A KAM SCHEME FOR $\mathrm{SL}(2,\mathbb{R})$ COCYCLES WITH LIOUVILLEAN FREQUENCIES

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ABSTRACT. We develop a new KAM scheme that applies to $\mathrm{SL}(2,\mathbb{R})$ cocycles with one frequency, irrespective of any Diophantine condition on the base dynamics. It gives a generalization of Dinaburg-Sinai's Theorem to arbitrary frequencies: under a closeness to constant assumption, the non-Abelian part of the classical reducibility problem can always be solved for a positive measure set of parameters.

1. INTRODUCTION

In this paper we are concerned with analytic quasiperiodic $\mathrm{SL}(2,\mathbb{R})$ cocycles in one frequency. Those are linear skew-products

$$(\alpha, A): \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2 \qquad (1.1)$$
$$(x, w) \mapsto (x+\alpha, A(x)\cdot w),$$

where $\alpha \in \mathbb{R}$ and $A : \mathbb{T} \to \mathrm{SL}(2,\mathbb{R})$ is analytic ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$). The main source of examples are given by Schrödinger cocycles, where

$$A(x) = S_{v,E}(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}, \qquad (1.2)$$

which are related to one-dimensional quasiperiodic Schrödinger operators

$$(Hu)_n = u_{n+1} + u_{n-1} + v(\theta + n\alpha)u_n. \qquad (1.3)$$

We are interested in the case where A is close to a constant. In this case, the classical question is whether (α, A) is reducible. This means that (α, A) is conjugate to a constant, that is, there exists $B : \mathbb{T} \to \mathrm{PSL}(2,\mathbb{R})$ analytic such that $B(x+\alpha)A(x)B(x)^{-1}$ is a constant.

Reducibility could be thought as breaking up into two different problems. One could first try to conjugate the cocycle into some Abelian subgroup of $\mathrm{SL}(2,\mathbb{R})$, and then conjugate to a constant inside the subgroup. It is easy to see that the second part involves the solution of a cohomological equation. Thus small divisor obstructions related to α must necessarily be present in the problem of reducibility. However, if one takes the point of view that finding a solution of the cohomological equation is an understood problem,[1] we should shift our focus to understanding the first problem.

We focus on the case where the Abelian subgroup is $\mathrm{SO}(2,\mathbb{R})$. Let us say that (α, A) is conjugate to a cocycle of rotations if there exists $B : \mathbb{T} \to \mathrm{SL}(2,\mathbb{R})$ analytic such that $B(x+\alpha)A(x)B(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$. We obtain the somewhat surprising

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[1] We must point out that the solution of the cohomological equation is an interesting problem in some concrete situations, see [AJ1].

result that conjugacy to a cocycle of rotations is frequent (under a closeness to constant assumption) irrespective of any condition on α. A particular case that ilustrates what we mean by frequent is the following:

Theorem 1.1. *Let $v : \mathbb{T} \to \mathbb{R}$ be analytic and close to a constant. For every $\alpha \in \mathbb{R}$, there exists a positive measure set of $E \in \mathbb{R}$ such that $(\alpha, S_{v,E})$ is conjugate to a cocycle of rotations.*[2]

Remark 1.1. The existence of a positive measure set of energies for which $(\alpha, S_{v,E})$ is conjugate to a cocycle of rotations implies the existence of some absolutely continuous spectrum for the corresponding Schrödinger operator (for stronger results, see [LS]). In this context and under the additional condition that v should be *even*, the existence of some absolutely continuous spectrum was first obtained by Yoram Last (unpublished), by different methods.

Another consequence of our new approach to the questions of reducibility is the following generalization of the global almost sure dichotomy between non uniform hyperbolicity and reducibility of Schrödinger cocycles obtained in [AK1] for recurrent Diophantine frequencies in the base (and extended in [AJ2, FK] to all Diophantine frequencies).

Theorem 1.2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $v : \mathbb{T} \to \mathbb{R}$ be analytic, then for almost every energy $E \in \mathbb{R}$ the cocycle $(\alpha, S_{v,E})$ is either non-uniformly hyperbolic or (analytically) conjugate to a cocycle of rotations.*

Remark 1.2. Theorem 1.2 has the following interesting consequence: in the essential support of the absolutely continuous spectrum, the generalized eigenfunctions are almost surely bounded. This property is in fact expected to hold for general ergodic Schrödinger operators (the so-called Schrödinger Conjecture [MMG]), but this is the first time it is verified in a Liouvillean context.

The proof of Theorem 1.2 starts from Kotani's theory that essentially asserts that there is an almost sure dichotomy between non-uniform hyperbolicity and L^2 rotations-reducibility (conjugacy to a cocycle with values in $\mathrm{SO}(2,\mathbb{R})$ of the cocycles $(\alpha, S_{v,E})$. Then it uses the convergence to constants of the renormalizations of cocycles that are L^2-conjugated to rotations, obtained in [K, AK1, AK2]. Finally, it is concluded using a precise version of the local result given in Theorem 1.1, namely Theorem 1.3 below.

The proof of Theorem 1.1 will be based on a new KAM scheme for $\mathrm{SL}(2,\mathbb{R})$ cocycles that is able to bypass the small divisors related to α. An important ingredient is the "cheap trick" recently developed in [FK], though our implementation is rather different (it does not involve renormalization, and it is applied without any Liouvillean assumption on α).

Before giving a more general statement, let us first recall some of the history of KAM methods applied to reducibility problems.

The first result of reducibility was due to Dinaburg-Sinai [DS]. It establishes that $(\alpha, S_{v,E})$ is reducible for a positive measure set E whenever α is Diophantine and v is sufficiently close to a constant (depending on α).

[2]Indeed the Lebesgue measure of the set $X(v, \alpha)$ of energies such that $(\alpha, S_{v,E})$ is conjugate to a cocycle of rotations converges (uniformly on α) to 4 as v converges to a constant. We recall that for v constant, $X(v, \alpha)$ is an open interval of length 4, and for v non-constant $X(v, \alpha)$ (and indeed the larger set of energies where the Lyapunov exponent vanishes) has Lebesgue measure strictly less than 4.

A more precise formulation (due to Herman [H]), gives an explicit condition on the topological dynamics of (α, A) which guarantees the reducibility (under closeness to constant assumptions). It is formulated in terms of the fibered rotation number $\rho = \rho(\alpha, A) \in \mathbb{T}$, a topological invariant which can be defined for all cocycles homotopic to the identity (this includes all Schrödinger cocycles and all cocycles which are close to constant). To define this invariant, first notice that (α, A) naturally acts on the torus $\mathbb{T} \times \mathbb{T}$ if one identifies the second coordinate with the set of arguments of non-zero vectors of $\mathbb{R}^2$. This torus map turns out to have a well defined rotation vector (if $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, otherwise one averages over the first coordinate). The first coordinate of the rotation vector is necessarily α and the second is, by definition the fibered rotation number ρ. As a Corollary of the classical KAM Theorem on $\mathbb{R}^2/\mathbb{Z}^2$, Herman concludes that if (α, ρ) is Diophantine and if A is sufficiently close to a constant (depending on α and ρ), then (α, A) is reducible.

Herman's Theorem suggests that, in the theory of reducibility, there are obstructions (small divisors) related not only to α, but also on the joint properties of the pair (α, ρ). That this is actually the case is not obvious. It follows from a result of Eliasson that for every α there exists a generic set B_α such that for every $\rho \in B_\alpha$ there exists A arbitrarily close to constant with $\rho(\alpha, A) = \rho$, and such that (α, A) is not reducible.[3]

We can now state the precise version of our Main Theorem. Let $\Delta_h = \{x \in \mathbb{C}/\mathbb{Z}, |\Im x| < h\}$. For a bounded holomorphic (possibly matrix valued) function ϕ on Δ_h, we let $\|\phi\|_h = \sup_{x \in \Delta_h} \|\phi(x)\|$. We denote by $C^\omega_h(\mathbb{T}, *)$ the set of all these $*$-valued functions ($*$ will usually denote $\mathbb{R}$, $\mathrm{SL}(2, \mathbb{R})$, or the space of 2-by-2 matrices $\mathbb{M}(2, \mathbb{R})$). In the theorem α is only supposed to be irrational and q_n denotes the sequence of denominators of its best rational approximations (see section 2.3). For $\tau > 0$, $0 < \nu < 1/2$ and $\varepsilon > 0$, let $\mathcal{Q}_\alpha(\tau, \nu, \varepsilon) \subset \mathbb{T}$ be the set of all ρ such that for every i we have

$$\|2 q_i \rho\|_{\mathbb{T}} > \varepsilon \max\{q_{i+1}^{-\nu}, q_i^{-\tau}\}. \tag{1.4}$$

Theorem 1.3. *For every $\tau > 0$, $0 < \nu < 1/2$, $\varepsilon > 0$, $h_* > 0$, there exists $\epsilon = \epsilon(\tau, \nu, \varepsilon) > 0$ with the following property. Let $h > h_*$ and let $A \in C^\omega_h(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ be real-symmetric such that $\|A - R\|_h < \epsilon$ for some rotation matrix R, and $\rho = \rho(\alpha, A) \in \mathcal{Q}_\alpha(\tau, \nu, \varepsilon)$. Then there exists real-symmetric $B : \Delta_{h-h_*} \to \mathrm{SL}(2, \mathbb{C})$ and $\phi : \Delta_{h-h_*} \to \mathbb{C}$ such that*

1. $\|B - \mathrm{id}\|_{h-h_*} < \varepsilon$ *and* $\|\phi - \rho\|_{h-h_*} < \varepsilon$,
2. $B(x + \alpha) A(x) B(x)^{-1} = R_{\phi(x)}$.

