TOPOLOGICALLY MIXING FLOWS WITH PURE POINT SPECTRUM.

Abstract. We are interested in the relationship between the measure theoretical behavior and the topological behavior of a smooth volume preserving system on a compact connected Riemannian manifold. The paper has two parts. In the first one, we give an overview of some old and new constructions showing how loose can this relationship be. The general approach in all the constructions is to start with linear Liouvillian translations or flows and perturb them using group extensions, reparametrizations or the successive conjugations techniques.

In the second part, we give a complete and selfcontained construction of smooth volume preserving flows on the torus that are topologically mixing and isomorphic to translation flows.

1. Introduction

In these notes we address the following question: *How far can the measure-theoretical behavior of a volume preserving diffeomorphism be from its topological behavior?*

We will ask this question for diffeomorphisms of compact connected smooth Riemannian manifolds that preserve a smooth measure μ, i.e., a measure which is smoothly equivalent to the Riemannian volume. We will denote such a dynamical system by (f, M, μ).

The ergodic properties that we will examine will range between discrete spectrum and Bernoulli and we will study their relationship with topological properties such as transitivity, minimality, topological mixing and orbit growth. In the first section we recall some definitions as well as some classical consequences of the ergodic properties on the topological ones, such as ergodicity implies transitivity, mixing implies topological mixing, etc.

Dynamicists might just skip this section and go directly to the following one where we overview some old and recent examples of systems with contrasting ergodic and topological properties, more precisely, systems whose ergodic features are weaker than their topological ones.

The overview is not in any means exhaustive and is rather focused on *Liouvillian constructions*, i.e. systems elaborated from translation maps or flows with Liouvillian frequencies via group extensions, reparametrizations or more generally via the successive conjugations techniques introduced in [1].

The constructions we outline display in general an *elliptic* behavior in the sense that their derivatives grow sublinearly, unlike the *parabolic* and *hyperbolic* paradigms for which $||Df^n||$ grow linearly or polynomially for the first and exponentially for the second. The latter systems are best represented by the horocycle flows and the Anosov maps for which it is known that the topological behavior and the measure-theoretical one are intimatly related. However, some of our constructions, namely the mixing or topologically mixing ones, stand at the borderline between the elliptic and the parabolic paradigm. Indeed, even if the derivatives grow sublinearly the growth of orbit segments of length n that can be distinguished with a fixed precision ϵ can be polynomial due to the dimension of the space.

Some of the recent examples are provided by reparametrizations of linear flows or by successive conjugations constructions in dimension greater than three. Placing ourselves in high dimensions, on one hand opens the way to obtaining the mixing properties (Cf. §3.2) and on the other hand leaves more space to the coexistence of exceptional sets of positive or full measure on which the topological properties can be violated (Cf. §3.4). To this respect, section 4 is a good example where both these advantages of working in high dimension are used to obtain a topologically mixing reparametrization of an irrational flow that is isomorphic to the original linear flow.

Content of the article. In section 2 we give the necessary notations and the definitons of the properties we will be interested in as well as the immediate implications between them. In section 3 we will review the following volume preserving constructions that were or will be published elsewhere and of which we only give brief outlines:

–Analytic minimal non-ergodic skew products over a Liouvillian rotation of the circle (Furstenberg's example).

–Analytic minimal and mixing flows obtained by reparametrization of irrational flows on $\mathbb{T}^3$.

–Analytic minimal and topologically mixing non-ergodic flows on $\mathbb{T}^5$ obtained by combining the two constructions above.

–Smooth transitive systems with a positive measure set of nondense orbits obtained using the successive conjugations techniques applied to the map on $\mathbb{T}^3$: $(x, y, z) \to (x + \alpha, y, z)$ where α is some Liouvillian number.

–Partially hyperbolic systems with the property of topological accessibility and not essential accessibility obtained by perturbing the direct product of the latter construction with a hyperbolic automorphism of $\mathbb{T}^2$.

–A minimal flow on $\mathbb{T}^3$ with a rotation set not reduced to a point obtained (almost straightforwardly) considering a special flow above a minimal volume preserving map on $\mathbb{T}^2$ with exactly two absolutely continuous ergodic components. The latter example was constructed in [16] using the successive conjugations techniques.

–Two isomorphic maps on $\mathbb{T}^2$ with different rotation sets. This is obtained considering a skew product over a Liouvillian rotation of the circle R_α whith a function φ such that $\varphi+\beta$ is a multiplicative coboundary for some $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}\int\varphi$.

In §3.6, starting again from the map of [16], we construct a diffeomorphism of $\mathbb{T}^3$ with two intermingled attractors.

Section 4 contains a complete proof of a new construction involving two properties at the opposite extremities in the spectrum of complexity for a dynamical system: topological mixing on one hand and pure point spectrum on the other hand. We will prove the following

THEOREM. *For any $r \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that there exists on $\mathbb{T}^d$ a volume preserving diffeomorphism of class C^r that is topologically mixing and isomorphic to a minimal translation on $\mathbb{T}^d$.*

This gives an answer in finite differentiability to the following question of Katok ([9] §8.2.e): *Is it possible to have a topologically mixing nonstandard smooth realization of some rotations or some linear flows on the torus?*

It would be interesting to answer Katok's question in the real analytic or in the C^∞ frame ($r = +\infty$ or ω in the beginning of the above statment), which should also be related to acheiving the isomorphism to rotations of the circle ($d = 1$ at te end of the statement).

2. Notations and definitions.

2.1. In all the text we will consider diffeomorphisms or flows on compact Riemannian manifolds. We will say that a diffeomorphism is volume preserving if it preserves a measure which is smoothly equivalent to the Riemannian volume.

2.2. *Topological tarnsitivity.* The dynamical system (f, M) is said to be topologically transitive if f has a dense orbit, i.e., if there exists $x \in M$ such that for any $y \in M$ there is a sequence $k_n \to \infty$ such that $f^{k_n}(x) \to y$. We recall that for complete separable metric spaces topological transitivity is equivalent to the fact that for any pair of non-empty open sets U and V of M, there exists an integer N such that $f^N(U) \bigcap V \neq \emptyset$, or equivalently if any open set invariant by f is dense.

2.3. *Minimality.* The dynamical system (f, M) is said to be minimal if the orbit of any point in M is dense, or equivalently if the only non-empty closed set invariant by f is M itself.

Transitivity and minimality are both, in ascending order, properties of irreducibility in topological dynamics. In a transitive system we can not isolate the orbit of an open set and in the minimal system we can not isolate the orbit of any given point.

A strengthening of topological mixing is the following property

2.4. *Topological mixing.* The dynamical system (f, M) is said to be topologically mixing if for any pair of non-empty open sets U and V there exists an integer N such that $f^n(U) \bigcap V \neq \emptyset$ for every $n \geq N$.

Both minimality and topological mixing are strengthening properties of transitivity but none of them implies the other as shown by the examples of minimal rotations on the circle that are not topologically mixing since they are isometries, and by hyperbolic linear automorphisms on the torus that are topologically mixing but clearly not minimal since they have periodic orbits.

2.5. *Ergodicity.* A system (f, M, μ) is said to be ergodic if every measurable invariant set is either of zero measure or of full measure. Ergodicity is a notion of irreducibility and in our case of volume preserving maps it implies topological transitivity (any open invariant set must have full measure and is hence dense).

Both minimality and ergodicity are strengthenings of transitivity but neither one of them implies the other. Hyperboliqu automorphisms on the torus are ergodic for the Haar measure but are not minimal while minimal maps can have invariant sets of positive and not full measure as long as these sets are dense in the space (Cf. §3.1). However when μ is the only probability measure on M invariant by f the system is said to be uniquely ergodic and is minimal on the support of μ (otherwise one would be able starting from the Dirac measure of a point and iterating it to construct an invariant measure distinct from μ). Hence, when μ is a volume, unique ergodicity implies minimality on the whole space, while in general this is not true as shown by the example of an irrational linear flow on the torus slowed down drastically at a single point: the only invariant measure is the Dirac measure at this point but the flow is clearly not minimal.

