm e^{2\pi i\beta(g)}$. Notice that if (α, A) is a cocycle with A constant and elliptic, then its fibered rotation number ρ equals $\beta(A)/2$.

If f is a periodic C^r function, we define its Fourier coefficients

$$\hat f(k) := \int_{\mathbb{T}} f(\theta) e^{-2\pi i k\theta}\, d\theta,$$

its truncation up to order N,

$$T_N f := \sum_{|k|\leq N} \hat f(k) e^{2\pi i k\theta},$$

and its remainder at order N, $R_N f := f - T_N f$. For any $s, s' \in \mathbb{N}$, $s' \geq s$ we have the following estimates:

$$|T_N f|_{s'} \leq C_{s,s'} N^{s'-s+1} |f|_s, \quad \text{and} \quad |R_N f|_s \leq C_{s,s'} \frac{|f|_{s'}}{N^{s'-s-1}}. \tag{6.1}$$

Also, if Q is quadratic in (f,g) (Q being C^2 and $Q(0,0) = 0$, $DQ(0,0) = 0$), we have

$$|Q(f,g)|_s \leq C_s(1 + |f|_0 + |g|_0)^{s+1}(|f|_0 + |g|_0)(|f|_s + |g|_s), \tag{6.2}$$

which simplifies to

$$|Q(f,g)|_s \leq C_s(|f|_0 + |g|_0)(|f|_s + |g|_s)$$

if $|f|_0 + |g|_0$ is *a priori* bounded (by 1 for example).

The main procedure at every step of the KAM scheme is given by the following proposition.

PROPOSITION 3. *If* $|A| \leq 1$ *and* N *and* K *satisfy* $K = \gamma(2N)^{3(\tau+1)}$ *and* $N^{6(\tau+1)}|F|_0$ *is small enough, then:*

(i) *if* $\beta(A) \in DS(N,K)$*, there are*

$$Y \in C^\infty\big(\mathbb{T}, sl(2,\mathbb{R})\big), A' \in sl(2,\mathbb{R}), F' \in C^\infty\big(\mathbb{T}, sl(2,\mathbb{R})\big)$$

such that

$$e^{Y(\cdot+\alpha)}(e^{F(\cdot)}A)e^{-Y(\cdot)} = e^{F'(\cdot)}A', \tag{6.3}$$

$$A' = e^{\hat{F}(0)}A,$$

$$|F'|_s \le C_s\gamma^2 N^{6(\tau+1)}|F|_s|F|_0 + C_{s,s'}\frac{|F|_{s'}}{N^{s'-s-1}},$$

$$|Y|_s \le \gamma C_s N^{6(\tau+1)}|F|_s, \quad \textit{and}$$

$$\rho(\alpha, e^{F'(\cdot)}A') = \rho(\alpha, e^{F(\cdot)}A);$$

(ii) if there exists $m \in \mathbb{Z}$, $0 < |m| \le N$, such that $\|\beta(A) - m\alpha\| < K^{-1}$, then there exist $B \in C^\infty(\mathbb{R}/2\mathbb{Z}, SL(2,\mathbb{R}))$, $A' \in sl(2,\mathbb{R})$, $F' \in C^\infty(\mathbb{T}, sl(2,\mathbb{R}))$ such that

$$|\beta(A')| < K^{-1},$$

$$B(\cdot+\alpha)(e^{F(\cdot)}A)B(\cdot)^{-1} = e^{F'(\cdot)}A',$$

$$|F'|_s \le C_{s,s'}\gamma^4 N^{20(\tau+1)}\Big(N^s|F|_0^2 + |F|_s|F|_0) + \frac{N^{s'}|F|_0 + |F|_{s'}}{N^{3(s'-s-1)}}\Big), \quad \textit{and}$$

$$\rho(\alpha, e^{F'(\cdot)}A') = \rho(\alpha, e^{F(\cdot)}A) - m\alpha/2.$$

Proof.