Proof of Theorem 1.1. We may assume that v has average 0. Recall (see, e.g., [H]) that the function $\rho : E \mapsto \rho(\alpha, S_{E,v})$ is continuous, non-increasing and onto $[0, 1/2]$. Fix $\tau > 0$, $0 < \nu < 1/2$ and $\varepsilon > 0$ so that $X = \mathcal{Q}_\alpha(\tau, \nu, \varepsilon) \cap [0, 1/2]$ has positive measure for every α, and let $Y = \rho^{-1}(X)$. If v is sufficiently close to constant

[3] More explicit examples can be given from the theory of the almost Mathieu operator. Let $v(x) = 2 \cos 2\pi x$, let $\alpha \in \mathbb{R}$ be arbitrary, and let $\rho \in [0, 1/2]$ be such that for every $k \in \mathbb{Z}$ $\|2\rho - k\alpha\|_{\mathbb{T}} > 0$, while $\liminf_{|k| \to \infty} \frac{1}{|k|} \ln \|2\rho - k\alpha\|_{\mathbb{T}} = -\infty$. It is well known that for every $\lambda \neq 0$, there exists a unique $E = E(\lambda, \alpha, \rho) \in \mathbb{R}$ such that $\rho(\alpha, S_{\lambda v, E}) = \rho$. On the other hand, it can be shown that $(\alpha, S_{\lambda v, E})$ is not reducible: indeed reducibility implies the existence of an eigenfunction for the "dual" operator $(Hu)_n = u_{n+1} + u_{n-1} + \lambda^{-1} v(\rho + n\alpha) u_n$ with energy $\lambda^{-1} E$ (this is an instance of Aubry duality), but H has no point spectrum by [JS].

then Theorem 1.3 implies that $(\alpha, S_{E,v})$ is analytically rotations reducible for every $E \in Y$. Moreover, at any E such that $(\alpha, S_{E,v})$ is analytically rotations-reducible, it is easy to see that the fibered rotation number has a bounded derivative. Since X has positive Lebesgue measure and $\rho(Y) = X$, it follows that Y has positive Lebesgue measure as well.[4]

Proof of Theorem 1.2. We just sketch the argument, which is a standard "global to local" reduction through convergence of renormalization (previously established in [AK2]).

Fix $\alpha \in (0,1) \setminus \mathbb{Q}$, and let $\alpha_n \in (0,1) \setminus \mathbb{Q}$, $n \geq 0$, be the n-th iterate of α by the Gauss map $x \mapsto \{x^{-1}\}$ (see section 2.3). Let $\beta_n = \prod_{k=0}^{n} \alpha_k$.

Lemma 1.4. *Let $\mathcal{P} \subset [0,1)$ be the set of all ρ such that there exist $\tau > 0$, $0 < \nu < 1/2$ and $\varepsilon > 0$ such that $\frac{\rho}{\beta_{n-1}} \in \mathcal{Q}_{\alpha_n}(\tau, \nu, \varepsilon)$ for infinitely many n. Let $A \in C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ be homotopic to a constant and such that (α, A) is L^2-conjugated to an $\mathrm{SO}(2,\mathbb{R})$-valued cocycle and the fibered rotation number of (α, A) belongs to $\mathcal{P}$. Then (α, A) is analytically rotations-reducible.*

Proof. Renormalization of cocycles (see [K], [AK1]) associates to each analytic conjugacy class $[(\alpha, A)]$ of cocycles in $(\mathbb{R} \setminus \mathbb{Q}) \times C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$, an analytic conjugacy class $[(\alpha_n, A_n)]$, such that if (α_n, A_n) is rotations-reducible (or reducible) then (α, A) is rotations-reducible (or reducible) in the same class of regularity.

If $A \in C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ is homotopic to a constant and (α, A) is L^2-conjugated to an $\mathrm{SO}(2,\mathbb{R})$-valued cocycle, then it follows from convergence of renormalization [AK2] that there are *renormalization representatives* (α_n, A_n) such that

- the fibered rotation number of (α_n, A_n) is $\rho_n = \frac{\rho}{\beta_{n-1}}$, where $\rho \in [0,1)$ is the fibered rotation number of (α, A) and $\beta_{n-1} = \prod_{j=0}^{n-1} \alpha_j$,
- there exists $h_0 > 0$ such that each A_n admits a bounded holomorphic extension to Δ_{h_0} and $\|A_n - R_{\rho_n}\|_{h_0} \to 0$.

By definition of $\mathcal{P}$, if the fibered rotation number of (α, A) belongs to $\mathcal{P}$ then we can find $\tau > 0$, $0 < \nu < 1/2$, $\varepsilon > 0$ and arbitrarily large $n \geq 0$ such that $\rho_n \in \mathcal{Q}_{\alpha_n}(\tau, \nu, \varepsilon')$. Let $\epsilon = \epsilon(\tau, \nu, \varepsilon)$ be as in Theorem 1.3, and choose such an n so large that $\|A_n - R_{\rho_n}\|_h < \epsilon$. By Theorem 1.3, (α_n, A_n), and hence (α, A), must be analytically rotations-reducible. □

On the other hand, Kotani Theory (interpreted in the language of cocycles, see [AK1], Section 2), yields:

Lemma 1.5. *Let $\mathcal{P} \subset [0,1)$ be any full measure subset. For every $v \in C^\omega(\mathbb{T}, \mathbb{R})$, for almost every $E \in \mathbb{R}$, $A = S_{v,E}$ satisfies:*

- *either (α, A) has a positive Lyapunov exponent, or*
- *(α, A) is L^2-conjugated to an $\mathrm{SO}(2,\mathbb{R})$-valued cocycle and the fibered rotation number of (α, A) belongs to $\mathcal{P}$.*

The result follows from a combination of Lemmas 1.4 and 1.5, since the set $\mathcal{P}$ specified in Lemma 1.4 has full measure in $[0,1)$ by a simple application of Borel-Cantelli. □

[4]In order to get that Y has measure close to 4, one should use that for $E \in Y$, Theorem 1.3 actually guarantees the existence of a conjugacy to rotations $B \in C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ which is close to a constant matrix conjugating $S_{0,E}$ to a rotation. The estimate can then be concluded in various ways, for instance as a consequence of the formula: $\frac{d\rho}{dE} = -\frac{1}{8\pi} \int_{\mathbb{T}} \|B\|_{\mathrm{HS}}^2$ (whose verification is straightforward).

1.1. **Further remarks.**

1.1.1. *Eliasson's Theory.* Eliasson developed a non-standard KAM scheme which enabled him to prove a much stronger version of Dinaburg-Sinai's Theorem. He showed that for every Diophantine α, there exists a full Lebesgue measure set $\Lambda(\alpha)$ (explicitly given in terms of a Diophantine condition) such that if A is sufficiently close to constant (depending on α) and if $\rho(\alpha, A) \in \Lambda(\alpha)$ then (α, A) is reducible. The natural question raised by our work is whether it is possible to combine the strength of Eliasson's Theorem and that of our main result:

Problem 1. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$[5]. Is there a full Lebesgue measure subset $\Lambda(\alpha)$ (explicitly given in terms of some Diophantine condition) such that for every A sufficiently close to constant such that $\rho(\alpha, A) \in \Lambda(\alpha)$, (α, A) is conjugate to a cocycle of rotations?

Such a result would be necessarily very subtly dependent on the analytic regularity of the cocycle since there are recent counterexamples in Gevrey classes [AK3]. For this reason, it is unlikely that a positive solution to the problem could ever be achieved by a KAM method (since KAM methods tend to generalize to weaker regularity). On the other hand, [AJ2] showed that the closeness quantifier in Eliasson's Theorem was α-independent, as long as α was Diophantine which leads us to believe that it is possible to obtain a result for all α in the analytic category. Unfortunately, the method of [AJ2] mixes up at its core the Abelian and non-Abelian subproblems of reducibility, so it seems unsuitable to prove results of the kind obtained here. A solution to Problem 1 has been obtained by the first author [A]: the proof provides (without restrictions on the fibered rotation number) a sequence of conjugacies which put the cocycle arbitrarily close to constants, so that (under a full measure condition on the fibered rotation number) the results of this paper eventually can be applied.

