Rigidity times for weakly mixing dynamical system which are not rigidity times for any irrational rotation

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Abstract

We construct an increasing sequence of natural numbers $(m_n)_{n=1}^{+\infty}$ with the property that $(m_n\theta[1])_{n\geqslant 1}$ is dense in $\mathbb{T}$ for any $\theta \in \mathbb{R}\setminus\mathbb{Q}$, and a continuous measure on the circle μ such that $\lim_{n\to+\infty}\int_{\mathbb{T}}\|m_n\theta\|d\mu(\theta)=0$. Moreover, for every fixed $k\in\mathbb{N}$, the set $\{n\in\mathbb{N}: k\nmid m_n\}$ is infinite.

This is a sufficient condition for the existence of a rigid, weakly mixing dynamical system whose rigidity time is not a rigidity time for any system with a discrete part in its spectrum.

1 Introduction

Let $\mathbb{T}$ denote the circle group with addition $mod 1$. For $\eta\in\mathbb{R}$ we denote by $\eta[1]$ the fractional part of η and $\|\eta\|$ its distance to integers. It follows that $\|\eta\| = \min(\eta[1],(1-\eta)[1])$. Therefore for any $\eta\in\mathbb{R}$, $\|\eta\|\leqslant\frac{1}{2}$.

In this note, we prove the following two results.

Theorem 1. *Fix rationally independent numbers* $\{\alpha_i\}_{i\in\mathbb{N}}\in\mathbb{T}$.[1] *There exists an increasing sequence* $(m_n)_{n=1}^{+\infty}$ *such that* $(m_n\theta[1])_{n\geqslant 1}$ *is dense in* $\mathbb{T}$ *for every irrational* θ, *and for every* $\epsilon>0$ *and* $k\in\mathbb{N}$ *there exists* $N_0\in\mathbb{N}$ *such that for every* $n\geqslant N_0$ *we have* $\|m_n\alpha_i\|<\epsilon$ *for at least* $k-1$ *choices of* $i\in\{1,...,k\}$. *Moreover for every* $k\in\mathbb{N}$ *the set* $\{n\in\mathbb{N}: k\nmid m_n\}$ *is infinite.*

Theorem 2. *Fix rationally independent numbers* $\{\alpha_i\}_{i\in\mathbb{N}}\in\mathbb{T}$ *and let* $(m_n)_{n\geqslant 1}$ *be the corresponding sequence from Theorem 1. There exists a continuous probability measure* μ *on* $\mathbb{T}$ *such that*

$$\lim_{n\to+\infty}\int_{\mathbb{T}}\|m_n\theta\|d\mu(\theta)=0.$$

[1]By this we mean that every finite collection is rationally independent.

Theorem 1 gives us an increasing sequence of natural numbers $(m_n)_{n=1}^{+\infty}$ which is not a rigidity time for any system with a discrete part in its spectrum. Indeed, if the system has an irrational eigenvalue then it has the irrational rotation as a factor. If it has a rational eigenvalue then it has a shift on a finite group as a factor. But a rigidity time for a dynamical system is also a rigidity time for its factors, and a sequence as in Theorem 1 cannot be a rigidity sequence for any rational or irrational rotation.

From Theorem 2, by the Gaussian measure space construction (see [3]), we deduce that there exists a weakly mixing dynamical system whose rigidity times contain the constructed sequence $(m_n)_{n=1}^{+\infty}$. This gives a full answer to the question stated in [2] of whether a rigidity times sequence of a system with discrete spectrum is a rigidity time for some weakly mixing and conversely whether a rigidity times sequence of a system with continuous spectrum is a rigidity times sequence for some discrete spectrum system. The first direction was established in [1] and later in [4], namely, any rigidity time of a system with discrete spectrum is also a rigidity time for some weakly mixing dynamical system.