Case (i): Since α is Diophantine and $\beta(A) \in DS(N,K)$, where $K = \gamma(2N)^{3(\tau+1)}$, it is easy to see, going through Fourier coefficients, that the so-called *linearized* equation

$$Y(\cdot+\alpha) - \mathrm{Ad}(A)Y(\cdot) = -(T_N F - \hat{F}(0))$$

has a unique solution $Y \in C^\infty(\mathbb{T}, sl(2,\mathbb{R}))$ such that $\hat{Y}(0) = 0$ and, for $k \in \mathbb{Z} - \{0\}$, one has $\hat{Y}(k) = 0$ if $|k| > N$ and

$$\hat{Y}(k) = \Big(e^{2\pi i k\alpha}\mathrm{Id} - \mathrm{Ad}(A)\Big)^{-1}\hat{F}(k)$$

if $|k| \le N$. The eigenvalues of $\Big(e^{2\pi i k\alpha}\mathrm{Id} - \mathrm{Ad}(A)\Big)^{-1}$ are

$$(e^{2\pi i k\alpha} - 1)^{-1}, \quad (e^{2\pi i k\alpha} - e^{2\pi i\beta(A)})^{-1} \text{ and } (e^{2\pi i k\alpha} - e^{-2\pi i\beta(A)})^{-1},$$

and from this it is not difficult to get

$$|Y|_s \le C_s\gamma^2 N^{6(\tau+1)}|F|_s. \tag{6.4}$$

If we define $A' = e^{\hat{F}(0)}A$ and F' by (6.3), one can see that F' is the sum of an expression that is at least quadratic in $(T_N F, Y, F)$ and of a remainder of size comparable to $R_N F = F - T_N F$. The required estimates then follow from the application of (6.1), (6.2) and (6.4). The equality on the fibered rotation numbers is due to the fact that this fibered rotation number is invariant under conjugations that are *homotopic to the identity*.

Case (ii): In this case, the constant matrix A is elliptic and can be conjugated to a rotation $\exp(\beta(A)\pi H)$, where $H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the constant conjugation being of size bounded by $C/\beta(A)^2 \le C\gamma^2 N^{2(\tau+1)}$ (because for some $0<|k_0|\le N$, $|||\beta - k_0\alpha||| \le 1/K < (\gamma N^{3(\tau+1)})^{-1}$ while $|||k_0\alpha||| \ge (\gamma N^{\tau+1})^{-1}$).

Applying this conjugation to $e^F A$ results in a new cocycle $e^{\bar F}\exp(\pi\beta(A)H)$, where $|\bar F|_s \le C\gamma^2 N^{2(\tau+1)}|F|_s$. To keep the notation simple, we will still write A in place of $\exp(\pi\beta(A)H)$ and F in place of $\bar F$. We now perform a conjugation $D(\cdot) = \exp(-\pi H m\cdot) \in C^\infty(\mathbb{R}/2\mathbb{Z}, SL(2,\mathbb{R}))$. Writing

$$D(\cdot+\alpha)(e^{F(\cdot)}A)D(\cdot)^{-1} = e^{\tilde F(\cdot)}\tilde A,$$

we see that $\tilde A = A - m\alpha H$, so $\beta(\tilde A) = \beta(A) - m\alpha$ and

$$|\tilde F|_s \le C_s(N^s|F|_0 + |F|_s).$$

Notice that although D is only $2\mathbb{Z}$-periodic, $\tilde F$ is $\mathbb{Z}$-periodic. But then we have seen that $\beta(\tilde A) \in DS(N^3,K)$, so *Case (i)* applies to $e^{\tilde F(\cdot)}\tilde A$ with N^3 in place of N: there exists a conjugation $\tilde Y \in C^\infty\big(\mathbb{T}, sl(2,\mathbb{R})\big)$ such that if we set $B(\cdot) = e^{\tilde Y(\cdot)}D(\cdot)$, one has the desired estimates

$$|F'|_s \le C_{s,s'}\gamma^2 N^{2(\tau+1)}\left(\gamma^2 N^{6(\tau+1)}(N^s|F|_0^2 + |F|_s|F|_0) + \frac{N^{s'}|F|_0 + |F|_{s'}}{N^{3(s'-s-1)}}\right).$$

The relation on the fibered rotation number is due to the fact that if $B\colon \mathbb{R}/2\mathbb{Z} \to SL(2,\mathbb{R})$ has topological degree $-m$, then

$$\rho(\alpha, B(\cdot+\alpha)e^{F(\cdot)}AB(\cdot)^{-1}) = \rho(\alpha, e^{F(\cdot)}A) - m\alpha/2. \qquad \square$$

