Quasi-periodic dynamics
&
the one dimensional Schrödingier operator

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Chapitre 1

Arithmetics in Quasi-Periodic Dynamics. The Diophantine and Liouville phenomena

1.1 Introduction

The dynamics of a minimal translation on the torus give the simplest example of ergodic transformations. These are called *quasi-periodic* dynamics and are denoted by T_α where $\alpha \in \mathbb{R}^d/\mathbb{Z}^d$ is the translation vector. The dynamics of a minimal translation T_α are simple yet rich : two points remain always at equal distance so that there is no mixing or dependance on initial conditions, while every orbit is dense and in fact equi-distributed on the torus with respect to the Haar measure (unique ergodicity). In addition, any translation can be perturbed into a translation with a rational translation vector, that is into periodic dynamics. In fact there exists a sequence of times $t_n \in \mathbb{Z}^*$ such that $T_\alpha^{t_n} \to \mathrm{Id}$ uniformly on the torus $\mathbb{T}^d$.

Beyond their paradigmatic importance, quasi-periodic systems are central to the study of dynamical systems due to their ubiquity in many situations where they persist after small perturbations, either on all of the space or on some part of it. In fact a single minimal translation can of course be perturbed in a way that makes it lose all its properties, from uniquely ergodic it becomes periodic, then mixing, then "chaotic", etc. But the fact is that in many situation and most importantly in many dynamics coming from classical mechanics or statistical mechanics they show some robustness after perturbation. One of the most far reaching and remarkable such phenomenon is the case of a completely integrable hamiltonian system that displays after perturbation a collection of invariant tori on which the dynamics are conjugated to minimal translations. A collection that fills most of the phase space as the perturbation is taken sufficiently small. This is essentially the content of the celebrated KAM (Kolmogorov Arnol'd Moser) theorem discovered in the midst of the 20^{th} century.

In other systems too, such as circle diffeomorphisms, holomorphic germs, skew-products above translation, time-change of translation flows on the torus, billiards inside convex bodies, outer billiards, as well as Schrödinger cocycles and other $\mathrm{SL}(n,\mathbb{R})$ cocylces above total translations, the quasi-periodic behavior appears to be persistent. We will call such systems *elliptic dynamical systems*.

One goal of our course is to show the persistence of quasi-periodic dynamics in the study of the Schrödinger operator (Diophantine influence), and to show on the other hand how these dynamics can bifurcate into completely different types of behavior (Liouville phenomena).

We define in this chapter Diophantine and Liouville vectors and we illustrate how, already in the frame of irrational rotations of the circle, they give rise to very sharply contrasting behaviors.

The most important ingredient in the study of elliptic dynamics is the simple yet fundamental analysis of the linear cohomological equation given in Section 1.6. We then give an application in the study of the ergodicity of the skew product applications : $\mathbb{T}^2 \to \mathbb{T}^2$, $(x,y) \mapsto (x+\alpha, y+\varphi(x))$.

1.2 Notations and definitions

- We will denote by $\mathbb{T}^d$ the torus $\mathbb{R}^d/\mathbb{Z}^d$
- For $r \in \mathbb{R}^+ \bigcup \{+\infty\}$, we denote by $C^r(\mathbb{T}^d,\mathbb{R})$ the set of real functions on $\mathbb{R}^d$ of class C^r and $\mathbb{Z}^d$-periodic. The set $C^r(\mathbb{T}^d,\mathbb{R})$ is hence a Baire space for the C^r topology. If there is no ambiguity we might just denote these spaces by C^r and we will denote their norms by $\|\cdot\|_r$. We use similar notations for the Bare subset $C^r_\beta(\mathbb{T}^d,\mathbb{R}) \subset C^r(\mathbb{T}^d,\mathbb{R})$ of functions having average $\beta \in \mathbb{R}$.
- For $x \in \mathbb{R}$ we denote $e(x) = e^{i2\pi x}$

• Given a map T on a space X and a function φ defined on X we denote

$$S_N^T\varphi(x) = \sum_{n=0}^{N-1} \varphi(T^n x)$$

if $N \geq 0$, and if $N < 0$

$$S_N^T\varphi(x) = \sum_{n=N}^{-1} \varphi(T^n x).$$

1.3 Irrational translations and their arithmetics

If T_α is a translation of vector α on the torus, $T_\alpha(x) = x + \alpha \bmod(1)$, we use the notation $S_N^\alpha = S_N^{T_\alpha}$, and when there is no ambiguity on the translation frequency we simply use the notation $S_N\varphi$.

• For $k = (k_1, ..., k_d) \in \mathbb{Z}^d$ and $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{R}^d$ we will use the following notations

$$\begin{aligned} (k, \alpha) &= \sum_{i=1}^{d} k_i \alpha_i, \\ |k| &= \sup_i |k_i|, \\ \|x\| &= \inf_{p \in \mathbb{Z}} |x + p|, \\ \{x\} &= x - [x]. \end{aligned}$$

• We will say that a vector $\alpha \in \mathbb{R}^d$ is irrational if the translation on $\mathbb{T}^d$ with frequency α is minimal, that is if $(k, \alpha) + l = 0$ for some $l \in \mathbb{Z}, k \in \mathbb{Z}^d$ implies that $k_1 = \ldots = k_d = l = 0$.

• Given a vector $\alpha \in \mathbb{R}^d$, we say it is τ *Diophantine* if there exists a constant $\gamma > 0$ such that for every $k \in \mathbb{Z}^d - \{0\}$ we have :

$$\|(k, \alpha)\| \geq \frac{\gamma}{\|k\|^{d+\tau}}.$$

In this case we say that α is Diophantine or satisfies a Diophantine condition, and denote $\mathrm{DC}(\tau, \gamma)$ the set of such vectors α. We denote $\mathrm{DC}(\tau) = \cup_{\gamma>0}\mathrm{DC}(\tau, \gamma)$.

If there is a constant $\gamma > 0$ such that α verifies the above condition with $\tau = 0$ we say that α is of *constant type.*

• We recall the arithmetic decomposition of $\mathbb{R}^d = \mathrm{DC} \sqcup L \sqcup Q$, where $\mathbb{Q}$ designates the vectors with rationally dependent coordinates, DC the Diophantine vectors, and L the Liouvillian vectors (vectors that do not satisfy any Diophantine condition and are not in $\mathbb{Q}$). For commodity in stating some results, we will use the notation α "is not $+\infty$ Diophantine" for Liouvillian vectors.

Exercise 1.3.1 *For $\alpha \in \mathbb{R}^d$, show that the following are equivalent*

(i) The translation T_α is transtivie
(ii) The translation T_α is minimal
(iii) The translation T_α is uniquely ergodic

Here are some exercises that show that Diophantine vectors are prevalent from the point of view of measure theory while Liouville vectors are prevalent from the point of view of topology. This duality of prevalence has fundamental consequences on the duality of behavior of elliptic systems depending on wether they are studied from point of view of measure or topology.

Exercise 1.3.2 a. (Dirichlet principle) *Show that for every $\alpha \in \mathbb{R}$ there exists infinitely many p, q such that $|\alpha - p/q| \leq 1/q^2$. Show that for any $d \geq 1$, there exists $C_d > 0$ such that if $\alpha \in \mathbb{R}^d$ is such that T_α is a minimal translation, then there exists infinitely many $k \in \mathbb{Z}^d$ such that $\|(k, \alpha)\| < C_d/|k|^d$.*

b. *Show that for every $\tau > 0$, $\mathrm{DC}(\tau)$ has full Lebesgue measure in $\mathbb{R}^d$, i.e. its complement has zero measure.*

c. *Show that $\mathrm{DC}(0)$ is a dense set that has zero Lebesgue measure.*

Exercise 1.3.3 *Show that the set of Liouville vectors is a G^δ dense set in $\mathbb{R}^d$.*

Exercise 1.3.4 *For $\alpha \in \mathbb{R}$, we say that α is of Roth type if $\alpha \in \cap_{\tau > 0} \mathrm{DC}(\tau)$. Show that numbers of Roth type have full measure in $\mathbb{R}$.*

1.4 Continued fraction algorithm

In the one dimensional case, there is a very powerful tool to study the arithmetics of real numbers. It is called the continued fraction algorithm and is related to the Gauss map Let $G : (0, 1) \to (0, 1)\theta \mapsto \{\frac{1}{\theta}\}$.

For $\alpha \in (0, 1)$ we write

$$\alpha = [a_1, a_2, \ldots] = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}$$

We let $\alpha_n = [a_1, \ldots, a_n]$.

For $\alpha \in \mathbb{R} - \mathbb{Q}$, we denote $\beta_n = G^n(\alpha)$.

We define p_n/q_n to be the sequence of best approximations or convergents of α as follows :

$$\|q_n\alpha\| < \|k\alpha\| \quad \forall 0 < k < q_{n+1}, k \neq q_n$$

Proposition 1.4.1 *Let $\alpha \in \mathbb{R} - \mathbb{Q}$, $\alpha \in (0, \frac{1}{2})$. The sequence of convergents of α satisfies*

1. $\dfrac{p_n}{q_n} = \alpha_n$
2. $|\alpha - \alpha_n| < \dfrac{1}{q_n q_{n+1}}$ *and*
3. $q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_0 = 1, p_0 = 0, q_1 = a_1, p_1 = 1$
4. $\|q_n\alpha\| = \beta_0 \ldots \beta_n$
5. *Let $I_n = (0, \|q_n\alpha\|)$ if n is even and $I_n = (-\|q_n\alpha\|, 0)$ if n is odd. Then the intervals $R^i_\alpha(I_n), i = 0, \ldots, q_{n+1} - 1$ and $R^i_\alpha(I_{n+1}), i = 0, \ldots, q_n - 1$ are all disjoint and their union cover the circle minus a finite number of points. We call this union a dynamical partition of the circle with two towers.*
6. *It holds that*
$$\frac{1}{q_n + q_{n+1}} \leq \|q_n\alpha\| \leq \frac{1}{q_{n+1}}$$

Proof. First of all, let p_n and q_n be defined as in 3). We use the relation (that can be proved inductively on n)

$$[a_1, \ldots, a_{n-1}, t] = \frac{tp_{n-1} + p_{n-2}}{tq_{n-1} + q_{n-2}}$$

and apply it to $t = a_n$ to conclude that 1 holds. It also follows by induction that $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n+1}$ which yields $\alpha_{n+1} - \alpha_n = \frac{(-1)^{n+1}}{q_n q_{n+1}}$ which yields 2) by summation of $\alpha_{n+1} - \alpha_n$.

Now we will show that the sequence of best denominators q'_n is actually equal to q_n. From 2) this we will then also have that α_n is the sequence of best rational approximations to α.

Observe first that in the circle ordering 0 falls between α and $q'_1\alpha$ otherwise $(q'_1 - 1)\alpha$ would be closer to the integers than α which contradicts the definition of q'_1. In the same way one shows that 0 falls between $q'_{n-1}\alpha$ and $q'_n\alpha$ for every n.

Next, q'_1 is the number of adjacent intervals $(0, \alpha)$, $(\alpha, 2\alpha)$,... that can fit on the circle before we cross 0, that is $q'_1 = [1/\alpha] = a_1 = q_1$ and $\|q_1\alpha\| = 1 - [1/\alpha]\alpha = \{1/\alpha\}\alpha = \beta_1\beta_0$.

Now q'_2 is obtained as follows. Let $\hat{a}_2$ be the number of intervals $((q_1 + 1)\alpha, \alpha), ((2q_1 + 1)\alpha, (q_1 + 1)\alpha), \ldots, ((\hat{a}_2 q_1 + 1)\alpha, ((\hat{a}_2 - 1)q_1 + 1)\alpha)$ fit inside $(0, \alpha)$. We have that $\hat{a}_2 = [\alpha/\|q_1\alpha\|] = [1/\beta_1] = a_2$. Hence $q'_2 = a_2 q_1 + 1 = q_2$, and $\|q_2\alpha\| = \alpha - a_2\|q_1\alpha\| = \alpha(1 - [1/\beta_1]\beta_1) = \alpha\beta_1\{1/\alpha_1\} = \beta_0\beta_1\beta_2$. This proves 3) for $n = 2$. The proof of 3) for $n \geq 2$ follows exactly the same lines.

By definition of $q_1 = [1/\alpha]$, we have that the disjoint union of $R^i_\alpha(I_0), i = 0, \ldots, q_1 - 1$ fills the circle minus the interval I_1 : this is 4) for $n = 0$. Next observe that the disjoint union $((q_1+1)\alpha, \alpha), ((2q_1+1)\alpha, (q_1+1)\alpha), \ldots, ((a_2 q_1 + 1)\alpha, ((a_2-1)q_1+1)\alpha)$ fills $(0, \alpha)$ minus the interval $(0, q_2\alpha)$. Similarly, for any $j \leq q_1 - 1$, the disjoint union $((q_1+j)\alpha, \alpha), ((2q_1+j)\alpha, (q_1+j)\alpha), \ldots, ((a_2 q_1 + j)\alpha, ((a_2 - 1)q_1 + j)\alpha)$ fills $((j-1)\alpha, j\alpha)$ minus $R^j_\alpha(0, q_2\alpha)$. This is exactly 4) for $n = 1$. The proof of 4) for $n \geq 2$ follows exactly the same lines.

From 4) we deduce that $q_{n+1}\|q_n\alpha\| + q_n\|q_{n+1}\alpha\| = 1$ from which 5) follows immediately. □

Exercise 1.4.1 *Complete the inductive proof of Proposition 1.4.1.*

Corollary 1.4.1 *One has the following relations between the sequence of convergents of $\alpha \in \mathbb{R} - \mathbb{Q}$ and the Diophantine properties of α*

1. *$\alpha \in \mathrm{DC}(\tau)$ if and only if there exists $C > 0$ such that $q_{n+1} \leq C q_n^{1+\tau}$*
2. *α is Liouville if and only if for any $a_j \to \infty$, there exists a subsequence q_{n_j} such that $q_{n_j+1} \geq q_{n_j}^{a_j}$*

Proof. 1) and 2) follow immediately from the fact that $\frac{1}{q_n + q_{n+1}} \leq \|q_n\alpha\| \leq \frac{1}{q_{n+1}}$ and the definition of best approximations. □

The Gauss map G is ergodic for the probability measure with density $\frac{1}{\ln 2(1+x)}$ (see for example the Book Introduction to modern dynamical systems by Katok Hasselblatt). This fact has important consequences on the measure theory of arithmetical properties of irrationals. For example one can show the following

Corollary 1.4.2 *For every $\gamma > 1$, we have that for almost every $\alpha \in \mathbb{R}$, there exists a constant $C(\alpha)$ such that for every n, $q_{n+1} \leq n^\gamma q_n$.*

Proof. Since q_{n+1}/q_n is of the order of $\|q_n\alpha\|/\|q_{n-1}\alpha\| = \beta_n = G^n(\alpha)$ we see that $\mathrm{Leb}\{\alpha : q_{n+1}(\alpha)/q_n(\alpha) \geq n^\gamma\} = \mathcal{O}(n^{-\gamma})$. Hence Borel Cantelli Lemma implies that for almost every α one has only finitely many n such that $q_{n+1}(\alpha)/q_n(\alpha) \geq n^\gamma$ which yields 3). $\square$

Exercise 1.4.2 *Give a proof of exercises 1.3.2–1.3.4 for $d = 1$ using the continued fractions and Gauss map.*

1.5 Discrepancies of the sequence $\{n\alpha\}$

If a real number α is irrational the sequence $n\alpha[1]$ is uniformly distributed on the circle. This follows for example from unique ergodicity of R_α. One can further push the study of statistical properties of the sequence $n\alpha[1]$ by evaluating the deviations from the average of the number of points of the sequence that belong to some fixed interval on the circle. We will see in this section how these deviations are intimately related to the Diophantine properties of α.

Define the discrepancy function

$$\Delta(\alpha, I, x, N) = S_N^\alpha \chi_I(x) - N|I|$$

where $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ otherwise.

Proposition 1.5.1 *If $\alpha \in \mathrm{DC}(0)$, then there exists C that only depends on α and the length of I such that*

$$|\Delta(\alpha, I, x, N)| \leq C \ln N.$$

If $\alpha \in \mathrm{DC}(\tau)$, with $\tau > 0$, then there exists C that only depends on α and the length of I such that

$$|\Delta(\alpha, I, x, N)| \leq C N^{\frac{\tau}{\tau+1}} \ln N.$$

For every $\epsilon > 0$ and almost every $\alpha \in \mathbb{R}$, there exists C that only depends on ϵ and α and the length of I such that

$$|\Delta(\alpha, I, x, N)| \leq C(\ln N)^{2+\epsilon}.$$

Proof. Observe first that for every p/q, $p \wedge q = 1$ such that $|\alpha - p/q| \leq 1/q^2$ we have that $\Delta(\alpha, I, x, q) \leq 2$.

Next, for $\alpha \in \mathbb{R} - \mathbb{Q}$ and its sequence p_n/q_n of convergents, any integer N writes as $N = \sum_{0 \leq s \leq S} b_s q_s$ with S such that $q_S \leq N \leq q_{S+1}$ and $b_s \leq q_{s+1}/q_s$ and $b_S \leq N/q_S$.

From corollary 1.4.1, we have that $\alpha \in \mathrm{DC}(0)$ iff there exists $C > 0$ such that $q_{n+1} \leq C q_n$. Hence $\Delta(\alpha, I, x, N) \leq 2CS$. But for every n, $q_{n+2} = a_{n+2} q_{n+1} + q_n \geq 2q_n$ hence $S \leq 2 \ln N$ which implies that $\Delta(\alpha, I, x, N) \leq 4C \ln N$.

Similarly, if $\alpha \in \mathrm{DC}(\tau)$ we observe that since $b_s \leq q_{s+1}/q_s \leq C q_s^\tau$, then

$$\begin{aligned} \Delta(\alpha, I, x, N) \leq \sum_{0 \leq s \leq S} 2b_s \leq 2C \sum_{0 \leq s \leq S} (b_s q_s)^{\tau/(1+\tau)} \\ \leq 2CS \max_{s \leq S} (b_s q_s)^{\tau/(1+\tau)} \leq C \ln N N^{\tau/(1+\tau)} \end{aligned}$$

Fix $\gamma = 1 + \epsilon$. Corollary 1.4.1 implies that for a.e. α

$$\Delta(\alpha, I, x, N) \leq C \sum_{s=0}^{S} s^\gamma \leq C' S^{\gamma+1} \leq \bar{C}(\ln N)^{2+\epsilon}.$$

□

Proposition 1.5.2 *For any $\eta : \mathbb{R} \to \mathbb{R}^+$ such that $\lim_{x \to +\infty} \eta(x) = 0$ there exist α and I such that*

$$\limsup_{N \in \mathbb{N}} |\Delta(\alpha, I, 0, N)| \geq N\eta(N)$$

Proof. Suppose the continued fraction of α is known up to a_n. By choosing a_{n+1} sufficiently large we can insure that $1/q_n \geq 4\eta(q_{n+1}/10)$ (it suffices to take $4\eta(a_{n+1}/10) \leq 1/q_n$). Let q_{n_j} be a subsequence with the latter property. Up to extracting a subsequence from n_j we can assume that there exists β be such that for all j, $1/(4q_{n_j}) < \beta - p_j/q_{n_j} < 3/(4q_{n_j})$ for some p_j.

For $a \leq q_{n_j+1}/(10q_{n_j})$ we have that $\Delta(\alpha, [0, \beta], 0, aq_{n_j}) = ap_j - aq_{n_j}\beta$. Take $N_j = [q_{n_j+1}/10]$. Then

$$\begin{aligned}\Delta(\alpha, [0, \beta], 0, N_j) &= [q_{n_j+1}/(10q_{n_j})]p_j - N_j\beta \\ &= N_j\theta_j, \quad \text{with } \theta_j \leq -1/(4q_{n_j}) \\ &\leq -N_j\eta(N_j)\end{aligned}$$

□

Exercise 1.5.1 *Let $\alpha \in (0, \frac{1}{2})$. Let $I_k = [0, \|k\alpha\|]$, and define χ_{I_k} such that $\chi_{I_k}(x) = 1$ if $x \in I_k$ and $\chi_{I_k}(x) = 0$ if $x \notin I_k$.*

- *Assume $k = 1$. Show that the function $\psi_1(x) = x - [x]$ satisfies $\chi_{I_1}(x) - \alpha = \psi_1(x - \alpha) - \psi_1(x)$.*
- *Show that for any k, there exists ψ_k defined and differentiable except at finitely many points and with constant derivative such that $\chi_I(x) - \|k\alpha\| = \psi_k(x + \alpha) - \psi_k(x)$*
- *What do you conclude for the discrepancy function $\Delta(\alpha, I_k, N, \cdot)$*

1.6 The linear cohomological equation

Given $\varphi \in C^r(\mathbb{T}^d, \mathbb{R})$ and $\alpha \in \mathbb{T}^d$ we call cohomological equation above T_α the search of solutions $\psi \in C^s(\mathbb{T}^d, \mathbb{R})$ to the equation

$$\varphi(x) - \int_{\mathbb{T}^d} \varphi(x)dx = \psi(x + \alpha) - \psi(x) \tag{1.1}$$

Exercise 1.6.1 *Show that if φ is a trigonometric polynomial then (1.1) has a trigonometrical polynomial solution.*

Exercise 1.6.2 *For which functions in $C^\infty(\mathbb{T}, \mathbb{R})$ does equation (1.1) have a solution in $C^\infty(\mathbb{T}, \mathbb{R})$ when $\alpha = p/q$ is rational ?*

Theorm 1.6.1 *If $\alpha \in \mathbb{T}^d$ is a Diophantine vector then equation (1.1) has a solution in $C^\infty(\mathbb{T}^d, \mathbb{R})$ for every $\varphi \in C^\infty(\mathbb{T}^d, \mathbb{R})$*

Proof. Let f be a measurable function defined on $\mathbb{T}^d$. Then $f \in C^\infty(\mathbb{T}^d, \mathbb{R})$ if and only if $\hat{f}_k = \int_{\mathbb{T}^d} f(x)e(-(k, x))dx$ satisfies $\|\hat{f}_k\| = \mathcal{O}(\|k\|^{-r})$ for any $r \in \mathbb{N}$.