2.6. *Mixing.* A system (f, M, μ) is said to be mixing if and only if given any two measurable sets A and B we have

$$\lim_{n \to \infty} \mu(A \bigcap T^{-n} B) = \mu(A)\mu(B)$$

with $n \in \mathbb{N}$ for diffeomorphisms and $n \in \mathbb{R}$ for flows. Clearly mixing implies ergodicity and for volume preserving systems it also implies topological mixing. Topological mixing does not imply mixing and we will even see in §3.3 that topologically mixing volume preserving systems might be non-ergodic.

A stronger notion than mixing is Bernoullicity for which the limit above becomes a plain equality as n is large enough for pairs (A, B) taken from some dense collection of sets. A Bernoulli system being a system isomorphic to an independent shift on a finite alphabet.

A weaker notion than mixing is that of weak mixing when for any pair (A,B) the limit above is required to hold along a subsequence of full density in $\mathbb{N}$ or $\mathbb{R}$.

2.7. *Isomorphism.* Two dynamical systems (f,M,μ) and (g,N,ν) are said to be *metrically isomorphic* if there exists an isomorphism $h:(M,\mu)\to(N,\nu)$ such that $foh=hog$. In the same way, two flows are said to be isomorphic if $f^t\circ h=h\circ g^t$ for every $t\in\mathbb{R}$.

Two isomorphic systems have the same ergodic properties. Nevertheless, unless h is a conjugacy (bicontinuous bijection), the topological properties of two isomorphic systems might be different. Isomorphisms that are not conjugacies are a source of systems with contrasting ergodic and topological properties as will be seen in the sequel.

2.8. *Special flows and reparametrizations.* If $R_{t\alpha}$ is a translation flow on $\mathbb{T}^n$ and ϕ is a strictly positive smooth real function on $\mathbb{T}^n$, we define the *reparametrization of $R_{t\alpha}$ with velocity ϕ* as the flow given by the vector field $\phi(x)\alpha$, that is, by the system

$$\frac{dx}{dt}=\phi(x)\alpha.$$

The new flow has the same orbits as $R_{t\alpha}$ and preserves a measure equivalent to the Haar measure given by the density $\frac{1}{\phi}$. Moreover, if $R_{t\alpha}$ is ergodic then so is the reparametrized flow.

Given a dynamical system (f,M,μ) and a function $\varphi\in L^1(M)$, $\varphi>c>0$, the *special flow constructed over the diffeomorphism f and under the function φ* is the quotient flow of the action

$$\begin{array}{ccc} M\times\mathbb{R} & \longrightarrow & M\times\mathbb{R} \\ (x,s) & \longrightarrow & (x,s+t) \end{array}$$

by the relation $(x,s+\varphi(x))\sim(f(x),s)$. This flow acts on the manifold $M_{f,\varphi}=\mathbb{T}^n\times\mathbb{R}/\sim$, and preserves the normalized Lebesgue measure on $M_{f,\varphi}$, i.e. the product of the invariant measure on the base μ with the Lebesgue measure on the fibers divided by the constant $\int_M\varphi(x)d\mu(x)$. The function φ, that measures the time needed by a point on the base to return to it, is called the *ceiling function*. We will denote the special flow over f and under φ by $T^t_{f,\varphi}$.

As an example, one can view the flow $R_{t(\alpha,1)}$ as a special flow over the translation R_α and under the constant function equal to one. There is a natural correspondance between reparametrizations of translation flows and special flows over discrete translations (see for example [2] Chapter 16).

2.9. *The successive conjugations techniques* [1]. We give very briefly and in a particular frame the general scheme of the thechnique introduced in [1]: M is a manifold admitting a circle action S^t, α is a Liouvillian number and we consider the map f_0 equal to S^α. Then we construct inductively a sequence of diffeomorphisms h_n such that $h_n \circ f_0 \circ h_n^{-1}$ converges to some diffeomorphism f in the C^∞ topology while the fact that the sequence h_n itself does not converge allows f to have an *exotic* behavior.

Applied to the rotation R_α on the circle this technique provides with examples of smooth diffeomorphisms of the circle with rotation number α with almost every possible degree of regularity of the conjugacy to R_α (Cf. [12] Chapter 6).

Applied on a general compact manifold M admitting a circle action (for example the disc) with a choice of volume preserving conjugacies h_n one can obtain volume preserving diffeomorphisms f of M with preselected metric behavior such as ergodicity or more specifically isomorphism to irrational translations on tori (with arbitrarilly fixed number of frequencies, even ∞) or weak mixing (Cf. [1]).

In relation with our subject, observe that the latter constructions show that the relation can be loose between the metric behavior of a diffeomorphism and the topological type of the manifold. Moreover, by the same techniques it was shown in [8] that on any smooth compact manifold admitting a non trivial circle action there exists a weak mixing diffeomorphism preserving a measurable Riemannian metric and a C^∞ measure.

Using the method of alternation of stretch of [5] one can now apply the successive conjugations techniques on manifolds admitting a $\mathbb{T}^2$ action and get smooth volume preserving diffeomorphisms that are mixing (Cf. §3.2).

3. Some examples.

3.1. *Minimal non-ergodic systems.* We describe in this paragraph the classical example of Furstenberg of a minimal non-ergodic skew product on $\mathbb{T}^2$. Denote by μ the Haar measure on $\mathbb{T}^2$. Given a function $\varphi : \mathbb{T}^1 \to \mathbb{R}$, we consider the dynamical system $(S_{\alpha,\varphi}, \mathbb{T}^2, \mu)$ given by $S_{\alpha,\varphi}(x, y) = (x+\alpha, y+\varphi(x))$. It is clear that the system $(S_{\alpha,0}, \mathbb{T}^2, \mu)$ is not ergodic. If α is irrational then the restriction of $S_{\alpha,\varphi\equiv 0}$ to any circle $y = y_0$ is uniquely ergodic. In general, if $\varphi(x) = \psi(x + \alpha) - \psi(x)$ for some measurable function $\psi : \mathbb{T}^1 \to \mathbb{R}$ then one can easily check that $h : \mathbb{T}^2 \to \mathbb{T}^2$, $h(x, y) = (x, y+\psi(x))$ is an isomorphism between $(S_{\alpha,0}, \mathbb{T}^2, \mu)$ and $(S_{\alpha,\varphi}, \mathbb{T}^2, \mu)$. It then follows that the latter system is not ergodic and that its invariant ergodic measures are supported on the graphs of $\psi + y_0$ for $y_0 \in \mathbb{R}$. Hence, to insure that $S_{\alpha,\varphi}$ is minimal one needs to check that h sends each circle $y = y_0$ into a dense subset of $\mathbb{T}^2$. This can be achieved if ψ is highly discontinuous, sending for instance any interval in $\mathbb{T}^1$ into a dense set of $\mathbb{R}$. Such a solution ψ to the above linear equation is sometimes called a wild coboundary (Cf. [7] or [12] Propositions 4.2.5, 4.2.6).

THEOREM. *Let α be an irrational number exponentially well approximated by rationals. Then there exists a real-analytic function φ over the circle with mean value zero such that the equation*

$$\varphi(x) = \psi(x + \alpha) - \psi(x)$$

admits a solution ψ that is measurable but not continuous.

In this case the skew product $S_{\alpha,\varphi} : (x, y) \to (x + \alpha, y + \varphi(x))$ is minimal and isomorphic to S_0 hence non-ergodic.

Since $S_{\alpha,\varphi}$ is a skew product over an isometry it is clearly not topologically mixing. It is easy to see that $S_{\alpha,\varphi}$ is even rigid, in the sense that $S^{q_n}_{\alpha,\varphi} \to \mathrm{Id}_{\mathbb{T}^2}$ uniformly along somme sequence q_n going to infinity (one can check that the sequence q_n of denominators of the best approximations of α is a rigidity sequence whenever φ is of class C^1).

3.2. *Minimality and mixing.* There are not too many examples of systems combining minimality and topological mixing. One can mention the classical examples of Horocyclic flows on homogeneous spaces and more recently some special reparametrizations of Liouvillian translation flows on $\mathbb{T}^d$, $d \geq 3$ [5]. Both these examples are volume preserving and uniquely ergodic.

The following is an open problem: *Does there exist a diffeomorphism of class C^r (≥ 1) on $\mathbb{T}^2$ that is minimal and topologically mixing?*

A harder and even more interesting problem is to find examples of diffeomorphisms on $\mathbb{T}^2$ that *preserve the Haar measure, are uniquely ergodic and have the mixing property* (Cf §2.6).