Our approach is inspired by the completely spectral approach adopted in [4]. First we prove the existence of a sequence m_n which is not a rigidity time for any circle rotation, but still satisfies that $\|m_n\alpha_i\|$ is small for most of the indices i of a family of rationally independent numbers $\{\alpha_i\}_{i\in\mathbb{N}} \in \mathbb{T}$ (see precise statement in Theorem 1).

This allows to construct a continuous probability measure on $\mathbb{T}$, that is a weak limit of discrete measures each supported on some finite set connected with the numbers $\alpha_1, \alpha_2, \ldots$, with a Fourier transform converging to 1 along this sequence.

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2 Proof of Theorem 1

Let there be given a family of rationally independent numbers $\{\alpha_i\}_{i\in\mathbb{N}} \in \mathbb{T}$. We will first state a lemma, which is a generalisation of Lemma 1 in [4].

Definition 3. [4] For an interval $I \subset \mathbb{T}$ and fixed $\epsilon > 0$ one says that $\theta \in \mathcal{A}(N_1, N_2, \epsilon, I, k)$ if for every $m \in [N_1, N_2]$ such that $\|m\alpha_i\| < \epsilon$ for $i = 1, ..., k$, we have $m\theta[1] \notin I$.

Lemma 4. *For every $l \geqslant 2$ there exists $L(l) \in \mathbb{N}$ such that for every $0 < \epsilon < \frac{1}{2l^2}$, for every $v > 0$, for every k, there exist $K(\epsilon) \in \mathbb{N}$ and $N' = N'(l, \epsilon, v, N, k) \in \mathbb{N}$ such that $\theta \in \mathcal{A}(N_1, N', I, \epsilon, k)$ for some interval I of size $\frac{1}{l}$, implies that $\|\sum_{i=1}^{k} r_i\alpha_i - l'\theta\| < v$ for some $|r_1|, ..., |r_k| < K(k, \epsilon)$ and some $|l'| < L(l)$.*

The proof is a repetition of the proof of Lemma 1 in [4]. Instead of considering $\phi_\epsilon : \mathbb{T} \to \mathbb{R}$ one needs to consider $\phi_\epsilon^k : \mathbb{T}^k \to \mathbb{R}$. It follows by the proof that the number $L(l, \{\alpha_i\}_{i=1}^k)$ does not depend on the numbers $\{\alpha_i\}_{i=1}^k$ and that is why we just had $L(l)$ in the statement. Indeed, similarly to Lemma 1 in [4], one considers a polynomial $\varphi_l : \mathbb{T} \to \mathbb{R}$, $\varphi_l(y) := \sum_{0<|k|<L(l)} \hat{\varphi}_k e^{i2\pi ky}$, where $L(l)$ is such that

- $\varphi_l(y) > 1$ for every $y \notin [0, \frac{1}{l}]$
- $|\varphi_l(y)| < l^2$ for every $y \in \mathbb{T}$.

Therefore $L(l) \in \mathbb{N}$ does not depend on $\{\alpha_i\}_{i=1}^k$.

Remark 5. Consider an ergodic rotation $T : T^j \to T^j$, $T(x_1, ..., x_j) = (x_1 + \gamma_1, ..., x_j + \gamma_j)$, for $\gamma_1, ..., \gamma_j \in \mathbb{T}$. It follows that for every $k \in \mathbb{N}$ and every $\epsilon > 0$, there exist (infinitely many) $m \in \mathbb{N}$ such that $\|m\gamma_i\| < \epsilon$ for $i = 1, ..., j$ and $k \nmid m$. Indeed, for every fixed $k \in \mathbb{N}$ there exist a sequence $(r_n)_{n\geqslant 1}$ such that $T^{r_n}(0) \to \frac{1}{k}T(0)$.