If we look for a solution to (1.1) under the form $\sum_{k\in\mathbb{Z}^*}\widehat{\psi}_k e((k,x))$ we get that

$$\begin{aligned}\widehat{\psi}_k &= \frac{\widehat{\varphi}_k}{e((k,\alpha))-1}\\ &= e(\frac{1}{2}(k,\alpha)+\frac{1}{4})\frac{\widehat{\varphi}_k}{2\sin(\pi(k,\alpha))}\end{aligned}$$

Since $\sin(x)\geq\frac{\pi}{x}$ for $x\in(0,\pi/2)$ we conclude that

$$|\widehat{\psi}_k|\leq\frac{|\widehat{\varphi}_k|}{\|(k,\alpha)\|}$$

so that if α is a Diophantine vector and $\varphi\in C^\infty(\mathbb{T}^d,\mathbb{R})$ we get that $\|\widehat{\psi}_k\| = \mathcal{O}(\|k\|^{-r})$ for any $r\in\mathbb{N}$, and thus that $\psi\in C^\infty(\mathbb{T}^d,\mathbb{R})$.

□

Exercise 1.6.3 *Let $r\geq 3$. Assume that $\alpha\in\mathrm{DC}(\tau,\gamma)\subset\mathbb{R}$ for some $0\leq\tau<r-2$, $\tau\notin\mathbb{N}$. Show that if $\varphi\in C^r(\mathbb{T},\mathbb{R})$, then there exists $\psi\in C^{[r-\tau]-1}(\mathbb{T},\mathbb{R})$ such that*

$$\psi(x+\alpha)-\psi(x)=\varphi(x)-\int_{\mathbb{T}}\varphi(\theta)d\theta$$

and

$$\|\psi\|_{C^{[r-\tau]-1}}\leq\frac{C}{\gamma}\|\varphi\|_{C^r}$$

where C is a universal constant. What can you say if $\tau\in\mathbb{N}$?

Theorm 1.6.2 *If $\alpha\in\mathbb{R}^d$ is a Liouville vector then for a set of functions that contains a G^δ dense set of $\varphi\in C^\infty(\mathbb{T}^d,\mathbb{R})$ equation (1.1) has no solution in $L^2(\mathbb{T}^d,\mathbb{R})$*

Proof. Since α is a Liouville vector, there exists a sequence $k_n\in\mathbb{Z}^d$ be such that $\|(k_n,\alpha)\|\leq\|k_n\|^{-3n}$. Fix $r\in\mathbb{N}$.

Let

$$E(N,M,r)=\left\{f\in C^r(\mathbb{T}^d,\mathbb{R}):\|S_N^\alpha(f-\int_{\mathbb{T}^d}f)\|_{L^2}>M\right\}$$

We have that $E(N,M,r)$ is an open set in the C^r topology. Since $\varphi\in\bigcap_{M\in\mathbb{N}^*}\bigcup_{N\in\mathbb{N}}E(N,M,r)$ satisfies $\sup_{N\in\mathbb{N}}\|S_N^\alpha(\varphi-\int_{\mathbb{T}^d}\varphi)\|_{L^2}=\infty$ equation

(1.1) has no solution in $L^2(\mathbb{T}^d, \mathbb{R})$ for the function φ. Since r was arbitrary, we finish if we prove that $\cup_{N\in N} E(N, M, r)$ is dense in the C^r topology.

Let $\epsilon > 0$ and $\varphi_0 \in C^r(\mathbb{T}^d, \mathbb{R})$. We first approximate φ_0 by a trigonometrical polynomial $P(x) = \sum_{|j|/leqJ} \widehat{P}_j e((j, x))$ such that $\|P - \varphi_0\|_r \leq \epsilon/2$. Then we consider the sequence $\varphi_n = P(x) + \|k_n\|^{-n}\mathrm{Re}(e((k_n, x)))$ for $n \geq 1$. For n sufficiently large we have $\|\|k_n\|^{-n}\mathrm{Re}(e((k_n, x)))\|_r \leq \epsilon/2$ and thus $\|\varphi_n - \varphi_0\|_r \leq \epsilon$.

On the other hand we claim that $\lim_{n\to\infty} \|S^\alpha_{N_n}(\varphi_n - \int_{\mathbb{T}^d} \varphi_n)\|_{L^2} = \infty$, where $N_n = \|k_n\|^{2n}$. Therefore $\varphi_n \in E(N_n, M, r)$ for n sufficiently large.

The proof of the claim comes from two observations. The first one is that $P(x) - \int_{\mathbb{T}^d} P = Q(x + \alpha) - Q(x)$ for a trigonometric polynomial Q so that $\|S^\alpha_N(P - \int_{\mathbb{T}^d} P)\|_{L^2} < 2\|Q\|_{L^2}$. The other observation is that up to $m \ll 1/\|(k_n, \alpha)\|$ iterations by T_α the sinusoid $\mathrm{Re}(e((k_n, x)))$ is essentially constant (Liouville phenomenon!). Namely

$$S^\alpha_m \mathrm{Re}(e((k_n, x))) = X_{n,m}\mathrm{Re}(e((k_n, x)))$$

with

$$\begin{aligned} X_{n,m} &= \frac{1 - e(m\|(k_n, \alpha)\|)}{1 - e(\|(k_n, \alpha)\|)} \\ &= e((m-1)\|(k_n, \alpha)\|/2)\frac{\sin(\pi m\|(k_n, \alpha)\|)}{\sin(\pi\|(k_n, \alpha)\|)} \end{aligned}$$

Since $\sin(x) \leq x$, $\sin(x) \geq \frac{\pi}{x}$ for $x \in (0, \pi/2)$ we get that for $m\|(k_n, \alpha)\| \leq 1/2$

$$|X_{n,m}| \geq \frac{m}{\pi}.$$

Finally the k_n^{th} Fourier coefficient a_{k_n} of $S^\alpha_{N_n}\varphi_n$ satisfies $|a_{k_n}| \geq \frac{\|k_n\|^n}{\pi}$ and the claim follows.

□

Exercise 1.6.4 *Show that for $\alpha \in \mathbb{R}^d$ irrational and $\varphi \in C^r(\mathbb{T}^d, \mathbb{R})$ equation (1.1) has a C^s solution with $s \leq r$ if and only if* $\sup_{N\in\mathbb{Z}} \|S^\alpha_N\varphi\|_r < +\infty$ *(use Gottshalk Hedlund theorem).*

Show that for any $\varphi \in C^\infty(\mathbb{T}, \mathbb{R})$ that is not a polynomial, there exists a dense G^δ set of $\alpha \in \mathbb{R}$ such that $\sup_{N\in\mathbb{Z}} \|S^\alpha_N\varphi\|_{C^0} = +\infty$.

1.7 Ergodicity of skew products above rotations

In this chapter, we study the display of ergodicity by skew products on the torus $\mathbb{T}^2$. For $\alpha \in \mathbb{R}$ and $\varphi \in C^\infty(\mathbb{T}, \mathbb{R})$, we define the skew product

$$W_{\alpha,\varphi} : \mathbb{T}^2 \to \mathbb{T}^2 : W_{\alpha,\varphi}(x, y) = (x + \alpha, y + \varphi(x))$$

We show that if the rotation angle α is Diophantine then $W_{\alpha,\varphi}$ is smoothly conjugated to $W_{\alpha,\beta}$, $\beta = \int_\mathbb{T} \varphi(\theta)d\theta$. In particular, if $\int_\mathbb{T} \varphi(\theta)d\theta = 0$ then $W_{\alpha,\varphi}$ is not ergodic for Lebesgue measure. In contrast we show that if α is Liouville, then for a residual set (in the C^∞ topology) of functions φ with zero Lebesgue average, the skew product $S_{W,\alpha}$ is ergodic for Lebesgue. This will be based on the analysis of the linear cohomological equation studied in section 1.6.

For the sequel we will use the notation $C^r_\beta(\mathbb{T}, \mathbb{R})$ to denote the subset of $C^r(\mathbb{T}, \mathbb{R})$ of functions with Lebesgue average equal to β. When we say that $W_{\alpha,\varphi}$ is ergodic we always mean with respect to Haar measure on $\mathbb{T}^2$.

Theorm 1.7.1 *Assume $\alpha \in \mathbb{R}$ is Liouville. Then, for any $r \in \mathbb{N}$, the set of $\varphi \in C^r_0(\mathbb{T}, \mathbb{R})$ such that $W_{\alpha,\varphi}$ is ergodic is a dense G_δ subset of $C^r_0(\mathbb{T}, \mathbb{R})$. The same conclusion is true in the sets $C^r(\mathbb{T}, \mathbb{R})$ and $C^r_\beta(\mathbb{T}, \mathbb{R})$ for any $\beta \in \mathbb{R}$.*

The result is the best one can have as shown by the following

Theorm 1.7.2 *Assume $\alpha \in \mathbb{R}$ is Diophantine. Then for any $\varphi \in C^\infty(\mathbb{T}, \mathbb{R})$ we have that $W_{\alpha,\varphi}$ is C^∞-conjugated to $W_{\alpha,\beta}$, $\beta = \int_\mathbb{T} \varphi(\theta)d\theta$. As a consequence $W_{\alpha,\varphi}$ is ergodic if and only if (α, β) are rationally independent.*

We recall that two maps (M_1, f_1) and (M_2, f_2) are said to be C^r- *conjugated* if there exists a homeomorphism h of class C^r from M_1 onto M_2 such that $h \circ f_1(x) = f_2 \circ h(x)$ for any $x \in M_1$.

Proof of Theorem 2.1.2. From Theorem 1.6.1 we know that there exists $\psi \in C^\infty(/T, \mathbb{R})$ such that

$$\psi(x + \alpha) - \psi(x) = \varphi(x) - \beta$$

Define the smooth diffeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2 : h(x, y) = (x, y - \psi(x))$. Observe that $h \circ W_{\alpha,\varphi}(x, y) = (x + \alpha, y + \varphi(x) - \psi(x + \alpha)) = (x + \alpha, y + \beta - \psi(x)) = W_{\alpha,\beta} \circ h(x, y)$. □

Note that if $\beta = 0$ for example then each graph $(x, y_0 - \psi(x))$ forms an invariant ergodic component for the decomposition of the Haar measure under the action of $W_{\alpha,\varphi}$.

The rest of this section is dedicated to the proof of Theorem 2.1.2.

We first give a general argument that shows that the set of φ such that $W_{\alpha,\varphi}$ is ergodic is G_δ.

Proposition 1.7.1 *Given any $\beta \in \mathbb{T}$, the set of $\varphi \in C^r_\beta(\mathbb{T},\mathbb{R})$ such that $W_{\alpha,\varphi}$ is ergodic is a G_δ subset of $C^r_\beta(\mathbb{T},\mathbb{R})$.*

Proof. Let f_j be a countable dense subset of $C^0_0(\mathbb{T}^2,\mathbb{C})$. Then for $r, j, N, k \in \mathbb{N}$ we define the set

$$A^r(i,j,N,k) = \left\{\varphi \in C^r_\beta(\mathbb{T},\mathbb{R}) : |\frac{1}{N}\int_{\mathbb{T}^2} S_N^{\alpha,\varphi} f_i(\theta) f_j(\theta) d\theta| < 1/k\right\}$$

where $S_N^{\alpha,\varphi} f$ denotes the Birkhoff sums of f above the map $W_{\alpha,\varphi}$. Observe that $A^r(i,j,N,k)$ is an open set in $C^r_\beta(\mathbb{T},\mathbb{R})$ since the quantity estimated is continuous in φ. On the other hand $\varphi \in C^r_\beta(\mathbb{T},\mathbb{R})$ is such that $W_{\alpha,\varphi}$ is ergodic if and only if

$$\varphi \in \mathbf{A} := \bigcap_i \bigcap_j \bigcap_k \bigcup_N A^r(i,j,N,k).$$

To prove the latter assertion suppose first that $W_{\alpha,\varphi}$ is ergodic then by Birkhoff theorem we see that $\frac{1}{N} S_N^{\alpha,\varphi} f_i$ converges almost surely to 0 as $N \to \infty$ therefore $|\frac{1}{N}\int_{\mathbb{T}^2} S_N^{\alpha,\varphi} f_i(\theta) f_j(\theta) d\theta|$ converges to 0 hence $\varphi \in \mathbf{A}$. Conversely suppose that $\varphi \in \mathbf{A}$ and assume that $f \circ W_{\alpha,\varphi} = f$ almost surely and $\frac{1}{N} S_N^{\alpha,\varphi} f = f$ and approaching f by linear combinations of the f_i's we obtain that $\int_{\mathbb{T}^2} f(\theta) f_j(\theta) = 0$ for every j which implies that $f = 0$. $\square$

Exercise 1.7.1 *Show that the set of $(\alpha,\varphi) \in \mathbb{T} \times C^r(\mathbb{T},\mathbb{R})$ such that $W_{\alpha,\varphi}$ is ergodic is a G_δ set.*

Exercise 1.7.2 *Show that for any fixed $\varphi \in \times C^r(\mathbb{T},\mathbb{R})$ the set of $\alpha \in \mathbb{R}$ such that $W_{\alpha,\varphi}$ is ergodic is a G_δ set.*

From Proposition 2.2.2 the only thing left to prove to get Theorem 2.1.1 is density of $\varphi \in C^r_\beta(\mathbb{T},\mathbb{R})$ such that $W_{\alpha,\varphi}$ is ergodic.

To do this we start with a criterion on φ that implies ergodicity of $W_{\alpha,\varphi}$.

Lemma 1.7.1 *The skew product $W_{\alpha,\varphi}$ is ergodic if and only for any $\lambda \in \mathbb{Z}$ the equation*

$$h(x+\alpha) = e^{i2\pi\lambda\varphi(x)} h(x) \tag{1.2}$$

does not have a non constant measurable complex solution h.

Proof. Assume that (2.7) has a measurable solution h for some choice of $\lambda \in \mathbb{Z}$. Let $H(x,y) = h(x)e(-\lambda y)$. Then $H(W_{\alpha,\varphi}(x,y)) = H(x,y)$ which implies that $W_{\alpha,\varphi}$ is not ergodic. Conversely, note that the subspaces V_k of functions of the form $g(x)e(ky)$ with g measurable are invariant under the action of $W_{\alpha,\varphi}$, hence the existence of a nontrivial eigenvalue for $W_{\alpha,\varphi}$ implies that there is a nontrivial solution to (2.7). □

Now we translate this criterion into a criterion on the Birkhoff sums of φ above R_α.

Lemma 1.7.2 (Criterion for ergodicity) *If for every $k \in \mathbb{Z}$ and for every λ in $\mathbb{Z}^*$, we have*

$$\inf_{m\in\mathbb{N}} \left| \int_{\mathbb{T}^d} e(\lambda S_m\varphi(\theta))e(k\theta)d\theta \right| = 0,$$

then $W_{\alpha,\varphi}$ is ergodic.

Proof of Lemma 2.2.2. We must show that (2.7) has no non constant measurable solutions. The case $\lambda = 0$ is easy since T_α is ergodic. Also since T_α is ergodic any measurable solution of (2.7) has constant modulus and is therefore in L^2. Suppose h is a solution of (2.7), we have

$$h(\theta + m\alpha) = e(\lambda S_m\varphi(\theta))h(\theta)$$

and for any $k \in \mathbb{Z}$

$$\begin{aligned} e(-mk\alpha)\int_{\mathbb{T}} h(\theta)e(k\theta)d\theta &= \int_{\mathbb{T}} h(\theta+m\alpha)e(k\theta)d\theta, \\ &= \int_{\mathbb{T}} e(\lambda S_m\varphi(\theta))h(\theta)e(k\theta)d\theta. \end{aligned}$$

Should the condition of the Lemma be satisfied we would have from the fact that the characters $e(k\theta)$ form an L_2 basis that

$$\inf_{m\in\mathbb{N}} \left| \int_{\mathbb{T}} e(\lambda S_m\varphi(\theta))h(\theta)e(k\theta)d\theta \right| = 0.$$

Which implies that $\int_{\mathbb{T}^d} h(\theta)e(k\theta)d\theta = 0$ for any $k \in \mathbb{Z}$, hence h is zero. □

In all the rest r will be an arbitrary integer. Given $\alpha, \beta \in \mathbb{T}$ define

$\mathbf{A^r}(\alpha, \beta) =$
$\left\{\varphi \in C^r_\beta(\mathbb{T}, \mathbb{R}) | \forall \lambda \in \mathbb{Z}^*, \forall k \in \mathbb{N}, \inf_{m \in \mathbb{N}} \left|\int_{\mathbb{T}} e(\lambda S_m \varphi(\theta)) e(k\theta) d\theta\right| = 0\right\}.$

In light of lemma 2.2.2 density in theorem 2.1.1 will follow from

Proposition 1.7.2 *Fix $r \in \mathbb{N}$ and $\beta \in \mathbb{T}$. If α is Liouville then $\mathbf{A}^r(\alpha, \beta)$ is a G_δ dense subset of $C^r_\beta(\mathbb{T}, \mathbb{R})$.*

Proof of proposition 2.2.3. For $j, \lambda, n, k \in \mathbb{N}^4$, define

$$\mathbf{A}^r_{j,\lambda,k,n}(\alpha, \beta) = \{\varphi \in C^r_\beta(\mathbb{T}, \mathbb{R}) \ : \ \left|\int_{\mathbb{T}} e(\lambda S_n \varphi(\theta)) e(k\theta) d\theta\right| < \frac{1}{j}\}.$$

We have

$$\mathbf{A}^r(\alpha, \beta) = \bigcap_{j \in \mathbb{N}} \bigcap_{\lambda \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \mathbf{A}^r_{j,\lambda,k,n}(\alpha, \beta).$$

The set $\mathbf{A}^r_{j,\lambda,k,n}(\alpha, \beta)$ is obviously open and we just have to prove that $\cup_{n \in \mathbb{N}} \mathbf{A}^r_{j,\lambda,k,n}(\alpha, \beta)$ is dense, which will be the goal of the rest of the section.

First we express the fact that α is not Diophantine :
There exists a sequence $k_n \in \mathbb{N}$ with $\lim_{n \to \infty} k_n = +\infty$ such that

$$\|k_n \alpha\| < \frac{1}{k_n^{3n}}. \tag{1.3}$$

We introduce now a sequence of real functions on $\mathbb{T}$

$$\psi^{(n)}(\theta) = \frac{\cos(2\pi k_n \theta)}{k_n^n}.$$

Finally let $m_n = |k_n|^{2n}$. The essential fact about m_n is that $|k_n|^n \ll m_n \ll \|k_n \alpha\|^{-1}$.

We will need the following lemmas (the first one is direct from the definition of $\psi^{(n)}$)

Lemma 1.7.3 *For any $r \in \mathbb{N}$ we have*

$$|\psi^{(n)}|_r \underset{n \to \infty}{\longrightarrow} 0.$$

Lemma 1.7.4 *Let $\tilde{\psi}^{(n)} = \cos(2\pi k_n \theta)$. Then we have for any m such that $m\|k_n\alpha\| < 1/2$*

$$S_m \tilde{\psi}^{(n)}(\theta) = X_{n,m} \cos\left(2\pi k_n \theta + \phi_{n,m}\right),$$

where $X_{n,m} \geq \frac{2}{\pi} m$ and $\phi_{n,m} \in [0, 2\pi)$.

Proof of Lemma 2.2.4. We have

$$\begin{aligned} S_{m_n} \tilde{\psi}^{(n)}(\theta) &= \operatorname{Re}\left(1 - e^{i2\pi m_n k_n \alpha} e^{i2\pi k_n \theta}\right), \\ &= \operatorname{Re}\left(e^{i\pi(m_n - 1)k_n\alpha} \frac{\sin\left(\pi m_n k_n \alpha\right)}{\sin\left(\pi k_n \alpha\right)} e^{i2\pi k_n \theta}\right), \\ &= X_{n,m} \cos\left(2\pi k_n \theta + \phi_{n,m}\right), \end{aligned}$$

if we let

$$X_{n,m} = \frac{\sin\left(\pi m k_n \alpha\right)}{\sin\left(\pi k_n \alpha\right)}.$$

Since $m\|(k_n, \alpha)\| < 1/2$, we have $\sin(\pi m(k_n, \alpha)) = \sin(\pi m\|(k_n, \alpha)\|) \geq 2m\|k_n\alpha\|$ while $\sin(\pi(k_n, \alpha)) = \sin(\pi\|(k_n, \alpha)\|) \leq \pi\|k_n\alpha\|$ hence

$$X_{n,m} \geq \frac{2}{\pi} m$$

□

Remark : The functions $\psi^{(n)}$ we introduced are "α" periodic (in the sense that $\psi^{(n)}(x + j\alpha) \sim \psi^{(n)}(x)$) as long as the number of iteration of T_α, is such that $m\|k_n\alpha\| = o(1)$. Consequently, the Birkhoff sums of $\psi^{(n)}$ up to m_n will look like $m_n\psi^{(n)}$. If m_n is moreover such that $\frac{m_n}{D_n} \to \infty$, where D_n is the denominator in the expression of $\psi^{(n)}$, then $\psi^{(n)}_{m_n}$ will have great oscillations. This essential phenomenon will allow us to check the criterion of lemma 2.2.2 by means of the stationnary phase method. The idea next, is to perturb a given function by adding to it a function $\psi^{(n)}$ that will "produce ergodicity" up to time m_n.

From the estimation in lemma 2.2.4 and a basic fact on stationnary phase, we have the following crucial result.

Lemma 1.7.5 *Fix $g \in C^\infty(\mathbb{T}^d, \mathbb{R})$ and $p > 0$, and assume that $\lambda \in [1/p, p]$. There exists a constant $C(g, p) > 0$ such that*

$$\left| \int_{\mathbb{T}^d} e^{i\lambda S_{m_n} \psi^{(n)}(\theta)} g(\theta) d\theta \right| \leq C Y_n^{-1/8}$$

where $Y_n = |k_n|^n$.

Exercise 1.7.3 Give a proof of lemma 2.2.5. *Hint :* Cut the integral into intervals where the derivative of $\psi_{m_n}^{(n)}$ is larger than $Y_n^{-1/4}$ and intervals where the derivative is less than $Y_n^{-1/4}$ then majorize the integral on each of these intervals.

Now, we are ready to prove that $\cup_{n\in\mathbb{N}}\mathbf{A}^r_{j,\lambda,k,n}(\alpha,\beta)$ is dense in $C^r_\beta(\mathbb{T}^d,\mathbb{R})$.