Mixing features can be gained by a transitive system if we consider special flows above the system. Even when the initial system is rigid it is possible that for the special flow a vertical shear appears and introduces mixing properties if the base is ergodic and topological mixing properties if the base is only transitive. Rigidity on the base becomes itself responsible for accumulation of shear in the vertical direction as time runs to infinity and large multiples of rigidity times of the map on the base become times of mixing for the special flow. In fact any rigid ergodic diffeomorphism admits a smooth weak mixing special flow above it (Cf. [11]). Nevertheless, the special flows over rigid transformations tend to conserve the rigidity property, which is not incompatible with being weak mixing (both rigidity and weak mixing are G^δ properties) Cf. [3], [11], [14] and Theorem 2 in [5].

More has to be satisfied by the map on the base if one requires mixing of the special flow. For a translation $R_{\beta,\beta'}$ on $\mathbb{T}^2$ we can choose (β, β') such that strong rigidity times alternate and never overlap in one and the other direction on $\mathbb{T}^2$, then by an adequate choice of the ceiling function ψ it is possible to obtain a mixing special flow $T^t_{R_{\beta,\beta'},\psi}$ (Cf. [17], [5]). Such a special flows can be viewed as a real-analytic time-change of the irrational linear flow $R_{t(\beta,\beta',1)}$ on $\mathbb{T}^3$. Summarizing, we have

THEOREM. *There exists real-analytic uniquely ergodic volume preserving mixing flows on* $\mathbb{T}^3$.

3.3. *Minimal and topologically mixing non-ergodic systems.* In order to obtain a minimal and topologically mixing non-ergodic system it is hence natural to consider suspensions over minimal non-ergodic systems. Such systems are provided by Furstenberg's skew products $S_{\alpha,\varphi}$ described in §3.1. With the fact that mixing special flows over minimal translations are also available (see the above section) our task becomes easier. Indeed one considers the direct product of $S_{\alpha,\varphi}$ with a minimal translation on $\mathbb{T}^2$ of vector β, β' chosen as above with the additional property of being independent with α over $\mathbb{Q}$. The product $f = S_{\alpha,\varphi} \times R_{\beta,\beta'}$ will then be minimal and not ergodic as well as the special flow $T^t_{f,\psi}$ where ψ depends only on two variables and is chosen as in [5] to guarantee that the flow is topologically mixing [4]. Summarizing, we have

THEOREM. *There exists a real-analytic volume preserving minimal and non-ergodic topologically mixing flow on* $\mathbb{T}^5$.

3.4. *Transitive systems with a positive measure set of nondense orbits.* If f is topologically transitive on a complete separable metric space M it is easy to see that the set of points with a dense orbit is a set of first category, i.e. a dense G^δ, call it $\mathcal{T}(f)$. On the other hand, if a volume preserving system (f, M, μ) is ergodic then $\mathcal{T}(f)$ has full measure. It is thus a natural question to ask how small can the measure of $\mathcal{T}(f)$ be for a non-ergodic volume preserving topologically transitive system. To answer this question, in a work in preparation with A. Windsor we use the techniques of successive conjugations introduced by [1] on a manifold where the transverse direction to the circle action has dimension greater than 2. In particular, it is possible to show the following

THEOREM. *For any $\epsilon > 0$, there exists a measure preserving diffeomorphism on $\mathbb{T}^3$ that is topologically transitive and such that the measure of the set of points with a dense orbit is less than ϵ.*

We think that with the same techniques of successive conjugations, it should be possible to produce a topoligcally transitive diffeomorphism such that $\mu(\mathcal{T}(f)) = 0$.

Examples of transitive maps with a set $\mathcal{T}(f)$ of zero measure were known to exist for diffeomorphisms that do not presevre volume

3.5. *Topological accessibility without essential accessibility for partially hyperbolic systems.* In this paragraph, we sketch an application of the foregoing construction 3.4 to the theory of partially hyperbolic systems.

A system (f, M) is said to be partially hyperbolic if there is a *df*-invariant continuous splitting of the tangent bundle of M: $T_xM = E^s(x) + E^c(x) + E^u(x)$, such that the map *df* is uniformly contracting in E^s, uniformly expanding in E^u and E^c is uniformly dominated by E^s and E^u, i.e., for every $x \in M$ and all unit vectors $v_s \in E^s(x), v_c \in E^c(x)$, and $v_u \in E^u(x)$, we have

$$||d_x f v_s|| < ||d_x f v_c|| < ||d_x f v_u||.$$

For each point $x \in M$, there are stable $W^s(x)$ and unstable $W^u(x)$ manifolds that are tangent respectively to $E^s(x)$ and $E^u(x)$ and such that the forward orbit of a point on $W^s(x)$ approaches exponentially the forward orbit of x while the backward orbit of a point on $W^u(x)$ approaches the backward orbit of the point x. These manifold form the so called stable and unstable foliations which are invariant under f. A point $z \in M$ is said to be accessible from a point $x \in M$ if there is a sequence $x = x_0, x_1, ..., x_n = z$ such that for every n we have $x_i \in W^s(x_{i-1})$ or $x_i \in W^u(x_{i-1})$. Accessibility is an equivalence relation and a partially hyperbolic system is said to have the accessibility property if there is only one class of accessibility (any point is accessible from any other point) and is said to have the *essential accessibility property* if there is a class of accessibiltiy of full measure. We will say that f has the *topological accessibility property* if there is a class of accessibility that is dense (given any two open sets we can access some point of one from some point of the other).

It is a highlight of the ergodic theory of volume preserving partially hyperbolic systems that under some restrictions on the central bundle E^c an accessibility class is up to a null set contained in a single ergodic component In particular, a topologically accessible volume preserving partially hyperbolic system is transitive.

Starting from a direct product of the map described in §3.4 and that we will denote by f with a hyperbolic automorphism of the torus A and applying a local perturbation that creates accessibility in some small open set of $\mathbb{T}^5$ we get a partially hyperbolic system that has an open and dense class of accessibility due to the transitivity of f but is not ergodic since f is not. The same technique can yield an example of a volume preserving diffeomorphism having an open and dense ergodic component of measure less than ϵ on which it is Bernoulli and has all its Lyapunov exponents non zero. We call such an ergodic component a non uniformly hyperbolic one. We just mention here, without proof, the following

THEOREM. *Given any $\epsilon > 0$, there exists a partially hyperbolic volume preserving system on $\mathbb{T}^5$ that is topologically accessible and has an open* *and dense non uniformly hyperbolic ergodic component of measure less than ϵ.*

3.6. *Intermingled attractors.* Given a system (f, M, μ), we call a set A an attractor for f if the set $B(A) = \{x \in M/\omega(x) = A\}$ has positive measure (the ω-limit set of x being the set of accumulation points of the forward orbit of the point x).

Two sets B and B' are said to be intermingled if their union has full measure and if any open set of M contains a Lebesgue density point of each of B and A.

A system is said to have two intermingled attractors A and A' if their basins of attraction $B(A)$ and $B(A')$ are intermingled [15]. Attractors with intermingled basins were constructed in [10].

Based on the existence of a measure preserving diffeomorphism on $\mathbb{T}^2$ with exactly two intermingled absolutely continuous ergodic components (Cf. [16]) we will deduce an easy construction of a system having two intermingled attractors. Recall from [16]

THEOREM. *There exists a minimal diffeomorphism of $\mathbb{T}^2$ preserving Haar measure which decomposes into exactly two absolutely continuous ergodic components.*

Consider now a function $\lambda \in C^\infty(\mathbb{T}^2, \mathbb{R})$ with average zero with respect to Haar measure and with distinct averages on the above ergodic decomposition μ_1 and μ_2. Clearly $\int_{\mathbb{T}^2} \lambda(x) d\mu_1(x) \int_{\mathbb{T}^2} \lambda(x) d\mu_2(x) < 0$.

