Exponential growth of product of matrices in $\mathrm{SL}(2,\mathbb{R})$.

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Abstract

In this note we investigate the exponential growth of products of two matrices $A, B \in \mathrm{SL}(2,\mathbb{R})$. We prove, assuming A is a fixed hyperbolic matrix, that for Lebesgue almost every B, products of length n involving less than n^α, $0 \leq \alpha < 1/2$ matrices B are uniformly bounded from below by γ^n for some $\gamma > 1$.

Throughout this note we denote by μ the Haar measure on $\mathrm{SL}(2,\mathbb{R})$, the group of 2×2 matrices with real entries and with determinant 1 and by λ the Lebesgue measure on $\mathbb{R}$.

In [5] the following problem was posed

Problem 1. *Describe the pairs of matrices A_0, $A_1 \in \mathrm{SL}(2,\mathbb{R})$, with A_0 hyperbolic (i.e. $|\mathrm{Tr}A_0| > 2$) and A_1 elliptic (i.e. $|\mathrm{Tr}A_1| < 2$) for which the following is true : there exist constants $0 < p < 1$ and $\gamma > 1$ such that, for every word $w \in \{0,1\}^{\mathbb{N}}$ which satisfies the following* frequency condition

$$\#\big\{j \in \{1,\ldots,k\};\ w_j = 1\big\} \leq pk, \tag{1}$$

we have for any $k \in \mathbb{N}$

$$\|A_{w_1}A_{w_2}\cdots A_{w_k}\| > \gamma^k \ ?$$

Furthermore, is it true that given any hyperbolic matrix $A_0 \in \mathrm{SL}(2,\mathbb{R})$, the set of matrices $A_1 \in \mathrm{SL}(2,\mathbb{R})$ for which the above holds for some constants $0 < p < 1$ and $\gamma > 1$ is a set of full Lebesgue measure?

The same question can of course be asked when A_1 can take a finite number $d \geq 1$ of values in $\mathrm{SL}(2,\mathbb{R})$.

Observe that a positive answer to Problem 1 would have the following dynamical consequence : given a measurabe map $A : X = [0,1] \to \{A_0, A_1\}$ that assumes the value A_0 on a set of measure greater than p, then given any ergodic automorphism T of X, the Lyapunov exponent of the cocycle $T_A : X \times \mathrm{SL}(2,\mathbb{R}) \to X \times \mathrm{SL}(2,\mathbb{R}), (x, y) \mapsto (Tx, A(x)y)$, is bounded from below by γ.

In this note, we give a positive answer to the above question but under the much restrictive frequency condition stating that the number of the unkown matrices A_1 in a product of length n is less than n^α, $\alpha < 1/2$. We conjecture that the same statement of Theorem 2 below should be true for $0 \le \alpha < 1$ but our method, that avoids any kind of combinatorics on words, requires in an essential way $\alpha < 1/2$. A positive answer to Problem 1, that correpsonds to the case $\alpha = 1$ plus a frequency condition, is naturally harder to conjecture since as we said before it would have a strong dynamical consequence.

Definition 1. Let $C_n = \{0,1\}^n$. For every $\alpha \in [0,1]$ we define the set $C_n^\alpha \subset C_n$ given by the words $W = \{w_1, \ldots, w_n\}$ satisfying

$$\#\big\{j \in \{1, \ldots, k\};\ w_j = 1\big\} \le n^\alpha. \tag{2}$$

Given two matrices A and B in $\mathrm{SL}(2,\mathbb{R})$, we denote for $W \in C_n$, $W(A,B)$ the product $A_{w_1} \ldots A_{w_n}$, with $A_{w_j} = A$ if $w_j = 0$ and $A_{w_j} = B$ if $w_j = 1$.

Definition 2 (α-hyperbolic pairs)**.** We say that a pair (A, B) is *α-hyperbolic with exponent* $\gamma > 1$ if there exists a constant $C > 0$ such that for any integer n, and any $W \in C_n^\alpha$ we have

$$\|W(A,B)\| \ge C\gamma^n.$$

We will prove the following

Theorem 2. *Let $\alpha < 1/2$ and H be a hyperbolic matrix in* $\mathrm{SL}(2,\mathbb{R})$*, that is* $\mathrm{Tr}(H) > 2$*. Then for any $1 < \gamma < \mathrm{Tr}(H)/2$ we have that for μ-almost every $B \in \mathrm{SL}(2,\mathbb{R})$ the pair (H, B) is α-hyperbolic with exponent γ.*

Remark 1. From the proof it will be clear that the same conclusion holds if we consider products of H and a finite number of matrices $B_1, \cdots, B_l$ if the frequency of each of the matrices B_i is less than n^α, $\alpha < 1/2$.