In view of the estimates of Proposition 3, we now apply Corollary 4 of Appendix C with $a = 20(\tau+1)$, $\sigma_0 = 1$, $M = 1$, $m = 0$, and $(\mu_p, \bar\mu_p) \in \{(1,0),(3,1)\}$. This provides us with $0<\sigma<\sigma_0$, $0<g$, $s_0\in\mathbb{N}$ (of the form $s_0 = O(\tau)$), and $L := \gamma^4\bar C_{s_0}$, $\bar C_{s_0} := \max_{s\le s_0, s'\le s_0} C_{s,s'}$ for which the conclusions of Corollary 4 are satisfied. Assuming that

$$|F_0|_{s_0} \le \gamma^{-d_0}\varepsilon,$$

where $d_0 = 4s_0/g$, $\varepsilon = (\bar C_{s_0})^{-s_0/g}$, setting $N_n = L^{(1+\sigma)^n}$ and inductively applying Proposition 3 (with, at step n, N_n, A_n, F_n in place of N, A, F) enables us to construct sequences $A_n \in sl(2,\mathbb{R})$, $F_n \in C^\infty\big(\mathbb{T}, sl(2,\mathbb{R})\big)$, $Z_n \in C^\infty(\mathbb{R}/2\mathbb{Z}, SL(2,\mathbb{R}))$, $\tilde m_n \in \mathbb{Z}$ such that

$$Z_n(\cdot+\alpha)e^{F_n(\cdot)}A_n Z_n(\cdot)^{-1} = e^{F_{n+1}(\cdot)}A_{n+1},$$

$$2\rho(\alpha, e^{F_{n+1}(\cdot)}A_{n+1}) = 2\rho(\alpha, e^{F_n(\cdot)}A_n) - \tilde m_n\alpha,$$

and satisfying the following estimates: setting $\varepsilon_{n,s} = |F_n|_s$ and $K_n = \gamma(2N_n)^{3(\tau+1)}$, for all $s\ge 0$,

$$\varepsilon_{n,s} = O(N_n^{-\infty}).$$

Also,

- if $\beta(A_n) \in DS(N_n,K_n)$ then $\tilde m_n = 0$ and Z_n is of the form e^{Y_n}, with $Y_n \in C^\infty\big(\mathbb{T}, sl(2,\mathbb{R})\big)$ satisfying $|Y_n|_s \le \gamma C_s N_n^{6(\tau+1)}\varepsilon_{n,s}$ and
- if $\beta(A_n) \notin DS(N_n,K_n)$, then $0 \ne |\tilde m_n| \le N_n$ and $|\beta(A_{n+1}| < K_n^{-1}$.

Notice that Z_n can be, depending on whether at step n we are in *Case (i)* or *Case (ii)* of Proposition 3, "small" or "big" according to the C^∞-topology.

We have thus proven *almost reducibility* without any condition on the fibered rotation number.

THEOREM 9. *If $\alpha \in \mathrm{DC}(\gamma,\tau)$ is Diophantine and $A \in SL(2,\mathbb{R})$, there exists ϵ (depending only on τ and A) and $s_0 \in \mathbb{N}$ such that for any $F \in C^\infty\big(\mathbb{T}, sl(2,\mathbb{R})\big)$ with $|F|_{s_0} \le \gamma^{-d_0}\varepsilon$, there is a sequence $W_n \in C^\infty(\mathbb{R}/2\mathbb{Z}, SL(2,\mathbb{R}))$ ($W_n = Z_{n-1}\cdots Z_n$) for which $W_n(\cdot+\alpha)(e^{F(\cdot)}A)W_n(\cdot)^{-1}$ is $\mathbb{Z}$-periodic and converges to a constant in the C^∞-topology.*

Now, to prove Theorem 5 under the assumption $|F|_{s_0} \le \gamma^{-d_0}\varepsilon$, we just have to prove that if $\rho(\alpha, e^{F(\cdot)}A)$ is Diophantine with respect to α then $\beta(A_n) \in DS(N_n, K_n)$ for all large enough n. We make the following remarks: at the n-th step of the iteration scheme, the fibered rotation number of $(\alpha, e^{F_n}A_n)$ satisfies

$$2\rho(\alpha, e^{F(\cdot)}A) = 2\rho(\alpha, e^{F_n(\cdot)}A_n) + (\tilde{m}_0 + \ldots + \tilde{m}_{n-1})\alpha,$$

and using the smallness of F_n we get

$$|2\rho(\alpha, e^{F(\cdot)}A) - \beta(A_n) - (\tilde{m}_0 + \ldots + \tilde{m}_{n-1})\alpha| \le \varepsilon_{n,0}.$$