Let $\Lambda : \mathbb{T}^2 \to \mathrm{SL}(2, \mathbb{R})$ be the diagonal matrix function $\Lambda(x) = \begin{pmatrix} e^{\lambda(x)} & 0 \\ 0 & e^{-\lambda(x)} \end{pmatrix}$ and consider on $\mathbb{T}^2 \times \mathbb{R}^2$ the fibered diffeomorphism F associated to this matrix and to the map S coming from the above proposition: $(x, v) \to (Sx, \Lambda(x)v)$. Since $SL(2, \mathbb{R}) \to PSL(2, \mathbb{R})$ acts canonically on $\mathbb{P}_1(\mathbb{R}^2)$ from the left we can associate to the skew product F the following diffeomorphism f of $\mathbb{T}^2 \times \mathbb{P}_1(\mathbb{R}^2)$: $(x, \theta) \to (S(x), \Lambda(x)\theta)$. Then we will have two points on $\mathbb{P}_1(\mathbb{R}^2)$ S and N such that

THEOREM. *The map S_λ has two intermingled attractors: $\mathbb{T}^2 \times \{S\}$ and $\mathbb{T}^2 \times \{N\}$.*

The reason is that for any $\theta \in \mathbb{P}_1(\mathbb{R}^2)$ and for μ_1-a.e. $x \in \mathbb{T}^2$ we have $\omega(x, \theta) = \mathbb{T}^2 \times \{0\}$ and a similar fact for μ_2 and $\mathbb{T}^2 \times \{1/2\}$.

Remark. It is not excluded that for some choice of λ the map f might in addition be transitive.

3.7. *Rotation set.* Assume f is a map on the torus $\mathbb{T}^d$ homotopic to identity. A topological invariant of f is its rotation set or more precisely all the accumulation points in $\mathbb{R}^d$ of $1/n(\tilde{f}^n(x) - x)$ as $x \in \mathbb{T}^d$ and $\tilde{f}$ is a lift of f to $\mathbb{R}^d$.

3.7.1. It is known that a minimal diffeomorphism on $\mathbb{T}^2$ must have its rotation set included in an interval but it is an open question whether it should be reduced to a single point or not. In this paragraph we sketch a construction of a minimal flow on $\mathbb{T}^3$ whose rotation set in $\mathbb{R}^3$ is not reduced to a single point. Like the foregoing construction, it is based on the existence of a volume preserving diffeomorphism S on $\mathbb{T}^2$ with exactly two absolutely continuous ergodic components [16]. This time we consider a special flow above S with the same function λ used in §3.6 and obtain the following

THEOREM. *There exists a minimal flow on $\mathbb{T}^3$ with a rotation set not reduced to a point in $\mathbb{R}^3$.*

In fact the rotation set of the special flow $T^t_{S,\lambda}$ contains an interval of the type $[(\alpha, 0, \int_{\mathbb{T}^2} \lambda(x) d\mu_1(x)), (\alpha, 0, \int_{\mathbb{T}^2} \lambda(x) d\mu_2(x))]$, where $(\alpha, 0)$ is the unique rotation vector of the map S (S has a unique rotation vector because it is rigid since it is obtained via the successive conjugations techniques applied to a circle action as in [1]).

3.7.2. The rotation set is invariant under topological conjugacy but can vary under isomorphism. This can be observed using the following construction by Fayad Katok and Windsor (to appear as a sequel for [6])

THEOREM. *There exists $\alpha \in \mathbb{R} - \mathbb{Q}$ and $\varphi \in C^\infty(\mathbb{T}, \mathbb{R})$ and $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}\int_{\mathbb{T}} \varphi(x)dx$ such that the equation*

$$e^{2\pi i(\varphi(x)+\beta)}\psi(x) = \psi(x+\alpha)$$

has a measurable solution $\psi : \mathbb{T} \to \mathbb{C}$.

Considering the skew product $S_{\alpha,\varphi}$ on $\mathbb{T}^2 : (x, y) \to (x+\alpha, y+\varphi(x))$, we get the following

COROLLARY. *The diffeomorphism $S_{\alpha,\varphi}$, whose rotation set is reduced to a single vector $(\alpha, \int_{\mathbb{T}} \varphi(x)dx)$, is isomorphic to the minimal translation on $\mathbb{T}^2$ $R_{\alpha,\beta}$.*

4. Topologically mixing realizations of linear flows.

A prototype example where the topological structure of a dynamical system determines its ergodic properties that appears in hyperbolic dynamics is the following: *A volume preserving smooth Anosov flow that is topologically mixing is Bernoulli.* We will show how this is no longer true in the elliptic frame by constructing a topologically mixing volume preserving flow of the torus that is uniquely ergodic and isomorphic to a linear translation flow.

The flow we will construct is a special flow $T^t_{R_\alpha,\phi}$ over a minimal translation R_α on a $d-$dimensional torus and under a function ϕ with any given regularity r. The dimension d required by the construction is of the order of $r \ln r$. The flow $T^t_{R_\alpha,\phi}$ will result to be isomorphic to the translation flow $T^t_{R_\alpha,1}$ because we choose α and ϕ such that ϕ

is cohomologous to 1 above R_α (i.e. $\phi(x) - 1 = \psi(x+\alpha) - \psi(x)$, with a transfer function $\psi \in L^2(\mathbb{T}^d, \mathbb{R})$. On the opposite hand, the flow will be topologically mixing because we choose α and ϕ such that the Birkhoff sums over R_α, $S_m\phi(x) := \phi(x) + \phi(x+\alpha) ... + \phi(x + (m-1)\alpha)$ are *stretching* as $m \to \infty$, for *all* $m \in \mathbb{N}$. Roughly speaking, we say that $S_m\phi$ are stretching on a subset $M \subset \mathbb{N}$ if as $m \to \infty$, $m \in I$, we have that $S_m\phi$ display large oscillations over an increasingly dense collection $\mathcal{I}_m$ of small open sets of $\mathbb{T}^d$. We can then show that M is a subset of topological mixing times of the flow $T^t_{R_\alpha,\phi}$, obtaining thus topologically mixing if $M = \mathbb{N}$. Naturally, the total measure of the sets in $\mathcal{I}_m$ must go to zero as m goes to infinity since ϕ is an L^2 coboundary; but this does not preclude topological mixing which is in this particular construction quite invisible from the metric point of vue.

4.0.3. *Plan of the construction.* We first fix the regularity $r \in \mathbb{N}$ of the examples we want to construct. The construction is split into two parts. In the first part, we fix a class of irrational numbers α for which we are able to construct a function ϕ over the circle such that: (i) $\varphi \in C^r(\mathbb{T}, \mathbb{R})$; (ii) φ is an L^2 coboundary over R_α; (iii)The Birkhoff sums over R_α $S_m\varphi$ are stretching on $\bigcup M_n$, where M_n is an increasing sequence of intervals of integers (the sequence being necessarily lacunary since we are working in dimension 1). The function φ is chosen so that the increasing stretch as $m \subset M_n$, $n \to \infty$ occurs only on sets with relative measure going to zero. The class of α considered essentially consists of irrationals for which the sequence of denominators q_n of their best approximations geometric with an exponent chosen such that the intervals M_n in (iii) be the largest possible under the constraints (i) and (ii).

4.1. *Notations.* Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $r \in \mathbb{N} \bigcup \{+\infty\}$. We denote by $C^r(\mathbb{T}^d, \mathbb{R})$ the set of real functions defined on $\mathbb{R}^d$ and $\mathbb{Z}^d$ periodic, of class C^r. By $||.||_{C^r}$ we denote a norm on $C^r(\mathbb{T}^d, \mathbb{R})$. By $||.||_{L^2}$ we denote the L^2 norm $\left(\int_{\mathbb{T}^d} |\varphi(x)|^2 dx\right)^{\frac{1}{2}}$.

For a real number x we denote by:

$-[x]$ the integer part of α,

$-\{x\} = x - [x]$ its fractional part,

$-|||x||| = \min(\{x\}, 1 - \{x\})$ the distance of x to the closest integer.

When we write $\frac{p}{q} \in \mathbb{Q}$, we assume that $q \in \mathbb{N}$, $q \geq 1$, $p \in \mathbb{Z}$ and that p and q are relatively prime.