Proposition 6. *Fix rationally independent numbers* $\{\alpha_i\}_{i\in\mathbb{N}} \in \mathbb{T}$. *There exists a sequence* (s_n) *such that* $\lim_{n\to+\infty} \|s_n\alpha_i\| = 0$ *for* $i = 1, ...$ *and* $(s_n\theta[1])_{n\geqslant 1}$ *is dense in* $\mathbb{T}$ *if and only if* $\theta \notin \mathbb{Q} + \mathbb{Q}\alpha_1 + ...$[2].

Proof.
We will use Lemma 4 for $k = 1, 2, ...$ Define for $n \geqslant 1$ the sequence $l_n = n+1$. Let $\epsilon_n = \frac{1}{2(n+1)^2}$ and $K_n := K(n, \epsilon_n)$. Define $v_n = \frac{1}{n}\inf_{0\leqslant|k_1|,...,|k_{n+1}|\leqslant K_{n+1}} \|\sum_{i=1}^{n+1} k_i\alpha_i\|$. Take $N_0 = 0$ and apply Lemma 4 with $k = 1, l = l_1, \epsilon = \epsilon_1, N = N_0, v = v_1$. Denote $N_1 = N'(l_1, \epsilon_1, v_1, N_0, 1)$. We apply Lemma 4 inductively for $k = n, l = l_n, \epsilon = \epsilon_n, N = N_n, v = v_n$ and choose $N_{n+1} > N'(l_n, \epsilon_n, v_n, N_n, n)$ sufficiently large. Then we define an increasing sequence $(s_n)_{n=1}^{+\infty}$ by taking, for every $i \in \mathbb{N}$, all integers $s \in [N_i, N_{i+1}]$ such that $\|s\alpha_t\| < \epsilon_i$ for every $t = 1, ..., i$ (we can choose N_{i+1} so that such $s \in [N_i, N_{i+1}]$ exists). Moreover by Remark 5 we can choose N_{i+1} so that for every $r = 1, ..., i$ there exists $s_r \in [N_i, N_{i+1}]$ with $r \nmid s_r$.

Notice first that for every $r \in \mathbb{N}$, $\lim_{n\to+\infty} \|s_n\alpha_r\| = 0$. Indeed, for every $j > r$ and every $t \in \mathbb{N}$ such that $s_t \in [N_j, N_{j+1}]$ we have $\|s_t\alpha_r\| < \epsilon_j$.

Now, let $\theta \in \mathbb{T}$ be such that $s_n\theta[1]$ is not dense in $\mathbb{T}$. Then there exists $I \subset \mathbb{T}$, $|I| = \frac{1}{l_s}$ for some s such that $s_n\theta[1] \notin I$. By the definition of the sequence $(s_n)_{n=1}^{+\infty}$ it follows that there exists n_0 such that $\theta \in \mathcal{A}(N_n, N_{n+1}, I, \epsilon, n)$ for every $n \geqslant n_0$. Therefore, by Lemma 4 it follows that there are integers $k_1^n, ..., k_n^n$ with $|k_i^n| < K_n$ for every $i = 1, ..., n$ such that $\|\sum_{i=1}^n k_i^n\alpha_i - l'\theta\| < v_n$ for some $|l'| < L = L(l_s)$. Therefore, $\|\sum_{i=1}^n h_i^n\alpha_i - L!\theta\| < L!v_n$, for some numbers $h_1^n, ..., h_n^n \in \mathbb{N}$ with $|h_i^n| < L!K_n$. It follows by triangle inequality that $\|\sum_{i=1}^{n+1} h_i^{n+1}\alpha_i - \sum_{i=1}^n h_i^n\alpha_i\| < L!v_n + L!v_{n+1} < 2L!v_n$. By the definition of v_n, we get that there exists $n_1 \in \mathbb{N}$ such that for $n \geqslant n_1$, these two combinations are equal. Therefore $|\sum_{i=1}^{n_1} h_i^{n_1}\alpha_i - L!\theta\| < L!v_n$ for every $n \geqslant n_1$. But $v_n \leqslant \frac{1}{n} \to 0$ and consequently $\theta \in \mathbb{Q} + \mathbb{Q}\alpha_1 + ... + \mathbb{Q}\alpha_{n_1}$.