Fix a function $\varphi \in C^r_\beta(\mathbb{T}^d,\mathbb{R})$ and take $\epsilon > 0$ arbitrarily small.. We let P be a trigonometrical polynomial with average β and such that $|\varphi - P|_r < \epsilon$. We let Q be the zero average solution of the linear cohomological solution associated to P, that is $P(x) - \beta = Q(x+\alpha) - Q(x)$.

Then for $n \in \mathbb{N}$ we define

$$\varphi^{(n)} = P + \psi^{(n)}.$$

Now, if we take n large enough, we will have, from Lemma 2.2.3 :

$$|\varphi - \varphi^{(n)}|_r < 2\epsilon. \tag{1.4}$$

On the other hand, up to extracting a subsequence from m_n, we can assume that $m_n\beta$ converges to some number $\bar{\beta}$ and thus that $S_{m_n}P$ converges to $\bar{\beta}$. Indeed $Q(x+m_n\alpha) - Q(x)$ converges in C^r norm to 0.

Then, applying lemma 2.2.5 we have

$$\int_{\mathbb{T}^d} e(\lambda S_{m_n}\varphi^{(n)}(\theta))\chi_j(\theta)d\theta = \int_{\mathbb{T}^d} e(\lambda S_{m_n}\psi^{(n)}(\theta))e(\lambda S_{m_n}P(\theta))e(j\theta)d\theta \underset{n\to\infty}{\longrightarrow} 0.$$

(We used lemma 2.2.5 for $g(\theta) = e(j\theta)$, and the fact that $e(\lambda S_{m_n}P(\theta))$ converges to a constant when n goes to infinity.)

So when n is sufficiently large, we have that $\varphi^{(n)} \in \mathbf{A}^r_{j,\lambda,k,m_n)}(\alpha,\beta)$. The real number ϵ being arbitrarily small this completes the proof of density of $\cup_{n\in\mathbb{N}}\mathbf{A}^r_{j,\lambda,k,n}(\alpha,\beta)$. Proposition 2.2.3 is thus proved. □

Proof of theorem 2.1.1. In light of lemma 2.2.2, Proposition 2.2.3 implies that for any $r \in \mathbb{N}$, the set of $\varphi \in C^r_\beta(\mathbb{T},\mathbb{R})$ for which $W_{\alpha,\varphi}$ is ergodic contains a G_δ dense subset of $C^r_\beta(\mathbb{T},\mathbb{R})$. Together with Proposition 2.2.2, this proves theorem 2.1.1. □

Exercise 1.7.4 *Generalize the results of Theorems 2.1.1 and 2.1.2 to skew products above d dimensional irrational translation of $\mathbb{T}^d$, i.e. to the maps $W_{\alpha,\varphi} : \mathbb{T}^d \times \mathbb{T} \to \mathbb{T}^d \times \mathbb{T} : W_{\alpha,\varphi}(\theta,y) = (\theta+\alpha, y+\varphi(\theta))$.*

Chapitre 2

Reparametrization of irrational flows on the torus

In this chapter, we study the display of weak mixing by reparametrized linear flows on the torus $\mathbb{T}^d$, $d \geq 2$. We show that if the vector of the translation flow is Liouville (i.e. well approximated by rationals), then for a residual set (in the C^∞ topology) of time change functions, the reparametrized flow is weak mixing. If the vector of the linear flow is Diophantine, we will show that any C^∞ reparametrization of the flow is C^∞ conjugated to a linear flow. This will be based on the existence of smooth solutions to the linear cohomological equation studied in chapter **??**.

In the case $d = 2$ we show that the reparametrized flows are rigid, in the sense that they converge to identity in the C^∞ norm along a subsequence of time.

2.1 Weak mixing and linearizability of reparametrized flows

Assume T_α is a minimal translation on $\mathbb{T}^d$ and consider on $\mathbb{T}^{d+1}$ the *irrational* translation flow :

$$\frac{dx}{dt} = (1, \alpha).$$

This flow is minimal and uniquely ergodic for the Haar measure on $\mathbb{T}^{d+1}$. Given $\phi \in C^r(\mathbf{T}^{d+1}, \mathbf{R}_+^*)$, $r \geq 1$, we define the *reparametrization*, or smooth time change, of this translation flow, with *speed* $\frac{1}{\phi}$, to be the flow given by

$$\frac{d\theta}{dt} = \frac{\alpha}{\phi(\theta, s)}, \qquad \frac{ds}{dt} = \frac{1}{\phi(\theta, s)}.$$

The reparametrized flow is strictly ergodic (minimal and uniquely ergodic) and the invariant measure is $\phi(x)dx$, where dx denotes the Haar measure on $\mathbb{T}^{d+1}$ (see exercise 2.1.1). We denote this reparametrized flow by by $T^t_{(\alpha,1),\phi^{-1}}$.

Considering a Poincaré section, such a flow can be viewed as a suspension flow constructed from T_α on $\mathbb{T}^d$ and a suspension function φ with the same regularity C^r than ϕ. This is called a special flow with base T_α and ceiling function φ. To obtain results on time change for flows, it is often more convenient to work with special flows that are easier to handle, and then transfer the properties to reparametrizations. The exact definition of special flows will be given in Section 2.2 and the natural correspondence with reparametrization will be explained in Section 2.3.

Exercise 2.1.1 Consider the reparametrized flow $T^t_{(\alpha,1),\phi^{-1}}$. Take a line segment ℓ of size ϵ in the direction of the vector $(\alpha, 1)$ and consider its image by an infinitesimal iteration of the reparametrized flow $\tilde{\ell} = T^{dt}_{(\alpha,1),\phi^{-1}}\ell$. Compute the length of $\tilde{\ell}$ up to first order in dt and ϵ and deduce that $\phi(x)dx$ is an invariant density for $T^t_{(\alpha,1),\phi^{-1}}$. Deduce unique ergodicity of $T^t_{(\alpha,1),\phi^{-1}}$.

We recall first the definition of weak mixing for flows and we introduce some notations.

-A measure preserving flow $\{T^t\}$ on (L, μ) is said to be weak mixing if for all $f, g \in L^2(L, \mu)$ we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \left| \int_L f(T^u x) g(x) d\mu - \int_L f d\mu \int_L g d\mu \right| du = 0. \tag{2.1}$$

An equivalent definition is that for all measurable sets A and B

$$\mu(T^{-t} A \bigcap B) \longrightarrow \mu(A)\mu(B), \tag{2.2}$$

when $|t|$ goes to infinity on a set of density one over $\mathbb{R}$.

One can also prove that a flow $\{T^t\}$ is weak mixing if and only if it does not have an eigenfunction, i.e. a measurable function h, not constant, such that $h(T^t x) = e^{i\lambda t} h(x)$, for some eigenvalue $\lambda \in \mathbb{R}$.

For the equivalence between the definitions, we refer to the book of Parry : *Introduction to Ergodic theory*, or the book *Introduction to Ergodic theory* of Confeld, Fomin and Sinai.

We can now state the results of this chapter. Let $d \in \mathbb{N}$, $d \geq 1$.

Theorm 2.1.1 *Assume the irrational vector α in $\mathbb{R}^d$ is Liouville. Then, for any $r \in \mathbb{N}$, the set of $\phi \in C^r(\mathbb{T}^{d+1}, \mathbb{R}_+^*)$ such that $T^t_{(\alpha,1),\phi^{-1}}$ is weak mixing is a dense G_δ subset of $C^r(\mathbf{T}^{d+1}, \mathbf{R}_+^*)$.*

The result is the best one can have as shown by the following proposition due to Kolmogorov (in dimension 2, i.e. $d = 1$) and generalized to any dimension by Herman :

Theorm 2.1.2 *Assume the irrational vector $\alpha \in \mathbb{T}^d$ is Diophantine. Then for any $\phi \in C^\infty(\mathbf{T}^{d+1}, \mathbf{R}_+^*)$ we have that $T^t_{(\alpha,1),\phi^{-1}}$ is C^∞ conjugate to a translation flow on $\mathbb{T}^{d+1}$.*

We recall that two flows (M_1, T_1^t) and (M_2, T_2^t) are said to be C^r- *conjugated* if there exists a homeomorphism h of class C^r from M_1 onto M_2 such that $h \circ T_1^t(x) = T_2^t \circ h(x)$ for any $x \in M_1$.

The dichotomy is therefore complete between weak mixing and conjugation to translation flows and we have the following combined statement of theorems 2.1.1 and 2.1.2 :

Corollary 2.1.1 *Let T_α be a minimal translation on $\mathbb{T}^d$. Then either one of two possibilities hold :*
(i) The vector α is Diophantine. Then, for any $\phi \in C^\infty(\mathbb{T}^{d+1}, \mathbb{R}_+^*)$, $T^t_{(\alpha,1),\phi^{-1}}$ is C^∞ conjugated to a translation flow on $\mathbb{T}^{d+1}$.
(ii) The vector α is Liouville. Then, for a dense G_δ of $\phi \in C^\infty(\mathbb{T}^{d+1}, \mathbb{R}_+^*)$, $T^t_{(\alpha,1),\phi^{-1}}$ is weak mixing (for its unique invariant measure).

The rest of this chapter is dedicated to the proofs of theorems 2.1.1 and 2.1.2. The essential results are obtained in Section 2.2 where we prove analogous theorems for special flows.

It is possible to obtain weak mixing as defined in (2.2), with a direct and geometrical method, but it is easier to prove the equivalent spectral characterization of weak mixing, i.e. the non-existence of eigenfunctions. This is the approach that we will adopt. In Lemma 2.2.2, we state a central criterion on the Birkhoff-sums of the ceiling function of a special flow that guarantees the non-existence of eigenfunctions. Then we will use Baire category arguments and the stationary phase method to study when this criterion is fulfilled in relation with the arithmetics of α and the regularity of the ceiling function φ. In Section 2.3 we derive the results for reparametrizations.

2.2 Special Flows above toral translations.

We recall the definition of a special flow : Given a Lebesgue space L, a measure preserving transformation T on L and an integrable strictly positive real function defined on L we define the special flow over T and under the *ceiling function* φ by inducing on $L \times \mathbb{R}/\sim$, where $\sim$ is the identification $(x, s+\varphi(x)) \sim (T(x), s)$,the action of

$$\begin{array}{rcl} L \times \mathbb{R} & \to & L \times \mathbb{R} \\ (x,s) & \to & (x, s+t). \end{array}$$

Exercise 2.2.1 *Show that if T preserves a unique probability measure μ then the special flow will preserve a unique probability measure that is the normalized product measure of μ on the base and the Lebesgue measure on the fibers diveded by the integral of φ.*

In this section α is an irrational vector in $\mathbb{R}^d$, $d \geq 1$, and we consider special flows constructed over the translation T_α of the torus $\mathbb{T}^d$ with a ceiling function φ. We denote these flows by $T^t_{\alpha,\varphi}$. We will prove the following :

Theorm 2.2.1 *If the vector $\alpha \in \mathbb{R}^d$ is Liouville then for any $r \in \mathbb{N}$ there exists a dense G_δ (for the C^r topology) of functions $\varphi \in C^r(\mathbb{T}^d, \mathbb{R}^*_+)$, such that the special flow $T^t_{\alpha,\varphi}$ is weak mixing for its unique invariant measure.*

To prove optimality of theorem 2.2.1 we need the counterpart statement for Diophantine frequencies.

Theorm 2.2.2 *If the vector $\alpha \in \mathbb{R}^d$ is Diophantine then for every $\varphi \in C^\infty(\mathbb{T}^d, \mathbb{R}^*_+)$, the special flow $T^t_{\alpha,\varphi}$ is C^∞ conjugated to the linear flow $T^t_{(\alpha/\int\varphi, 1/\int\varphi)}$.*

Proof of theorem 2.2.2. The proof is based on the existence of solutions to the linear cohomological equation above Diophantine vectors. There is a general fact about conjugacies of special flows that states as follows.

Proposition 2.2.1 *If $\varphi_1, \varphi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R}_+^*)$ are such that there exists $\chi \in C^\infty(\mathbb{T}^d, \mathbb{R})$ such that*

$$\varphi_1(\theta) - \varphi_2(\theta) = \chi(\theta + \alpha) - \chi(\theta) \tag{2.3}$$

then the special flows T^t_{α,φ_1} and T^t_{α,φ_2} are C^∞ conjugated.

On the other hand we saw in chapter 1 that the linear cohomological equation

$$\varphi(\theta) - \int_{\mathbb{T}^d} \varphi = \chi(\theta + \alpha) - \chi(\theta), \tag{2.4}$$

has a C^∞ solution χ if α is Diophantine and φ is of call C^∞, hence the special flow $T^t_{\alpha,\varphi}$ is C^∞ conjugate to the special flow $T^t_{\alpha,\int \varphi}$ which is nothing but $T^t_{(\alpha/\int \varphi, 1/\int \varphi)}$.

We still need to give a proof of proposition 2.2.1. Let M_i be the spaces where T^t_{α,φ_i} act for $i = 1, 2$. That is $M_i = \mathbb{T} \times \mathbb{R}/\sim_i$ where $(x, s + varphi_i(x)) \sim_i (x + \alpha, s)$. Define $h : M_1 \to M_2 : (x, s) \mapsto (x, s + \chi(x))$. To see that h is a well defined C^∞ diffeomorphism we just need to check that $h(x, s+\varphi_1(x)) = h(x + \alpha, s)$ for any $(x, s) \in M_1$. But $h(x, s + \varphi_1(x)) = (x, s + \varphi_1(x) + \chi(x))$ and due to (2.3) we have that $h(x + \alpha, s) = (x + \alpha, s + \chi(x + \alpha)) = (x+\alpha, s+\chi(x)+\varphi_1(x)-\varphi_2(x)) = (x, s+\chi(x)+\varphi_1(x))$, and the proposition follows.

□

We start the proof of theorems 2.1.1 and 2.2.1 with a very general fact about weak mixing applied in our particular context.

Proposition 2.2.2 *Let $\alpha \in \mathbb{R}^d$ and $r \in \mathbb{N}$, The set of $\phi \in C^r(\mathbb{T}^{d+1}, \mathbb{R}_+^*)$ such that $T^t_{(\alpha,1),\phi^{-1}}$ is weak mixing for its unique invariant probability measure is a G_δ set. Similarly, the set of $\varphi \in C^r(\mathbb{T}^d, \mathbb{R}_+^*)$ such that $T^t_{\alpha,\varphi}$ is weak mixing for its unique invariant probability measure is a G_δ set.*

Proof. We use the following characterization of weak mixing. An ergodic flow (T^t, M, μ) is weak mixing if and only if for any $f \in L^2_0(M, \mu)$ we have

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \left| \int_M f(T^u x)\bar{f}(x) d\mu \right| du = 0. \tag{2.5}$$

Indeed if T^t is weak mixing the above follows from the definition (1) of weak mixing given in section 2.1. Conversely, if T^t is not weak mixing there is a measurable eigenfunction g such that $g(T^t x) = e(\lambda)g(x)$. Ergodicty implies that g is of constant modulus thus g is in L^2 but (2.5) clearly does not hold for g.

Now, if f_j is a countable base in $L^2(M,\mu)$ then (2.5) is equivalent to

$$\lim \inf_{N\to\infty, N\in\mathbb{N}} \frac{1}{N}\int_0^N \left| \int_M f_j(T^u x)\bar{f}_j(x)d\mu \right| du = 0. \tag{2.6}$$

Since the unique invariant probability measure by $T^t_{(\alpha,1),\phi^{-1}}$ μ_ϕ is proportional to the measure of density $\phi(x)dx$ and since ϕ is a smooth strictly positive function, an L^2 countable basis of complex functions $\{f_j\}$ for the Lebesgue measure is also a countable basis for $L^2(\mathbb{T}^{d+1},\mathbb{C})$. We fix such a family $\{f_j\}$. For $r, j, N, k \in \mathbb{N}$ we define

$$A^r(j,N,k) = \Big\{\phi \in C^r(\mathbb{T}^{d+1},\mathbb{R}^*_+) \ : \\ \frac{1}{N}\int_0^N \left| \int_{\mathbb{T}^{d+1}} f_j(T^u_{(\alpha,1),\phi^{-1}}x)\bar{f}_j(x)d\phi(x)dx \right| du < 1/k \Big\}$$

and observe that $A^r(j,N,k)$ is an open set since the quantity estimated is continuous in ϕ. On the other hand $\phi \in C^r(\mathbb{T}^{d+1},\mathbb{R}^*_+)$ satisfies (2.6) if and only if

$$\phi \in \bigcap_{j\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \bigcap_{M\in\mathbb{N}} \bigcup_{N\geq M} A^r(j,N,k)$$

whence the set of $\phi \in C^r(\mathbb{T}^{d+1},\mathbb{R}^*_+)$ such that $T^t_{(\alpha,1),\phi^{-1}}$ is weak mixing for its unique invariant probability measure is a countable intersection of open sets, that is a G_δ set.

□

Considering proposition 2.2.2, to prove theorem 2.2.1, it is enough to show that the set of strictly positive functions φ for which the flow is weak mixing is dense in $C^r(\mathbb{T}^d,\mathbb{R}^*_+)$, for any $r\in\mathbb{N}$.
Still, the way we prove density, we obtain that the set contains a G_δ dense subset. We then use proposition 2.2.2 to conclude that it is exactly a G_δ dense set.

First we state a classical general lemma on weak mixing for ergodic special flows. In this lemma $\{T^t\}$ will be the special flow constructed from an ergodic automorphisim T of a Lebesgue space L and a ceiling function $f > 0$.

Lemma 2.2.1 *The flow $\{T^t\}$ is weak mixing if and only if, for any λ in $\mathbb{R}^*$ the equation*

$$h(T(x)) = e^{i2\pi\lambda f(x)}h(x) \tag{2.7}$$

does not have a non zero measurable complex solution h.

Proof. Assume that (2.7) has a measurable solution h. Let $H(x,s) = h(x)e(\lambda s)$. To see that H is well defined just note that $H(x,f(x)) = h(x)e(\lambda f(x)) = h(x+\alpha) = H(x+\alpha,0)$. Observe then that $H(T^t(x,s)) = H(x,s+t) = e(\lambda t)H(x,s)$ which confirms that T^t has an eigenfunction and is thus not weak mixing.

Conversely, if H is an eigenfunction of T^t with eigenvalue λ, we observe that $h(x) = H(x,0)$ satisfies $h(Tx) = H(Tx,0) = H(x,f(x)) = H(T^{f(x)}(x,0)) = e(\lambda f(x))H(x,0) = e(\lambda f(x))h(x)$.

□

Now we return to the special flow constructed over the rotation automorphism T_α of the torus $\mathbb{T}^d$ with the ceiling function φ and from the Lemma we just stated we obtain a criterion, on the Birkhoff sums of φ, which guarantees weak mixing for the special flow. We will sometimes denote the Birkhoff sums by

$$\varphi_m(\theta) = \sum_{k=0}^{m-1} \varphi(\theta + k\alpha).$$

Lemma 2.2.2 (Criterion for weak mixing) *If for every g in $L^2(\mathbb{T}^d,\mathbb{C})$, and for every λ in $\mathbb{R}^*$, we have*

$$\inf_{m\in\mathbb{N}} \left| \int_{\mathbb{T}^d} e^{i\lambda\varphi_m(\theta)} g(\theta)d\theta \right| = 0,$$

then the flow over T_α with the ceiling function φ is weak mixing.

Proof of Lemma 2.2.2. Since T_α is ergodic any measurable solution of (2.7) has constant modulus and is therefore in L^2. Suppose h is a solution of (2.7), we have

$$h(\theta + m\alpha) = e^{i\lambda\varphi_m(\theta)}h(\theta)$$

and for any $k \in \mathbb{Z}^d$

$$\begin{aligned} e^{-i2\pi m(k,\alpha)} \int_{\mathbb{T}^d} h(\theta) e^{i2\pi<k,\theta>} d\theta &= \int_{\mathbb{T}^d} h(\theta + m\alpha) e^{i2\pi<k,\theta>} d\theta, \\ &= \int_{\mathbb{T}^d} e^{i\lambda\varphi_m(\theta)} h(\theta) e^{i2\pi(k,\theta)} d\theta. \end{aligned}$$

Should the condition of the Lemma be satisfied we would have

$$\inf_{m\in\mathbb{N}} \left| \int_{\mathbb{T}^d} e^{i\lambda\varphi_m(\theta)} h(\theta) e^{i2\pi<k,\theta>} d\theta \right| = 0.$$

Which implies that $\int_{\mathbb{T}^d} h(\theta) e^{i2\pi<k,\theta>} d\theta = 0$ for any $k \in \mathbb{Z}^d$, hence h is zero. □

Remark : It is enough to check the condition of the lemma for the characters $\chi_j(\theta) = e^{i2\pi<j,\theta>}$ that form a basis of L^2.

In all the rest r will be an arbitrary integer. Given the irrational vector $\alpha \in \mathbb{T}^d$ define

$\mathbf{A}^{\mathbf{r}}(\alpha) =$
$\left\{\varphi \in C^r(\mathbb{T}^d, \mathbb{R}) | \forall\lambda \in \mathbb{R}^*, \forall j \in \mathbb{N}, \inf_{m\in\mathbb{N}} \left| \int_{\mathbb{T}^d} e^{i\lambda\varphi_m(\theta)} \chi_j(\theta) d\theta \right| = 0\right\}.$

In light of lemma 2.2.2 density in theorem 2.2.1 will follow from

Proposition 2.2.3 *Fix $r \in \mathbb{N}$. If α is Liouville then $\mathbf{A}^r(\alpha)$ is a G_δ dense subset of $C^r(\mathbb{T}^d, \mathbb{R})$.*

Proof of proposition 2.2.3. For $j, p, k \in \mathbb{N}^3$, define

$$\mathbf{A}^r_{(j,p,k,m)}(\alpha) = \{\varphi \in C^r(\mathbb{T}^d, \mathbb{R}) \ : \ \forall\lambda \in [\frac{1}{p}, p], \left| \int_{\mathbb{T}^d} e^{i\lambda\varphi_m(\theta)} \chi_j(\theta) d\theta \right| < \frac{1}{k}\}.$$

We have

$$\mathbf{A}^r(\alpha) = \bigcap_{j\in\mathbb{N}} \bigcap_{p\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \mathbf{A}^r_{(j,p,k,m)}(\alpha).$$

The set $\mathbf{A}^r_{(j,p,k,m)}(\alpha)$ is obviously open (we took $\lambda \in [\frac{1}{p}, p]$ for this purpose), and we just have to prove that it is dense, which will be the goal of the rest of the section.