For any irrational real number α there exists a sequence of rationals $\{\frac{p_n}{q_n}\}_{n\in\mathbb{N}}$, called the convergents of α defined by $q_0 = 1$ and

$$|||q_n\alpha||| < |||k\alpha|||, \quad \text{for all } k < q_{n+1}. \tag{1}$$

For any n we have

$$(2) \qquad \frac{1}{q_n(q_n+q_{n+1})} \le (-1)^n(\alpha - \frac{p_n}{q_n}) \le \frac{1}{q_nq_{n+1}}.$$

Furthermore, if we assume that n is even and that in this case $\alpha - \frac{p_n}{q_n} > 0$ then we must have

$$(3) \qquad \left|\alpha - \frac{p_n}{q_n} - \frac{1}{q_nq_{n+1}}\right| < \frac{1}{q_{n+1}q_{n+2}}.$$

4.2. **Wild coboundaries above rotations of the circle.**

In this section we assume that that the class of differentiability r in which we want to build our example is fixed, $r \ge 4$, and that α is an irrationnal number with its sequence of rational approximations p_n/q_n satisfying

$$(4) \qquad q_n^{2r+\frac{5}{2}-0.1} \le q_{n+1} \le q_n^{2r+\frac{5}{2}}.$$

The goal of the section is to prove the following

4.2.1. PROPOSITION. *Under condition (4) on α, there exists a function $\varphi \in C^r(\mathbb{T}^1,\mathbb{R})$ satisfying:*
i) There exists a solution $\psi \in L^2(\mathbb{T}^1,\mathbb{R})$ to the equation

$$\varphi(x) = \psi(x+\alpha) - \psi(x).$$

ii) There exists a sequence of positive numbers $K_n \to \infty$ such that for every $m \in [q_n^{2r+1+0.1}, q_n^{2r+3/2-0.2}]$ and every interval $I_{n,k} = [k/q_n + 1/4q_n^2, k/q_n + 1/2q_n^2]$, $k \le q_n - 1$, we have for every $x \in I_{n,k}$

$$S_m\varphi'(x)|I_{n,k}| = S_m\varphi'(x)\frac{1}{4q_n^2} \ge K_n.$$

4.2.2. *Construction of the function φ.* Let θ be a smooth real function defined on $\mathbb{R}$ with the following properties:

$$(5) \qquad \theta(x) = 0 \text{ for } x \in (-\infty,0]\cup[1,\infty),$$

$$(6) \qquad \theta'(x) = 1 \text{ for } x \in [\frac{1}{8},\frac{3}{4}],$$

$$(7) \qquad \int_0^1 \theta(x)dx = 0,$$

For every $n \in \mathbb{N}$, define

$$\varphi_n(x) := \frac{1}{e^{2n}q_n^{2r}}\theta(q_n^2x)$$

for $x \in [0,1)$ that we extend by periodicity to a function $\varphi_n \in C^\infty(\mathbb{T}^1,\mathbb{R})$.
The factor q_n^2 will be explained in §4.2.3.

Let

$$\varphi(x) := \sum_{n=1}^{\infty} \varphi_n(x).$$

A straightforward derivation gives $||\varphi_n||_{C^r} \leq e^{-2n}||\theta||_{C^r}$ which implies that $\varphi \in C^r(\mathbb{T}^1, \mathbb{R})$. The rest of this section is devoted to prove that φ satisifies Proposition 4.2.1.

4.2.3. *Plan of the proof.* The right hand side of (4), which is a Diophantine condition on α, and the fact that $\varphi_n \in C^\infty(\mathbb{T}^1, \mathbb{R})$ imply that there exists a solution $\psi_n \in C^\infty(\mathbb{T}^1, \mathbb{R})$ to

$$\varphi_n(x) = \psi_n(x+\alpha) - \psi_n(x). \tag{8}$$

The factor q_n^2 in the expression of φ_n allows to localize the oscillations of the Birkhoff sums $S_m\varphi_n$ on essentially q_n intervals of length $1/q_n^2$, hence on a set of relative measure $1/q_n$. With this choice for φ_n, the arithmetic condition (4) will allow to have an upper bound on $||\psi_n||_{L^2}$ that will imply (i) of Proposition 4.2.1 together with an optimal range of stretch.

We set $\psi_n(0) = 0$. To bound $||\psi_n||_{L^2}$, one might turn to Fourier expansions and try to use (4) but the nature of the construction is such that φ is *at the boundary* of being L^2 since we required that the sums $S_m\varphi$ have large oscillations for large intervals of time (condition (ii)). For example, a direct and rough estimation of the Fourier coefficient $\hat{\psi}_n(q_n)$ gives

$$\begin{aligned} |\hat{\psi}_n(q_n)| &= \frac{1}{|1 - e^{i2\pi q_n \alpha}|}|\hat{\varphi}_n(q_n)| \\ &\leq q_{n+1}q_n^{-2r-2}, \end{aligned}$$

which using (4) does not even imply that $|\hat{\psi}_n(q_n)|$ is small.

Instead, we will use the fact that $\psi_n(m\alpha) = S_m\varphi_n(0)$. With the shape of φ_n it will be easy to estimate the latter sums *by hand* up to $m = q_{n+1}$ (Lemma 4.2.6). Since the first q_{n+1} points of the orbit of 0 under R_α give a subdivision of the circle with essentially intervals of size $1/q_{n+1}$ (Cf. (11)) we can then interpolate to estimate ψ_n over all the circle (Proposition 4.2.7) using the fact that $||\psi_n||_{C^1}$ is bounded away from q_{n+1}, a fact that easily follows this time from Fourier expansions (Cf. §4.2.4).

To prove (ii) we will first obtain a lower bound on $S_m\varphi_n'$ on the corresponding intervals $I_{n,k}$ and for the corresponding range on m (Lemma 4.2.8). This will again be obtained *by hand* using the shape of φ_n and (11). At this level the higher frequencies $S_m\varphi_l'$, $l > n$ are still small and we only need to avoid interferences with the lower frequencies $S_m\varphi_l'$ $l < n$. For this, we use the bound on $||\psi_l||_{C^1}$ obtained in §4.2.4 which clearly implies a uniform bound (independent on m) of $||S_m\varphi_l||_{C^1}$.

4.2.4. LEMMA. *For every n large enough, we have*

$$||\psi_n||_{C^1} \leq \frac{q_{n+1}}{q_n^2}.$$

Proof. Using Fourier expansions

$$\begin{aligned}||\psi_n||_{C^1} &\leq \sum_{k\in\mathbb{Z}} 2\pi|k\hat{\psi}_n(k)| \\ &\leq \sum_{k\in\mathbb{Z}} \frac{2\pi|k|}{|1-e^{i2\pi k\alpha}|}|\hat{\varphi}_n(k)| \\ &\leq \sum_{|k|<q_{n+1}} a_k + \sum_{|k|\geq q_{n+1}} a_k,\end{aligned}$$

where $a_k := \frac{2\pi|k|}{|1-e^{i2\pi k\alpha}|}|\hat{\varphi}_n(k)|$.

For $k < q_{n+1}$, (1) and (2) imply $|||k\alpha||| \geq |||q_n\alpha||| \geq \frac{1}{2q_{n+1}}$ hence

$$\begin{aligned}\sum_{|k|<q_{n+1}} a_k &\leq 2\pi q_{n+1} \sum_{|k|<q_{n+1}} |k||\hat{\varphi}_n(k)| \\ &\leq Cq_{n+1}||\varphi_n||_{C^3}, \qquad (9)\end{aligned}$$

where C is some constant independent on n.

For $k \geq q_{n+1}$, we use the Diophantine condition in (4) that implies for every $k \in \mathbb{N}$

$$|||k\alpha||| \geq |k|^{-(2r+\frac{5}{2})},$$

hence

$$\begin{aligned}\sum_{|k|\geq q_{n+1}} a_k &\leq \sum_{|k|\geq q_{n+1}} \pi|k|^{2r+\frac{7}{2}}|\hat{\varphi}_n(k)| \\ &\leq \sum_{|k|\geq q_{n+1}} \frac{\pi}{|k|^{\frac{5}{2}}}|k|^{2r+6}|\hat{\varphi}_n(k)| \\ &\leq C'\frac{1}{(q_{n+1})^{\frac{3}{2}}}||\varphi_n||_{C^{2r+6}}, \qquad (10)\end{aligned}$$

where C' is some constant independent on n.