On the other hand, it follows by construction of $(s_n)_{n\geqslant 1}$ that $(s_n\theta[1])_{n\geqslant 1}$ is not dense in $\mathbb{T}$ if $\theta \in \mathbb{Q} + \mathbb{Q}\alpha_1 + ...$ □

Proof of Theorem 1. For every $i \in \mathbb{N}$, let $\{s_n^{(i)}\}_{n\in\mathbb{N}}$ be a sequence as in Proposition 6 applied to the family of rationally independent numbers $\{\alpha_j\}_{j\in\mathbb{N}, j\neq i} \in \mathbb{T}$. Let $(N_s(i))_{s\geqslant 1}$ be the corresponding sequence of natural numbers given in the proof of Proposition 6, that is $\|s_t^{(i)}\alpha_r\| < \frac{1}{2(j+1)^2}$ for every $t \geqslant N_j(i)$ (this implies that $s_t > N_j(i)$) and every $r < j$. Then define the sequence $\tilde{s}_n^{(i)} := s_{n+N_i(i)}^{(i)}$

I made some explanation.

[2]By this we mean that there does not exist n_0 such that $\theta \in \mathbb{Q} + \mathbb{Q}\alpha_1 + ... + \mathbb{Q}\alpha_{n_0}$.

Then define m_n to be the sequence $\tilde{s}_1^{(1)}, \tilde{s}_2^{(1)}, \tilde{s}_1^{(2)}, \tilde{s}_3^{(1)}, \tilde{s}_2^{(2)}, \tilde{s}_1^{(3)}, \tilde{s}_4^{(1)}, \tilde{s}_3^{(2)}, \tilde{s}_2^{(3)}, \tilde{s}_1^{(4)}, \ldots$. The sequence m_n satisfies the conditions of Theorem 1. Indeed, first note that for any irrational θ there exists i such that $\theta \notin \bigcup_{i=1}^{+\infty}(\mathbb{Q}+\ldots+\mathbb{Q}\alpha_{i-1}+\mathbb{Q}\alpha_{i+1}+\ldots)$, hence $(m_n\theta[1])_{n\geqslant 1}$ is dense by just considering the subsequence $\tilde{s}_l^{(i)}$. Secondly fix $\epsilon > 0$ and $k \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $\frac{1}{2(r+1)^2} < \epsilon$. Define $N_0 := (\max\{N_r(1), \ldots, N_r(r)\})^2$. Then, by definition of the sequence $(m_n)_{n\geqslant 1}$, $\|m_n\alpha_i\| < \frac{1}{2(r+1)^2} < \epsilon$, for every $n > N_0$ and every $i \in \{1, \ldots, k\}$ except for at most one i that satisfies $m_n = \tilde{s}_{l_n}^{(i)}$. □

Remark 7. It follows that for every $\epsilon > 0, i \in \mathbb{N}$ there exist $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$, $\sum_{s=1}^{i} \|m_n\alpha_s\| < \frac{1}{2} + \epsilon$.

3 Proof of Theorem 2.

Fix rationally independent numbers $(\alpha_i)_{i\geqslant 1} \in \mathbb{T}$ and let $(m_n)_{n=1}^{+\infty}$ be the corresponding sequence given by Theorem 1.
For the construction of the measure μ we will proceed similary to [4] (and we borrow notation from there). For a probability measure ν on $\mathbb{T}$ we denote by $\nu^n = |\int_{\mathbb{T}} \|m_n\theta\| d\mu(\theta)|$. We will define inductively a sequence $(k_n)_{n\geqslant 1}$ so that the measure μ will be a weak limit of discrete measures $\mu_p := \frac{1}{2^p}\sum_{i=1}^{2^p} \delta_{k_i\alpha_i}$ for some numbers $k_i \in \mathbb{N}$ such that there exists a sequence $(N_p)_{p=1}^{+\infty}$ for which

(i) For every $p \geqslant 1$, for every $j \in [1, p-1]$, for every $n \in [N_j, N_{j+1}]$, $\mu_p^n < \frac{1}{2^{j-1}}$ (for $j = 0$ $\mu_p^n < 1$).