First we write the fact that α is not Diophantine :
There exist a sequence $k_n \in \mathbb{Z}^d$ with $\lim_{n\to\infty} |k_n| = +\infty$ such that

$$\|(k_n,\alpha)\| < \frac{1}{|k_n|^{3n}}. \tag{2.8}$$

We introduce now a sequence of real functions on $\mathbb{T}^d$

$$\psi^{(n)}(\theta) = \frac{\cos(2\pi(k_n,\theta))}{|k_n|^n}.$$

Finally let $m_n = |k_n|^{2n}$. The essential fact about m_n is that $|k_n|^n \ll m_n \ll \|(k_n,\alpha)\|^{-1}$.

We will need the following lemmas (the first one is direct from the definition of $\psi^{(n)}$)

Lemma 2.2.3 *For any $r \in \mathbb{N}$ we have*

$$|\psi^{(n)}|_r \underset{n\to\infty}{\longrightarrow} 0.$$

Lemma 2.2.4 *Let $\tilde{\psi}^{(n)} = \cos(2\pi(k_n,\theta))$. Then we have for any m such that $m\|(k_n,\alpha)\| < 1/2$*

$$\tilde{\psi}^{(n)}_m(\theta) = X_{n,m}\cos\left(2\pi(k_n,\theta)+\phi_{n,m}\right),$$

where $X_{n,m} \geq \frac{2}{\pi}m$ and $\phi_{n,m} \in [0,2\pi)$.

Proof of Lemma 2.2.4. We have

$$\begin{aligned}
\tilde{\psi}^{(n)}_{m_n}(\theta) &= \mathrm{Re}\left(1 - e^{i2\pi m_n(k_n,\alpha)}e^{i2\pi(k_n,\theta)}\right), \\
&= \mathrm{Re}\left(e^{i\pi(m_n-1)(k_n,\alpha)}\frac{\sin\left(\pi m_n(k_n,\alpha)\right)}{\sin\left(\pi(k_n,\alpha)\right)}e^{i2\pi(k_n,\theta)}\right), \\
&= X_{n,m}\cos\left(2\pi(k_n,\theta)+\phi_{n,m}\right),
\end{aligned}$$

if we let

$$X_{n,m} = \frac{\sin\left(\pi m(k_n,\alpha)\right)}{\sin\left(\pi(k_n,\alpha)\right)}.$$

Since $m\|(k_n,\alpha)\| < 1/2$, we have $\sin(\pi m(k_n,\alpha)) = \sin(\pi m\|(k_n,\alpha)\|) \geq 2m\|(k_n,\alpha)\|$ while $\sin(\pi(k_n,\alpha)) = \sin(\pi\|(k_n,\alpha)\|) \leq \pi\|(k_n,\alpha)\|$ hence

$$X_{n,m} \geq \frac{2}{\pi}m$$

□

Remark : The functions $\psi^{(n)}$ we introduced are "α" periodic (in the sense that $\psi^{(n)}(x+j\alpha) \sim \psi^{(n)}(x)$) as long as the number of iteration of T_α, is such that $m\|(k_n,\alpha)\| = o(1)$. Consequently, the Birkhoff sums of $\psi^{(n)}$ up to m_n will look like $m_n\psi^{(n)}$. If m_n is moreover such that $\frac{m_n}{D_n} \to \infty$, where D_n is the denominator in the expression of $\psi^{(n)}$, then $\psi^{(n)}_{m_n}$ will have great oscillations. This essential phenomenon will allow us to check the criterion of lemma 2.2.2 by means of the stationnary phase method. The idea next, is to perturb a given function by adding to it a function $\psi^{(n)}$ that will "produce mixing" at time m_n.

From the estimation in lemma 2.2.4 and a basic fact on stationnary phase, we have the following crucial result.

Lemma 2.2.5 *Fix $g \in C^\infty(\mathbb{T}^d,\mathbb{R})$ and $p > 0$, and assume that $\lambda \in [1/p,p]$. There exists a constant $C(g,p) > 0$ such that*

$$\left| \int_{\mathbb{T}^d} e^{i\lambda\psi^{(n)}_{m_n}(\theta)} g(\theta)d\theta \right| \leq CY_n^{-1/8}$$

where $Y_n = |k_n|^n$.

Exercise 2.2.2 *rm Give a proof of lemma 2.2.5. Hint : Assume $d = 1$, the general case being similar. Cut the integral into intervals where the derivative of $\psi^{(n)}_{m_n}$ is larger than $Y_n^{-1/4}$ and intervals where the derivative is less than $Y_n^{-1/4}$ then majorize the integral on each of these intervals.*

Now, we are ready to prove that $\mathbf{A}^r_{(j,p,k)}(\alpha) = \cup_{m\in\mathbb{N}}\mathbf{A}^r_{(j,p,k,m)}(\alpha)$ is dense in $C^r(\mathbb{T}^d,\mathbb{R})$.

Fix a function $\varphi \in C^r(\mathbb{T}^d,\mathbb{R})$ and take $\epsilon > 0$ arbitrarily small. We can assume the integral of φ is zero. We let P be a trigonometrical polynomial with zero average and such that $|\varphi - P|_r < \epsilon$. We let Q be the zero average solution of the linear cohomological solution associated to P, that is $P(x) = Q(x+\alpha) - Q(x)$.

Then for $n \in \mathbb{N}$ we define

$$\varphi^{(n)} = P + \psi^{(n)}.$$

Now, if we take n large enough, we will have, from Lemma 2.2.3 :

$$|\varphi - \varphi^{(n)}|_r < 2\epsilon. \tag{2.9}$$

On the other hand, up to extracting a subsequence from m_n, we can assume that $m_n\alpha$ converges to some vector $\bar{\alpha}$ on $\mathbb{T}^d$ and thus that P_{m_n} converges to a fixed function $Q(x+\bar{\alpha}) - Q(x)$.

Then, applying lemma 2.2.5 we have

$$\int_{\mathbb{T}^d} e^{i\lambda\varphi^{(n)}_{m_n}(\theta)}\chi_j(\theta)d\theta = \int_{\mathbb{T}^d} e^{i\lambda\psi^{(n)}_{m_n}(\theta)}e^{i\lambda P_{m_n}(\theta)}\chi_j(\theta)d\theta \underset{n\to\infty}{\longrightarrow} 0.$$

(We used lemma 2.2.5 for $g(\theta) = e^{i\lambda(Q(\theta+\bar{\alpha})-Q(\theta))}\chi_j(\theta)$, and the fact that $e^{i\lambda P_{m_n}(\theta)}\chi_j(\theta) \to e^{i\lambda(Q(\theta+\bar{\alpha})-Q(\theta))}\chi_j(\theta)$ uniformly when n goes to infinity.)

So when n is sufficiently large, we have that $\varphi^{(n)} \in \mathbf{A}^r_{(j,p,k)}(\alpha)$. The real number ϵ being arbitrarily small this completes the proof of density of $\mathbf{A}^r_{(j,p,k)}(\alpha)$.
proposition 2.2.3 is proved. □

Proof of theorem 2.2.1. In light of lemma 2.2.2, proposition implies that for any $r \in \mathbb{N}$, the set of $\varphi \in C^r(\mathbb{T}^d, \mathbb{R}^*_+)$ for which the flow is weak mixing contains a G_δ dense subset of $C^r(\mathbb{T}^d, \mathbb{R}^*_+)$. Together with proposition 2.2.2, this proves theorem 2.2.1. □

2.3 From special flows to reparametrizations.

In this section we see how the theorems for the special flows $T^t_{\alpha,\varphi}$ translate into theorems for the reparametrized flows $T^t_{(\alpha,1),\phi^{-1}}$.

Consider the submanifold of $\mathbb{T}^{d+1}$, $X = \mathbb{T}^d \times \{0\}$. This is a global Poincaré section to the flow, in the sense that any orbit of $T^t_{(\alpha,1),\phi^{-1}}$ (which are the same orbits as those of the linear flow $T^t_{(\alpha,1)}$) hits X in finite time. Therefore, $T^t_{(\alpha,1),\phi^{-1}}$ can be viewed as a special flow constructed from T_α on $\mathbb{T}^d$ and from the ceiling function given by the time that a point from X takes to come back to X. This time is simple to compute.

$$R(\phi)(\theta) = \varphi(\theta) = \int_0^1 \phi(\theta + \xi\alpha, \xi)d\xi. \tag{2.10}$$

Remark : This formula just translates the fact that the return time to that section is the integral of one over the speed along the fibres. This explains why we used the reparametrizing function in the form ϕ^{-1} instead of ϕ.

From this correspondence between reparametrizations and special flows, theorem 2.1.2 follows immediately from theorem 2.2.2.

Proof of theorem 2.1.1. Fix a Liouville vector $\alpha \in \mathbb{R}^d$ and $r \in \mathbb{N}$.

Introduce the set

$$\mathbf{B}^r(\alpha) = \{\phi \in C^r(\mathbb{T}^{d+1}, \mathbb{R}_+^*) \mid R(\phi) \in \mathbf{A}^r(\alpha)\},$$

and

$$\mathbf{B}^r_{(j,p,k)}(\alpha) = \{\phi \in C^r(\mathbb{T}^{d+1}, \mathbb{R}_+^*) \mid R(\phi) \in \mathbf{A}^r_{(j,p,k)}(\alpha)\}.$$

The function $R(\phi)$ being the ceiling function obtained from ϕ by (2.10).

As in the proof of theorem 2.2.1, we need only to prove that $\mathbf{B}^r_{(j,p,k)}(\alpha)$ is C^r dense in $C^r(\mathbb{T}^{d+1}, \mathbb{R}_+^*)$. We fix ϕ and we take $\widehat{\phi}$ a trigonometrical polynomial close to ϕ. Obviously, $\widehat{\varphi} = R(\widehat{\phi})$ (given by (2.10)) will be a trigonometrical polynomial and, by the proof of theorem 2.2, for n sufficiently large, $\widehat{\varphi} + \psi^{(n)} \in \mathbf{A}^r_{(j,p,k)}(\alpha)$

Recall that

$$\psi^{(n)}(\theta) = \mathrm{Re}\left(\frac{e^{i2\pi(k_n,\theta)}}{|k_n|^n}\right).$$

And define

$$\Psi^{(n)}(\theta, s) = \frac{i2\pi\left((k_n,\alpha) + l_n\right)}{e^{i2\pi((k_n,\alpha)+l_n)} - 1}\mathrm{Re}\left(\frac{e^{i2\pi(k_n,\theta)}e^{i2\pi l_n s}}{|k_n|^n}\right),$$

where l_n is chosen to be the closest integer to $-(k_n, \alpha)$. A straightforward computation implies

$$R\left(\Psi^{(n)}\right) = \psi^{(n)}.$$

Before we conclude, we need to check that $\Psi^{(n)}$ is small.

Lemma 2.3.1 *We have*

$$|\Psi^{(n)}|_r \underset{n\to\infty}{\longrightarrow} 0.$$

Proof of lemma 2.3.1. The choice of l_n such that $|(k_n, \alpha) + l_n| < \frac{1}{2}$ implies

$$\left|\frac{i2\pi\left((k_n,\alpha) + l_n\right)}{e^{i2\pi((k_n,\alpha)+l_n)} - 1}\right| < \frac{\pi}{2}.$$

Since $|l_n| \leq |k_n|(\sum |\alpha_j|) + 1$, $\Psi^{(n)}$, just like $\psi^{(n)}$, goes to 0 in the C^r topology when n goes to infinity. $\square$

Now, because the correspondence in (2.10), $\phi \to R(\phi)$, is linear we will have $R(\widehat{\phi}+\Psi^{(n)}) = \widehat{\varphi}+\psi^{(n)} \in \mathbf{A}_{(j,p,k)}(\alpha)$, or equivalently $\widehat{\phi}+\Psi^{(n)} \in \mathbf{B}_{(j,p,k)}(\alpha)$. By Lemma 2.3.1 and the choice of $\widehat{\phi}$ this last function is close to ϕ which proves density of $\mathbf{B}_{(j,p,k)}(\alpha)$. Hence the set of $\phi \in C^r(\mathbb{T}^{d+1}, \mathbb{R}_+^*)$ such that $T^t_{(\alpha,1),\phi^{-1}}$ is weak mixing for its unique invariant probability measure contains a dense G^δ subset of $C^r(\mathbb{T}^{d+1}, \mathbb{R}_+^*)$. Theorem 2.1.1 on reparametrization now follows from the combination of the latter result with proposition 2.2.2. □

2.4 Rigidity of the reparametrized flows

In this section, we will see that in the case $d = 1$ the reparametrized flows $T^t_{(\alpha,1),\phi^{-1}}$ are always rigid in the following sense.

Definition 2.4.1 *We say that a flow T^t (or a map T) defined on a smooth manifold M is C^r rigid if there exists a sequence t_n such that T^{t_n} converges uniformly to the Identity map in the C^r topology.*

In the case α is a Diophantine number this is not surprising and comes directly from the fact that $T^t_{(\alpha,1),\phi^{-1}}$ is conjugate to a linear irrational flow (any irrational α can be approxiamted by rationals p_n/q_n faster than $1/q_n^2$ which automatically yields rigidity of $T^t_{(\alpha,1)}$ since $T^{q_n}_{(\alpha,1)} \to \mathrm{Id}$). But in the Liouvillean case we get that there exists a sequence q_n such that $T^{q_n}_{(\alpha,1),\phi^{-1}} \to \mathrm{Id}$ while the $T^t_{(\alpha,1),\phi^{-1}}$ is in general weak mixing, in which case the general sequence of times t_n is a mixing sequence for $T^t_{(\alpha,1),\phi^{-1}}$.

Theorm 2.4.1 *For any $\alpha \in \mathbb{R} - \mathbb{Q}$ and any $\phi \in C^\infty(\mathbb{T}^2, \mathbb{R}_+^*)$, the flow $T^t_{(\alpha,1),\phi^{-1}}$ is C^∞ rigid.*

Proof. We will prove the equivalent fact that $T^t_{\alpha,\varphi}$ is C^∞-rigid for any $\alpha \in \mathbb{R} - \mathbb{Q}$ and any $\varphi \in C^\infty(\mathbb{T}, \mathbb{R}_+^*)$ The proof is based on the so called improved Denjoy-Koksma inequality, a variant of which we now state.

Lemma 2.4.1 *For any $\alpha \in \mathbb{R} - \mathbb{Q}$ and any $\varphi \in C^\infty(\mathbb{T}, \mathbb{R}_+^*)$, there exists q_n such that $\|q_n\alpha\| \to 0$ and for any $r \in \mathbb{N}$*

$$\lim_{n\to\infty} \|S^\alpha_{q_n}\varphi - q_n \int_{\mathbb{T}} \varphi\|_r = 0. \tag{2.11}$$

Before we prove this technical lemma let us observe first how theorem 2.4.1 can be derived from it. Without loss of generality we assume that $\int_{\mathbb{T}} \varphi = 1$. Fix now $\epsilon > 0$ such that $2\epsilon < \inf_{x\in\mathbb{T}} \varphi(x)$. For $x \in \mathbb{T}$ let us look at $T^{q_n}_{\alpha,\varphi}(x,\epsilon)$. We know that

$$T^{q_n}_{\alpha,\varphi}(x,\epsilon) = (x + m\alpha, \epsilon + q_n - S^{\alpha}_{m}\varphi(x))$$

where m is the unique integer such that $0 < \epsilon + q_n - S^{\alpha}_{m}\varphi(x) < \varphi(x + m\alpha)$. Since for any r, $\lim_{n\to\infty} \|S^{\alpha}_{q_n}\varphi - q_n\|_r = 0$ we get that for any $x \in \mathbb{T}$ $m = q_n$ and since $\|q_n\alpha\| \to 0$ we have that $T^{q_n}_{\alpha,\varphi}(x,\epsilon) \to (x,\epsilon)$ in any C^r norm. Finally any $(x,s) \in M$ can be written as $T^{s'}_{\alpha,\varphi}(x,\epsilon)$ for some $-\epsilon < s' < \max_{x\in\mathbb{T}} \varphi(x)$, thus

$$T^{q_n}_{\alpha,\varphi}(x,s) = T^{s'}_{\alpha,\varphi}T^{q_n}_{\alpha,\varphi}(x,\epsilon) \sim T^{s'}_{\alpha,\varphi}(x,\epsilon) = (x,s)$$

and the proof of theorem 2.4.1 will be complete once we give the

Proof of lemma 2.4.1. For $k \in \mathbb{Z}^*$ and $\chi_k(x) = e(x)$ we write for $m \in \mathbb{Z}$: $S^{\alpha}_{m}\chi_k = X_{m,k}\chi_k$, $|X_{m,k}| = |\sin(\pi m k\alpha)/\sin(\pi k\alpha)|$. From the properties of the sequence q_n of denominators of convergents of α (see proposition 1.4.1 of chapter **??**) we get that $|X_{q_n,k}| \leq \frac{\pi}{2}|k|q_l/q_{n+1}$ for every $l \leq n$ and $|k| < q_l$. On the other hand we trivially have $|X_{m,k}| \leq m$ for all $m \in \mathbb{Z}$.

Let us now compute $\|S^{\alpha}_{q_n}\varphi - q_n\int\varphi\|_r$ using Fourier series.

$$\begin{aligned}\|S^{\alpha}_{q_n}\varphi - q_n\|_r &\leq C\sum_{k\in\mathbb{Z}^*} |\widehat{\varphi}_k||k|^{r+1}|X_{q_n,k}| \\ &\leq C\frac{q_l}{q_{n+1}}\sum_{|k|<q_l} |\widehat{\varphi}_k||k|^r + Cq_n\sum_{|k|\geq q_l} |\widehat{\varphi}_k||k|^r\end{aligned}$$

Now since $\varphi \in C^\infty(\mathbb{T},\mathbb{R})$ we know that $\sum_{|k|\geq q_l} |\widehat{\varphi}_k||k|^{r+2}$ converges therefore

$$\|S^{\alpha}_{q_n}\varphi - q_n\|_r \leq C\frac{q_l}{q_{n+1}} + C\frac{q_n}{q_l^2} \tag{2.12}$$

To finish we need to consider two cases.

Case 1 : α is not of constant type. That means that there exists a subsequence q_{n_i} of its denominators such that $q_{n_i+1}/q_{n_i} \to 0$ as $i \to \infty$. If we take $n = l = n_i$ in (2.12) we then get

$$\|S^{\alpha}_{q_{n_i}}\varphi - q_{n_i}\|_r \leq C\frac{q_{n_i}}{q_{n_i+1}} + \frac{C}{q_{n_i}}$$

whence $\|S^{\alpha}_{q_{n_i}}\varphi - q_{n_i}\|_r \to 0$ as required.

Case 2 : α is not of constant type. That means that there exists K such that $q_{l+1} \leq K q_l$ for any l. We deduce that for any n sufficiently large there exists $l_n \leq n$ such that $q_{l_n} \in [q_n/n^2, q_n/n]$. If we take $l = l_n$ in (2.12) we then get

$$\|S^{\alpha}_{q_n}\varphi - q_n\|_r \leq \frac{C}{n} + \frac{Cn^4}{q_n}.$$

Since $q_{l+2} \geq 2q_l$ for every l this means that $q_n \geq 2^{n/2}$ and we get the required $\|S^{\alpha}_{q_n}\varphi - q_n\|_r \to 0$.

□

Exercise 2.4.1 What is the minimal regularity of φ you can give to insure that $T^t_{\alpha,\varphi}$ is C^0-rigid ?

Chapitre 3

The Poincaré Siegel theorem on linearization of holomorphic germs

In this chapter, we see how holomorphic germs $f : \mathbb{C} \to \mathbb{C} : z \mapsto \lambda z + \mathcal{O}(z^2)$ with $|\lambda| \neq 1$ or with $\lambda = e(\alpha) := e^{i2\pi\alpha}$ with α Diophantine are linearizable in the neighborhood of the origin, which means that they are conjugated by a holomorphic map to the map $\Lambda : z \mapsto z$: there exists h biholomorphic bijection defined on a neighborhood of 0 such that $h\phi = \Lambda h$. This result will be based on the solvability of the linearized equation $\Lambda\Delta h - \Delta h\Lambda = \Delta f$ for $f = \mathcal{O}(z^2)$ and on a quadratically convergent scheme of successive conjugations known as the KAM scheme (Kolmogorov Arnol'd Moser). The proof for the case $|\lambda| \neq 1$ was obtained by Poincaré by direct computation of the coefficients of the linearizing germ h from those of f. In this case there are no small divisors and the majorization of the coefficients of h is relatively simple (see for example section 2.1.B in the book of Katok Hasselblatt : Introduction to the modern theory of dynamical systems).

When $= e(\alpha)$ with α Diophantine the linearization result was first discovered by Siegel (in 1952)who handled delicate estimates involving the small divisors that appear in the coefficients of h if one computes them directly from those of f.

The proof we give is an example of application of the KAM quadratic scheme to reach the conjugacy and follows the lines of an article published by Moser in 1966 and can also be found in the aforementioned book of Katok Hasselblatt, in section 2.8.

This is introduction to KAM theory will be helpful in understanding the analysis of the quasi-periodic Schrödinger operator for small real analytic potentials.

A proof of a more general theorem based on majorizing series can be found for example at

3.1 Notations and result

For $\Delta > 0$ we denote $B_\Delta = \{z \in \mathbb{C} : |z| \leq r\}$. We will denote C^ω_Δ the set of holomorphic maps f such that $f(0) = 0$ and the radius of convergence of f is larger than r. We will denote $\bar{C}^\omega_\Delta$ the set of $f \in C^\omega_\Delta$ and the power series of f is convergent and continuous for $|z| \leq \Delta$. We then define a norm on $\bar{C}^\omega_\Delta$ given by $|f|_\Delta := \sup_{|z|\leq\Delta} |f(z)| < \infty$.