By definition of φ_n we have $||\varphi_n||_{C^l} \leq e^{-2n}q_n^{2l-2r}||\theta||_{C^l}$. Hence (9) and (10) together with (4) imply

$$||\psi_n||_{C^1} \leq Cq_{n+1}e^{-2n}q_n^{6-2r}||\theta||_{C^3} + C'q_{n+1}e^{-2n}q_n^{2r+12}q_n^{-\frac{5}{2}(2r+\frac{5}{2}-0.1)}||\theta||_{C^{2r+6}}$$

which implies the bound on $||\psi_n||_{C^1}$ claimed in the Lemma since we took $r \geq 4$. □

4.2.5. We will always assume n even, the other case being similar. From (3) we have for any $m \leq q_{n+1}$, if we write $m = lq_n + j$, $j < q_n$

$$|||m\alpha - j\frac{p_n}{q_n} - \frac{l}{q_{n+1}}||| < \frac{1}{q_{n+1}}. \tag{11}$$

Define

$$\begin{aligned} C_n &= [-\frac{2}{q_{n+1}}, \frac{2}{q_{n+1}}], \\ L_n &= \bigcup_{j \leq q_n - 1} [\frac{j}{q_n}, \frac{j}{q_n} + \frac{1}{q_n^2}], \\ R_n &= \bigcup_{j \leq q_n - 1} [\frac{j}{q_n} + \frac{1}{q_n^2}, \frac{j+1}{q_n}]. \end{aligned}$$

LEMMA. *We have*

$$L_n \subset \bigcup_{0 \leq m \leq q_{n+1}/q_n} R_\alpha^m(C_n) \tag{12}$$

and

$$R_n \subset \bigcup_{q_{n+1}/q_n \leq m \leq q_{n+1}} R_\alpha^m(C_n). \tag{13}$$

Proof. From (11) we have

$$\begin{aligned} &[jp_n/q_n + l/q_{n+1} - 1/q_{n+1}, jp_n/q_n + l/q_{n+1} + 1/q_{n+1}] \subset R_\alpha^{lq_n+j}(C_n) \\ &\qquad \subset [jp_n/q_n + l/q_{n+1} - 3/q_{n+1}, jp_n/q_n + lq_{n+1} + 3/q_{n+1}], \end{aligned}$$

and the lemma easily follows. □

4.2.6. LEMMA. *For n large enough, we have*
For any $m \in \mathbb{N}$,

$$|S_m\varphi_n(0)| \leq (\frac{m}{q_n} + 1)e^{-2n}q_n^{-2r}||\theta||_{C^0} \tag{14}$$

For any $\frac{q_{n+1}}{q_n} \leq m \leq q_{n+1}$,

$$|S_m\varphi_n(0)| \leq e^{-2n}q_n^{2-2r}||\theta||_{C^1} \leq e^{-2n}. \tag{15}$$

Proof. Since φ_n is identically zero outside the segment $[0, 1/q_n^2]$, the first equation follows from the fact that if $\{k_i\alpha\} \in [0, 1/q_n^2]$, for $i = 1, 2$, then $|k_1 - k_2| \geq q_n$ since $|||l\alpha||| \geq 1/2q_n$ if $l < q_n$.

Using (5) and (11) again, we get that $\varphi_n(m\alpha) = 0$ for any $q_{n+1}/q_n \leq m \leq q_{n+1}$, hence $S_m\varphi_n(0) = S_{q_{n+1}}\varphi_n(0)$.

But Denjoy-Koksma inequality implies that

$$|S_{q_{n+1}}\varphi_n(0) - q_{n+1}\int_{\mathbb{T}}\varphi_n(x)dx| \leq ||\varphi_n||_{C^1},$$

and (15) follows since $\int_{\mathbb{T}}\varphi_n(x)dx = 0$. □

4.2.7. We can give now the proof of
PROPOSITION. *For every n large enough we have*

$$||\psi_n||_{L^2} \leq e^{-n}. \tag{16}$$

Recall the definition of L_n and R_n in Lemma 4.2.5. We have
CLAIM.

$$|\psi_n(x)| \leq 2e^{-2n}q_n^{\frac{1}{2}} \quad \text{for every } x \in L_n, \tag{17}$$

$$|\psi_n(x)| \leq e^{-2n} \quad \text{for every } x \in R_n. \tag{18}$$

Proof of the claim. From Lemma 4.2.4 we have for $x \in R_\alpha^m C_n$

$$|\psi_n(x) - \psi_n(m\alpha)| \leq \frac{4}{q_{n+1}}||\psi_n||_{C^1} \leq \frac{4}{q_n^2}. \tag{19}$$

Recall that since we took $\psi_n(0) = 0$ we have

$$\psi_n(m\alpha) = S_m\varphi_n(0). \tag{20}$$

Now, if $x \in L_n$, we know from (12) that $x \in R_\alpha^m(C_n)$ for some $m \leq q_{n+1}/q_n$, and (14) then implies

$$|\psi_n(m\alpha)| \leq (\frac{q_{n+1}}{q_n^2} + 1)e^{-2n}q_n^{-2r}||\theta||_{C^0} \leq e^{-2n}q_n^{\frac{1}{2}},$$

hence with (19) we get (17).

If $x \in R_n$, (13) implies that $x \in R_\alpha^m(C_n)$ for some $q_{n+1}/q_n \leq m \leq q_{n+1}$, then (15) implies

$$|\psi_n(m\alpha)| \leq e^{-2n}q_n^{2-2r}||\theta||_{C^1} = o(e^{-2n}),$$

which yields (18) due again to (19). □

Proof of Proposition 4.2.7. Since $L_n \cup R_n = \mathbb{T}^1$, we have

$$\begin{aligned} ||\psi_n||_{L^2}^2 = \int_{\mathbb{T}^1}\psi_n^2(x)dx &= \int_{L_n}\psi_n^2(x)dx + \int_{R_n}\psi_n^2(x)dx \\ &\leq \frac{4}{q_n}4e^{-4n}q_n + e^{-4n} \\ &\leq 17e^{-4n}. \end{aligned}$$

□

We now turn to assessing the stretching of $S_m\varphi$ on the intervals $I_{n,k} = [k/q_n + 1/4q_n^2, k/q_n + 1/2q_n^2]$, $k \leq q_n - 1$.

4.2.8. Lemma. *For $m \le \frac{1}{10}q_{n+1}/q_n$, and $x \in I_{n,k}$, we have*

$$S_m\varphi_n'(x) \ge (\frac{m}{q_n} - 1)e^{-2n}q_n^{2-2r}. \tag{21}$$

Proof. For every $m = lq_n + j \le \frac{1}{10}q_{n+1}/q_n$, i.e. $l \le \frac{1}{10}q_{n+1}/q_n^2$, it follows from (11) that

$$R_\alpha^m I_{n,k} \subset \left[\frac{jp_n + k}{q_n} + \frac{1}{8q_n^2}, \frac{jp_n + k}{q_n} + \frac{3}{4q_n^2}\right]. \tag{22}$$

From the definition of φ_n in §4.2.2 we have that the derivative of φ_n is constant equal to $e^{-2n}q_n^{2-2r}$ on the interval $[\frac{1}{8q_n^2}, \frac{3}{4q_n^2}]$ and equal to zero outside the interval $[0, \frac{1}{q_n^2}]$. Hence, (22) implies that for $x \in I_{n,k}$, $\varphi_n'(x + m\alpha)$ is equal to $e^{-2n}q_n^{2-2r}$ if $jp_n + k$ is a multiple of q_n and to 0 if not, which yields (21). □

4.2.9. Proposition *For every $m \in [q_n^{2r+1+0.1}, \frac{1}{10}q_{n+1}/q_n]$, for every $x \in I_{n,k}$, we have for n large enough*

$$S_m\varphi'(x) \ge \frac{1}{2}e^{-2n}q_n^{2.1}. \tag{23}$$

Proof. In (21), $m \ge q_n^{2r+1+0.1}$ implies

$$S_m\varphi_n'(x) \ge e^{-2n}q_n^{2.1} - e^{-2n}q_n^{2-2r}. \tag{24}$$

But on the one hand, since $\varphi_l(x) = \psi_l(x + \alpha) - \psi_l(x)$, Lemma 4.2.4 implies for any $l, m \in \mathbb{N}$

$$||S_m\varphi_l||_{C^1} \le 2||\psi_l||_{C^1} \le 2\frac{q_{l+1}}{q_l^2},$$

hence for any $m \in \mathbb{N}$ we have

$$\sum_{l \le n-1} ||S_m\varphi_l||_{C^1} \le \sum_{l \le n-1} 2\frac{q_{l+1}}{q_l^2} \le q_n, \tag{25}$$

and on the other hand the direct estimation of $||\varphi_l||_{C^1}$ gives

$$\sum_{l \ge n+1} ||S_m\varphi_l||_{C^1} \le m \sum_{l \ge n+1} e^{-2l}q_l^{2-2r}||\theta||_{C^1} = o(1) \tag{26}$$

for $m \le q_{n+1}$.