(ii) For $p_0 \in \mathbb{N}$ denote by $\eta_{p_0} = \frac{1}{4}\inf_{1\leqslant i<j\leqslant 2^{p_0}} \|k_i\alpha_i - k_j\alpha_j\|$. Then for every $l \in \mathbb{N}$ and every $r \in [1, 2^{p_0}]$, $\|k_{l2^{p_0}+r}\alpha_{l2^{p_0}+r} - k_r\alpha_r\| < \eta_{p_0}$.

In fact, similarly to [4], we get that any weak limit μ of a sequence μ_p as above, satisfies the conclusion of Theorem 2. Indeed, by (i) $\mu^n \to 0$. By (ii) it follows that for each p_0, the intervals $I_r = [-\eta_{p_0} + k_r\alpha_r, \eta_{p_0} + k_r\alpha_r]$, $r = 1, \ldots, 2^{p_0}$ are disjoint and $\mu_p(I_r) = \frac{1}{2^{p_0}}$ for every $p \geqslant p_0$ and therefore the limit measure μ is continuous.

Therefore, we just have to construct the measures μ_p as in (i) and (ii). We will do an inductive construction, in which we will additionally require that for every p

$$\mu_p^n < \frac{1}{2^{p+1}} + \frac{1}{2^{p+3}} \text{ for } n \geqslant N_p. \tag{1}$$

For $p = 0$ let $k_1 = 1$, then μ is the Dirac measure at α_1. Let $N_0 = 0$. For $p = 1$, $k_2 = 1$, then μ_1 is the average of Dirac measures at α_1 and α_2. We choose $N_1 = 1$. This satisfies (i) and (1) for $p = 1$.
Assume that we have constructed k_i for $i = 1, \ldots, 2^p$, N_l for $1 \leqslant l \leqslant 2^p$ such that (i) and (1) is satisfied up to p and (ii) is satisfied for every $p_0 \leqslant p$ and $0 \leqslant l \leqslant 2^{p-p_0} - 1$. We now choose k_{2^p+1} so that $k_{2^p+1}\alpha_{2^p+1}$ is sufficiently close to $k_1\alpha_1$ so that

$$\nu_{p,1} = \frac{1}{2^p}\sum_{i=1}^{2^p} \delta_{k_i\alpha_i} + \frac{1}{2^{p+1}}(\delta_{k_{2^p+1}}\alpha_{2^p+1} - \delta_{k_1\alpha_1})$$

satisfies $\nu^n_{p,1} < \frac{1}{2^{j-1}}$ for $n \in [N_j, N_{j+1}]$ and $j \in [0, p-1]$ ($\nu^n_{p,1} = \mu^n_p + \frac{1}{2^{p+1}}(\|m_n k_{2^p+1}\alpha_{2^p+1} - m_n k_1\alpha_1\|)$). Moreover it follows that for $n \geqslant N_p$ we have $\nu^n_{p,1} < \mu^n_p + \frac{1}{2^{p+1}} < \frac{1}{2^{p+1}} + \frac{1}{2^{p+3}} + \frac{1}{2^{p+1}} < \frac{1}{2^{p-1}}$. Let $N_{p,1} > N_p$ be sufficiently large so that $\nu^n_{p,1} < \frac{1}{2^p}$ for $n \geqslant N_{p,1}$ ($\nu^n_{p,1} < \mu^n_p + \frac{1}{2^{p+1}}$ and μ^n_p can be arbitrary close to $\frac{1}{2^{p+1}}$ by Remark 7).
Now construct idnuctively for $s = 1, ..., 2^p$ the numbers $k_{2^p+s}, N_{p,s} \in \mathbb{N}$ for the measures $\nu_{p,s}$ given by $\nu_{p,s} = \mu_p + \frac{1}{2^{p+1}}(\sum_{i=1}^{s}(\delta_{k_{2^p+i}\alpha_{2^p+i}} - \delta_{k_i\alpha_i}))$. It follows that by choosing k_{2^p+s} so that $k_{2^p+s}\alpha_{2^p+s}$ is sufficiently close to $k_s\alpha_s$ and $N_{p,s}$ large enough, we can insure that