Remark 2. Problem 1 can also be posed in the frame of one-dimensional Schrödinger operators in $l^2(\mathbb{Z})$,

$$\big(H_{(V_n)}u\big)_n = u_{n+1} + u_{n-1} + V_n u_n$$

where V_n is the potential at site n. The $\mathrm{SL}(2,\mathbb{R})$ matrices involved in the study of the state $\left(H_{(V_n)}u\right)_n = E(u)_n$ (the so-called Schrödinger matrices) are then of the form

$$A_{V_n}(E) = \begin{pmatrix} V_n - E & 1 \\ -1 & 0 \end{pmatrix}$$

In this context, Problem 1 corresponds to taking V_n in a finite set of values $\{v_0, \ldots, v_d\}$, fix E such that $|v_0 - E| > 2$, and see whether almost surely in $v_1, \ldots, v_d$ the conclusion of Problem 1 holds. A stronger version of Problem 1 (we call this Problem 1') in this setting is the following. Given $v_0, v_1, \ldots, v_d$, is it true that for almost every E such that $|v_0 - E| > 2$, the set $\{A_{v_0}(E), \ldots, A_{v_d}(E)\}$, satisfies the conclusion of Problem 1? Although Schrödinger matrices form a zero measure set in $\mathrm{SL}(2,\mathbb{R})$, it will be clear from our proof of Theorem 2 that given a finite number of values $v_i \in \mathbb{R}$, and if we consider the products of the matrices

$$A_i(E) = \begin{pmatrix} v_i - E & 1 \\ -1 & 0 \end{pmatrix}$$

where in a product of size n every matrix other than $A_0(E)$ appears less then n^α times, $\alpha < 1/2$, then for almost every energy E such that $\gamma_0 = |v_0 - E| > 2$, these products will be hyperbolic with exponent $\gamma_0/2$.

In the *dynamical* case, that is when V_n is of the form $V(T^n w)$ where $T : (\Omega, \mu) \to (\Omega, \mu)$ $(T_*\mu = \mu)$ is a measurable dynamics on the probability space (Ω, μ), $V \in L^\infty(\Omega)$ assumes a finite number of values and $\omega \in \Omega$ is in a set of μ-measure 1, a classical and natural counterpart of dynamical nature to Problems 1' is the problem of positivity of Lyapunov exponent of the corresponding cocycle for Lebesgue a.e. energy $E \in \mathbb{R}$, which is solved by a celebrated theorem of Kotani [7] (see also [2]). Kotani's result, which applies under some aperiodicity condition on the cocycle, has, as usual in the theory, consequences on the spectrum Σ of the operator $H_{(V_n)}$ $(V_n = V(T^n\omega))$: its almost absolutely continuous part Σ_{ac} is empty. In some cases one can even prove that the spectrum Σ itself is of zero Lebesgue measure and that the corresponding Schrödinger cocycle is uniformly hyperbolic. For potentials given by evaluation of certain finite valued functions over an irrational rotation on the circle this has been proved by Damanik and Lenz [3], [4]. We refer the reader to the survey paper by Damanik [1] for further informations and references.

We now prove Theorem 2.

We first observe that up to conjugating we can always assume that H is of the form

$$H = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \quad a > 2. \tag{3}$$

On the other hand it is easy to check that any matrix $B \in \mathrm{SL}(2, \mathbb{R})$ such that

$$B = \begin{pmatrix} r & s \\ t & u \end{pmatrix}, \quad u \neq 0,$$

can be written as a product

$$B = B(b_1, b_2, b_3) = \begin{pmatrix} b_1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_3 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

with

$$b_1 = -(s+1)/u, \quad b_2 = -u, \quad b_3 = (t-1)/u \tag{5}$$

The matrices in $\mathrm{SL}(2, \mathbb{R})$ that cannot be represented as above, that is with $u = 0$, have zero Haar measure. Moreover, zero Lebesgue measure sets in $\mathbb{R}^3$ of (b_1, b_2, b_3) correspond by (5) to zero Haar measure sets in $\mathrm{SL}(2, \mathbb{R})$. Theorem 2 hence follows from the following

Theorem 3. *Let $\alpha < 1/2$, $a > 2$ and H be given by (3). Then for any $1 < \gamma < a/2$ we have that for any $(\theta, \theta') \in \mathbb{R}^2$, for Lebesgue-almost every $b \in \mathbb{R}$ the pair $(H, B(b, b+\theta, b+\theta'))$ is α-hyperbolic with exponent γ.*

Proof. Fix $(\theta, \theta') \in \mathbb{R}^2$, and let M be an arbitrarilly large number. Define for $W \in C_n^\alpha$

$$\begin{aligned} \mathcal{E}_W &= \left\{ b \in [-M, M] \mid \|W(H, B(b, b+\theta, b+\theta'))\| \leq \gamma^n \right\}, \\ \mathcal{E}_n &= \bigcup_{W \in C_n^\alpha} \mathcal{E}_W. \end{aligned}$$

By the Borel-Cantelli lemma, we finish if we prove that

$$\sum_{n \in \mathbb{N}} \lambda(\mathcal{E}_n) < \infty.$$

The latter follows if we prove that

$$\sum_{n \in \mathbb{N}} \#(C_n^\alpha) \max_{W \in C_n^\alpha} \lambda(\mathcal{E}_W) < \infty \tag{6}$$

which we will show holds if $\alpha < 1/2$.