1.1.2. *Parabolic behavior.* Besides the elliptic subgroup $\mathrm{SO}(2, \mathbb{R})$, there are essentially two other Abelian subgroups of $\mathrm{SL}(2, \mathbb{R})$. Those are the hyperbolic subgroup (diagonal matrices) and the parabolic subgroup (stabilizer of a non-zero vector). It has been understood for quite some time that a relatively simple and open condition (uniform hyperbolicity of the cocycle) ensures that the cocycle is conjugate to a diagonal cocycle (this does not involve any KAM scheme). This condition is satisfied for an open set of cocycles. The parabolic case is, on the other hand, in the frontier of elliptic and hyperbolic behavior and so it has "positive codimension" (for instance, it happens at most for a countable set of energies for Schrödinger cocycles). If α is Diophantine, Eliasson showed that if A is close to constant (depending on α) then (α, A) is conjugate degenerate) if and only if (α, A) is not uniformly hyperbolic and $\|2\rho(\alpha, A) - k\alpha\|_{\mathbb{T}} = 0$ for some $k \in \mathbb{Z}$.

Unfortunately, no result of the kind obtained in this paper can be proved in the parabolic case.[6] In fact even the less ambitious goal of analytically conjugating a cocycle to a triangular form can not be obtained without some arithmetic restriction

[5]The result is obviously false for $\alpha \in \mathbb{Q}$.

[6] For an explicit counterexample, consider $v(x) = 2\cos 2\pi x$, and let α be such that $\liminf_{q\to\infty} \frac{1}{q} \ln \|q\alpha\|_{\mathbb{T}} = -\infty$. Then for every $\lambda \neq 0$ and every $E \in \mathbb{R}$, the cocycle $(\alpha, S_{\lambda v, E})$ is not conjugate to a parabolic subgroup of $\mathrm{SL}(2, \mathbb{R})$. Indeed, as in footnote 3, such a conjugacy implies the existence of an eigenvalue for a dual operator (by Aubry duality), which is known to have no point spectrum (by Gordon's Lemma, [AS]).

on α: consider, for instance, a cocycle of the form (α, A) with $A = R_{\lambda\phi}$, where $\phi \in C^\omega(\mathbb{T}, \mathbb{R})$ has average 0 but is not a coboundary, i.e., the equation $\phi(x) = \psi(x+\alpha) - \psi(x)$ admits no solution $\psi \in C^\omega(\mathbb{T}, \mathbb{R})$ (such a ϕ exists as long as α can be exponentially well approximated by rational numbers). Then $\rho(\alpha, A) = 0$, but (α, A) can not be analytically conjugated to a cocycle in triangular form (from such a conjugacy it is easy to construct a solution of the cohomological equation).

1.1.3. *Optimality.* We do not claim optimality of the condition we impose on ρ. Our proof uses a single procedure, rational approximation, independent of the Diophantine properties of α. This has the advantage of providing a unified argument, but is clearly unoptimal in the Diophantine case (it is actually surprising that the procedure works at all in the case of bounded type). More precise estimates should be obtainable by interpolation arguments with classical KAM schemes.

2. Notations and preliminaries

2.1. **The fibered Lyapunov exponent.** Given a cocycle (α, A), for $n \in \mathbb{Z}$, we denote the iterates of (α, A) by $(\alpha, A)^n = (n\alpha, A^{(n)}(\cdot))$ where for $n \geq 1$

$$\begin{cases} A^{(n)}(\cdot) = A(\cdot + (n-1)\alpha) \cdots A(\cdot) \\ A^{(-n)}(\cdot) = A(\cdot - n\alpha)^{-1} \cdots A(\cdot - \alpha)^{-1} \end{cases}$$

We call fibered products of (α, A) the matrices $A^{(n)}(\cdot)$ for $n \in \mathbb{N}$.

The fibered Lyapunov exponent is defined as the limit

$$L(\alpha, A) := \lim_{|n|\to\infty} \frac{1}{n} \int_{\theta\in\mathbb{T}} \log \|A^{(n)}(\theta)\| d\theta,$$

which by the subbadditive theorem always exists (similarly the limit when n goes to $-\infty$ exists and is equal to $L(\alpha, A)$).

2.2. **The fibered rotation number.** Assume that $A(\cdot) : \mathbb{T} \to SL(2, \mathbb{R})$ is continuous and homotopic to the identity; then the same is true for the map

$$F : \mathbb{T} \times \mathbb{S}^1 \to \mathbb{T} \times \mathbb{S}^1$$
$$(\theta, v) \mapsto (\theta + \alpha, \frac{A(\theta)v}{\|A(\theta)v\|}),$$

therefore F admits a continuous lift $F : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ of the form $\tilde{F}(\theta, x) = (\theta + \alpha, x + f(\theta, x))$ such that $f(\theta, x+1) = f(\theta, x)$ and $\pi(x + f(\theta, x)) = A(\theta)\pi(x)/\|A(\theta)\pi(x)\|$, where $\pi : \mathbb{R} \to \mathbb{S}^1$, $\pi(x) = e^{i2\pi x} := (\cos(2\pi x), \sin(2\pi x))$. In order to simplify the terminology we shall say that $\tilde{F}$ is a lift for (α, A). The map f is independent of the choice of the lift up to the addition of a constant integer $p \in \mathbb{Z}$. Following [H] and [JM] we define the limit

$$\lim_{n\to\pm\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tilde{F}^k(\theta, x)),$$

that is independent of (θ, x) and where the convergence is uniform in (θ, x). The class of this number in $\mathbb{T}$, which is independent of the chosen lift, is called the *fibered rotation number* of (α, A) and denoted by $\rho(\alpha, A)$. Moreover $\rho(\alpha, A)$ is continuous as a function of A (with respect to the uniform topology on $C^0(\mathbb{T}, SL(2, \mathbb{R}))$, naturally restricted to the subset of A homotopic to the identity).

2.3. **Continued fraction expansion.** Define as usual for $0 < \alpha < 1$,

$$a_0 = 0, \qquad \alpha_0 = \alpha,$$

and inductively for $k \geq 1$,

$$a_k = [\alpha_{k-1}^{-1}], \qquad \alpha_k = \alpha_{k-1}^{-1} - a_k = G(\alpha_{k-1}) = \{\frac{1}{\alpha_{k-1}}\},$$

We define

$$\begin{array}{ll} p_0 = 0 & q_1 = a_1 \\ q_0 = 1 & p_1 = 1, \end{array}$$

and inductively,

$$\left\{ \begin{array}{l} p_k = a_k p_{k-1} + p_{k-2} \\ q_k = a_k q_{k-1} + q_{k-2}. \end{array} \right. \tag{2.1}$$

Recall that the sequence (q_n) is the sequence of best denominators of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ since it satisifies

$$\forall 1 \leq k < q_n, \quad \|k\alpha\| \geq \|q_{n-1}\alpha\|$$

and

$$\|q_n\alpha\| \leq \frac{1}{q_{n+1}} \tag{2.2}$$

where we used the notation

$$\|x\| = \|x\|_{\mathbb{T}} = \inf_{p \in \mathbb{Z}} |x - p|.$$

3. Growth of cocycles of rotations

Let be given a number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We will fix in the sequel a particular subsequence (q_{n_k}) of the denominators of α, that we will denote for simplicity by (Q_k) and denote $(\overline{Q}_k)$ the sequence (q_{n_k+1}). The goal of this section is to introduce a subsequence (Q_k) for which we will have a nice control on the Birkhoff sums of real analytic functions above the irrational rotations of frequency α. The properties required from our choice of the sequence (Q_k) are summarized in the following statement, and are all what will be needed from this section in the sequel. The rest of Section 3 is devoted to the construction of the sequence (Q_k) and can be skipped in a first reading.

If $f \in C^0(\mathbb{T}, \mathbb{R})$ we define its k-th Fourier coefficient by $\hat{f}(k) = \int_{\mathbb{T}} f(x) e^{-2\pi i k x} dx$ and the n-th Birkhoff sum of f over $x \mapsto x + \alpha$ by $S_n f = \sum_{k=0}^{n-1} f(\cdot + k\alpha)$. We also introduce the notation $\eta_n = \eta / n^2$, for any $\eta > 0$.

Proposition 3.1. *Given any $\eta \in (0,1)$, $h_* > 0$ and $M > 1$, there exists $C(h_*, \eta, M) > 0$ such that for any irrational α, there exists a subsequence (Q_k) of denominators of α such that $Q_0 = 1$, and for any $h \geq h_*$ and any function $\varphi \in C_h^\omega(\mathbb{T}, \mathbb{R})$, it holds for all $k > 0$ and $h_k := h(1 - \eta_k)$*

- $Q_{k+1} \leq \overline{Q}_k^{16M^4}$
- $\|S_{Q_k}\varphi - Q_k\widehat{\varphi}(0)\|_{h_k} \leq C\|\varphi - \widehat{\varphi}(0)\|_h (Q_k^{-M} + \overline{Q}_k^{-1+\frac{1}{M}})$.
- *for any $l \leq Q_{k+1}$,* $\|S_l\varphi - l\widehat{\varphi}(0)\|_{h_k} \leq C\|\varphi - \widehat{\varphi}(0)\|_h \left(\overline{Q}_k Q_k^{-M} + \overline{Q}_k^{\frac{1}{M}}\right)$.