A. $\nu^n_{p,s} < \frac{1}{2^{j-1}}$ for every $n \in [N_j, N_{j+1}]$, and $j \leqslant p-1$.

B. $\nu^n_{p,s} < \frac{1}{2^{p-1}}$ for $n \geqslant N_p$.

C. $\nu^n_{p,s} < \frac{1}{2^p}$ for $n \geqslant N_{p,s}$.

Indeed, for $s = 1$ the above conditions are satisfied, assume that for some $s \geqslant 1$, they hold. We will prove that they hold for $s+1$. First note that $\upsilon_{p,s} - \upsilon_{p,s-1} = \frac{1}{2^{p+1}}(\delta_{k_{2^p+s}\alpha_{2^p+s}} - \delta_{k_s\alpha_s})$. Therefore by choosing k_{2^p+s} so that $k_{2^p+s}\alpha_{2^p+s}$ is sufficienlty close to $k_s\alpha_s$ and by induction hypothesis, we get that $\nu^n_{p,s} < \frac{1}{2^{j-1}}$ for every $n \in [N_j, N_{j+1}]$ with $j \leqslant p-1$. The same arguments gives us $\nu^n_{p,s} < \frac{1}{2^{p-1}}$ for $N_{p,s-1} \geqslant n \geqslant N_p$. For $n > N_{p,s-1}$ we use the fact that $\nu^n_{p,s-1} < \frac{1}{2^p}$ to get $\nu^n_{p,s} < \nu^n_{p,s-1} + \frac{1}{2^{p+1}} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}}$. For the third point we use the fact that for n sufficiently large, $\|m_n\alpha_i\|$ is arbitrary small for all but one $i \in \{1, ..., 2^p+s\}$ (compare with Remark 7), to get that for $N_{p,s}$ large enough, $\nu^n_{p,s} < \frac{1}{2^{p+1}} + \frac{1}{2^{p+2}} + \frac{1}{2^{p+2}} = \frac{1}{2^p}$, for $n \geqslant N_{p,s}$.

Finally we define $\mu_{p+1} = \nu_{p,2^p}$ and observe that μ_{p+1} satisfies (i). Moreover, by definition $\mu_{p+1} = \frac{1}{2^{p+1}}\sum_{i=1}^{2^{p+1}}\delta_{k_i\alpha_i}$ and using the properties of the sequence $(m_n)_{n\geqslant 1}$ ($\|m_n\alpha_i\|$ is arbitrary small for all but one $i = 1, ..., 2^{p+1}$, see also Remark 7) we get that if N_{p+1} is sufficiently large, then (1) is satisfied for μ_{p+1}.

Moreover, for $l = 2^{p-p_0} + l' - 1$ we have $\|k_{l2^{p_0}+r}\alpha_{l2^{p_0}+r} - k_r\alpha_r\| \leqslant \|k_{l2^{p_0}+r}\alpha_{l2^{p_0}+r} - k_{l'2^{p_0}+r}\alpha_{l'2^{p_0}+r}\| + \|k_{l'2^{p_0}+r}\alpha_{l'2^{p_0}+r} - k_r\alpha_r\| < \eta_{p_0}$ By induction hypothesis and the choice of $k_{l2^{p_0}+r}$. Therefore (ii) is satisfied for $p+1$ and every $l \leqslant 2^{p+1}$. This finishes the proof. □

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