Theorm 3.1.1 *Let $f(z) = \lambda z + \sum_{k\geq 2} f_k z^k$ be a holomorphic map in a neighborhood of 0. Assume that $|\lambda| \neq 1$ or $\lambda = e(\alpha)$ with α Diophantine, then there exists $\Delta > 0$ and a $h \in C^\omega_\Delta : h(z) = z + \sum_{k\geq 2} h_k z^k$ such that h is a biholomorphic bijection from B_Δ onto its image and such that $h \circ f \circ h^{-1}$ is defined on $B_{\Delta/2}$ and*

$$h \circ f \circ h^{-1}(z) = \lambda z, \text{ for } |z| \leq \Delta/2.$$

3.2 Proof of the Poincaré Siegel theorem

Exercise 3.2.1 *Let u_n be a sequence of positive real numbers such that $u_{n+1} \leq 2^n u_n^2$. Show that if $u_0 < 1/2$ then u_n converges to 0 and $u_n = o(a^{-n})$ for any $a > 1$.*

We will always assume that $|| \leq 1$, the case $|| > 1$ being reduced to that case by consideration of f^{-1}. We will denote f by $f = \Lambda + \Delta f$ where $\Delta f = \sum_{k\geq 2} f_k z^k$ will always correspond to holomorphic functions in a neighborhood of zero such that $\Delta f = \mathcal{O}(z^2)$. We look for the conjugacy under the form $I + \Delta h$ with $I : z \mapsto z$.

Observe that $|\Delta f'|_\Delta \to 0$ as $\Delta \to 0$ which will allow us to treat f as a perturbation of Λ in the neighborhood of 0.

Lemma 3.2.1

a) *Let $f \in \bar{C}^\omega_\Delta$, then $|f_k|\Delta^k \leq |f|_\Delta$.*

b) *Let $d \in \mathbb{N}$ and assume $|f_k| \le K|k|^d \Delta^{-k}$ then $f \in C^\omega_\Delta$ and for any $\delta > 0$, $|f|_{\Delta(1-\delta)} \le C(d)\frac{K}{\delta^{d+1}}$.*

c) *There exists $C_0 > 0$ such that if $f \in \bar{C}^\omega_\Delta$, then $f' \in C^\omega_\Delta$ and for any $\delta > 0$: $|f'|_{\Delta(1-\delta)} \le \frac{C_0}{\delta^2}|f|_\Delta$.*

Proof.

a) $f_k \Delta^k = \int_{\mathbb{T}} f(\Delta e(\theta)) e(-k\theta) d\theta$.

b) $|f|_{\Delta(1-\delta)} \le \sum |f_k| \Delta^k (1-\delta)^k \le K \sum |k|^d (1-\delta)^k \le C(d)\frac{K}{\delta^{d+1}}$. The last inequality is obtained by observing that $|k|^d$ is smaller than the k coefficient that appears in the power series expansion of the absolute value of the d^{th} derivative of the function $\delta \mapsto 1/\delta$ (expanded around $(1-\delta) \sim 0$, that is write the expansion of $1/(1-u)$ with $u = 1-\delta$).

c) Apply **a**) to bound the coefficients of the series of f then apply **b**) to bound $|f'|_{\Delta(1-\delta)}$.

□

Hereafter we will assume that $\alpha \in \mathrm{DC}(d-1)$, that is there exists $\gamma > 0$ such that for any $k \in \mathbb{Z}^*$

$$\|k\alpha\| \ge \frac{\gamma}{|k|^d}$$

Proposition 3.2.1 *Let $\Delta f \in \bar{C}^\omega_\Delta$, then there exists $\Delta h \in C^\omega_\Delta$ such that*

$$\Lambda \Delta h - \Delta h \Lambda = \Delta f \tag{3.1}$$

and

$$|\Delta h'|_{\Delta(1-\delta)} \le \frac{C(d)}{\gamma}\delta^{-d-2}|\Delta f|_\Delta \tag{3.2}$$

Proof. We simply have that

$$\Delta h_k = \frac{\Delta f_k}{k \ \_}$$

hence $|\Delta h'_k| \le \frac{C}{\gamma}|k|^{d+1}|\Delta f_k|$ which implies the estimate of the proposition in light of lemma 3.2.1. □

Proposition 3.2.2 *Let $\Delta, \delta > 0$ be sufficiently small and such that $|\Delta f'|_\Delta \le \epsilon \le \delta^{d+10}$. Then there exists $\Delta h \in C^\omega_\Delta$ such that $f_1 = h \circ f \circ h^{-1}$, with $h = I + \Delta h$, is holomorphic on $B_{\Delta(1-\delta)^4}$ and $f_1 = \Lambda + \Delta f_1$ with*

$$|\Delta f_1'|_{\Delta(1-\delta)^5} \le C\delta^{-d-4}\epsilon^2.$$

Moreover Δh is as in proposition 3.2.1, and thus

$$|\Delta h'|_{\Delta(1-\delta)} \leq \frac{C(d)}{\gamma}\delta^{-d-2}\epsilon$$

Proof. Because Δh solves (5.2.2), it satisfies (3.2), and because we assumed that $|\Delta f'|_{\Delta} \leq \epsilon \leq \delta^{d+10}$ it is easy to show that $h^{-1}(B_{\Delta(1-\delta)^4}) \subset B_{\Delta(1-\delta)^3}$ while $f(B_{\Delta(1-\delta)^3}) \subset B_{\Delta(1-\delta)^2}$ so that f_1 is well defined.

We now write $h \circ f = f_1 \circ h$. Because Δh solves (5.2.2) we get that

$$\Delta f_1 \circ h = \Delta h \circ f - \Delta h \circ \Lambda$$

Hence

$$\Delta f_1 = (\Delta h \circ f - \Delta h \circ \Lambda) \circ h^{-1}.$$

But $h^{-1}(B_{\Delta(1-\delta)^4}) \subset B_{\Delta(1-\delta)^3}$ and $\Lambda(B_{\Delta(1-\delta)^3}) \subset B_{\Delta(1-\delta)^2}$ as well as $f(B_{\Delta(1-\delta)^3}) \subset B_{\Delta(1-\delta)^2}$ so

$$|\Delta f_1|_{\Delta(1-\delta)^4} \leq |\Delta h'|_{\Delta(1-\delta)^2}|\Delta f|_{\Delta(1-\delta)^3} \leq C\delta^{-d-2}\epsilon^2$$

and the estimate of $|\Delta f_1'|_{\Delta(1-\delta)^5}$ follows from lemma 3.2.1. □

The proof of theorem 3.1.1 will be obtained by an inductive application of proposition 3.2.2.

For $\eta > 0$, define $\delta_n = \frac{\eta}{2^n}$ and $\Delta_n = \Delta_{n-1}(1-\delta_n)^5$. We fix $\eta > 0$ sufficiently small so that $\Delta_n \geq \Delta_0/2$ for every $n \in \mathbb{N}$. Then fix Δ_0 sufficiently small so that $\epsilon_0 := |\Delta f'|_{\Delta_0} \leq \delta_0^{d+10}$.

We apply proposition 3.2.2 inductively : $f_0 = f$, $f_1 = h_0 \circ f_0 \circ h_0^{-1}$, $f_2 = h_1 \circ f_1 \circ h_1^{-1} \ldots$

We denote $H_n = h_{n-1} \circ \ldots \circ h_0$, so that $f_n = H_n \circ f \circ H_n^{-1}$. We also introduce $\epsilon_n := |\Delta f_n'|_{\Delta_n}$.

To justify the possibility of an inductive application of proposition 3.2.2 and the well definiteness of f_n we will need to verify that for every n : $\epsilon_n \leq \delta_n^{d+10}$.

But the proposition says that $\epsilon_n \leq C\delta_{n-1}^{-d-4}\epsilon_{n-1}^2$ (with $C := C(d,\gamma)$), or equivalently that $\epsilon_n \leq C'a^{n-1}\epsilon_{n-1}^2$ if we let $C' = C\eta^{-d-4}$ and $a = 2^{d+4}$. If we define $\bar{\epsilon}_n = C'a^{n+1}\epsilon_n$ we observe that $\bar{\epsilon}_n \leq \bar{\epsilon}_{n-1}^2$ and thus $\bar{\epsilon}_n \leq (\bar{\epsilon}_0)^{2^n}$.

This immediately gives us that if we chose Δ_0 sufficiently small so that ϵ_0 and therefore $\bar{\epsilon}_0$ be sufficiently small we will have that ϵ_n converges quadratically to 0 (like b^{2^n} for some $b \ll 1$), and by the same token that the inductive hypothesis $\epsilon_n \leq \delta_n^{d+10}$ is indeed satisfied.

We proved that for $\Delta_0 > 0$ sufficiently small $|f_n|_{\Delta_0/2} \to 0$. It remains to show that H_n converges.

It is clear that since $|\Delta h_n|_{\Delta_0/2} \le C(d)\delta_n^{-d-2}\epsilon_n$ then if ϵ_n converges quadratically to 0 then $\epsilon'_n := C(d)\delta_n^{-d-2}\epsilon_n$ also converges quadratically to 0. Hence if Δ_0 is chosen sufficiently small it will follow that $H_n(B_{\Delta_0/4}) \subset B_{\Delta_0/2}$, so that $H_n \in \bar{C}^\omega_{\Delta_0/8}$ and there is a constant $C > 0$ such that $|H_{n+1} - H_n|_{\Delta_0/8} = |\Delta h_n \circ H_n|_{\Delta_0/4} \le |\Delta h_n|_{\Delta_0/2}$. The latter implies that H_n is a Cauchy sequence in the Banach space $\bar{C}^\omega_{\Delta_0/8}$ and yields its convergence to some $H_\infty \in \bar{C}^\omega_{\Delta_0/8}$.

The same argument shows that if Δ_0 is chosen sufficiently small then H_n^{-1} is a Cauchy sequence in the Banach space $\bar{C}^\omega_{\Delta_0/8}$ and converges to some $H_\infty^{-1} \in \bar{C}^\omega_{\Delta_0/8}$.

Moreover, we have $H_\infty \circ f \circ H_\infty^{-1}(z) = \Lambda(z)$ for $|z| \le \Delta_0/16$. The proof of theorem 3.1.1 is thus completed. $\square$

3.3 The Liouville phenomenon. Cremer cycles.

In the next exercise, we show that if α is super-Liouville, then $f(z) = e^{i2\pi\alpha}z + z^2$ is not linearizable since it has infinitely many cycles accumulating the origin.

Exercise 3.3.1 Let $f : \mathbb{C} \to \mathbb{C} : f(z) = e^{i2\pi\alpha}z + z^2$. Assume that α is such that there exist infinitely many q_n such that $\|q_n\alpha\| \le n^{-2^{q_n}}$

1. Show that there exists a sequence $z_n \to 0$ such that $f^{q_n}(z_n) = z_n$ (here $f^2 = f \circ f,\ f^3 = f \circ f \circ f, \ldots$). *Hint :* Compute the coefficient of z in $f^{q_n}(z) - z$ and then express this coefficient in terms of the roots of the polynomial $f^{q_n}(z) - z$.

2. Conclude that f is not topologically conjugated to $\Lambda(z) = e^{i2\pi\alpha}z$.

Chapitre 4

$\mathrm{SL}(2,\mathbb{R})$ cocycles

4.1 Introduction

Given a diffeomorphism f acting on a manifold M, a fundamental approach to understand the dynamics of f is to first analyze the dynamics of its tangent map, that is the geometry of $Df^n(x)$ where f^n is the n times composition of f. For example if the norm of $Df^n(x)$ grows geometrically at some point x then this means that some nearby points in the neighborhood of x may get split exponentially fast (in n) under the action of f^n : this is called sensitive dependance on initial conditions and is the basis phenomenon behind what is commonly called chaotic dynamics. To the contrary if the norm of Df^n remains bounded at all points then this means that the dynamics of f is related to that of translations on tori or sphere rotations.

Since $Df^n(x) = Df(f^{n-1}(x)).Df(f^{n-2}(x))\dots Df(x)$, the dynamics of Df^n can be thought as a product of matrices (or composition of linear operators) "above" the orbits of f. This leads to a generalization of the problem of understanding $Df^n(x)$ to that of understanding the product of linear maps or matrices "above" the dynamics of f. By this we mean that given a matrix map $x \in X \mapsto A(x) \in M(d,\mathbb{R})$, we are interested in the products $A(f^n(x))A(f^{n-1}(x))\dots A(x)$. The map $(x,v) \in M \times \mathbb{R}^d \mapsto (f(x), A(x)v)$ is called a linear cocycle above f with the cocycle map being $A(\cdot)$. Linear cocycles above f are skew products above f that act linearly on the second coordinate (a *skew product* over T is a dynamical system F acting on a product space $X \times Y$ such that $\pi \circ F = T$, where $\pi : X \times Y \to X$ is the projection on the first coordinate). They can also be viewed as a multiplicative version of the linear skew products that act by translation on the second coordinate that we have seen in the previous chapter. Notice that the case $A(\cdot) = Df(\cdot)$ is a particular case and that some general results that are obtained for linear cocycles may be applied to the particular case of Df but that it turns out that the study of the behavior of linear cocycles when the map A is independent on f is often much simpler then the special case $A = Df$.

Let us first understand the picture when a single linear map $A \in \mathrm{GL}(m,\mathbb{R})$ is iterated. Let $v \in \mathbb{R}^m$. We are interested in the rate of growth of $\|A^n v\|$ as $n \to \infty$. It is easy to see, by writing A in its real Jordan form, for instance, that $\mathbb{R}^m$ can be decomposed into A-invariant subspaces $E_1 \oplus \ldots \oplus E_r$ in such a way that for each i,there is a number λ_i s.t. $\forall v \in E_i - \{0\}$,

$$\lim \frac{1}{n} \ln \|A^n v\| = \pm\lambda_i, n \to \pm\infty$$

The λ_i are the log of the moduli of the eigenvalues of A.

It is of course easy to construct diffeomorphisms f on a manifold M and matrix maps $A : M \to \mathrm{GL}(m,\mathbb{R})$ so that the product $A(f^n(x))A(f^{n-1}(x)) \ldots A(x)$ behaves erratically on many points x. But the Oseledts theory, which can actually be viewed as a multiplicative version of the Birkhoff Ergodic Theorem, tells us that in the case of invertible maps f that preserve a probability measure μ, then the asymptotic behavior of $A(f^n(x))A(f^{n-1}(x)) \ldots A(x)$ as $n \to \infty$ is, for μ-almost every point x, similar to that of a single linear map that is iterated.

We will state and prove Oseledets theorem only in the case of two dimensional matrices of determinant 1 ($\mathrm{SL}(2,\mathbb{R})$ matrices) and we refer to L. Arnold *Random dynamical systems*, Springer 1998 for general statements and proofs. The $\mathrm{SL}(2,\mathbb{R})$ case already contains most of the ideas and is easier to expose.

We recall that a matrix $A \in \mathrm{SL}(2,\mathbb{R})$ is called *hyperbolic* if its trace is strictly larger than 2 which is also equivalent to A having one real eigenvalue strictly larger than 1 and one eigenvalue strictly smaller than 1. A matrix with trace strictly less than two is called *elliptic* and it is conjugated to a matrix in the group $\mathrm{SO}(2,\mathbb{R})$ of rotation matrices. The "degenerate" case of matrices with trace exactly two is called *parabolic.*

For $v \in \mathbb{R}^2$ we denote $\|v\|$ the Euclidean norm of v, and use a similar notation for its induced operator norm on $A \in \mathrm{SL}(2,\mathbb{R})$, that is $\|A\| = \sup_{\|v\|=1} \|Av\|$.

We will very frequently the following geometric fact from the linear algebra of $\mathrm{SL}(2,\mathbb{R})$ matrices.

Any matrix $A \in \mathrm{SL}(2,\mathbb{R})(2,R)$ can be written as $A = RDS$, where R and S belong to $\mathrm{SO}(2,\mathbb{R})$ (the group of rotations), and $D \in \mathrm{SL}(2,\mathbb{R})$ is diagonal with non-negative entries. These entries are called the *singular values* of A. Notice then that $\|A\| = \|A^{-1}\| = \|D\|$. The matrix A is a rotation if and only if D is the identity matrix.

Lemma 4.1.1 *If A is not a rotation, then there are unit vectors $u(A) \perp s(A)$ unique modulo sign, such that*

$$\|A \cdot u(A)\| = \|A\|, \quad \|A \cdot s(A)\| = \|A\|^{-1}\,.$$

Moreover $A(u(A))$ and $A(s(A))$ are collinear respectively to $s(A^{-1})$ and $u(A^{-1})$. Notice $u(A)$ and $s(A)$ are the eigenvectors of the symmetric matrix A^*A.

$\mathrm{SL}(2,\mathbb{R})$ **cocycles. Notations.**

Given a dynamical system (T, X, μ) and a measurable map $A : X \to \mathrm{SL}(2,\mathbb{R})$, the group of real 2×2 matrices with determinant 1, we define an $\mathrm{SL}(2,\mathbb{R})$-*cocycle* over (T, X, μ) with in the fibres the matrix map A by

$$F\colon (x, v) \in X \times \mathbb{R}^2 \mapsto (Tx, A(x) \cdot v) \in X \times \mathbb{R}^2, \quad \text{where} \quad A : X \to \mathrm{SL}(2,\mathbb{R})\,.$$

Since the pair (T, A) specifies $F = F_{T,A}$, we also call it a cocycle. Sometimes (when the underlying T is fixed) we also call the map A a cocycle.

The powers F^n can be written as $F^n(x, v) = (T^n x, A^n_T(x) \cdot v)$, where

$$A^n_T(x) = A(T^{n-1}x) \cdots A(x) \quad (\text{for } n > 0).$$

In the case T is invertible, F is also invertible since we can define

$$A^{-n}_T(x) = A(T^{-n}x)^{-1} \cdots A(T^{-1}x)^{-1} \quad (\text{for } n > 0), \quad A^0_T(x) = \mathrm{Id}.$$

The following identity '

$$A^{m+n}_T(x) = A^m_T(T^n x)\, A^n_T(x) \quad \text{for all } x \in X,\ m,\ n \in \mathbb{Z}. \tag{4.1}$$

is called the "cocyle identity".

Most of the time T will be fixed and we write for simplicity $A^n(x)$ instead of $A^n_T(x)$.

As we said, Oseledets theory will apply to cocycles where the base map preserves a probability measure μ. So we will always assume that in the base we have a dynamical system (T, X, μ) and since we will only be intersted in the behavior of the products $A^n(x)$ (and $A^{-n}(x)$ in the case T is invertible) only on a set of full μ-measure, we will only assume that the map A is μ-measurable. Sometimes, especially when the base dynamics is regular, it is interesting to consider regular maps A : analytic, smooth, continuous...

Let us mention two basic examples where the $\mathrm{SL}(2,\mathbb{R})$ cocycles appear :

Example 4.1.1 Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. Given a diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ that preserves area and orientation, we can define a $\mathrm{SL}(2,\mathbb{R})$-cocycle (T, A) by taking $X = \mathbb{T}^2$, μ as Lebesgue measure, $T = f$, and A as the derivative of f (because the tangent bundle is trivial, i.e., $TX = X \times \mathbb{R}^2$).

Example 4.1.2 Let H_1 and H_2 be two matrices in $\mathrm{SL}(2,\mathbb{R})$. Suppose we multiply these matrices randomly with probability p_1 we multiply by H_1 (on the left) and with probability p_2 we multiply by H_2 (always on the left). We want to understand what is the statistical behavior of the products when many trials are repeated. We look for results of the kind of the law of large numbers but in this multiplicative context. The good setting to study this problem is to consider the space $X = \{0,1\}^{\mathbb{N}}$ of infinite "one-sided words" formed with the "letters" 0 and 1. We equip X with the product measure $\mu = p_1\delta_0 + p_2\delta_1$ and consider on X the measure preserving "shift" map $\sigma(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. Then we consider the matrix map defined over X as follows $A(x_0, x_1, \ldots) = H_1$ if $x_0 = 0$ and $A(x_0, x_1, \ldots) = H_2$ if $x_0 = 1$. Our problem becomes to understand the behavior of the cocycle $A^n(x)$ for μ-a.e. $x \in X$. A celebrated result of Furstenberg states that unless in some obvious cases such as H_1 and H_2 are commuting matrices in $\mathrm{SO}(2,\mathbb{R})$, the products $A^n(x)$ will grow exponentially for μ-almost every x and will actually behave similarly to the case of a single hyperbolic matrix H that is multiplied.

In Sections 4.2–4.4 below we follow very closely the presentation of Avila and Bochi text *Trieste Lecture Notes on Lyapunov Exponents Part I.*

4.2 The Lyapunov Exponent

Given a cocycle $F = F_{T,A}$, we want to understand the behavior of typical orbits $F^n(x,v) = (T^n x, A^n(x)\cdot v)$, as it is usual in Ergodic Theory. Thus we aim to obtain information about the sequence of matrices $A^n(x)$, for a full-measure set of points x. The most basic information of this kind concerns asymptotic growth.

Theorm 4.2.1 (Furstenberg–Kesten Theorem [?] for $\mathrm{SL}(2,\mathbb{R})$) *Assume $T : X \to X$ is a μ-preserving transformation and $A : X \to \mathrm{SL}(2,\mathbb{R})$ is a measurable map such that :*

$$\int \log \|A(x)\|\, d\mu(x) < \infty. \tag{4.2}$$

Then for μ-almost every $x \in X$, the following limit exists :

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|. \tag{4.3}$$

The function $\lambda : X \to [0,\infty)$ is T-invariant $\lambda(Tx) = \lambda(x)$ for μ-a.e. $x \in X$, λ is μ-integrable, and its integral is given by

$$\Lambda = \int \lambda(x)\, d\mu(x) = \lim_{n\to\infty} \frac{1}{n} \int \log \|A^n(x)\|\, d\mu(x) = \inf_{n\geq 1} \frac{1}{n} \int \log \|A^n(x)\|\, d\mu(x)\,. \tag{4.4}$$

We call (4.2) the *integrability condition*. The number $\lambda(x)$ is called the *(upper) Lyapunov exponent* at the point x, and Λ is called the *integrated (upper) Lyapunov exponent*. If T is ergodic then λ is constant equal to Λ almost everywhere, so we write $\lambda = \Lambda$ for simplicity.

Also notice that any norm on $\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)$ would work equally well in the statement of Theorem 4.2.1 and would yield the same results with the same function $\lambda(x)$ and the same number Λ.