Definition 3.1. Let $0 < \tilde{\mathcal{A}} \leq \tilde{\mathcal{B}} \leq \tilde{\mathcal{C}}$. We say that the pair of denominators (q_l, q_n) forms a $\mathrm{CD}(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ bridge if

- $q_{i+1} \leq q_i^{\tilde{\mathcal{A}}}, \quad \forall i = l, \ldots, n-1$
- $q_l^{\tilde{\mathcal{C}}} \geq q_n \geq q_l^{\tilde{\mathcal{B}}}$

Lemma 3.2. *For any $\tilde{\mathcal{A}}$ there exists a subsequence Q_k such that $Q_0 = 1$ and for each $k \geq 0$, $Q_{k+1} \leq \overline{Q}_k^{\tilde{\mathcal{A}}^4}$, and either $\overline{Q}_k \geq Q_k^{\tilde{\mathcal{A}}}$, or the pairs $(\overline{Q}_{k-1}, Q_k)$ and (Q_k, Q_{k+1}) are both $\mathrm{CD}(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}, \tilde{\mathcal{A}}^3)$ bridges.*

Proof. Assume the sequence Q_l is constructed up to k. Let $q_n > Q_k$ (if it exists) be the smallest denominator such that $q_{n+1} > q_n^{\tilde{\mathcal{A}}}$. If $q_n \leq \overline{Q}_k^{\tilde{\mathcal{A}}^4}$ then we let $Q_{k+1} = q_n$. If $q_n \geq \overline{Q}_k^{\tilde{\mathcal{A}}^4}$ (or if q_n does not exist) it is possible to find $q_{n_0} := \overline{Q}_k, q_{n_1}, q_{n_2}, \ldots, q_{n_j}$ (or an infinite sequence $q_{n_1}, q_{n_2}, \ldots$) such that : $j \geq 2$, $q_{n_j} = q_n$, and for each $0 \leq i \leq j-1$, the pairs $(q_{n_i}, q_{n_{i+1}})$ and $(q_{n_i+1}, q_{n_{i+1}})$ are $\mathrm{CD}(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}, \tilde{\mathcal{A}}^3)$ bridges. We then let $Q_{k+1} := q_{n_1}$, $Q_{k+2} := q_{n_2}, \ldots, Q_{k+j} := q_{n_j}$ and it is straightforward to check that $Q_{k+1}, Q_{k+2}, \ldots, Q_{k+j-1}$ (or $Q_{k+1}, Q_{k+2}, \ldots$) satisfy the second condition of the lemma while $Q_{k+j} = q_n$, in case q_n exists, satisfies the first one. □

Proof of proposition 3.1.

Lemma 3.3. *Let $h_*, \eta, U > 0$. There exists $C(h_*, \eta, U) > 0$ such that for any $\varphi \in C_h^\omega(\mathbb{T}, \mathbb{R})$ with $h \geq h_*$ and for any irrational α and any pair $q_{s_1} \leq q_{s_2}$ of denominators of α, we have if $\delta \geq \max(1/\sqrt{q_{s_1}}, \eta/(10 s_2^2))$*

$$\|S_{q_{s_2}}\varphi - q_{s_2}\widehat{\varphi}(0)\|_{h(1-\delta)} \leq C\|\varphi - \widehat{\varphi}(0)\|_h \left(s_2^4 \frac{q_{s_1}}{q_{s_2+1}} + \frac{q_{s_2}}{q_{s_1}^U} \right)$$

Proof. Write $\varphi(z) = \sum_{l \in \mathbb{Z}} \widehat{\varphi}(l) e^{i2\pi l z}$. For any $l \in \mathbb{Z}^*$, we have that $|\widehat{\varphi}(l)| e^{2\pi |l| h} \leq \|\varphi - \widehat{\varphi}(0)\|_h$. Now

$$S_{q_{s_2}}\varphi(z) - q_{s_2}\widehat{\varphi}(0) = \sum_{l \in \mathbb{Z}^*} \frac{1 - e^{i2\pi q_{s_2} l\alpha}}{1 - e^{i2\pi l\alpha}} \widehat{\varphi}(l) e^{i2\pi l z}$$

hence

$$\begin{aligned}
\|S_{q_{s_2}}\varphi - q_{s_2}\widehat{\varphi}(0)\|_{h(1-\delta)} &\leq \|\varphi - \widehat{\varphi}(0)\|_h \sum_{l \in \mathbb{Z}^*} \left| \frac{1 - e^{i2\pi q_{s_2} l\alpha}}{1 - e^{i2\pi l\alpha}} \right| e^{-2\pi h |l| \delta} \\
&\leq 2\pi \|\varphi - \widehat{\varphi}(0)\|_h \frac{q_{s_1}}{q_{s_2+1}} \sum_{0 < |l| < q_{s_1}} |l| e^{-2\pi h |l| \delta} \\
&+ \|\varphi - \widehat{\varphi}(0)\|_h \sum_{|l| \geq q_{s_1}} q_{s_2} e^{-2\pi h |l| \delta}
\end{aligned}$$

where we have used the facts that $\|l\alpha\|_{\mathbb{T}} \geq \frac{1}{2q_{s_1}}$ for $0 < |l| < q_{s_1}$ and $\|q_{s_2}\alpha\|_{\mathbb{T}} \leq 1/q_{s_2+1}$. The result now follows from the condition $\delta \geq \max(1/\sqrt{q_{s_1}}, \eta/(10 s_2^2))$ □

In all the sequel we will let $\tilde{\mathcal{A}} := 2M$ and $U = 16M^4$ in Lemmas 3.2 and 3.3.

A consequence of Lemma 3.3 is the following.

Corollary 3.4. *Given $h_*, M > 0$, there exists $T_0(h_*, M)$ such that, for any $\varphi \in C_h^\omega(\mathbb{T}, \mathbb{R})$ with $h \geq h_*$, and for any irrational α and any denominator q_n of α such that $q_n \geq T_0$, we have*

(1) *If* $q_{n+1} \geq q_n^{\tilde{\mathcal{A}}}$ *then*

$$\|S_{q_n}\varphi\|_{h(1-\eta_n)} \leq \|\varphi - \widehat{\varphi}(0)\|_h \frac{1}{q_n^M}.$$

(2) *If there exists* l *such that* $q_n^{1/\tilde{\mathcal{A}}^3} \leq q_l \leq q_n^{1/\tilde{\mathcal{A}}}$, *then*

$$\|S_{q_n}\varphi\|_{h(1-\eta_n)} \leq \|\varphi - \widehat{\varphi}(0)\|_h \left(q_{n+1}^{-1+1/M} + q_n^{-M}\right).$$

As a consequence we get for $Q_k \geq T_0$

$$\|S_{Q_k}\varphi - Q_k\widehat{\varphi}(0)\|_{h(1-\eta_k)} \leq \|\varphi - \widehat{\varphi}(0)\|_h (Q_k^{-M} + \overline{Q}_k^{-1+\frac{1}{M}})$$

Proof. To prove 1 apply Lemma 3.3 to $s_1 = s_2 = n$: indeed if q_n is sufficiently large we have $\eta_n \geq 1/\sqrt{q_n}$, whence

$$\|S_{q_n}\varphi - q_n\widehat{\varphi}(0)\|_{h(1-\eta_n)} \leq C\|\varphi - \widehat{\varphi}(0)\|_h \left(n^4 \frac{q_n}{q_{n+1}} + \frac{1}{q_n^{U-1}}\right)$$

and 1 follows if q_n is sufficiently large form the fact that $q_{n+1} \geq q_n^{\tilde{\mathcal{A}}}$ and from the choices $\tilde{\mathcal{A}} = 2M$ and $U = 16M^4$.

To prove 2 we let $s_1 = l$, $s_2 = n$, whence for q_n sufficiently large

$$\|S_{q_n}\varphi - q_n\widehat{\varphi}(0)\|_{h(1-\eta_n)} \leq C\|\varphi - \widehat{\varphi}(0)\|_h \left(n^4 \frac{q_l}{q_{n+1}} + \frac{q_n}{q_l^U}\right).$$

Since $q_n \leq q_l^{\tilde{\mathcal{A}}^3}$ implies that $n = \mathcal{O}(\ln q_l)$ (the denominators grow at least geometrically) we get 2 from $q_n^{1/\tilde{\mathcal{A}}^3} \leq q_l \leq q_n^{1/\tilde{\mathcal{A}}}$ and the choices of $\tilde{\mathcal{A}}$ and U.

The conclusion of the corollary follows from 1 if $\overline{Q}_k \geq Q_k^{\tilde{\mathcal{A}}}$ and from 2 if not since in this case $Q_k^{1/\tilde{\mathcal{A}}^3} \leq \overline{Q}_{k-1} \leq Q_k^{1/\tilde{\mathcal{A}}}$ as $(\overline{Q}_{k-1}, Q_k)$ is a CD$(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}, \tilde{\mathcal{A}}^3)$ bridge (we also use the fact that $n_k \geq k$ in $Q_k = q_{n_k}$). □

Another consequence of lemma 3.3 is the following.