Fix $W \in C_n$. Let $m = \#\{j \in \{1,\ldots,k\};\ w_j = 0\} + 3\#\{j \in \{1,\ldots,k\};\ w_j = 1\}$. In the sequel we will only need to remember that $m \geq n$. It is an easily checkable classical fact that for $W = W(H, B(b, b+\theta, b+\theta'))$

$$W = \begin{pmatrix} \Delta(W,b,\theta,\theta') & * \\ * & * \end{pmatrix}$$

with

$$\Delta(W,b,\theta,\theta') = \det \begin{pmatrix} z_1 & 1 & 0 & & & 0 \\ 1 & z_2 & 1 & 0 & & 0 \\ & & & & & \\ & & & & & \\ 0 & & 0 & 1 & z_{m-1} & 1 \\ 0 & & & 0 & 1 & z_m \end{pmatrix} \tag{7}$$

and $z_i = a$ corresponds to the appearance of a letter H in W while a string $z_i = b, z_{i+1} = b+\theta, z_{i+2} = b+\theta'$ corresponds to the appearance of a letter B. In particular

$$\|W(H, B(b, b+\theta, b+\theta'))\| \geq |\Delta(W,b,\theta,\theta')|. \tag{8}$$

Observe that for fixed θ and θ'

$$P_W(b) := a^{-m}\Delta(W,b,\theta,\theta') \tag{9}$$

is a polynomial of degree $3\#\{j \in \{1,\ldots,k\};\ w_j = 1\} \leq 3n^\alpha$ if $W \in C_n^\alpha$.

The following simple computational lemma is useful :

Lemma 4. *Let $l \geq 1$ and*

$$\Delta_l = \det \begin{pmatrix} d_1 & 1/a & 0 & & & 0 \\ 1/a & d_2 & 1/a & 0 & & 0 \\ & & & & & \\ & & & & & \\ 0 & & 0 & 1/a & d_{l-1} & 1/a \\ 0 & & & 0 & 1/a & d_l \end{pmatrix}$$

and assume that $a \geq 2$ and for every $i = 1,\ldots,l$, $d_i \geq 1$. Then

$$\Delta_l \geq \frac{1}{2^l}$$

Proof. Denote by Δ_l the previous $l \times l$ determinant. It is a classical easy fact that for $l \geq 2$, $\Delta_l = d_l \Delta_{l-1} - (1/a)^2 \Delta_{l-2}$ (setting $\Delta_0 = 1$). Introducing $u_l = \Delta_l / \Delta_{l-1}$ and using the fact that $a \geq 2$ and $d_i \geq 1$ ($i \geq 1$), we get $u_l \geq 1 - 1/(4u_{l-1})$. Since $u_1 \geq 1/2$ we have by induction $u_l \geq 1/2$ and consequently $\Delta_l = u_l \cdots u_1 \geq (1/2)^l$. $\square$

Applying this lemma to (7) we get

Corollary 5. *For P_W of (9) we have*

$$P_W(a + |\theta| + |\theta'|) \geq \frac{1}{2^m}.$$

Now we need the following

Proposition 6 ([6], Prop. 3.2)**.** *Let $P(x)$ be a polynomial of degree $\leq n$. For $I \subset \mathbb{R}$, denote by $\|P\|_I := \max_{x \in I} |P(x)|$. Then for any open interval I and any $\epsilon > 0$ we have*

$$\lambda\left(\{x \in I : |P(x)| \leq \epsilon\}\right) \leq 2n(n+1)^{1/n} \left(\frac{\epsilon}{\|P\|_I}\right)^{1/n} \lambda(I).$$

Without loss of generality, we can assume that $M \geq a + |\theta| + |\theta'|$ and we deduce from Corollary 5 and Proposition 6 that for $\varepsilon > 0$

$$\lambda\left(\left\{b \in [-M, M] \mid |P_W(b)| < \varepsilon\right\}\right) \leq 24Mn^\alpha (2^m \varepsilon)^{\frac{1}{3n^\alpha}}. \tag{10}$$

Finally, we pose

$$\varepsilon = \left(\frac{\gamma}{a}\right)^m.$$

Applying (10), we get

$$\begin{aligned} \lambda(\mathcal{E}_W) &\leq 24Mn^\alpha \left(\frac{2\gamma}{a}\right)^{\frac{m}{3n^\alpha}} \\ &\leq 24Mn^\alpha \tau^{n^{1-\alpha}}, \end{aligned} \tag{11}$$

where we used $m \geq n$ and $\tau = (2\gamma/a)^{1/3} < 1$. But clearly $\#(C_n^\alpha) \leq n^\alpha n^{n^\alpha}$ while

$$\sum_{n \in \mathbb{N}} n^{2\alpha} n^{n^\alpha} \tau^{n^{1-\alpha}} < \infty$$

as soon as $\alpha < 1/2$ which gives (6) and ends the proof of Theorem 3. $\square$

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