Now assume that there is an infinite sequence n_k such that $\tilde{m}_{n_k-1} \neq 0$; then $|\beta(A_{n_k})| < K_{n_k}^{-1}$ and we have

$$|2\rho(\alpha, e^{F(\cdot)}A) - (\tilde{m}_0 + \ldots + \tilde{m}_{n_k-1})\alpha| \le K_{n_k}^{-1} + \varepsilon_{n_k,0}.$$

But since $\rho(\alpha, e^{F(\cdot)}A)$ is Diophantine with respect to α,

$$\frac{C}{(N_0 + \cdots N_{n_k-1})^2} \le K_{n_k}^{-1} + \varepsilon_{n_k,0}.$$

This is clearly impossible for k large enough, since $K_n = \gamma(2N_n)^{3(\tau+1)}$ and $\varepsilon_{n,0} = O(N_n^{-\infty})$.

We conclude that for n big enough, $\beta(A_n) \in DS(N_n, K_n)$, and consequently the conjugation $Z_n(\cdot)$ is of the form $e^{Y_n(\cdot)}$ with $Y_n \in C^\infty\big(\mathbb{T}, sl(2,\mathbb{R})\big)$ satisfying $|Y_n|_s \le \gamma C_s N_n^{6(\tau+1)}\varepsilon_{n,s}$ (*Case (i)* of the iteration scheme). The product $Z_n(\cdot)\cdots Z_0(\cdot)$ clearly converges in the C^∞-topology; this is the required conjugation that transforms $(\alpha, e^{F(\cdot)}A)$ to a constant cocycle.

7. APPENDIX C
CONVERGENCE OF A GENERALIZED KAM SCHEME

In this section we present a proof of the convergence of a generalized KAM scheme. We insist on the fact that nonconstant losses of derivatives are allowed when solving the linearized equations; typically, a loss of ms derivatives is possible ($m < 1$) if we are dealing with C^s-norms. Also, a loss in the truncated part is allowed (in the sequel of the text, this is governed by the constants $\mu, \bar{\mu}$). We essentially follow the presentation of [13, Chap. 5] and [8, Sec. 6] with some simplifications and generalizations (and corrections of some minor errors).

Let $a > 0$, $0 < \sigma_0 \le 1$, $M, m, \mu, \bar\mu > 0$ such that

$$\frac{M}{\mu} < 1 + \sigma_0, \qquad \frac{m}{\mu} < 1, \qquad 0 < 2\bar\mu < \mu.$$

Then there exists g such that

$$\frac{1}{\mu - \bar\mu} < g < \min(\frac{1+\sigma_0}{M}, \frac{1}{m}, \frac{1}{\bar\mu}).$$

and $0 < \sigma < \sigma_0$ such that

$$1 + \sigma < (\mu - \bar\mu) g,$$

that is

$$\frac{1+\sigma}{\mu - \bar\mu} < g < \min\left(\frac{1+\sigma_0}{M}, \frac{1}{m}, \frac{1}{\bar\mu}\right). \tag{7.1}$$

For $s, \bar s$, let $C_{s,\bar s} \colon [0,\infty) \to [1,\infty]$ be a family of continuous functions on $[0,2]$ such that $C_{s,\bar s}(t) = \infty$ if $t > 2$, increasing with respect to $s, \bar s \ge 0$, and let $\bar C_s = \max_{t \le 2} C_{s,s}(t)$.

THEOREM 10. *There exist $s_0 > 0$ such that if $\varepsilon_{p,s}$ is a double sequence satisfying, for any $s, \bar s, p \in \mathbb{N}$,*

$$\varepsilon_{p+1,s} \le C_{s,\bar s}(1 + \lambda_p^a \varepsilon_{p,0}) \times (\lambda_p^{a+Ms} \varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^{a+ms} \varepsilon_{p,s} \varepsilon_{p,0} + \lambda_p^{a-(\bar s - s)\mu} (\varepsilon_{p,\bar s} + \lambda_p^{\bar\mu \bar s} \varepsilon_{p,0})), \tag{7.2}$$

where $\lambda_p = L^{(1+\sigma)^p}$, $L = \overline{C}_{s_0}$, and if

$$\varepsilon_{0,0} \le (\bar C_{s_0})^{-s_0/g}, \quad \varepsilon_{0,s_0} \le 1,$$

then, $\varepsilon_{p,s} = O(\lambda_p^{-\infty})$ for any $s \in \mathbb{N}$. Also, s_0 does not depend on the sequence $(\bar C_s)_s$ and can be taken of the form $(a+1)\xi(\sigma_0, M, m, \mu, \bar\mu)$.