The proof of Theorem 4.2.1 is based on Kingman's Subadditive Ergodic Theorem of which we just recall the statement

Theorm 4.2.2 (Kingman's Subadditive Ergodic Theorem) *Let $f_n : X \to \bar{\mathbb{R}}$ be a sequence of measurable functions such that f_1^+ is μ-integrable and*

$$f_{m+n} \leq f_m + f_n \circ T^m \quad \textit{for all } m,\ n \geq 1.$$

Then $\frac{1}{n} f_n$ *converges a.e. to a function* $f : X \to \bar{\mathbb{R}}$*. Moreover,* f^+ *is* μ*-integrable and*

$$\int f = \lim_{n\to\infty} \frac{1}{n} \int f_n = \inf_{n\geq 1} \frac{1}{n} \int f_n \in \mathbb{R} \cup \{-\infty\}$$

A sequence of functions as in the hypotheses of the theorem is called *subadditive.*

Proof. [Proof of Theorem 4.2.1] The sequence of functions $f_n(x) = \log \|A^n(x)\|$ is subadditive, and f_0 is integrable. Therefore Theorem 4.2.2 assures that f_n/n converges almost everywhere to a function λ. Since $f_n \geq 0$, $\lambda \geq 0$. Theorem 4.2.2 also gives (4.4). □

4.3 Oseledets Theorem

??

Theorem 4.2.1 gives information about the growth of the matrices $A^n(x)$, while the Oseledets Theorems below describe the asymptotic behavior of vectors $A^n(x) \cdot v$.

Theorm 4.3.1 (One–Sided Oseledets) *Let* $T : X \to X$ *be a* μ*-preserving transformation and* $A : X \to \mathrm{SL}(2, \mathbb{R})$ *satisfy the integrability condition* (4.2)*. Let* $\lambda(\cdot)$ *be the Lyapunov exponent of the cocycle* (T, A)*.*

For a.e. x *such that* $\lambda(x) = 0$ *we have*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = 0 \quad \text{for every } v \in \mathbb{R}^2 \setminus \{0\}. \tag{4.5}$$

For a.e. $x \in X$ *such that* $\lambda(x) > 0$*, there exists a one-dimensional vector space* $E^-_x \subset \mathbb{R}^2$ *such that :*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = \begin{cases} \lambda & \text{for all } v \in \mathbb{R}^2 \setminus E^-_x \\ -\lambda & \text{for all } v \in E^-_x \setminus \{0\} \end{cases} \tag{4.6}$$

Moreover, the spaces E^-_x *depend measurably on the point* x *and are invariant by the cocycle.*

Measurability of the spaces E^-_x means that they give a measurable map from the set $\{x;\ \lambda(x) > 0\}$ to $\mathbb{P}^1$ (the projective space of $\mathbb{R}^2$), while invariance means that $A(x) \cdot E^-_x = E^-_{Tx}$.

Thus Theorem 4.3.1 says that if $\lambda(x) > 0$ then $\|A^n(x)\cdot v\|$ grows like $e^{n\lambda(x)}$ for v in all directions in $\mathbb{R}^2$, except for one direction for which the growth is like $e^{-n\lambda(x)}$.

For invertible cocycles we have :

Theorm 4.3.2 (Two-sided Oseledets) *Let T be an invertible bimeasurable transformation of the probability space (X,μ), and let $A : X \to \mathrm{SL}(2,\mathbb{R})$ satisfy the integrability condition. For a.e. point x where the Lyapunov exponents is positive, there exists a splitting $\mathbb{R}^2 = E_x^+ \oplus E_x^-$ into two linear one-dimensional subspaces such that* (4.6) *holds and*

$$\lim_{n\to-\infty} \frac{1}{|n|} \log \|A^n(x)\cdot v\| = \begin{cases} \lambda & \textit{for all } v \in \mathbb{R}^2 \setminus E_x^+ \\ -\lambda & \textit{for all } v \in E_x^+ \setminus \{0\} \end{cases} \tag{4.7}$$

Moreover, the spaces E_x^+ and E_x^- are invariant by the cocycle, depend measurably on the point x, and satisfy :

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \angle\big(E^+_{T^n x}, E^-_{T^n x}\big) = 0\,. \tag{4.8}$$

We call E_x^+ and E_x^- the *Oseledets spaces.* In view of (4.8), we say that the angle between them decreases at most *subexponentially.* Notice that E_x^- can only be distinguished when iterating in the future. In fact, when iterating in the future only E^- can be distinguished since all other vectors grow at the same exponential rate. This is why only in the invertible case can we have the complete decomposition of $\mathbb{R}^2$ similar to the case of a single matrix : E_x^- is the only vector that does not grow exponentially when multiplied by $A_n(x), n \geq 0$, and E_x^+ is the only vector that does not grow exponentially when multiplied by $A_n(x), n \leq 0$.

Proof of Oseledets Theorem.

The following lemma will be used a few times :

Lemma 4.3.1 *Let $f : X \to \mathbb{R}$ be a measurable function such that $f\circ T - f$ is integrable in the extended sense*[1]. *Then*

$$\lim_{n\to\infty} \frac{1}{n} f(T^n x) = 0 \quad \textit{for a.e. } x \in X.$$

1. A measurable function g is said to be integrable in the extended sense if g^+ or g^- are integrable. Notice that the Birkhoff Theorem still applies.

Proof. Let $g = f \circ T - f$, and assume $g^+ \in L^1(\mu)$. By Birkhoff's Theorem, there is a function $\tilde{g}$ with $\tilde{g}^+ \in L^1(\mu)$ such that

$$\frac{f \circ T^n}{n} = \frac{f}{n} + \frac{1}{n} \sum_{j=0}^{n-1} g \circ T^j \to \tilde{g} \text{ a.e.}$$

For every point x where convergence above holds and $\tilde{g}(x) \neq 0$, we have $|f(T^n x)| \to \infty$. But, by the Poincaré's Recurrence Theorem, the set of points x which satisfy the latter condition has zero measure. Therefore $\tilde{g} = 0$ a.e., as we wanted to show. □

Proof. [Proof of Theorem 4.3.1] Let $\lambda(\cdot)$ be given by Theorem 4.2.1. For each point such that (4.3) holds and $\lambda(x) = 0$. Then, for every non-zero $v \in \mathbb{R}^2$,

$$\|(A^n(x))^{-1}\|^{-1} \|v\| \leq \|A^n(x) \cdot v\| \leq \|A^n(x)\| \|v\| .$$

Taking log's, dividing by n, and making $n \to +\infty$ gives (4.5).

Now consider the T-invariant set $[\lambda > 0] = \{x \in X;\ \lambda(x) > 0\}$. For a.e. $x \in [\lambda > 0]$, the orthogonal directions $s_n(x) = s(A^n(x))$, $u_n(x) = u(A^n(x))$ are defined for sufficiently large n. We are going to show that they converge to (necessarily measurable) maps $[\lambda > 0] \to \mathbb{P}^1$, and that $\lim s_n(x)$ is exactly the E^-_x space we are looking for.

Fix some x with $\lambda(x) > 0$. We may write λ, s_n instead of $\lambda(x)$, $s_n(x)$ etc. Take unit vectors in the directions of s_n and u_n that by simplicity of notation we indicate by the same symbols.

Let $\alpha_n > 0$ be the angle between s_n and s_{n+1}. That is, $s_n = \pm \cos \alpha_n \, s_{n+1} \pm \sin \alpha_n \, u_{n+1}$. Since the vectors s_{n+1}, u_{n+1} are orthogonal and so are their images by $A^{n+1}(x)$, we get :

$$\|A^{n+1}(x) \cdot s_n\| \geq \|A^{n+1}(x) \cdot (\sin \alpha_n \, u_{n+1})\| = (\sin \alpha_n) \|A^{n+1}(x)\| .$$

On the other hand :

$$\|A^{n+1}(x) \cdot s_n\| \leq \|A(T^n x)\| \, \|A^n(x) s_n\| = \|A(T^n x)\| \, \|A^n(x)\|^{-1} .$$

So it follows that

$$\sin \alpha_n(x) \leq \frac{\|A(T^n x)\|}{\|A^n(x)\| \, \|A^{n+1}(x)\|} . \tag{4.9}$$

From the definition (4.3) of λ, the integrability condition (4.2), and Lemma 4.3.1 it follows that for a.e. x,

$$\lim_{n \to \infty} \frac{1}{n} \log \sin \alpha_n = -2\lambda .$$

Thus, for a.e. x such that $\lambda(x) > 0$ we have that $\alpha_n(x)$ goes exponentially fast to zero, and, in particular, $s_n(x)$ is a Cauchy sequence in $\mathbb{P}^1$, for a.e. x. Let $s(x)$ be the limit. As the tail of a geometric series goes to zero with the same speed as the summand, we have :

$$\lim_{n\to\infty} \frac{1}{n} \log \sin \angle(s_n, s) = -2\lambda\,. \tag{4.10}$$

Now write $\beta_n = \angle(s_n, s)$. Then

$$A^n(x)\cdot s = \pm\|A^n(x)\|^{-1}\cos\beta_n \pm \|A^n(x)\|\sin\beta_n$$

Therefore, using (4.10),

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\log\|A^n(x)\cdot s\| &\le \limsup_{n\to\infty}\frac{1}{n}\log\max\left(\|A^n(x)\|^{-1}, \|A^n(x)\|\sin\beta_n\right)\\ &= \max(-\lambda, \lambda - 2\lambda) = -\lambda\,.\end{aligned}$$

On the other hand, $\frac{1}{n}\log\|A^n(x)\cdot s\| \ge \frac{1}{n}\log\|A^n(x)\|^{-1} \to -\lambda$, so it follows that $\frac{1}{n}\log\|A^n(x)\cdot s\| \to -\lambda$. Now, if v is a unit vector not collinear to s then

$$\|A^n(x)\cdot v\| \ge \|A^n(x)\|\sin\angle(v, s),$$

which implies that $\frac{1}{n}\log\|A^n(x)\cdot v\| \to \lambda$. So we have proved that (4.6) holds taking E^-_x as the s direction. Finally, notice that if $v \in A(x)\cdot E^-_x \setminus \{0\}$ then $\frac{1}{n}\log\|A^n(Tx)\cdot v\| \to -\lambda$. It follows that $v \in E^-_{Tx}$ almost surely. So invariance holds and the proof of Theorem 4.3.1 is completed. □

We now consider the invertible case :

Proof. [Proof of Theorem 4.3.2] Let E^- and E^+ be the spaces given by Theorem 4.3.1 applied respectively to $F = F_{T,A}$ and F^{-1}. Then (4.6) and (4.7) hold.

We are left to show (4.8) which in particular implies that $E^-_x \neq E^+_x$ for a.e. x such that $\lambda(x) > 0$. Fix $\epsilon \ll 1$. For a.e. x such that $\lambda(x) > 0$ we have for a subsequence of n that goes to $+\infty$ that $A^n E^-_x = \lambda_n E^-_{T^n x}$ with $|\lambda_n| \le \lambda^{-n}(1+\epsilon)^n$ and $\|A^{-n}(T^n x)E^+_{T^n x}\| \le \lambda^{-n}(1+\epsilon)^n$ and $\|A^{-n}(T^n x)v\| \le \lambda^n(1+\epsilon)^n$ for any unitary vector v (as an exercise, show that these properties hold for μ-a.e. x). Let θ_n be the angle between $E^+_{T^n x}$ and $E^-_{T^n x}$. We then have that $E^-_x = A^{-n}(T^n x)A_n E^-_x = \lambda_n A^{-n}E^-_{T^n x}$ and projecting $E^-_{T^n x}$ on $E^+_{T^n x}$ we get that

$$1 = \|E^-_x\| \le \lambda^{-2n}(1+\epsilon)^{2n} + \theta_n(1+\epsilon)^{2n}$$

hence $\frac{1}{n}\ln(\theta_n) \ge -3\epsilon$ and since $\epsilon > 0$ can be made arbitrary small we are done. □

4.4 Uniform Hyperbolicity

A whole class of examples that deserves to be studied in some detail is that of the uniformly hyperbolic cocycles.

In this subsection we assume X is a compact Hausdorff space.

Let $T : X \to X$ and $A : X \to \mathrm{SL}(2, \mathbb{R})$ be continuous maps. We say the cocycle in *uniformly hyperbolic* if there exist constants $c > 0$ and $\tau > 0$ such that

$$\|A^n(x)\| > ce^{\tau n}, \quad \text{for all } n \geq 0. \tag{4.11}$$

This definition of uniform hyperbolicity is apparently weaker than the more usual one ; but we will establish their equivalence in Theorem 4.4.1 and Corollary 4.4.1 below.

Exercise 4.4.1 *For any $k \geq 1$, a cocycle (T, A) is uniformly hyperbolic if and only if so is its power (T^k, A_T^k).*

Theorm 4.4.1 *If (T, A) is a uniformly hyperbolic cocycle then there exist a map $E^s : X \to \mathbb{P}^1$ and constants $C > 0$ and $\sigma > 0$ such that*

$$\|A^n(x)|E_x^s\| < Ce^{-\sigma n}, \quad \textit{for all } x \in X \textit{ and } n \geq 0. \tag{4.12}$$

Moreover, the map E^s is unique, invariant by the cocycle, and continuous.

Proof. Assume (T, A) is uniformly hyperbolic, and fix $\tau > 0$ so that (4.11) is satisfied. We will use Oseledets Theorem 4.3.1 and its proof. Let $s_n(x)$ be the direction the most contracted by $A^n(x)$. We have the estimate (4.9) for the angle $\alpha_n(x) = \angle(s_n(x), s_{n+1}(x))$. Here A is uniformly bounded, so we obtain from (4.11) that $\alpha_n(x)$ goes exponentially fast to zero, uniformly in x. In particular, $E^s(x) = s(x) = \lim s_n(x)$ exists and is a continuous function.

We wish to use again (4.9)

$$\sin \alpha_n(x) \leq \frac{\|A(T^n x)\|}{\|A^n(x)\| \, \|A^{n+1}(x)\|}$$

and from uniform geometric increase of $\|A^n(x)\|$ we wish to get

$$\beta_n = \angle(s_n, s) \leq C \frac{1}{\|A^n(x)\| \, \|A^{n+1}(x)\|}$$

and then

$$A^n(x) \cdot s = \pm \|A^n(x)\|^{-1} \cos \beta_n \pm \|A^n(x)\| \sin \beta_n \leq Ce^{-n\tau}$$

as desired, but the problem is that the bound on β_n would hold only on typical points of some invariant measure of T such that the rate of increase of $\|A_n(x)\|$ becomes almost uniform after some large n.

We turn therefore to an ergodic-theoretic argument.

Observe first that the same proof of convergence of s_n shows that

$$\angle(s_n(Tx), A(x)s_{n+1}(x)) = O(e^{-2n\tau})$$

which implies the cocycle invariance of $E^s : E^s(Tx) = A(x)E^s(x)$.

Let now μ be any T-invariant Borel probability measure. By the proof of Theorem 4.3.1, we know that $E^s(x)$ is the Oseledets contracting direction for μ-a.e. $x \in X$. Consider the continuous function $\phi(x) = \log \|A(x)|E^s(x)\|$. Its n-th Birkhoff average is

$$B_n(x) = \frac{1}{n}(\phi + \phi \circ T + \cdots + \phi \circ T^{n-1}) = \frac{1}{n} \log \|A^n(x)|E^s(x)\| .$$

By Oseledets' Theorem, for μ-a.e. $x \in X$, $\lim B_n(x)$ exists and equals $-\lambda(x) = -\lim \frac{1}{n} \log \|A^n(x)\|$. By the hypothesis (4.11), $\lambda(x) \geq \tau$. Now we need the following :

Lemma 4.4.1 *Let $\phi : X \to \mathbb{R}$ be a continuous function, and let B_n denote the n-Birkhoff average of ϕ under T. Assume that there is $a \in \mathbb{R}$ such that for every T-invariant measure μ, we have $\lim_{n\to\infty} B_n(x) \leq a$ for μ-a.e. $x \in X$. Then* $\limsup_{n\to\infty} B_n(x) \leq a$ uniformly. *That is, for every $a' > a$ there exists $n_0 \in \mathbb{N}$ such that $B_n(x) < a'$ for every $n \geq n_0$ and every $x \in X$.*

Proof. This is a standard Krylov–Bogoliubov argument. If the conclusion is false then there is $a' > a$ and sequences $n_i \to \infty$ and $x_i \in X$ such that $B_{n_i}(x_i) \geq a'$. Consider the sequence of measures $\mu_i = \frac{1}{n_i} \sum_{j=0}^{n_i-1} \delta_{T^j x_i}$. Passing to a subsequence, we can assume that μ_i converges weakly to a measure μ. Then μ is T-invariant and $\int \phi \, d\mu = \lim \int \phi \, d\mu_i = \lim B_{n_i}(x_i) \geq a'$. So by Birkhoff's Theorem, the set of points x such that $\lim B_n(x) \geq a'$ has positive μ measure. This contradicts the assumption. $\square$

Coming back to the proof of Theorem 4.4.1, it follows from the lemma that $\limsup_{n\to\infty} B_n(x) \leq -\tau$ uniformly. In particular, for any $\sigma < \tau$, there exist n_0 such that $B_{n_0}(x) < -\sigma$ for every $x \in X$. Thus (by the same argument as for Remark 4.4.1), (4.12) holds for appropriate C.

Let us show uniqueness of E^s. If for some x there existed two linearly independent vectors v_1, v_2 in $\mathbb{R}^2$ such that $\lim_{n\to\infty} A^n(x) \cdot v_i = 0$ for both $i = 1, 2$ then we would have $\|A^n(x)\| \to 0$, which is impossible.

$\square$

Corollary 4.4.1 *If $T : X \to X$ is a homeomorphism and (T, A) is uniformly hyperbolic then there is a continuous invariant splitting $\mathbb{R}^2 = E^u_x \oplus E^s_x$ such that*

$$\|A^{-n}(x)|E^u_x\| < Ce^{-\sigma n}, \quad \|A^n(x)|E^s_x\| < Ce^{-\sigma n}, \quad \textit{for all } x \in X \textit{ and } n \geq 0,$$

where $C > 0$ and $\sigma > 0$ are constants. The spaces E^u_x and E^s_x are uniquely defined and are invariant by the cocycle.

Proof. Let E^s and E^u be given by Theorem 4.4.1 applied respectively to the cocycle and its inverse. Since $\|A^{-n}(x)|E^s_x\| = \|A^n(T^{-n}x)|E^s_{T^{-n}x}\|^{-1} \to \infty$ as $n \to +\infty$, we see that $E^s_x \neq E^u_x$. $\square$

The spaces E^u and E^s are called respectively the *unstable* and *stable* directions. By continuity, the angle between them has a positive lower bound.

Proposition 4.4.1 *Let (T, A) be a uniformly hyperbolic cocycle. Then for every continuous map $B : X \to \mathrm{SL}(2,\mathbb{R})$ sufficiently close to A, the cocycle (T, B) is uniformly hyperbolic.*

Proof. Let $E^s : X \to \mathbb{P}^1$ be the stable direction. For $\alpha > 0$, define the following *cone field* :

$$C^s_\alpha(x) = \left\{ v \in \mathbb{R}^2;\ \angle(v, E^s_x) < \alpha \text{ or } v = 0 \right\}.$$

It is easy to see that there is α and $k \geq 1$ such that for every $x \in X$,

$$v \in C_\alpha(T^k(x)), \quad w = [A^k(x)]^{-1} \cdot v \quad \Rightarrow \quad \begin{cases} w \in C_{\alpha/2}(x) \\ \|w\| > 2\|v\| \end{cases}$$

Therefore if B is sufficiently close to A then

$$v \in C_\alpha(T^k(x)), \quad w = [B^k(x)]^{-1} \cdot v \quad \Rightarrow \quad \begin{cases} w \in C_{\alpha}(x) \\ \|w\| > 2\|v\| \end{cases}$$

It follows that for any $m \geq 1$ and $v \in C_\alpha(T^{km}x)$ we have $\|[B^{km}(x)]^{-1} \cdot v\| > 2^m\|v\|$. So $\|B^{km}(x)\| > 2^m$. This proves that (T^k, B^k_T) is uniformly hyperbolic, and thus by Remark 4.4.1, so is (T, B). $\square$

Example 4.4.1 *Let $T : X \to X$ be a homeomorphism. Let $f : X \to \mathbb{R}$ be a continuous positive function, and define diagonal matrices $B(x) = \exp(f(x)\mathrm{Id})$. For any continuous $C : X \to \mathrm{SL}(2,\mathbb{R})$, the cocycle (T, A) with $A(x) = C(Tx)^{-1} B(x) C(x)$ is uniformly hyperbolic. However, it is not true that all uniformly hyperbolic cocycles (T, A) are of this form, because topological obstructions may arise.*

4.5 The fibered rotation number

Let T be a homeomorphisms of a compact metric space X. We associate to a *cocycle* (T, A) a *projective cocycle* given by the map F_A (same notation as that of the cocycle map) from $X \times \mathbb{S}^1 \circlearrowleft : (x, v) \mapsto (Tx, \frac{A(x)v}{\|A(x)v\|})$.

If $A : X \to SL(2,\mathbb{R})$ is homotopic to Identity, that is, if there exists an application $\tilde{A} : [0, 1] \times X \to SL(2,\mathbb{R})$ such that $\tilde{A}(0, \cdot) = A(\cdot)$ et $\tilde{A}(1, \cdot) = \mathrm{Id}$, then the projective cocycle F_A is itself homotopic to Identity : There exists a *lift* $\tilde{F}_A$ of F_A to $X \times \mathbb{R}$ such that $\tilde{F}_A(x, y) = (Tx, \tilde{f}_A(x, y))$ where $\tilde{f}_A : X \times \mathbb{R} \to \mathbb{R}$ is a continuous application such that

(i) $\tilde{f}_A(x, y + 1) = \tilde{f}_A(x, y) + 1$;
(ii) for every $x \in X$, $\tilde{f}_A(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is a strictly increasing homeomorphism ;
(iii) if π_2 is the projection map $X \times \mathbb{R} \to X \times \mathbb{S}^1 : (x, y) \mapsto (x, e^{2\pi i y})$, then $F_A \circ \pi_2 = \pi_2 \circ \tilde{F}_A$.