Corollary 3.5. *Given* $h_*, \eta, M > 0$, *there exists* $T_0(h_*, \eta, M)$ *such that, for any* $\varphi \in C_h^\omega(\mathbb{T}, \mathbb{R})$ *with* $h \geq h_*$, *and for any irrational* α *and for* $Q_k \geq T_0$, *we have for* $m \leq Q_{k+1}$

$$\|S_m\varphi - l\widehat{\varphi}(0)\|_{h(1-\eta_k)} \leq \|\varphi - \widehat{\varphi}(0)\|_h \left(\overline{Q}_k Q_k^{-M} + \overline{Q}_k^{\frac{1}{M}}\right). \tag{3.1}$$

Proof. We distinguish two cases.

Case 1: $\overline{Q}_k \geq Q_k^{\tilde{\mathcal{A}}}$. We let u, v be such that $q_u = Q_k$ and $q_{v+1} = Q_{k+1}$ and write $m < Q_{k+1}$ as $m = \sum_{s=u}^{v} a_s q_s + b, a_s < q_{s+1}/q_s$, $b \leq q_u$ and apply lemma 3.3 to each s with $s_1 = s_2 = s$ and obtain

$$\begin{aligned}\|S_m\varphi - m\widehat{\varphi}(0)\|_{h(1-\eta_u)} &\leq C\|\varphi - \widehat{\varphi}(0)\|_h \left(\sum_{s=u}^{v}\left(s^4 + \frac{q_{s+1}}{q_s^U}\right) + q_u\right) \\ &\leq C\|\varphi - \widehat{\varphi}(0)\|_h (\frac{\overline{Q}_k}{Q_k^U} + q_u + v^5)\end{aligned}$$

where we used that for every $s \in [u+1, v]$ we have that $q_{s+1} \leq q_s^{\tilde{\mathcal{A}}}$ (by construction of the sequence Q_k). Since $\overline{Q}_k \geq Q_k^{\tilde{\mathcal{A}}}$ and $v = \mathcal{O}(\ln \overline{Q}_k)$ (because $Q_{k+1} \leq \overline{Q}_k^{\tilde{\mathcal{A}}^4}$) we obtain (3.1).

Case 2: $\overline{Q}_k \leq Q_k^{\tilde{\mathcal{A}}}$. By definition of the sequence Q_k we then have that $(\overline{Q}_{k-1}, Q_k)$ and (Q_k, Q_{k+1}) are $\mathrm{CD}(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}, \tilde{\mathcal{A}}^3)$ bridges. We let $u = n_{k-1}+1$, $v = n_{k+1}-1$ and write $m < Q_{k+1}$ as $m = \sum_{s=u}^{v} a_s q_s + b, a_s < q_{s+1}/q_s$, $b \leq q_u$ and apply lemma 3.3 to each s with $s_1 = s_2 = s$ and obtain as before

$$\begin{aligned}\|S_m\varphi - m\widehat{\varphi}(0)\|_{h(1-\eta_u)} &\leq C\|\varphi - \widehat{\varphi}(0)\|_h \left(\sum_{s=u}^{v}\left(s^4 + \frac{q_{s+1}}{q_s^U}\right) + q_u\right)\\ &\leq C\|\varphi - \widehat{\varphi}(0)\|_h (q_u + v^5)\end{aligned}$$

because we have that for each $s \in [u, v]$ that $q_{s+1} \leq q_s^{\tilde{\mathcal{A}}}$. From there we obtain (3.1) since $v = \mathcal{O}(\ln q_u)$ (from $q_v \leq q_u^{\tilde{\mathcal{A}}^4}$) and $q_u \leq Q_k^{1/\tilde{\mathcal{A}}}$. □

Proposition 3.1 is enclosed in the definition of the sequence Q_k given in lemma 3.2 and in the conclusions of corollary 3.4 and corollary 3.5. □

4. The inductive step.

We denote by $\Omega(h)$ the set of cocycles (α, A) with $A \in C_h^\omega$, and $\Omega(h, \varepsilon, \tau, \nu) \subset \Omega(h)$ the set of cocycles with α irrational and the fibered rotation number ρ satisfying

$$\|q_n\rho\| \geq \varepsilon \max(q_n^{-\tau}, q_{n+1}^{-\nu}). \tag{4.1}$$

4.1. In this section we show that a close to constant cocycle, but not too close to $\pm\,\mathrm{id}$, can be reduced to become essentially $e^{-\rho^2/\alpha}$-close to rotations, where ρ is its rotation frequency in the base and ρ its fibered rotation number.

For $\phi \in \mathbb{C}$ we denote by R_ϕ the matrix $\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}$.

Proposition 4.1. *For every $D \geq 0, h_* > 0$, there exist $\epsilon_0 = \epsilon_0(h_*, D) > 0$ and $C_0 = C_0(h_*, D) > 0$ with the following properties. Let $h > h_*$, $0 < \delta < 1$, $\bar{\alpha} \in \mathbb{R}$, $\bar{A} \in C_h^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$, $\bar{\phi} \in C_h^\omega(\mathbb{T}, \mathbb{R})$ satisfy $\|\bar{\phi} - \hat{\bar{\phi}}(0)\|_h \leq D$,*

$$\rho^{-1} = \|(R_{2\bar{\phi}} - \mathrm{id})^{-1}\|_h < \epsilon_0^{-1/4}, \tag{4.2}$$

$$\|R_{-\bar{\phi}}\bar{A} - \mathrm{id}\,\|_h < \epsilon_0. \tag{4.3}$$

Then:

(1) *There exist $B \in C^\omega_{e^{-\delta/3}h}(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ with*

$$\|B - \mathrm{id}\,\|_{e^{-\delta/3}h} \leq \frac{C_0}{\rho^2}\|R_{-\bar{\phi}}\bar{A} - \mathrm{id}\,\|_h \tag{4.4}$$

and $\tilde{\bar{\phi}} \in C^\omega_{e^{-\delta/3}h}(\mathbb{T}, \mathbb{R})$ such that, letting $\tilde{\bar{A}}(x) = B(x+\bar{\alpha})A(x)B(x)^{-1}$, we have

$$\|R_{-\tilde{\bar{\phi}}}\tilde{\bar{A}} - \mathrm{id}\,\|_{e^{-\delta/3}h} \leq C_0 e^{-\frac{h\delta\rho^2}{C_0|\bar{\alpha}|}}\|R_{-\bar{\phi}}\bar{A} - \mathrm{id}\,\|_h. \tag{4.5}$$

(2) *If $\alpha \in \mathbb{R}$ and $A \in C_h^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ satisfy $\|A\|_h \leq D$ and $A(x+\bar{\alpha})\bar{A}(x) = \bar{A}(x+\alpha)A(x)$ (i.e., (α, A) commutes with $(\bar{\alpha}, \bar{A})$) then there exists $\tilde{\phi} \in C^\omega_{e^{-\delta}h}(\mathbb{T}, \mathbb{R})$ such that, letting $\tilde{A}(x) = B(x+\alpha)A(x)B(x)^{-1}$ we have*

$$\|R_{-\tilde{\phi}}\tilde{A} - \mathrm{id}\,\|_{e^{-\delta}h} \leq C_0 e^{-\frac{h\delta\rho^2}{C_0|\bar{\alpha}|}}. \tag{4.6}$$

Proof. We will need a preliminary lemma.

Lemma 4.2. *For every $D > 0$, there exists $C > 0$, $\epsilon > 0$ with the following properties. Let $W \subset \mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}$ be the set of all (A,θ) such that $\|A\| < 2D$ and $\|R_\theta^{-1}A - \mathrm{id}\,\| < \epsilon \max\{1, \|R_{2\theta} - \mathrm{id}\,\|^2\}$. There exists a real symmetric holomorphic function $F : W \to \mathrm{SL}(2,\mathbb{C})$ such that $B = F(A,\theta)$ satisfies $BAB^{-1} = R_{\theta'}$ and $\|B - \mathrm{id}\,\| < C\frac{\|R_\theta^{-1}A - \mathrm{id}\,\|}{\|R_{2\theta} - \mathrm{id}\,\|^2}$.*