The proof of (23) follows from (24)–(26). □

4.2.10. *Proof of Proposition 4.2.1.* Point (i) follows from Proposition 4.2.7 since $\psi := \sum_{n\in\mathbb{N}} \psi_n$ is then an L^2 solution to $\psi(x + \alpha) - \psi(x) = \varphi(x)$. Since (4) implies that $q_n^{2r+3/2-0.2} \le \frac{1}{10}q_{n+1}/q_n$, point (ii) follows from Proposition 4.2.9 if we let $K_n := \frac{1}{8}e^{-2n}q_n^{0.1}$. □

4.3. **High dimensional tori and alternation of stretch.**

4.3.1. *A criterion for topologically mixing.* Consider a special flow $T_{\alpha,\phi}$ constructed over a minimal translation R_α of $\mathbb{T}^d$, $d \geq 2$ and under a function $\phi \in C^1(\mathbb{T}^d, \mathbb{R}_+^*)$. We will give a criterion involving the Birkhoff sums of ϕ above the translation R_α that guaranties topological mixing for the flow $T_{\alpha,\phi}$. It is a mild version of the one used in [5] to guarantee mixing. Likewise, it is based on a mechanism of alternating *stretch* of the Birkhoff sums of the function ϕ, i.e., on the existence of a covering of $\mathbb{N}$ by intervals on each one of which at least one of the quantities $\partial S_m\phi/\partial x_i$ is large. The altrenation of stretch is necessary to obtain topological mixing since the Denjoy Koksma inequality implies that a special flow built over over an irrational rotation of the circle and under a function of bounded distortion is not mixing [13].

PROPOSITION. *If there exist sequences u_l and K_l going to infinity, such that for every $l = nd + i$, $1 \leq i \leq d$ there exists a familly $\mathcal{I}_l$ of intervals of $[0, 1)$ satisfying*

i) *Given any interval $J \in [0, 1)$, there exists for every l large enough $I \in \mathcal{I}_l$ such that $I \subset J$.*
ii) *For every $m \in [\frac{1}{2}u_l, 2u_{l+1}]$, for every $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_d) \in \mathbb{T}^{d-1}$ and for every $I \in \mathcal{I}_l$ we have for any $x \in I$*

$$|\frac{\partial S_m\phi}{\partial x_i}(x_1, ..., x_{i-1}, x, x_{i+1}, x_d)||I| \geq K_l$$

then $T_{\alpha,\phi}$ is topologically mixing.

Proof. The flow $T_{\alpha,\phi}$ is topologically mixing if given any two open sets $\mathcal{O}$ and $\mathcal{V}$ in M_ϕ, there exists t_0 such that for every $t \geq t_0$, $T^t_{\alpha,\phi}\mathcal{O} \bigcap \mathcal{V} \neq$. Let $t \in [u_{nd+i}, u_{nd+i+1}]$ and assume i is equal to 1, the other cases being similar. From i) and ii), if n is large enough there exist an interval $I \subset [0, 1)$, $(x_2, ..., x_d) \in \mathbb{T}^{d-1}$, and $s \in \mathbb{R}$ such that the interval $\overline{I} := I \times \{x_2, ..., x_d, s\}$ satisfies $\overline{I} \in \mathcal{O}$ and for every $m \in [u_{nd+1}/2, 2u_{nd+2}]$, for every $x \in I$

$$|\frac{\partial\phi}{\partial x_1}(x, x_2, ..., x_d)||I| \geq K_{nd+1}.$$

We will finish if we prove that if t is large enough (or equivalently n) then $T^t_{\alpha,\phi}(\overline{I}) \bigcap \mathcal{V}$ is not empty.

Since $\overline{I} = T^s I \times \{x_2, ..., x_d, 0\}$ we can put $s = 0$ in our proof. Recall that for a point $(y, 0) \in \mathbb{T}^d$ we have

$$T^t_{\alpha,\phi}(y, 0) = (R^m_\alpha y, t - S_m\phi(y)) \tag{27}$$

for the unique integer m such that $0 \leq t - S_m\phi(y) < \phi(y + m\alpha)$. We call this integer $m(y, t)$. The following lemma is an immediate consequence of the unique ergodicty of R_α and the fact that ϕ is assumed to be continuous

LEMMA *For t large enough, we have for every $y \in \mathbb{T}^d$, $m(y, t) \in [t/2, 2t]$.*

An immediate consequence of this lemma is that when n is large enough and $t \in [u_{nd+i}, u_{nd+i+1}]$ then $m(y,t) \in [\frac{1}{2}u_{nd+i}, 2u_{nd+i+1}]$ for any point $y \in \mathbb{T}^d$. This holds in particular for the endpoints of $\overline{I}$ that we denote by y_0 and y_1. We will assume that $\partial S_{m(y_0,t)}/\partial x_1$ is negative, the other case being exactly similar.

We claim that $m(y_1,t) - m(y_0,t) \geq CK_{nd+1}$ where C is a constant depending only on $||\phi||$. Indeed, by definition of $m(y,t)$ we have

$$|S_{m(y_0,t)}\phi(y_0) - S_{m(y_1,t)}\phi(y_1)| \leq 2||\phi||,$$

which implies

$$|S_{m(y_0,t)}\phi(y_0) - S_{m(y_0,t)}\phi(y_1) - S_{m(y_1,t)-m(y_0,t)}\phi(y_0 + m(y_0,t)\alpha)| \leq 2||\phi||,$$

but $m(y_0,t) \in [\frac{1}{2}u_{nd+1}, 2u_{nd+2}]$, so (ii) implies

$$|S_{m(y_0,t)}\phi(y_0) - S_{m(y_0,t)}\phi(y_1) \geq K_{nd+1},$$

hence

$$S_{m(y_1,t)-m(y_0,t)}\phi(y_0 + m(y_0,t)\alpha) \geq K_{nd+1} - 2||\phi||,$$

and finally

$$m(y_1,t) - m(y_0,t) \geq \frac{1}{||\phi||}K_{nd+1} - 2$$

as claimed.

For every $m \in \mathbb{N}$ denote by

$$I_m = \{y \in \overline{I} : m(y,t) = m\}.$$

By (27) we have that $T^t_{\alpha,\phi}I_m$ if I_m is not empty is an arc that lies over $R^m_\alpha(I_m) \subset R^m_\alpha\overline{I}$ on the base. For every $m \in [m(y_0,t), m(y_1,t)]$ the set I_m is not empty, furthermore for every $m(y_0,t) < m < m(y_1,t)$ we have that the arc $T^t_{\alpha,\phi}$ over R^m_α connects the base to the ceiling function (it is an almost vertical arc as t goes to infinity). Since from our claim $m(y_1,t) - m(y_0,t) \to \infty$ as $t \to \infty$ and since the rotation on the base is minimal one of these arcs will intersect $\mathcal{U}$ whenever t (or equivalently n) is large enough. □

4.3.2. *Choosing the dimension of the torus.*

To complete the construction the idea is the following: Assume $\alpha = (\alpha_1, ..., \_d)$ is a d-dimensional vector with all its coordinates satisfying (4). Then we can use the precedent section to construct above each α_i a function φ_{α_i} that is an L^2 coboundary and that guarantees topological mixing for $t \in [[q_n(\alpha_i)]^{\gamma_1}, [q_n(\alpha_i)]^{\gamma_2}]$ ($\gamma_1 < \gamma_2$ being given by Proposition 4.2) of the special flow over R_α and under $\phi = \varphi_{\alpha_1} + ... + \varphi_{\alpha_d}$. All we need is that the union of the above intervals cover $\mathbb{R}$. One way is to require that $[q_n(\alpha_{i+1})]^{\gamma_1} \leq [q_n(\alpha_i)]^{\gamma_2}$ with the convetion that $q_n(\alpha_{d+1}) = q_{n+1}(\alpha_1)$. But then we will have $q_{n+1}(\alpha_i) \leq [q_n(\alpha_i)]^{(\frac{\gamma_2}{\gamma_1})^d}$ and the dimension d have to be chosen such that the latter is compatible with (4).