Exercise 4.5.1 *Show the following*

- *A lift $\tilde{F}_A$ is not unique. It is unique up to an additive integer : if $\tilde{G}_A : (x, y) \mapsto (Tx, \tilde{g}_A(x, y))$ is another lift of F_A then $\tilde{g}_A \equiv \tilde{f}_A + p$ for some $p \in \mathbb{Z}$.*
- *$\tilde{F}^n(x, y)$ is given by $(x, y) \mapsto (T^n, \tilde{f}_A^n(x, y))$.*
- *The $(x, y) \mapsto \tilde{f}_A(x, y) - y$ is $\mathbb{Z}$-periodic in y and defines thus a map from $X \times \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.*

Theorm 4.5.1 *Suppose (X, T) is uniquely ergodic with its unique invariant probability measure and suppose that A is homotopic to identity. There exists then $\tilde{\rho} \in \mathbb{R}$ such that*

$$\frac{\tilde{f}_A^n(x, y) - y}{n}$$

converges uniformly in $(x, y) \in X \times \mathbb{R}/\mathbb{Z}$ *to* $\tilde{\rho}$*. This limit is independent on the lift of* F_A*, up to an addition of an integer (see first item of exercise 4.5.1). We denote therefore* $\rho_A = \tilde{\rho}[1] \in \mathbb{R}/\mathbb{Z}$ *the fibered rotation number of the cocycle* (T, A)*. Moreover the following holds :*

- *For any probability measure on* $X \times \mathbb{R}/\mathbb{Z}$ *invariant by* F_A *we have*

$$\int_{X\times\mathbb{R}/\mathbb{Z}} (\tilde{f}_A(x, y) - y)dm(x, y) = \tilde{\rho}.$$

- *the map* $C^0(X, \mathrm{SL}(2, \mathbb{R})) \to \mathbb{R}/\mathbb{Z} : A \mapsto \rho_A$ *is continuous for the uniform convergence norm on* $C^0(X, \mathrm{SL}(2, \mathbb{R}))$*.*

Proof.

Lemma 4.5.1 *For every* $x \in X$*,* $y, z \in \mathbb{R}$ *such that* $|y - z| < 1$*, we have* $\tilde{f}_A^n(x, y) - \tilde{f}_A^n(x, z)| < 1$*.*

Proof. From (i) and (ii) above we get that for $y < z < y + 1 : \tilde{f}_A(x, y) < \tilde{f}_A(x, z) < \tilde{f}_A(x, y + 1) = \tilde{f}_A(x, y) + 1$, thus $|\tilde{f}_A(x, y) - \tilde{f}_A(x, z)| < 1$ and the Lemma follows by iteration of the latter inequality. □

Lemma 4.5.2 *There exists* $\tilde{\rho}$*, such that for any probability measure* m *on* $X \times \mathbb{R}/\mathbb{Z}$ *invariant by* F_A *we have that*

$$\int_{X\times\mathbb{R}/\mathbb{Z}} (\tilde{f}_A(x, y) - y)dm(x, y) = \tilde{\rho}.$$

Proof. Denote the projection on the first variable by $\pi^1(x, y) = x$. Observe first that the projection of m on the first variable $\pi^1_* m(B) := m(B \times \mathbb{T})$ for $B \subset X$ is invariant by T therefore $pi^1_* m =$ by unique ergodicity of T.

Let $\varphi(x, y) = \tilde{f}_A(x, y) - y$. By the Birkhoff ergodic theorem, there exists $\tilde{\varphi} \in L^1(X \times \mathbb{R}/\mathbb{Z})$ such that $\tilde{\varphi}(F_A(x, y)) = \tilde{\varphi}(x, y)$ and $\int_{X\times\mathbb{R}/\mathbb{Z}} \tilde{\varphi}(x, y)dm = \int_{X\times\mathbb{R}/\mathbb{Z}} \varphi(x, y)dm$ and

$$\lim_{N\to\infty} \frac{1}{N} S_N^{F_A} \varphi(x, y) = \tilde{\varphi}(x, y). \tag{4.13}$$

But Lemma 4.5.2 implies that for (x, y) such that (4.13) holds then (4.13) holds for (x, y') for any $y' \in \mathbb{T}$ and is independent of y. The latter implies in particular that $\tilde{\varphi}$ is a function of only one variable x, and therefore

$\tilde{\varphi}(Tx) = \tilde{\varphi}(x)$ for $\pi^1_* m =$ -a.e. $x \in X$, thus $\tilde{\varphi}$ is constant by unique ergodicity of $(T, X,)$. Denote $\rho(m)$ the latter constant and note that $\rho(m) = \int_{X\times\mathbb{R}/\mathbb{Z}} \varphi(x, y)dm$. From what preceded we conclude that for almost every $x \in X$ and for every $y \in \mathbb{T}$ we have that

$$\lim_{N\to\infty} \frac{1}{N} S_N^{F_A} \varphi(x, y) = \rho(m). \tag{4.14}$$

Hence $\rho(m)$ does not depend on m and Lemma 4.5.2 is proved. □

The uniform convergence of $\frac{\tilde{f}_A^n(x,y)-y}{n}$ then follows from Lemma 4.5.2 and the following Lemma, similar to Lemma 4.4.1 of Section 4.4.

Lemma 4.5.3 *Let Z be compact metric space, and $G : Z \circlearrowleft$ a continuous map and $\phi : Z \to \mathbb{R}$ a continuous function such that for any probability measure m on Z that is G invariant we have that $\int_Z \phi dm = L$ is independent of m, then $\lim_{N\to\infty} \frac{1}{N} S_N^G \phi(z) = L$ uniformly in $z \in Z$.*

Proof. Let $L = \int_Z \phi dm$ for some probability measure m on Z that is G invariant. Suppose that we do not have that $\lim_{N\to\infty} \frac{1}{N} S_N^G \phi(z) = L$ uniformly in $z \in Z$. Then there exists $\epsilon > 0$ such that for any $n > 0$ there exists z_n such that $|\frac{1}{N} S_N^G \phi(z_n) - L| > \epsilon$. Consider then the sequence of probability measures $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{G^i z_n}$ satisfies $|\int_Z \phi d\mu_n - L| > \epsilon$ and by extraction one obtains a limit measure μ_∞ that is invariant by G and satisfies $|\int_Z \phi d\mu_\infty - L| \geq \epsilon$ which contradicts the assumption of the Lemma. □

To finish the proof of Theorem 4.5.1, we still need to show that the rotation number $\rho(A)$ is continuous in $A \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$. For this observe that by the uniform convergence of $\frac{\tilde{f}_A^n(x,y)-y}{n}$ we have for every $\epsilon > 0$ that there exists $N > 0$ such that for every $(x, y) \in X \times \mathbb{R}/\mathbb{Z}$

$$|\tilde{f}_A^N(x, y) - y - N\tilde{\rho}(A)| < N\epsilon$$

and since $\tilde{f}_A^N(x, y) - y$ is continuous in A (for the uniform convergence norm) we have by compacity of $X \times \mathbb{R}/\mathbb{Z}$ that for B sufficiently close to A (for the uniform convergence norm) for every $(x, y) \in X \times \mathbb{R}/\mathbb{Z}$

$$|\tilde{f}_B^N(x, y) - y - N\tilde{\rho}(A)| < N\epsilon$$

(here, we chose the same "integer" in the lift of F_A and F_B) hence for every $p \geq 1$

$$|\tilde{f}_B^{pN}(x, y) - y - pN\tilde{\rho}(A)| < pN\epsilon$$

thus $|\tilde{\rho}(B) - \tilde{\rho}(A)| \leq \epsilon$. The proof of Theorem 4.5.1 is now complete. □

Chapitre 5

Reducible cocycles

5.1 Reducibility of Uniformly Hyperbolic cocycles

Definition 5.1.1 *A cocycle (f, A) is said to be C^r reducible if there exists $B(\cdot) \in C^r(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ and a constant matrix A_0 such that*

$$B(f(x))A(x)B(x)^{-1} = A_0$$

We also say that A is C^r cohomologous to the constant matrix A_0.

More generally, two cocycles (f, A) and (f, A') are said to be C^r cohomologous if if there exists $B(\cdot) \in C^r(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ such that

$$B(x+\alpha)A(x)B(x)^{-1} = A'(x).$$

The maps $(x, v) \in M \times \mathbb{R}^2 \mapsto (f(x), A(x)v)$ and $(x, v) \mapsto (f(x), A'(x)v)$ are then C^r conjugated. We call the map $x \mapsto B(x)$ a *fibered conjugacy.*

Note that two cocycles that are C^0 cohomologous share many dynamical features : same Lyapunov exponent, Oseledet's decomposition of one is sent into the other's by B, if the products A_n are bounded then the same holds for A'_n, etc.

Exercise 5.1.1 *Show that two cocycles that are C^0 cohomologous share many dynamical features : same Lyapunov exponent, Oseledet's decomposition of one is sent into the other's by B, if the products A_n are bounded then the same holds for A'_n. Try to find other invariants for the C^0 cohomology class* *between cocycles with the same base dynamics. Find invariants for the C^r cohomology class between cocycles with the same base dynamics that are not invariants for the C^0 cohomology class.*

Exercise 5.1.2 *Show that (α, A) with $A(\cdot)$ not homotopic to Identity cannot be reducible. Give an example of non reducible cocycle.*

Theorm 5.1.1 *If (α, A) is of class C^r and (α, A) is uniformly hyperbolic then the stable and unstable directions of (α, A) are of class C^r. Equivalently (α, A) is C^r cohomologous to a diagonal matrix $A'(\theta) = . \begin{pmatrix} e^{\varphi(\theta)} & 0 \\ 0 & e^{-\varphi(\theta)} \end{pmatrix}$, with $\varphi \in C^r(\mathbb{T}, \mathbb{R}_+^*)$.*

Proof. We will use the following lemma

Lemma 5.1.1 *Let $I, J \subset \mathbb{R}$ be two intervals. Let $(f^{(n)})_{n\in\mathbb{N}}$ be a sequence of maps $f^{(n)} : I \times J \to J$ (we write $f_\theta^{(n)}(x) := f^{(n)}(\theta, x)$) such that $f^{(n)}$ is of class C^r ($r \in \mathbb{N}$, $r = \infty$, $r = \omega$) and there exists $M > 0$ and $\lambda \in (0, 1)$ such that*

- $\|f^{(n)}\|_r \le M$
- $\|\partial_2 f^{(n)}\| < \lambda$

then there exists $x^\infty : I \to J$ such that

$$x^\infty(\theta) = \lim_{n\to\infty} f_\theta^{(1)} \circ \ldots \circ f_\theta^{(n)}(x)$$

for any $x \in J$. Moreover, $x^\infty(\cdot)$ is of class C^r.

Let us see how this Lemma applies to show that the continuous maps $\theta \mapsto E^s(\theta)$ and $\theta \mapsto E^u(\theta)$ (from Corollary 4.4.1) are actually C^r. WLOG we can assume in the argument of Proposition 4.4.1 that $k = 1$ in that there exists a cone field

$$C_\alpha^s(\theta) = \left\{v \in \mathbb{R}^2;\ \angle(v, E_\theta^s) < \alpha \text{ or } v = 0\right\}.$$

such that for every $\theta \in X$,

$$v \in C_\alpha^s(\theta + \alpha), \quad w = A(\theta)^{-1} \cdot v \quad \Rightarrow \quad \begin{cases} w \in C_{\alpha/2}^s(\theta) \\ \|w\| > 2\|v\| \end{cases}$$

This means that the projective maps on $\mathbb{P}^1$ (the 1 dimensional space of directions of $\mathbb{R}^2$) $f_m(\theta)(\cdot)$ associated to the matrices $(A^m(\theta))^{-1}$ are uniformly contracting on any small interval around $\theta + m\alpha$.

Fix two some small interval of $\theta : I \subset I'$ and let m_n be such that $\theta \in I$ implies $\theta + m_n\alpha \in I'$ and $m_{n+1} - m_n$ is large but bounded. Then there exists an interval J such that the projective maps $f^{(n)}(\theta)(\cdot)$ associated to $[A(\theta + m_n\alpha) \dots A(\theta + (m_{n+1} - 1)\alpha)]^{-1}$ for any $\theta \in I$ are uniform contractions on J.

Since $f_\theta^{(1)} \circ \dots \circ f_\theta^{(n)}(x)$ converges to the direction $E^s(\theta)$ for any $x \in J$, Lemma 5.1.1 applies and gives the required smoothness of $E^s(\theta)$. $\square$

Theorm 5.1.2 *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is Diophantine and if (α, A) is a smooth uniformly hyperbolic cocycle then (α, A) is reducible to $A_0 = \begin{pmatrix} e^L & 0 \\ 0 & e^{-L} \end{pmatrix}$ where $L = L(\alpha, A)$ is the Lyapunov exponent of (α, A).*

Proof. From Theorem 5.1.1 we first conjugate using a smooth matrix $B_1(\cdot)$ the cocycle (α, A) to $A'(\theta) = \begin{pmatrix} e^{\varphi(\theta)} & 0 \\ 0 & e^{-\varphi(\theta)} \end{pmatrix}$ with $\varphi \in C^\infty(\mathbb{T}, \mathbb{R}_+^*)$

By the Birkhoff ergodic theorem (the usual additive ergodic theorem) applied to the function φ we get that the (integrated) Lyapunov exponent

$$L := L(\alpha, A) = L(\alpha, A') =_{a.s.} \lim_{n\to\infty} S_n^\alpha \varphi(\theta) = \int_{\mathbb{T}} \varphi(u) du.$$

Then we apply Theorem 1.6.1 to get a smooth solution $b(\cdot)$ to the linear cohomological equation

$$b(\theta + \alpha) - b(\theta) = \ln \varphi(\theta) - L$$

Thus the matrix $B_2(\theta) = \begin{pmatrix} e^{-b(\theta)} & 0 \\ 0 & e^{b(\theta)} \end{pmatrix}$, satsifies

$$B_2(\theta + \alpha) A'(\theta) B_2(\theta)^{-1} = A_0 = \begin{pmatrix} e^L & 0 \\ 0 & e^{-L} \end{pmatrix}.$$

Hence, $A(\cdot)$ is reducible to A_0 *via* the smooth fibered conjugacy $B_2(\cdot)B_1(\cdot)$. $\square$

Exercise 5.1.3 *Show that Theorem 5.1.2 holds for $\alpha \in \mathrm{DC} \subset \mathbb{R}^d$, $d \geq 2$.*

The following exercise shows that the reducibility conclusion holds in finite regularity with the crucial *loss of derivatives phenomenon* that was discussed in the previous chapters.

Exercise 5.1.4 *As in Exercise 1.6.3, show that if* $\alpha \in \mathrm{DC}(\tau, \gamma) \subset \mathbb{R}$ *and* (α, A) *is a UH cocycle of class* C^r, $r > \tau + 2$ *then* (α, A) *is* $C^{[r-\tau]-1}$ *reducible. Write down the corresponding statement in higher dimension.*

Exercise 5.1.5 *Show that if* α *is Liouville then for a* G^δ *dense set of* $A \in C^\infty(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ *the cocycle* (α, A) *is not reducible.*

Proposition 5.1.1 *If* (α, A) *is uniformly hyperbolic then* $2\rho(\alpha, A) \in \mathbb{Z}\alpha$.

Definition 5.1.2 *We say that a cocycle* (α, A) *is non uniformly hyperbolic or NUH if it has positive Lyapunov exponent but it is not uniformly hyperbolic.*

The following is straightforward from the definitions

Proposition 5.1.2 *A non uniformly hyperbolic cocycle* (α, A) *is not* C^0 *reducible. In particular a cocycle* (α, A) *with* $LE(\alpha, A) > 0$ *and* $2\rho(\alpha, A) \notin \mathbb{Z}\alpha$ *is not reducible.*

Exercise 5.1.6 *Proof Proposition 5.1.2.*

Corollary 5.1.1 *Let* $\alpha \in \mathbb{R} - \mathbb{Q}$. *Consider the almost-Mathieu cocycle with large potential, i.e.* $\lambda > 2$

$$A_E(\theta) = \begin{pmatrix} \lambda\cos(\theta) - E & -1 \\ 1 & 0 \end{pmatrix}$$

then (α, A) *is NUH, thus not reducible, for an uncountable set of energies* E.

Proof. By Herman subharmonicity trick, we know that $LE(\alpha, A_E) > 0$ for every E.

Notice that $\rho(\alpha, A_E) \to 0$ as $E \to -\infty$ while $\rho(\alpha, A_E) \to 1$ as $E \to +\infty$. But $\rho(\alpha, A_E)$ is continuous in E and from Proposition 5.1.1, (α, A) is NUH if $2\rho(\alpha, A) \notin \mathbb{Z}\alpha$. $\square$

5.2 Reducibility in a perturbative setting.

In this section we will prove a reducibility result on cocycles (α, A) in the following setting

– We start form a *perturbative* setting where we suppose that $A(\cdot)$ is close to a constant matrix $A_0 \in \mathrm{SL}(2,\mathbb{R})$. Since the uniformly hyperbolic case was already solved, even in the global setting, we will assume that A_0 is an elliptic matrix, that is : A_0 is complex conjugated to $\begin{pmatrix} e^{i2\pi\beta_0} & 0 \\ 0 & e^{-i2\pi\beta_0} \end{pmatrix}$.
– We assume that $A(\cdot)$ is sufficiently smooth. We will work in the real analytic category but matrix functions $A(\cdot)$ with sufficient regularity (compared to the Diophantine condition satisfied by α) could also be reduced following a similar scheme. We denote by $C_h^\omega(\mathbb{T},\mathbb{R})$ the set of functions $f : \mathbb{T} \to \mathbb{R}$ such that $f(\theta) = \sum_{k\in\mathbb{Z}} f_k e(k\theta)$ and such that the series $f(\theta + is) := \sum_{k\in\mathbb{Z}} f_k e(k(\theta + is))$ converges in $\mathbb{C}$ for every $|s| \leq h$. The space $C_h^\omega(\mathbb{T},\mathbb{R})$ is a complete Banach space for the norm $|f|_h := \sup_{\theta\in\mathbb{T},|s|\leq h} |f(\theta+is)|$. A matrix map $A(\cdot) \in C_h^\omega$ if all its entries are in $C_h^\omega(\mathbb{T},\mathbb{R})$ and the C_h^ω norm of A is the maximum of the C_h^ω norms of the coefficients.
For $f \in C_h^\omega(\mathbb{T},\mathbb{R})$ we denote $\hat{f} := \int_{\mathbb{T}} f(\theta)d\theta$. We define similarly $\hat{A}$.
– We assume the frequency α is Diophantine
– We assume that the fibered rotation number is *Diophantine with respect to* α in the following sense : there exists $\kappa, \tau > 0$ such that $2\rho(\alpha, A) \in DS_\alpha(\tau,\kappa)$ with

$$DS_\alpha(\tau,\kappa) := \left\{ \beta \in \mathbb{R} \ : \ \forall k \in \mathbb{Z}\setminus\{0\},\ \min_{l\in\mathbb{Z}} |k\alpha - \beta - l| \geq \frac{\kappa}{|k|^\tau} \right\}.$$

Exercise 5.2.1 *Fix α and let $\tau > 1$. Show that* $\mathrm{Leb}(\mathbb{T}\setminus DS_\alpha(\tau,\kappa)) \to 0$ *as* $\kappa \to 0$.
Let $DS_\alpha(\tau) = \bigcup_{\kappa>0} DS_\alpha(\tau,\kappa)$. Show that $\mathrm{Leb}(\mathbb{T}\setminus DS_\alpha(\tau)) = 0$.
Generalize these definitions and results to $\alpha \in \mathbb{T}^d$ and $\beta \in \mathbb{R}$.

We will obtain the reducibility result using a quadratic scheme similar to what we have seen in the Poincaré Siegel Theorem for holomorphic germs.

To fix notations we pose $A(\cdot) = e^{F_0(\cdot)}A_0$ where $F_0 \in C_h^\omega(\mathbb{T}^d, sl(2,\mathbb{R}))$ and $A_0 = e^{U_0} \in SL(2,\mathbb{R})$ $(U_0 \in sl(2,\mathbb{R}))$.

Theorm 5.2.1 (Dinaburg-Sinai) *Let $\alpha \in DC(\sigma,\gamma)$, $\tau,\kappa > 0$, $A_0 \in SL(2,\mathbb{R})$, $h > 0$. There exists $\epsilon^*(\gamma,\sigma,\kappa,\tau,d,A_0)$ such that for every $F \in C_h^\omega(\mathbb{T}^d, sl(2,\mathbb{R}))$ satisfying*
(i) $|F|_h \leq \epsilon^*$

(ii) $2\rho(\alpha, e^{F(\cdot)}A_0) \in DS_\alpha(\kappa, \tau)$
the cocycle $(\alpha, e^{F(\cdot)}A_0)$ *is reducible on an analytic band of width* $h' = h/2$.

5.2.1 Reduction up to quadratic terms.

Denote $\epsilon_0 := |F_0|_h \leq \epsilon^*$. Let us look for a *fibered* conjugacy $B_1(\cdot)$ *close to Identity* that brings $A(\cdot)$ closer to a constant. Namely we look for B_1 of the form $B_1 = e^{Y_1}$ where $Y_1 \in C^\omega_{h'}(\mathbb{T}^d, sl(2,\mathbb{R}))$ (Y_1 will be of order ϵ_0) and for a constant $A_1 \in \mathrm{SL}(2,\mathbb{R})$ such that

$$e^{Y_1(\cdot+\alpha)}(e^{F_0(\cdot)}A_0)e^{-Y_1(\cdot)} = e^{F_1(\cdot)}A_1,$$

with F_1 much smaller then F_0.

Since $e^M = I + M + O(M^2)$, $(I+M)^{-1} = I - M + O(M^2)$ (and $O(\cdot)$ being uniform for M in a neighborhood of 0) we see that if we want F_1 to be zero we must take

$$Y_1(\cdot+\alpha) - A_0 Y_1(\cdot) A_0^{-1} = -F_0 + A_1 A_0^{-1} - \mathrm{Id} + O_2(|Y_1|_{h'}, |F_0|_h).$$

Where we use the notation $O_2(a,b) = O(a^2 + b^2 + ab)$.