Proof. In view of (7.1), we choose κ such that

$$\frac{1+\sigma+\kappa}{\mu - \bar\mu} < g. \tag{7.3}$$

The proof consists of several lemmas. We assume that $(\varepsilon_{p,s})$ is a sequence satisfying (7.2), where $\lambda_p = L^{(1+\sigma)^p}$ for some $L > 1$.

LEMMA 11. *Let $\gamma_0 > 0$, $s_0 = g\gamma_0$, $b = \kappa\gamma_0$. Then there exists*

$$\gamma_0^{\text{ref}}(a, \sigma_0, M, m, \mu, \bar\mu, g, \sigma, \kappa)$$

such that if $\gamma_0 > \gamma_0^{\text{ref}}$ and $\lambda_p = L^{(1+\sigma)^p}$, $L = \bar C_{s_0}$, then

$$\varepsilon_{p,0} \le \lambda_p^{-\gamma_0} \quad \textit{and} \quad \varepsilon_{p,s_0} \le \lambda_p^b$$

for any $p \in \mathbb{N}$, provided these inequalities hold for $p = 0$. Also, there exists

$$\Omega^{\text{ref}}(\sigma_0, M, m, \mu, \bar\mu, g, \sigma, \kappa) > 0$$

such that one can choose $\gamma_0^{\text{ref}} = (a+1) \times \Omega^{\text{ref}}$.

Proof. In view of (7.2), where we make $s = 0$, $\bar{s} = s_0$, and $s = s_0, \bar{s} = s_0$ we just have to check that

$$\lambda_p^a \varepsilon_{p,0} \le 1$$

and

$$3\bar{C}_{s_0}\lambda_p^a\lambda_p^{-(1+\sigma_0)\gamma_0} \le \lambda_p^{-\gamma_0(1+\sigma)}, \quad 3\bar{C}_{s_0}\lambda_p^{a+Ms_0}\lambda_p^{-(1+\sigma_0)\gamma_0} \le \lambda_p^{b(1+\sigma)},$$

$$3\bar{C}_{s_0}\lambda_p^a\lambda_p^{-2\gamma_0} \le \lambda_p^{-\gamma_0(1+\sigma)}, \quad 3\bar{C}_{s_0}\lambda_p^{a+ms_0}\lambda_p^{-\gamma_0}\lambda_p^b \le \lambda_p^{b(1+\sigma)}, \tag{7.4}$$

$$3\bar{C}_{s_0}\lambda_p^{a-\mu s_0}(\lambda_p^b + \lambda_p^{\bar{\mu}s_0-\gamma_0}) \le \lambda_p^{-\gamma_0(1+\sigma)}, \quad 3\bar{C}_{s_0}\lambda_p^a(\lambda_p^b + \lambda_p^{\bar{\mu}s_0-\gamma_0}) \le \lambda_p^{b(1+\sigma)}, \tag{7.5}$$

These inequalities are satisfied if

$$a < \gamma_0 \tag{7.6}$$

and

$$a < \gamma_0(\sigma_0 - \sigma), \qquad Ms_0 - (1+\sigma_0)\gamma_0 < b(1+\sigma) - a, \tag{7.7}$$

$$a < \gamma_0(1-\sigma), \qquad ms_0 - \gamma_0 < b\sigma - a, \tag{7.8}$$

$$\begin{cases} a + b < \mu s_0 - \gamma_0(1+\sigma), \\ a < (\mu - \bar{\mu})s_0 - \sigma\gamma_0, \end{cases} \qquad \begin{cases} a < b\sigma, \\ \bar{\mu}s_0 - \gamma_0 < b(1+\sigma) - a, \end{cases} \tag{7.9}$$

provided L is larger than some $L^{\text{ref}}(a, \sigma_0, M, m, \mu, \bar{\mu}, g, \sigma, \kappa, \gamma_0, \bar{C}_{s_0})$, where in the limit of large γ_0 it is possible to take $L^{\text{ref}} = \bar{C}_{s_0}$. Also in the limit of large γ_0, the previous conditions (7.6), (7.7)–(7.9) will be satisfied if

$$\begin{aligned} &\kappa < \mu g - (1+\sigma), && Mg - (1+\sigma_0) < \kappa(1+\sigma), \\ &mg - 1 < \kappa\sigma, && \sigma < g(\mu - \bar{\mu}), \\ &\bar{\mu}g - 1 < (1+\sigma)\kappa, \end{aligned}$$