Proof. Let $A^{(0)} = A$, $\theta^{(0)} = \theta$. Assuming $A^{(n)}$ and $\theta^{(n)}$ defined and satisfying the inductive estimates $|\theta^{(n)} - \theta| \leq (1 - 2^{-n})\epsilon_0^{1/2}\|2\theta\|^2_{\mathbb{C}/\mathbb{Z}}$ and $\|A^{(n)} - R_{\theta^{(n)}}\| < \epsilon_0^{n/2}\|A - R_\theta\|$, define $v^{(n)} \in \mathrm{sl}(2,\mathbb{C})$ small such that $A^{(n)} = e^{v^{(n)}}R_{\theta^{(n)}}$. Let $v^{(n)} = \begin{pmatrix} x^{(n)} & y^{(n)} - 2\pi z^{(n)} \\ y^{(n)} + 2\pi z^{(n)} & -x^{(n)} \end{pmatrix}$. Let $\begin{pmatrix} \tilde{x}^{(n)} \\ \tilde{y}^{(n)} \end{pmatrix} = (R_{2\theta^{(n)}} - \mathrm{id})^{-1}\begin{pmatrix} x^{(n)} \\ y^{(n)} \end{pmatrix}$ and let $w^{(n)} = \begin{pmatrix} \tilde{x}^{(n)} & \tilde{y}^{(n)} \\ \tilde{y}^{(n)} & -\tilde{x}^{(n)} \end{pmatrix}$. Let $A^{(n+1)} = e^{w^{(n)}}A^{(n)}e^{-w^{(n)}}$ and $\theta^{(n+1)} = \theta^{(n)} + z^{(n)}$. Then $A^{(n+1)}$ and $\theta^{(n+1)}$ satisfy the inductive estimates as well, so all the objects can be defined for every n. Then one can take $B = \lim_{n\to\infty} e^{w^{(n-1)}} \cdots e^{w^{(0)}}$. □

In what follows we fix $C_2 > 10$ and $C(D)$ as in Lemma 4.2 and set $N = [\frac{\delta h \rho^2}{C_1|\bar{\alpha}|}]$, where $C_1 \gg CC_2$ is a sufficiently large constant. Let $h_j = e^{-\delta\frac{j}{3N}}h$, $j \geq 0$.

Claim 4.3. *If ϵ_0 is sufficiently small, there exist sequences $B_i, A_i \in C^\omega_{h_i}(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$, $\xi_i \in C^\omega_{h_i}(\mathbb{T}, \mathbb{M}(2,\mathbb{R}))$ and $\phi_i \in C^\omega_{h_i}(\mathbb{T},\mathbb{R})$, $0 \leq i \leq N$, such that*

1. $A_0 = \bar{A}$, $\phi_0 = \bar{\phi}$,
2. $R_{\phi_i} = B_{i-1}A_{i-1}B_{i-1}^{-1}$ *for* $1 \leq i \leq N$
3. $A_i(x) = B_{i-1}(x + \bar{\alpha})B_{i-1}(x)^{-1}R_{\phi_i(x)}$ *for* $1 \leq i \leq N$,
4. $\xi_i = R_{\phi_i}^{-1}A_i - \mathrm{id}$ *for* $0 \leq i \leq N$,
5. $\|B_i - \mathrm{id}\,\|_{h_i} \leq \frac{C}{\rho^2}\|\xi_i\|_{h_i}$ *for* $0 \leq i \leq N - 1$,
6. $\|R_{\phi_i}\|_{h_i} \leq (1 + \frac{i}{N})D$ *for* $0 \leq i \leq N$,
7. $\|(R_{2\phi_i} - \mathrm{id})^{-1}\|_{h_i} \leq (1 + \frac{i}{N})\rho^{-1}$ *for* $0 \leq i \leq N$.
8. $\|\xi_i\|_{h_i} \leq \frac{1}{C_2}\|\xi_{i-1}\|_{h_{i-1}}$ *for* $1 \leq i \leq N$.

Proof. We proceed by induction on i. Assume that $B_0, ..., B_{i-1}$ have been already defined and hence $A_0, ..., A_i$, $\phi_0, ..., \phi_i$ and $\xi_0, ..., \xi_i$ are automatically defined by properties (1-4). Assume that estimates (5-8) have been also established for the objects so far defined.

Let F be as in the previous lemma. We want to define $B_i(z) = F(A_i(z), \phi_i(z))$. For this, we must check that $(A_i(z), \phi_i(z))$ is in the domain of F, which is in fact a consequence of estimates (6-8). Thus B_i is well defined, and satisfies estimate (5) as well by the previous lemma. If $i < N$, we now define A_{i+1}, ϕ_{i+1} and ξ_{i+1} by (2-4). Estimates (6) and (7) immediately follow from (2) and (5) (from (8) we have that $\|\xi_i\|_{h_i} \leq \epsilon_0$ and (4.2) then yields $\|\xi_i\|_{h_i}/\rho^2 \leq \epsilon_0^{1/2}$).

We obviously have $\|\xi_{i+1}(x)\| \leq \|B_i(x)\|^3\|A_i(x)\|^2\|B_i(x + \bar{\alpha}) - B_i(x)\|$. Thus, estimating the derivative of $B_i - \mathrm{id}$ with the Cauchy formula, we get $\|\xi_{i+1}\|_{h_{i+1}} \leq \frac{CN\bar{\alpha}}{\delta h}\|B_i\|^3_{h_{i+1}}\|A_i\|^2_{h_{i+1}}\|B_i - \mathrm{id}\,\|_{h_i}$, which immediately implies (8) since $N = [\frac{\delta h \rho^2}{C_1|\bar{\alpha}|}]$ with $C_1 \gg CC_2$. □

We now set $B = B_{N-1} \cdots B_0$, $\tilde{\bar{\phi}} = \phi_N$, $\tilde{\bar{A}} = A_N$ so that (1-4) give $\tilde{\bar{A}}(x) = B(x + \bar{\alpha})A(x)B(x)^{-1} = R_{\tilde{\bar{\phi}}}(\mathrm{id} + \xi_N)$. By (5) and (8) we have (4.4), while (8) gives (4.5). This proves the first statement.

For convenience of notation, let us write $\phi = \phi_N$ and $\xi = \xi_N$ (so that $\tilde{\tilde{A}} = R_\phi(\mathrm{id} + \xi)$).

Claim 4.4. *If $C_3(D)$ is sufficiently large and if $\epsilon_0 > 0$ is sufficiently small and $|\bar{\alpha}| < C_3^{-1} \delta h \rho^2$ then*

$$\|(R_{\phi(x+\alpha)+\phi(x)} - \mathrm{id})^{-1}\|_{e^{-2\delta/3}h} < 3\rho^{-1}. \tag{4.7}$$

Proof. By the Cauchy formula, $\|\tilde{A}(x+\bar{\alpha}) - \tilde{A}(x)\|_{e^{-2\delta/3}h} \leq \frac{C}{\delta h}|\bar{\alpha}| \ll \rho^2$. The commutation relation gives $\|\tilde{A}(x) R_{\phi(x)} \tilde{A}(x)^{-1} - R_{\phi(x+\alpha)}\|_{e^{-2\delta/3}h} \leq \frac{C}{\delta h}|\bar{\alpha}| + C\epsilon_0$, hence $\|\cos 2\pi\phi(x+\alpha) - \cos 2\pi\phi(x)\|_{e^{-2\delta/3}h} \ll \rho^2$. Thus for each x with $|\Im x| < e^{-2\delta/3}h$, one of $\|R_{\phi(x+\alpha)\pm\phi(x)} - \mathrm{id}\| \ll \rho$. Since $\|(R_{2\phi(x)} - \mathrm{id})^{-1}\| \leq \rho^{-1}$, one of $\|R_{\phi(x+\alpha)\pm\phi(x)} - \mathrm{id}\|_{e^{-2\delta/3}h} \ll \rho$. Clearly $\|R_{\phi(x+\alpha)+\phi(x)} - \mathrm{id}\|_{e^{-2\delta/3}h}$ is of order at least ρ, since

$$\inf_{x\in\mathbb{T}} \|R_{\phi(x+\alpha)+\phi(x)} - \mathrm{id}\| \geq \inf_{x\in\mathbb{T}} \|R_{2\phi(x)} - \mathrm{id}\|, \tag{4.8}$$

and we must have $\|R_{\phi(x+\alpha)-\phi(x)} - \mathrm{id}\|_{e^{-2\delta/3}h} \ll \rho$. Since $\|(R_{2\phi} - \mathrm{id})^{-1}\|_{e^{-\delta/3}h}$ is at most $2\rho^{-1}$, the claim follows. □

Notice that the second statement holds trivially unless $|\bar{\alpha}| \ll \delta h \rho^2$, so we will assume from now on that the estimate of the claim holds.

Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For $M \in \mathbb{M}(2,\mathbb{C})$, let $\mathcal{Q}(M) = \frac{M + JMJ}{2}$. Then $M - \mathcal{Q}(M)$ is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, so that it commutes with all R_θ, $\theta \in \mathbb{C}$. Notice that for $\theta \in \mathbb{C}$ we have

$$R_{-\theta}\mathcal{Q}(M) = \mathcal{Q}(MR_\theta), \quad R_\theta \mathcal{Q}(M) = \mathcal{Q}(R_\theta M), \tag{4.9}$$

so for $\theta_1, \theta_2 \in \mathbb{C}$ we have

$$(R_{-\theta_1} - R_{\theta_2})\mathcal{Q}(M) = \mathcal{Q}(MR_{\theta_1} - R_{\theta_2} M), \tag{4.10}$$

Let $L(x) = \mathcal{Q}(\tilde{A}(x))$, $L_1(x) = L(x+\bar{\alpha}) - L(x)$ and $L_2(x) = \mathcal{Q}(\tilde{A}(x)R_{\phi(x)} - R_{\phi(x+\alpha)}\tilde{A}(x))$. Then for $2N \leq n \leq 3N-1$ the Cauchy formula gives

$$\|L_1\|_{h_{n+1}} \leq \frac{C|\bar{\alpha}|}{h_n - h_{n+1}} \|L\|_{h_n}, \tag{4.11}$$

for some absolute C, a direct computation bounds $\|L_2(x)\|$ by

$$\|L_1(x)\|\|R_{\phi(x)}\| + \|\tilde{A}(x+\bar{\alpha})\|\|\tilde{\tilde{A}}(x) - R_{\phi(x)}\| + \|\tilde{\tilde{A}}(x+\alpha) - R_{\phi(x+\alpha)}\|\|\tilde{A}(x)\| \tag{4.12}$$

(here we have used the commutation relation), giving

$$\|L_2\|_{h_{n+1}} \leq C(\|L_1\|_{h_{n+1}} + \|\xi\|_{h_{n+1}}), \tag{4.13}$$

where C only depends on D, and we have used that ϵ_0 is small, and (4.10) gives

$$\|L\|_{h_{n+1}} \leq \|(R_{-\phi(x)} - R_{\phi(x+\alpha)})^{-1}\|_{h_{n+1}} \|L_2\|_{h_{n+1}}. \tag{4.14}$$

Thus

$$\|L\|_{h_{n+1}} \leq C\frac{\rho^{-1}|\bar{\alpha}|}{h_n - h_{n+1}} \|L\|_{h_n} + C\rho^{-1}\|\xi\|_{h_N}, \tag{4.15}$$

so that

$$\|L\|_{h_{3N}} \leq Ce^{-\frac{h\delta}{C\rho|\bar{\alpha}|}} + C\rho^{-1}\|\xi\|_{h_N} \leq Ce^{-\frac{h\delta\rho^2}{C|\bar{\alpha}|}} \tag{4.16}$$

where the second inequality uses that $\|\xi\|_{h_N} \leq C\rho^4 e^{-\frac{h\delta\rho^2}{C|\bar\alpha|}}$.

Since we are assuming that $\bar\alpha \ll \delta h\rho^2$, $L = \mathcal{Q}(\tilde A)$ is small so that $\frac{\tilde A - L}{(\det(\tilde A - L))^{1/2}}$ defines a map in $C^\omega_{e^{-\delta}h}(\mathbb{T}, \mathrm{SO}(2,\mathbb{R}))$ which is $C\|L\|_{e^{-\delta}h}$ close to $\tilde A$.

Notice that $\tilde A$ is itself homotopic to a constant since $(\alpha, \tilde A)$ and $(\bar\alpha, \tilde{\tilde A})$ commute and $\tilde{\tilde A}$ is homotopic to R_ϕ which is homotopic to a constant (since ϕ takes values on $\mathbb{R}$). Thus we can write $\frac{\tilde A - L}{(\det(\tilde A - L))^{1/2}} = R_{\tilde\phi}$ for some function $\tilde\phi \in C^\omega_{e^{-\delta}h}(\mathbb{T}, \mathbb{R})$. We then get $\|R_{-\tilde\phi}\tilde A - \mathrm{id}\|_{e^{-\delta}h} \leq C\|L\|_{e^{-\delta}h}$, which together with (4.16) gives the second statement. □

4.2. The following lemma shows that if a cocycle in $\Omega(h, \varepsilon, \tau, \nu)$ is sufficiently close to rotations, then it is possible to iterate it and use the Diophantine property on the fibered rotation number and the bounds obtained for the growth of Birkhoff sums involving the special denominators Q_k to end up with the required conditions for the reduction step of Proposition 4.1.

In the following statement and all through the rest of the paper we assume ε, τ, ν given and let

$$M = \max(4\overline{\tau}, \frac{2}{1-2\bar\nu}), \quad \overline{\tau} = \tau + 1, \quad \bar\nu = \frac{1}{2}(\nu + \frac{1}{2}) \tag{4.17}$$

and define the sequence (Q_k) as in proposition 3.1 and set

$$U_k = e^{-\overline{Q}_k Q_k^{-b} - \overline{Q}_k^a}, \quad a = \frac{2}{M} \quad b = \frac{M}{2}.$$

Recall that for $\eta > 0$, $\eta_k = \eta/k^2$.

Lemma 4.5. *Fix $D, \eta, h_* > 0$. There exists $J(h_*, D, \eta, \varepsilon, \nu, \tau)$ such that if $\overline{Q}_k \geq J$ and if $(\alpha, A) \in \Omega(h, \varepsilon, \tau, \nu)$ for some $h > h_*$ is such that $A = R_\varphi(\mathrm{id} + \xi)$ with $\|\varphi - \widehat\varphi(0)\|_h \leq D$ and $\|\xi\|_h \leq U_k$, then we can express $A^{(Q_{k+1})}$ as $R_{\varphi^{(Q_{k+1})}}(\mathrm{id} + \xi^{(Q_{k+1})})$ with $\varphi^{(Q_{k+1})} := S_{Q_{k+1}}\varphi$ and*

(1) $\|\varphi^{(Q_{k+1})} - \widehat{\varphi^{(Q_{k+1})}}(0)\|_{h(1-\eta_{k+1})} \leq D$
(2) $\|(R_{2\varphi^{(Q_{k+1})}} - \mathrm{id})^{-1}\|_{h(1-\eta_{k+1})} < \rho_k, \quad \rho_k = \frac{\varepsilon}{4(Q_{k+1}^{-\tau} + \overline{Q}_{k+1}^{-\nu})} < U_k^{-\frac{1}{10}}$
(3) $\|\xi^{(Q_{k+1})}\|_{h(1-\eta_{k+1})} \leq U_k^{\frac{1}{2}}$

Proof. Given $C = C(h_*, \eta, M)$ of Proposition 3.1, let $J > 0$ be such that if $\overline{Q}_k \geq J$ then

$$CD(\overline{Q}_k Q_k^{-M} + \overline{Q}_k^{\frac{1}{M}}) \leq \frac{1}{100}(\overline{Q}_k Q_k^{-b} + \overline{Q}_k^a) \tag{4.18}$$

$$CD(Q_{k+1}^{-M} + \overline{Q}_{k+1}^{-1+\frac{1}{M}}) \leq \frac{\varepsilon}{100}(Q_{k+1}^{-\tau} + \overline{Q}_{k+1}^{-\nu}) \tag{4.19}$$

$$U_k^{\frac{1}{10}} \leq \frac{\varepsilon}{100} Q_{k+1}^{-\tau} \leq \frac{1}{100}\|Q_{k+1}\rho\| \tag{4.20}$$

the last requirement being possible since by definition of the sequence (Q_k) we have that $Q_{k+1} \leq \overline{Q}_k^{16M^4}$. It is henceforth assumed that $\overline{Q}_k \geq J$. From Proposition 3.1 and (4.18), it follows that for all $l \leq Q_{k+1}$,

$$\|S_l\varphi - l\widehat\varphi(0)\|_{h(1-\eta_{k+1})} \leq \frac{1}{100}(\overline{Q}_k Q_k^{-b} + \overline{Q}_k^a). \tag{4.21}$$

To prove (3) we need the following straightforward claim on the growth of matrix products that can be proved following the lines of Lemma 3.1. of [AK1]

Claim 4.6. *We have that*

$$M_l(\mathrm{id}+\xi_l)\dots M_0(\mathrm{id}+\xi_0)=M^{(l)}(\mathrm{id}+\xi^{(l)})$$

with $M^{(l)}=M_l\dots M_0$ *and* $\xi^{(l)}$ *satisfying*

$$\|\xi^{(l)}\|\leq e^{\sum_{k=0}^{l}\|M^{(k)}\|^2\xi_k}-1$$

To prove (3) apply the claim to $M_k=R_{\varphi(x+k\alpha)}$ and $\xi_k=\xi(x+k\alpha)$ and observe that $\|R_\theta\|\leq e^{|\mathrm{Im}\theta|}$, and use (4.21).

On the other hand, Proposition 3.1 together with (4.19) and the Diophantine condition (4.1) on the fibered rotation number imply that

$$\|\varphi^{(Q_{k+1})}-Q_{k+1}\widehat{\varphi}(0)\|_{h(1-\eta_{k+1})}\leq\frac{\varepsilon}{100}(Q_{k+1}^{-\tau}+\overline{Q}_{k+1}^{-\nu})\leq\frac{1}{50}|||Q_{k+1}\rho|||. \tag{4.22}$$

Clearly (4.22) implies (1).

It comes from (4.22) and (4.20) and (3) that

$$\|\varphi^{(Q_{k+1})}-Q_{k+1}\widehat{\varphi}(0)\|_{h(1-\eta_{k+1})}+\|\xi^{(Q_{k+1})}\|_{h(1-\eta_{k+1})}\leq\frac{1}{25}|||Q_{k+1}\rho|||.$$

But the fibered rotation number of the cocycle $(Q_{k+1}\alpha,A^{(Q_{k+1})})$ is $Q_{k+1}\rho$, hence

$$\|\varphi^{(Q_{k+1})}-Q_{k+1}\rho\|_{h(1-\eta_{k+1})}\leq\frac{1}{10}|||Q_{k+1}\rho|||$$

so that using (4.20) again and the Diophantine property on the fibered rotation number ρ we get (2) and the proof of the lemma is over.