LEMMA. *For r large enough, there exists $d \in \mathbb{N}$ and $\sigma \in \mathbb{R}$ such that*

$$1 < \quad \sigma \quad < \frac{2r + \frac{3}{2} - 0.2}{2r + 1 + 0.1}, \tag{28}$$

$$2r + \frac{5}{2} - 0.09 < \quad \sigma^d \quad < 2r + \frac{5}{2}. \tag{29}$$

Moreover d and σ can be chosen such that

$$d \leq 400r \ln r. \tag{30}$$

Proof. Take $\sigma := 1 + 0.01/2r$. It is easy to check that $1 < \sigma < (2r + \frac{5}{2})/(2r + \frac{5}{2} - 0.09)$ and that this is enough for the existence of d. Then we have

$$d \leq \frac{\ln(2r + \frac{5}{2})}{\ln \sigma} \leq 400r \ln r.$$

□

In all the section we consider $r \geq 4$ fixed and $d \in \mathbb{N}$ such that (28) and (29) are satisfied.

4.3.3. *Choosing a translation on $\mathbb{T}^d$.* Given a vector $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{R}^d$, and for every $1 \leq i \leq d$, we denote by $q_n(\alpha_i)$, $n \in \mathbb{N}$, the sequence of denominators of the convergents of α_i.

PROPOSITION *There exists a vector in $\alpha \in \mathbb{R}^d$ such that for every n*

$$\frac{1}{2}[q_n(\alpha_{i-1})]^\sigma \leq q_n(\alpha_i) \leq [q_n(\alpha_{i-1})]^\sigma \tag{31}$$

for $2 \leq i \leq d$, and

$$\frac{1}{2}[q_n(\alpha_d)]^\sigma \leq q_{n+1}(\alpha_1) \leq [q_n(\alpha_d)]^\sigma. \tag{32}$$

Moreover, under these conditions and (29) it will follow that for every n large enough

$$[q_n(\alpha_i)]^{2r+\frac{5}{2}-0.1} \leq q_{n+1}(\alpha_i) \leq [q_n(\alpha_i)]^{2r+\frac{5}{2}} \tag{33}$$

for every $i = 1, ..., d$.

Proof. We will define the irrational numbers α_i from their sequence of convergents $a_n(\alpha_i)$, $n \in \mathbb{N}$. We recall that the denominators of the convergents of α_i are given by

$$q_{n+1}(\alpha_i) = a_n q_n(\alpha_i) + q_{n-1}(\alpha_i), \quad q_{-1}(\alpha_i) = q_0(\alpha_i) = 1. \tag{34}$$

Using this, we choose $a_0(\alpha_1)$ arbitrarilly and construct the sequence $(a_n(\alpha_2), ..., a_n(\alpha_d), a_{n+1}(\alpha_1))$ inductively so that (31) and (32) be satisfied. Assuming that the inequalities hold at step n we choose in light of (34) $a_{n+1}(\alpha_2), ..., a_{n+1}(\alpha_d), a_{n+2}(\alpha_1)$ successively in order to keep the inequalities true at step $n+1$. Now the right hand side in (31) and (32) imply that

$$q_{n+1}(\alpha_1) \leq [q_n(\alpha_d)]^{\sigma} \leq [q_n(\alpha_{d-1})]^{\sigma^2} \leq ... \leq [q_n(\alpha_1)]^{\sigma^d} \leq [q_n(\alpha_1)]^{2r+\frac{5}{2}}$$

In the same way, using now the left hand side in (31) and (32) we get

$$\begin{aligned} q_{n+1}(\alpha_1) &\geq \frac{1}{2^{1+\sigma+\sigma^2+...+\sigma^{d-1}}}[q_n(\alpha_1)]^{\sigma^d} \\ &\geq \frac{1}{2^{1+\sigma+\sigma^2+...+\sigma^{d-1}}}[q_n(\alpha_1)]^{2r+\frac{5}{2}-0.09} \\ &\geq [q_n(\alpha_1)]^{2r+\frac{5}{2}-0.1} \end{aligned}$$

for n large enough. We have hence proved (33) for $i=1$. The proof for $i>1$ is identical. □

4.3.4. Consider now the special flow over R_α with the ceiling function

$$\phi(x_1,...,x_d) := 1 + \varphi_{\alpha_1}(x_1) + ... + \varphi_{\alpha_d}(x_d),$$

where each of the φ_{α_i} is defined as φ in Section 4.2 with $q_n(\alpha_i)$ instead of q_n. Indeed, Proposition 4.2.1 does apply since each α_i satisfies the condition (4). We will prove the following

THEOREM *The special flow $T_{\alpha,\phi}$ constructed over R_α and under the function ϕ is topologically mixing and L^2 isomorphic to the linear flow $R_{t(\alpha,1)}$ on $\mathbb{T}^{d+1}$.*

4.3.5. *Isomorphism.* Since every φ_{α_i} is an L^2 coboundary then so is $\phi - 1$. We recall the following well known result, the proof of which can be found for example in [12], Chapter 4:

PROPOSITION. *If there exists an L^2 solution to $\phi(x)-1 = \psi(x+\alpha)-\psi(x)$, where $x, \alpha \in \mathbb{T}^d$ then the special flow $T_{\alpha,\phi}$ is L^2 isomorphic to the linear flow $R_{t(\alpha,1)}$.*

4.3.6. *Toplogical mixing.* It remains to prove that $T_{\alpha,\phi}$ is topologically mixing. We will check that the criterion in Proposition 4.3.1 holds.

Proposition 4.2.1 implies that for any $m \in \left[[q_n(\alpha_i)]^{2r+1+0.1}, [q_n(\alpha_i)]^{2r+3/2-0.2}\right]$, the sums $S_m\varphi_{\alpha_i}$ are stretching over the intervals $[k/q_n(\alpha_i)+1/4[q_n(\alpha_i)]^2, k/q_n(\alpha_i)+1/2[q_n(\alpha_i)]^2]$.

Therefore, define

$$u_{nd+i} := 2[q_n(\alpha_i)]^{2r+1.1}, \quad 1 \leq i \leq d.$$

For $1 \leq i < d$, (31) states that $q_n(\alpha_{i+1}) \leq [q_n(\alpha_i)]^{\sigma}$, hence

$$u_{nd+i+1} = 2[q_n(\alpha_{i+1})]^{2r+1.1} \leq 2[q_n(\alpha_i)]^{\sigma(2r+1.1)},$$

which in light of (28) yields for large n

$$[\frac{1}{2}u_{nd+i}, 2u_{nd+i+1}] \subset \left[[q_n(\alpha_i)]^{2r+1+0.1}, [q_n(\alpha_i)]^{2r+3/2-0.2}\right].$$

Likewise, using (32) this time, we get

$$[\frac{1}{2}u_{nd+d}, 2u_{(n+1)d+1}] \subset \left[[q_n(\alpha_d)]^{2r+1+0.1}, [q_n(\alpha_d)]^{2r+3/2-0.2}\right].$$

4.3.7. *End of the proof of Theorem 4.3.4.* We are ready now to check the conditions of Proposition 4.3.1 that guarantee topological mixing. Let u_l be the sequence defined above and define

$$K_{nd+i} := [q_n(\alpha_i)]^{0.1}, \quad 1 \leq i \leq d.$$

For each $l = nd+i$, $1 \leq i \leq d$, define $\mathfrak{I}_l$ to be the familly of intervals

$$I_{nd+i,k} := \left[\frac{k}{q_n(\alpha_i)} + \frac{1}{4[q_n(\alpha_i)]^2}, \frac{k}{q_n(\alpha_i)} + \frac{1}{2[q_n(\alpha_i)]^2}\right], \quad 0 \leq k \leq q_n(\alpha_i)-1.$$

Clearly, we have that given any interval $J \in \mathbb{T}^1$ and any l large enough there exists $I \in \mathfrak{I}_l$ such that $I \subset J$. On the other hand Proposition 4.2.1 applied to each φ_{α_i} implies that (ii) in Proposition 4.3.1 holds for $\phi = \sum \varphi_{\alpha_i}$ with our choice of u_l, K_l and $\mathfrak{I}_l$. □

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