or equivalently if

$$\max\left(\frac{1+\sigma+\kappa}{\mu}, \frac{\sigma}{\mu - \bar{\mu}}\right) < g < \min\left(\frac{(1+\sigma)\kappa + (1+\sigma_0)}{M}, \frac{1+\kappa\sigma}{m}, \frac{1+(1+\sigma)\kappa}{\bar{\mu}}\right),$$

which follows from (7.1) and (7.3). For later use (see (7.12)), we impose in addition that γ_0 is big enough so that

$$1 < \min\left(\frac{\mu g - \kappa - (a/\gamma_0)}{1+\sigma}, \quad \frac{(\mu - \bar{\mu})g - (a/\gamma_0)}{1+\sigma}\right) \tag{7.10}$$

and

$$\frac{a}{\sigma\kappa} \le \gamma_0. \tag{7.11}$$

It is clear that in order for (7.6), (7.7)-(7.9), and (7.10) to be satisfied, it is enough to choose γ_0 larger than some $(a+1) \times \Omega^{\text{ref}}(\sigma_0, M, m, \mu, g, \sigma, \kappa) > 0$. □

LEMMA 12. *Let $a > 0$, $0 < \sigma < \sigma_0$, and $\gamma_0 > \frac{a}{\sigma_0 - \sigma}$. If $\gamma_0 < \gamma < \frac{c}{1+\sigma}$ and if $u_p = O(\lambda_p^{-\gamma_0})$ satisfies*

$$u_{p+1} \le C(\lambda_p^a u_p^{1+\sigma_0} + \lambda_p^{-c})$$

for some constant $C > 0$, then $u_p = O(\lambda_p^{-\gamma})$.

Proof. We can assume $0 < u_p < 1$. Observe that

$$2C\lambda_p^a \lambda_p^{-\gamma(1+\sigma_0)} \leq \lambda_p^{-\gamma(1+\sigma)}, \qquad 2C\lambda_p^{-c} \leq \lambda_p^{-\gamma(1+\sigma)}$$

if p is big enough (since $\gamma > a/(\sigma_0 - \sigma)$ and $c/\gamma > 1+\sigma$). Now either of the following holds.

(i) For any p, the inequality $u_p > \lambda_p^{-\gamma}$ is true, and then

$$\lambda_p^{-c} < u_p^{c/\gamma}.$$

We have

$$u_{p+1} \leq 2C\lambda_p^a u_p^{\min((c/\gamma),(1+\sigma_0))}.$$

If we define ν_p by $u_p = \lambda_p^{-\nu_p}$, we see that

$$\nu_{p+1} \geq \frac{\min((c/\gamma),(1+\sigma_0))}{1+\sigma}\nu_p - \frac{a}{1+\sigma} - \frac{\log(2C)}{(1+\sigma)^{p+1}\log L}.$$

We now introduce $\sigma < \rho < \sigma_0$ and $\delta > 0$ such that $\frac{c}{1+\sigma} > \gamma_0 > \frac{a+\delta}{\rho-\sigma}$; it is plain that for $p \geq p_0$ big enough (remember that by assumption $\inf_p \nu_p > 0$ and $1+\rho < \min((c/\gamma),(1+\sigma_0))$), one has

$$\nu_{p+1} \geq \frac{1+\rho}{1+\sigma}\nu_p - \frac{a+\delta}{1+\sigma},$$

and thus

$$\nu_{p+1} - \frac{a+\delta}{\rho-\sigma} \geq \left(\frac{1+\rho}{1+\sigma}\right)^{p-p_0}\left(\nu_{p_0} - \frac{a+\delta}{\rho-\sigma}\right).$$

Since by assumption $\liminf \nu_p \geq \gamma_0$, this implies $u_p = O(\lambda_p^{-\infty})$.

(ii) There exists $p_k \to \infty$ such that $u_{p_k} \leq \lambda_{p_k}^{-\gamma}$ and then the induction can be initiated. □

The next lemma is similar to but easier than Lemma 12, so we leave the proof to the reader.

LEMMA 13. *Suppose the sequence $u_p \geq 0$ satisfies*

$$u_{p+1} \leq C(\lambda_p^{-\gamma_1} u_p + \lambda_p^{-\gamma_2})$$

for some $C > 0$, $\gamma_2 > 0$, $\gamma_1 \in \mathbb{R}$.