□

4.3. As a consequence of Proposition 4.1 and Lemma 4.5 we get the following inductive step, fundamental in our reduction.

Proposition 4.7. *Given* $h_*,\eta,D>0$, *there exists* $T(h_*,D,\eta,\varepsilon,\nu,\tau)$ *such that if* $k\geq 1$, $\overline{Q}_k\geq T$ *and if* $(\alpha,A_k)\in\Omega(h,\varepsilon,\tau,\nu)$ *for some* $h\geq h_*$, *can be written as* $A_k=R_{\varphi_k}(\mathrm{id}+\xi_k)$ *with*

- $\|\varphi_k-\widehat{\varphi}_k(0)\|_h\leq D-\eta_k$
- $\|\xi_k\|_h\leq U_k$

then, if we denote $h_{k+1}:=h(1-\eta_{k+1})^2$, *there exists* B_k *with* $\|B_k-\mathrm{id}\,\|_{h_{k+1}}\leq U_k^{\frac{1}{4}}$ *such that* $A_{k+1}(x):=B_k(x+\alpha)A_k(x)B_k(x)^{-1}$ *can be written as* $A_{k+1}=R_{\varphi_{k+1}}(\mathrm{id}+\xi_{k+1})$ *with*

- $\|\varphi_{k+1}-\widehat{\varphi}_{k+1}(0)\|_{h_{k+1}}\leq D-\eta_{k+1}$
- $\|\xi_{k+1}\|_{h_{k+1}}\leq U_{k+1}$

Proof. Let $\epsilon_0(h_*/2,2D)$ be as in Proposition 4.1. Assume k is such that $\overline{Q}_k\geq J$ where $J(h_*,D,\eta,\varepsilon,\nu,\tau)$ is given by Lemma 4.5 and $U_k\leq\epsilon_0(h_*/2,1)^2$.

By Lemma 4.5 we can apply Proposition 4.1 to the cocycle $(\bar\alpha,\bar A)$ with $\bar\alpha=Q_{k+1}\alpha$ and $\bar A=A_k^{(Q_{k+1})}$, and to the cocycle (α,A_k) that commutes with $(\bar\alpha,\bar A)$. We thus get a conjugacy B_k such that $\|B_k-\mathrm{id}\,\|_{h(1-\eta_{k+1})}\leq C_0U_k^{1/2-1/5}$ (see (4.4)) while (4.6) yields that $A_{k+1}(x) = B_k(x+\alpha)A_k(x)B_k(x)^{-1}$ can be expressed as $A_{k+1} = R_{\varphi_{k+1}}(\mathrm{id} + \xi_{k+1})$ with

$$\|\xi_{k+1}\|_{h_{k+1}} \leq C_0 e^{-\frac{h(1-\eta_{k+1})\eta_{k+1}\rho_k^2}{C_0 \|\!|Q_{k+1}\alpha|\!\|}}$$

with $\rho_k = \frac{\varepsilon}{4(Q_{k+1}^{-\tau} + \overline{Q}_{k+1}^{-\nu})}$.

Since $\|\!|Q_{k+1}\alpha|\!\| \leq 1/\overline{Q}_{k+1}$, (4.17) implies that $\overline{Q}_{k+1}^{2/M} + \overline{Q}_{k+1} Q_{k+1}^{-M/2} = o(\eta_{k+1}\rho_k^2 / \|\!|Q_{k+1}\alpha|\!\|)$ as $\overline{Q}_k$ tends to infinity. Hence, there exists $T(h_*, D, \eta, \varepsilon, \nu, \tau)$, such that for $\overline{Q}_k \geq T$ we have $\|\xi_{k+1}\|_{h_{k+1}} \leq U_{k+1}$, while the bound on φ_{k+1} follows immediately from the bounds on φ_k and B_k. □

Corollary 4.8. *Let $\varepsilon, \tau > 0$, $h > 2\varepsilon$, $0 < \nu < 1/2$, and let $T(h-\varepsilon, \varepsilon, \varepsilon/10, \varepsilon, \nu, \tau)$ be as in Proposition 4.7. Suppose that $(\alpha, A') \in \Omega(h-\varepsilon, \varepsilon, \tau, \nu)$ with $A' = R_{\varphi'}(\mathrm{id} + \xi')$ satisfying*

$$\|\varphi' - \widehat{\varphi}'(0)\|_{h-\varepsilon/2} \leq \frac{\varepsilon}{2} \tag{4.23}$$

$$\|\xi'\|_{h-\varepsilon/2} \leq U_{n_0} \tag{4.24}$$

with n_0 such that $\overline{Q}_{n_0} \geq T$. Then there exist $B : \Delta_{h-\varepsilon} \to \mathrm{SL}(2,\mathbb{C})$ and $\varphi : \Delta_{h-\varepsilon} \to \mathbb{C}$ such that

1. *B and φ are real-symmetric,*
2. *$\|B - \mathrm{id}\|_{h-\varepsilon} < \varepsilon/2$ and $\|\varphi - \widehat{\varphi}(0)\|_{h-\varepsilon} < \varepsilon$,*
3. *$B(x+\alpha)A'(x)B(x)^{-1} = R_{\varphi(x)}$.*

5. Proof of Theorem 1.3.

Let $h, \varepsilon, \tau > 0$ and $0 < \nu < 1/2$ be given (we assume that $\varepsilon < h/2$).

In order to obtain Theorem 1.3, it is sufficient to show that there exists B' with

$$\|B' - \mathrm{id}\|_{h-\varepsilon/2} \leq \varepsilon/10 \tag{5.1}$$

and (α, A') satisfying the conditions of Corollary 4.8 such that $A'(x) = B'(x+\alpha)A(x)B'(x)^{-1}$.

We first claim that there exists n_0 such that $Q_{n_0} \leq T^{\tilde{A}^4}$ while $\overline{Q}_{n_0} \geq T$. Indeed, let m_0 be such that $Q_{m_0} \leq T \leq Q_{m_0+1}$. There are two possibilities: either $\overline{Q}_{m_0} \geq T$ (and $n_0 = m_0$ satisfies the claim) or $\overline{Q}_{m_0} \leq T$ and by definiton of the sequence (Q_k) it then holds that $Q_{m_0+1} \leq T^{\tilde{A}^4}$ while by definition of m_0 it holds that $\overline{Q}_{m_0+1} \geq T$ (so $n_0 = m_0 + 1$ satisfies the claim).

Let $\epsilon_0(h-\varepsilon, \varepsilon)$ be as in Proposition 4.1. Define $\epsilon_1 := \min(\epsilon_0, (\frac{\varepsilon}{4} Q_{n_0}^{-\tau})^4)$. Since $Q_{n_0} \leq T^{\tilde{A}^4}$, by taking ϵ sufficiently small (depending on $h, \varepsilon, \tau, \nu$) then the hypothesis that A is ϵ-close to a constant rotation through Δ_h implies that $\xi_0 = R_{-Q_{n_0}\rho} A^{(Q_{n_0})}$ satisfies $\|\xi_0\|_h < \epsilon_1$, (where ρ is the fibered rotation number of the cocycle (α, A)).

Now, if $(\alpha, A) \in \Omega(h, \varepsilon, \tau, \nu)$, then $\|\!|2Q_{n_0}\rho|\!\| \geq \varepsilon Q_{n_0}^{-\tau} \geq 4\epsilon_1^{1/4}$. This implies that $\|(R_{2Q_{n_0}\rho} - \mathrm{id})^{-1}\| \leq 2/\|\!|2Q_{n_0}\rho|\!\| \leq \|\xi_0\|_h^{-1/4}/2$. We are thus in position to apply Proposition 4.1 to $(\overline{\alpha}, \overline{A}) = (\alpha, A)^{Q_{n_0}}$ and get a conjugacy B' such that

$\|B' - \mathrm{id}\|_{h_0-\varepsilon/4} \leq C_0\epsilon_0^{\frac{1}{2}}$ and $A'(x) := B'(x+\alpha)A(x)B'(x)^{-1}$ can be written as $A' = R_{\varphi'}(\mathrm{id} + \xi')$ where

$$\|\xi'\|_{h_0-\varepsilon/2} \leq C_0 e^{-\frac{h\varepsilon \|\|Q_{n_0}\rho\|\|^2}{16C_0\|\|Q_{n_0}\alpha\|\|}}$$

and as shown in the proof of Proposition 4.7 it follows from the fact that $\overline{Q}_{n_0} \geq T$ that $\|\xi'\|_{h-\varepsilon/2} \leq U_{n_0}$.

Moreover, it follows from the bound on B' that $\|\varphi' - \widehat{\varphi'}(0)\|_{h-\varepsilon/2} \leq \frac{\varepsilon}{2}$. □

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