(i) If $\gamma_1 < 0$, then $u_p = O(\lambda_p^b)$ for any $b > |\gamma_1|/\sigma$.

(ii) If $\gamma_1 > 0$, then $u_p = O(\lambda_p^{-b})$ for any $b < \min(|\gamma_1|/\sigma, \gamma_2/(1+\sigma))$.

The next lemma shows that, in fact, we can improve the estimates on $\varepsilon_{p,0}$ significantly without altering the one on ε_{p,s_0}, but rather extending it to $\varepsilon_{p,s}$ for any s.

LEMMA 14. *In view of (7.1), let us choose*

$$0 < \omega < \min\left(\frac{1+\sigma_0}{Mg} - 1, \frac{1}{\bar{\mu}g} - 1, \frac{1}{mg} - 1\right),$$

and define for $k \geq 1$ sequences γ_k, s_k such that

$$s_k = [(1+\omega)s_{k-1}], \qquad \gamma_k = \frac{s_k}{g}.$$

Then, for any $p \in \mathbb{N}$,

$$(P_k) \qquad \varepsilon_{p,0} = O(\lambda_p^{-\gamma_k}), \qquad \varepsilon_{p,s_k} = O(\lambda_p^b).$$

Proof. We prove by induction on k that (P_k) holds for k, assuming it is true for $k-1 \geq 0$ (the case $k=0$ is the content of Lemma 11). Observe that the sequences γ_k, s_k are increasing. By making $s = \bar{s} = s_k$ in (7.2),

$$\varepsilon_{p+1,s_k} \leq C_{s_k}\left(\lambda_p^{a+Ms_k}\varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^{a+ms_k}\varepsilon_{p,0}\varepsilon_{p,s_k} + \lambda_p^a(\varepsilon_{p,s_k} + \lambda_p^{\bar{\mu}s_k}\varepsilon_{p,0})\right),$$

and since (P_{k-1}) holds, $\lambda_p^{a+Ms_k}\varepsilon_{p,0}^{1+\sigma_0} = O(\lambda_p^{a+Ms_k-(1+\sigma_0)\gamma_{k-1}})$ with

$$a + Ms_k - (1+\sigma_0)\gamma_{k-1} = a + (Mg(1+\omega) - (1+\sigma_0))\gamma_{k-1} < 0,$$

since $\gamma_k > \gamma_0$, $Mg(1+\omega) < 1+\sigma_0$, and γ_0 was chosen big enough. We have, furthermore,

$$\bar{\mu}s_k - \gamma_{k-1} = (\bar{\mu}g(1+\omega) - 1)\gamma_{k-1} < 0,$$

since $\bar{\mu}g(1+\omega) < 1$. The inductive hypothesis (P_{k-1}) also yields $\lambda_p^{a+ms_k}\varepsilon_{p,0}\varepsilon_{p,s_k} = O(\lambda_p^{a+ms_k-\gamma_{k-1}}\varepsilon_{p,s_k})$, and it follows from $mg(1+\omega) < 1$ that

$$a + ms_k - \gamma_{k-1} = a + (mg(1+\omega) - 1)\gamma_{k-1} < a.$$

Finally, Lemma 13(a) applies and gives $\varepsilon_{p,s_k} = O(\lambda_p^b)$ with $b > a/\sigma$, hence for $b = \kappa\gamma_0$ due to (7.11).

Also, taking $s = 0, \bar{s} = s_k$ in (7.2),

$$\varepsilon_{p+1,0} \leq C_{s_k}\left(\lambda_p^a\varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^a\varepsilon_{p,0}^2 + \lambda_p^{a-\mu s_k}\left(\varepsilon_{p,s_k} + \lambda_p^{\bar{\mu}s_k}\varepsilon_{p,0}\right)\right),$$

hence

$$\varepsilon_{p+1,0} = O\left(\lambda_p^a\varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^{a+b-\mu s_k} + \lambda_p^{\bar{\mu}s_k-\gamma_{k-1}+a-\mu s_k}\right).$$

In order to be able to apply Lemma 12, we have to check that

$$\gamma_0 < \gamma_k < \frac{\mu s_k - (b+a)}{1+\sigma}, \quad \text{and} \quad \gamma_0 < \gamma_k < \frac{(\mu-\bar{\mu})s_k + \gamma_{k-1} - a}{1+\sigma},$$

or equivalently

$$\gamma_0 < \gamma_k < \min\left(\frac{\mu g\gamma_k - \kappa\gamma_0 - a}{1+\sigma}, \quad \frac{(\mu-\bar{\mu})g\gamma_k + \gamma_{k-1} - a}{1+\sigma}\right). \tag{7.12}$$

But the condition (7.10) ensures that this is the case. Lemma 12 then applies. □

We now conclude the proof of Theorem 10. Fix $s > 0$. Making $\bar{s} = s_k$ in (7.2), we get

$$\varepsilon_{p+1,s} \leq C_s\left(\lambda_p^{a+Ms_k}\varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^{a+ms}\varepsilon_{p,0}\varepsilon_{p,s} + \lambda_p^{a-\mu(s_k-s)}\left(\varepsilon_{p,s_k} + \lambda_p^{\bar{\mu}s_k}\varepsilon_{p,0}\right)\right)$$

From Lemma 14, we know that $\varepsilon_{p,0} = O(\lambda_p^{-\Gamma})$ for any $\Gamma > 0$, so

$$\varepsilon_{p+1,s} = O\left(\lambda_p^{a+ms-\Gamma}\varepsilon_{p,s} + \lambda_p^{\max(a+Ms_k-\Gamma(1+\sigma_0),\, a-\mu(s_k-s)+b,\, a-\mu(s_k-s)+\bar{\mu}s_k-\Gamma)}\right).$$

Now, we can choose Γ_k such that (remembering $\lim_{k\to\infty} s_k = \infty$)

$$\inf_k \max\Big(a-\mu(s_k-s)+b, a+Ms_k-\Gamma_k(1+\sigma_0), a+ms-\Gamma_k, a-(\mu-\bar{\mu})s_k+\mu s-\Gamma_k\Big) = -\infty.$$

Lemma 13(b) then implies that $\varepsilon_{p,s} = O(\lambda_p^{-\infty})$ for any $s \in \mathbb{N}$. □

We now assume that $a > 0$, $0 < \sigma_0 \le 1$, $M, m > 0$ are fixed and we assume that we are given sequences $(\mu_p), (\bar{\mu}_p)$, with $0 < 2\bar{\mu}_p < \mu_p$ taking a finite number of values (to simplify) and such that for any p,

$$\frac{1}{\mu_p - \bar{\mu}_p} < \min\left(\frac{1+\sigma_0}{M}, \frac{1}{m}, \frac{1}{\bar{\mu}_p}\right).$$

Now take g, σ with $0 < \sigma < \sigma_0$ such that

$$\frac{1}{\mu_p - \bar{\mu}_p} < g < \min\left(\frac{1+\sigma_0}{M}, \frac{1}{m}, \frac{1}{\bar{\mu}_p}\right), \qquad 1+\sigma < (\mu_p - \bar{\mu}_p)g.$$

As before, for any $s, \bar{s}$, let $C_{s,\bar{s}} \colon [0,\infty) \to [1,\infty]$ be a family of continuous functions on $[0,2]$ such that $C_{s,\bar{s}}(t) = \infty$ if $t > 2$, increasing with respect to $s, \bar{s} \ge 0$, and let $\bar{C}_s = \max_{t\le 2} C_{s,s}(t)$.

The following extension of Theorem 10 is clear from the previous proof.

COROLLARY 4. *There exists $s_0 > 0$ such that if $\varepsilon_{p,s}$ is a double sequence satisfying, for any $s, \bar{s}, p \in \mathbb{N}$,*

(7.13)

$$\varepsilon_{p+1,s} \le C_{s,\bar{s}}(1+\lambda_p^a \varepsilon_{p,0}) \times (\lambda_p^{a+Ms}\varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^{a+ms}\varepsilon_{p,s}\varepsilon_{p,0} + \lambda_p^{a-(\bar{s}-s)\mu_p}(\varepsilon_{p,\bar{s}} + \lambda_p^{\bar{\mu}_p\bar{s}}\varepsilon_{p,0})),$$

where $\lambda_p = L^{(1+\sigma)^p}$, $L = \bar{C}_{s_0}$ and if

$$\varepsilon_{0,0} \le (\bar{C}_{s_0})^{-\frac{s_0}{g}}, \quad \varepsilon_{0,s_0} \le 1,$$

then, for any $s \in \mathbb{N}$ $\varepsilon_{p,s} = O(\lambda_p^{-\infty})$. Also, s_0 does not depend on the sequence $(\bar{C}_s)_s$ and can be taken of the form $(a+1)\xi(\sigma_0, M, m, \mu, \bar{\mu})$.

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