Conversely, we have the following consequence of the resolution of the linearized conjugacy equation

Proposition 5.2.1 *If we solve the linearized equation*

$$Y_1(\cdot+\alpha) - A_0 Y_1(\cdot) A_0^{-1} = -F_0 + \widehat{F_0} \tag{5.1}$$

with $Y_1 \in C^\omega_{h'}(\mathbb{T}, sl(2,\mathbb{R}))$ *then*

$$e^{Y_1(\cdot+\alpha)}(e^{F_0(\cdot)}A_0)e^{-Y_1(\cdot)} = e^{F_1(\cdot)}A_1, \tag{5.2}$$

with $A_1 := e^{\widehat{F_0}}A_0$ *and* $|F_1|_{h'} = O_2((|Y_1|_{h'}, |F_0|_h)$.

Exercise 5.2.2 *Complete the proof of Proposition 5.2.1.*

5.2.2 The analysis of the linearized equation

Under the two Diophantine conditions on $\alpha \in DC(\sigma,\gamma)$ and on $\rho(\alpha, A_0) \in DS_\alpha(\tau,\kappa)$ we will show that as in the proof of the Diophantine holomorphic germs linearization Theorem, the linearized equation of Proposition 5.2.1

can be solved with good control on the solution up to a *loss in the domain of analyticity width*. Namely, we have the following, if we suppose that A_0 is complex conjugated to $D_{\beta_0} := \begin{pmatrix} e^{i2\pi\beta_0} & 0 \\ 0 & e^{-i2\pi\beta_0} \end{pmatrix}$. Since the fibered rotation number of A_0 above any system is equal to $\beta_0[1]$ we use the notation $\rho(A_0) = \beta_0$.

Proposition 5.2.2 *If $\alpha \in DC(\sigma,\gamma)$ and $2\beta_0 \in DS_\alpha(\kappa,\tau)$ then the equation*

$$Y(\cdot+\alpha) - A_0Y(\cdot)A_0^{-1} = F - \hat{F}$$

has a unique solution $Y \in C^\omega_{h'}(\mathbb{T}^d, sl(2,\mathbb{R}))$, defined for every $h' < h$, such that

$$\|Y\|_{h'} \leq C(\kappa,\gamma,\|A_0\|)\frac{\|F\|_h}{(h-h')^a}$$

with $a = d+1+\sigma+\tau$.

Proof. We need the following

Exercise 5.2.3 *If $2\beta_0 \in DS_\alpha(\kappa,\tau)$ and $\rho(A_0) = \beta_0$ then there exists P such that $\|P\| \leq C(\kappa,\|A_0\|)$ such that $P^{-1}A_0P = D_{\beta_0}$*

From Exercise 5.2.6, replacing $F - \hat{F}$ by $P(F-\hat{F})P^{-1}$, we can assume WLOG that $A_0 = D_{\beta_0}$. Indeed, in the required bound on the solution Y we already have a constant that depends on κ and $\|A_0\|$.

Define the following base of $\mathrm{sl}(2,\mathbb{R})$.

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad g_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The crucial observation is that the action $Ad(A_0) : \mathrm{sl}(2,\mathbb{R}) \to \mathrm{sl}(2,\mathbb{R})$, $Ad(A_0)(g) = A_0gA_0^{-1}$ is diagonal on the base (g_1,g_2,g_3) with

$$Ad(A_0)(g_1) = g_1, \quad Ad(A_0)(g_2) = e(2\beta)g_2, \quad Ad(A_0)(g_3) = e(-2\beta)g_3.$$

Hence if we write $F - \hat{F} = f_1(\cdot)g_1 + f_2(\cdot)g_2 + f_3(\cdot)g_3$ with $f_i \in C^\omega_h(\mathbb{T},\mathbb{R})$ such that $\int_{\mathbb{T}} f_i(\theta)d\theta = 0$, and if we look for the solution Y under the form $y_1(\cdot)g_1 + y_2(\cdot)g_2 + y_3(\cdot)g_3$, we get that the y_i are solutions of

$$y_1(\theta+\alpha) - y_1(\theta) = f_1(\theta) \tag{5.3}$$
$$y_2(\theta+\alpha) - e(2\beta)y_2(\theta) = f_2(\theta) \tag{5.4}$$
$$y_3(\theta+\alpha) - e(-2\beta)y_3(\theta) = f_3(\theta) \tag{5.5}$$

Passing to Fourier series coefficients, (5.6)–(5.8)

$$y_{1,k} = \frac{f_{1,k}}{e(k\alpha) - 1} \tag{5.6}$$

$$y_{2,k} = \frac{f_{2,k}}{e(k\alpha) - e(2\beta)} \tag{5.7}$$

$$y_{3,k} = \frac{f_{3,k}}{e(k\alpha) - e(-2\beta)} \tag{5.8}$$

From (5.6)–(5.8), the Diophantine conditions on α for (5.6), and the Diophantine condition $\beta \in DS_\alpha(\sigma, \kappa)$ for (5.7) and (5.8), and similar Cauchy estimates as in Lemma 3.2.1 we get the real analyticity and the required bound on Y. □

5.2.3 Adjusting the eigenvalues of the constant part

Lemma 5.2.1 *If $A = e^{F(\cdot)}A_0$ is such that $\rho(\alpha, A) \in DS_\alpha(\tau, \kappa)$ and if $\max_{\theta\in\mathbb{T}} \|F(\theta)\| \leq \epsilon$ and $\|A_0\| \leq M$ then there exists A_0' such that*
- $\|A_0' - A_0\| \leq C\epsilon$
- $\rho(A_0') = \rho(\alpha, A) \in DS_\alpha(\tau, \kappa)$

Here C is a constant that depends only on M and κ.

Exercise 5.2.4 *Prove Lemma 5.2.1. Observe first that $|\rho(\alpha, A_0) - \rho(\alpha, A)| \leq C\epsilon$, then use Exercise 5.2.6.*

5.2.4 The iterative KAM main step

We now use Proposition 5.2.2 and Lemma 5.2.1 to build the main step of the KAM iterative conjugacy scheme.

Let $\alpha \in DC(\sigma, \gamma)$. Fix $\tau, \kappa > 0$. Fix $\epsilon_0 > 0$ small.

Let h_n be the sequence $h_0 = h$ and $h_{n+1} = h_n(1 - \frac{1}{4^{n+1}})$ for $n \geq 0$. Let M_n be the sequence $M_0 = M$ and $M_{n+1} = M_n(1 + \frac{1}{4^{n+1}})$ for $n \geq 0$.

Proposition 5.2.3 *There exists $C(\sigma, \gamma, \tau, \kappa, M, h) > 0$ such that the following holds. Let A_n, F_n be such that*

(a) $A_n \in SL(2, \mathbb{R})$ is such that $\rho(A_n) \in DS_\alpha(\kappa, \tau)$ and $\|A_n\| \leq M_n$
(b) $F_n \in C^\omega_{h_n}(\mathbb{T}^d, sl(2, \mathbb{R}))$
(c) $\rho(\alpha, e^{F_n(\cdot)}A_n) \in DS_\alpha(\kappa, \tau)$

Then there exists A_{n+1}, Y_{n+1}, F_{n+1} such that

(a') $A_{n+1} \in SL(2,\mathbb{R})$ is such that $\rho(A_{n+1}) \in DS_\alpha(\kappa,\tau)$ and $\|A_{n+1}\| \leq M_{n+1}$

(b') $F_{n+1} \in C^\omega_{h_{n+1}}(\mathbb{T}^d, sl(2,\mathbb{R}))$,

(c') $e^{Y_{n+1}(\cdot)}e^{F_n(\cdot)}A_n e^{-Y_{n+1}(\cdot)} = e^{F_{n+1}(\cdot)}A_{n+1}$

(d') Let $\epsilon_n = |F_n|_{h_n}$ then

$$\|Y_{n+1}\|_{h_{n+1}} \leq C4^{na}\epsilon_n \tag{5.9}$$

$$\|A_{n+1} - A_n\| \leq C\epsilon_{n+1} \tag{5.10}$$

$$\epsilon_{n+1} \leq C4^{na}\epsilon_n^2 \tag{5.11}$$

provided that $C4^{na}\epsilon_n \leq \frac{1}{10^{n+1}}$

Proof. Due to (a) and (b) we can apply Proposition 5.2.2 and find a solution Y_{n+1} to the linearized equation

$$Y_{n+1}(\cdot+\alpha) - A_nY_{n+1}(\cdot)A_n^{-1} = -F_n + \hat{F}_n$$

with the bound (5.9). Next Proposition 5.2.1 implies

$$e^{Y_{n+1}(\cdot)}e^{F_n(\cdot)}A_n e^{-Y_n(\cdot)} = e^{F'_{n+1}(\cdot)}A'_{n+1}$$

with $A'_{n+1} = e^{\hat{F}_n}A_n$ and F'_{n+1} satisfying (b') and (5.11). We then apply Lemma 5.2.1 to write $e^{F'_{n+1}(\cdot)}A'_{n+1} = e^{F_{n+1}(\cdot)}A_{n+1}$ with A_{n+1} and F_{n+1} as in (a')–(d').

□

5.2.5 Convergence of the KAM scheme

We will now see how Proposition 5.2.4 yields the proof of Theorem 5.2.1. The following simple Lemma is nevertheless crucial to insure that the condition $C4^{na}\epsilon_n \leq \frac{1}{10^{n+1}}$ holds during the induction, and to show that the iterative conjugation scheme converges.

Lemma 5.2.2 *If ϵ_n is such that $\epsilon_{n+1} \leq C4^{na}\epsilon_n^2$ and if $\epsilon_0 \leq \epsilon^*(C,h_0)$ then for every n $C4^{na}\epsilon_n \leq \frac{1}{10^{n+1}}$*

Exercise 5.2.5 *Prove Lemma 5.2.2.*

Proof of Theorem 5.2.1.

From Lemma 5.2.1, we can assume to start with that $\rho(A_0) \in DS_\alpha(\kappa, \tau)$. Let $M_0 = M = \|A_0\|$. Let $\epsilon_0 = |F_0|_{h_0}$ that we will suppose to be sufficiently small.

Then we apply Proposition 5.2.4 to obtain A_1, Y_1, F_1 satisfying (a')–(d'). Note that the conjugacy equation (c') implies that $e^{F_1(\cdot)}A_1$ satisfies (c) by invariance of the fibered rotation number under conjugation of the cocycle.

If ϵ_0 is sufficiently small, we of course have $Ch_0^{-a}\epsilon_0 \leq \frac{1}{10^1} \ll 1$. We are then in condition to apply Proposition 5.2.4 again and again. Indeed, if ϵ_0 is sufficiently small, Lemma 5.2.2 allows to check the necessary inductive condition $C4^{na}\epsilon_n \leq \frac{1}{10^{n+1}}$ for every n. The outcome (a')–(c') of step (n) of the induction allows to check the hypothesis (a)–(c) at step $(n+1)$.

We get therefore a sequence A_n, Y_n, F_n such that the product $e^{Y_n(\cdot)} \dots e^{Y_1(\cdot)}$ converges in $C^\omega_{\frac{h}{2}}(\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$ to some $B(\cdot)$ such that

$$B(\cdot + \alpha)e^{F_0(\cdot)}A_0 B(\cdot)^{-1} = A_\infty$$

where $A_\infty := \lim_{n\to\infty} A_n$

$\square$

5.2.6 Eliasson's theorem

Theorm 5.2.2 *Fix $\tau, \gamma, h, \sigma > 0$ and $A_0 \in \mathrm{SL}(2,\mathbb{R})$. There exists $\epsilon := \epsilon(\tau, \gamma, h, \sigma, \|A_0\|)$ such that if $A \in C^\omega_h(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ satisfies*

- $|A - A_0|_h \leq \epsilon$
- $\rho(\alpha, A) \in DS_\alpha(\sigma)$

then the cocycle (α, A) is $C^\omega_{\frac{h}{2}}$ reducible.

Note that if we fix some $\sigma > 1$ then $DS_\alpha(\sigma)$ is of full measure. Hence the difference with Theorem 5.2.1 is that with just one closeness condition to A_0 a full measure set of fibered rotation numbers is covered, while in the former Theorem we needed to make ϵ smaller and smaller to cover more and more fibered rotation numbers.

As a consequence of Eliasson's Theorem, one can show that

Corollary 5.2.1 *Let $\alpha \in DC(\tau, \gamma)$. Consider the Schrd̈inger cocycle with potential $V \in C^\omega_h(\mathbb{T}, \mathbb{R})$. There exists $\epsilon := \epsilon(\tau, \gamma, h)$ such that if $|V|_h \leq \epsilon$ then then $(\alpha, A_{V,E})$ is $C^\omega_{\frac{h}{2}}$ is reducible for Lebesgue almost every E in the spectrum.*

The proof of Theorem 5.2.2 is based on a KAM scheme with truncation, and on the elimination of the resonances (uncontrolled small divisors) *via* conjugations that are not close to the Identity. The key point is that these conjugacies are required only a finite number of times after which the proof proceeds similarly to Theorem 5.2.1.

The resonance sequence.

We start with a Lemma that says that due to the Diophantine condition on α if the fibered rotation number (or any number ρ) has some resonance with the multiples of α then these resonances occur only very sporadically, and after "eliminating" them by changing ρ to $\rho + k\alpha$ for the resonant frequency k the new number becomes non resonant for a long time.

The following straightforward lemma on resonances between the fibered rotation number and the base frequency α will be crucial in the sequel.

Fix $U = \max(\tau + 1, \sigma + 1)$ and define

$$DS_\alpha^U(N) := \{\beta \in \mathbb{T} : |\beta - k\alpha - l| > \frac{1}{N^U}; \forall l \in \mathbb{Z}, \forall 0 < |k| \leq N\}$$

Lemma 5.2.3 *There exists $N_0(\tau,\gamma)$ if $\alpha \in DC(\tau,\gamma)$ and if $\beta \notin DS_\alpha^U(N)$ for some $U \geq \tau + 1$ and $N \geq N_0$ then there exists a unique $k \in [-N, N]$ such that $\|\beta - k\alpha\| \leq \frac{1}{N^U}$. Moreover, we have that the latter k satisfies and $\beta - k\alpha \in DS_\alpha^U(1000N)$.*

Proof. If $\beta \notin DS_\alpha^U(N)$ then there exists $k \in [-N, N]$ such that $\|\beta - k\alpha\| \leq \frac{1}{N^U}$. If there exists another $k' \in [-N, N] - \{0\}$ such that $\|\beta - k\alpha\| \leq \frac{1}{N^U}$, then $\|k'\alpha - k\alpha\| \leq \frac{1}{N^U}$ which contradicts $\alpha \in DC(\tau,\gamma)$ if N_0 is sufficiently large. The fact that $\beta - k\alpha \in DS_\alpha^U(1000N)$ follows similarly. $\square$

Given a sequence $N_n \to \infty$, we define and $\rho \in \mathbb{T}$, we define the *resonance sequence* k_n, $n \geq 1$, associated to ρ and $\{N_n\}_{n\geq 1}$ as follows : we let $\rho_0 = \rho$ and define inductively $\rho(A_n) = \rho - \sum_{i=1}^n k_i\alpha$, and let $k_{n+1} = 0$ if $\rho_n \in DS_\alpha^U(N_{n+1})$, and if $\rho_n \notin DS_\alpha^U(N_{n+1})$ then we take k_{n+1} to be the unique integer in $[-N_{n+1}, N_{n+1}]$ such that $\|\rho_n - k\alpha\| \leq \frac{1}{N_{n+1}^U}$ and $\rho_n - k\alpha \in DS_\alpha^U(1000N_{n+1})$.

Truncation procedure.

Let h_n be the sequence $h_0 = h$ and $h_{n+1} = h_n/100$ for $n \geq 0$. Note that $h_n \to 0$! Note also that $h_{n+1} - h_n = \delta_n = \frac{99}{100^{n+1}}$ and $h_{n+1} = \frac{1}{100^{n+1}}$.

Let $N_n = [\frac{1}{50\pi}(\frac{3}{2})^n 100^n]$.

What is N_n ?. The KAM scheme to prove Eliasson's Theorem is based on solving the linearized cohomological equation after truncation of the right hand side term. N_n is the order of truncation. Namely given $f \in C^\omega_h(\mathbb{T}, \mathbb{R})$ we define $T_N f(\theta) = \sum_{|k|\leq N} f_k e(k\theta)$, and $R_N f := f - T_N f$.

Exercise 5.2.6 *Show that $T_N f \in C^\omega_h$ and*

$$|T_N f|_{h'} \leq C(h-h')^{-a}|F|_h$$
$$|R_N f|_{h'} \leq C(h-h')^{-a}e^{-2\pi(h-h')N_n}$$

We write $A(\cdot) = e^{F(\cdot)}A$ with A constant, and where we can also assume that $\rho(A) = \rho(\alpha, A(\cdot))$.

The fact that $\rho(\alpha, A) \in DS_\alpha(\sigma)$ does not allow to start a KAM procedure where at each step we solve the linearized equation $Y(\cdot + \alpha) - Ad(A)Y(\cdot) = F - \hat{F}$ (indeed $\rho(\alpha, A) \in DS_\alpha(\sigma, \kappa)$ but maybe κ is too small compared to the norm of F and the procedure explodes from the first step ! !). Instead, we will truncate F at N_n and solve the linearized equation with $T_{N_n}F$ in the RHS instead of F, under the condition that $\rho(A) \in DS^U_\alpha(N_n)$. We will then obtain a conjugacy e^Y to $e^{\bar{F}}\bar{A}$ with $|\bar{F}\|_{h'} = N_n^{2U+2}(h-h')^{-a}\left(O^2(|F|_h) + O(e^{-2\pi(h-h')N_n}|F|_{h'})\right)$. Since N_n grows geometrically the quadratic part of the first error term guarantees convergence while the $e^{-2\pi(h-h')N_n}$ in front of the second term makes it very small.

The elimination of the resonance.

If the condition $\rho(A) \in DS^U_\alpha(N_n)$ we "correct" the fibered rotation number using a special conjugacy that is not close to identity that is chosen according to Lemma 5.2.3. This procedure is called *elimination of the resonance.*

This will guarantee quasi-reducibility of an analytic cocycle above a Diophantine rotation, and to get reducibility in the case $\rho(\alpha, A) \in DS_\alpha(\sigma)$ we just need to observe that the resonance phenomenon $\rho(A_n) \notin DS^U_\alpha(N_n)$ can happen only finitely many times (the fibered rotation number becomes always good after a certain order of the induction ! !).

For $k \in \mathbb{Z}$ let $E_k : \mathbb{R}/2\mathbb{Z} \to \mathrm{SL}(2, \mathbb{C}) := \begin{pmatrix} e(-\frac{k}{2}\cdot) & 0 \\ 0 & e(\frac{k}{2}\cdot) \end{pmatrix}$

From exercise , we know that if A is an elliptic matrix with $\rho(A) = \beta$ then there exists P such that $\|P\| \leq C(\|\beta\|, \|A_0\|)$ such that $P^{-1}A_0P = D_\beta$. Moreover, the dependence of $C(\|\beta\|, \|A_0\|)$ on $\|\beta\|$ is bounded by $1/\|\beta\|^2$.

When A is an elliptic matrix with $\beta = \rho(A)$ we have a matrix P such that $P^{-1}AP = \begin{pmatrix} e^{i2\pi\beta} & 0 \\ 0 & e^{-i2\pi\beta} \end{pmatrix}$

For such A and P, if we define the fibered conjugacy $D(\cdot) := PE_k(\cdot)P^{-1}$ then $D(\cdot + \alpha)AD(\cdot)^{-1} = \bar{A}$ with $\rho(\bar{A}) = \beta - k\alpha$.

If we apply this argument to a matrix cocycle $(\alpha, e^{F(\cdot)}A)$ we obtain $D(\cdot + \alpha)e^{F(\cdot)}AD(\cdot)^{-1} = e^{\bar{F}(\cdot)}\bar{A}$ with $\rho(\bar{A}) = \rho(A) - k\alpha$ and $|\bar{F}|_h \leq C(\|\beta\|, \|A_0\|)|F|_h$.

The main KAM step.

In this Proposition C is a constant that depends on $h, \gamma, \tau, \|A\|$

Proposition 5.2.4 *Let $A(\cdot) = e^{F(\cdot)}A$ with A constant, and $\rho := \rho(A) = \rho(\alpha, A(\cdot))$. Let k_n be the resonance sequence associated to ρ and $\{N_n\}$. There exists $V > 0$ and a sequence of matrices P_n such that $\|P_n\| \leq \gamma^{-2}N_n^{2(\tau+1)}$ and a sequence $Y_n \in C^\omega_{h_n}(\mathbb{T}, sl)$ such that if we denote $C_n = e^{Y_n}P_nE_{k_n}P_n^{-1}$, $B_n := C_n \circ \ldots C_1$ then*

$$B_n(\cdot + \alpha)A(\cdot)B_n^{-1}(\cdot) = e^{F_n}A_n$$

with $\epsilon_n := |F_n|_{h_n}$ satisfying

(i) $\rho_n := \rho(A_n) = \rho(\alpha, A) - \sum_{i=1}^n k_i\alpha$

(ii) In case $k_n = 0$ we have

$$(ii)_1 \qquad \epsilon_{n+1} \leq CN_{n+1}^V(\epsilon_n^2 + e^{-2\pi\delta_{n+1}N_{n+1}}\epsilon_n)$$

In case $k_n \neq 0$ we have

$$(ii)_2 \qquad \epsilon_{n+1} \leq CN_{n+1}^V e^{4\pi h_{n+1}N_{n+1}}(\epsilon_n^2 + e^{-20\pi\delta_{n+1}N_{n+1}}\epsilon_n)$$

(iii) In case $k_n = 0$ we have

$$(iii)_1 \qquad |Y_n|_{h_n} \leq CN_{n+1}^V\epsilon_n$$

In case $k_n \neq 0$ we have

$$(iii)_2 \qquad |Y_n|_{h_n} \leq CN_{n+1}^V e^{4\pi h_{n+1}N_{n+1}}\epsilon_n$$

We note that the proposition holds under the condition that $CN_{n+1}^V e^{4\pi h_{n+1}N_{n+1}}\epsilon_n$ *remains small in the induction.*

□

Proof. By induction : suppose given the step n. Then we have two cases
Case 1 : $\rho_n \in DS_\alpha^U(N_{n+1})$. Then we let $k_{n+1}=0$. We solve with Y_{n+1} the linearized equation with $T_{N_{n+1}}F_n$ in the RHS. We then use Lemma 5.2.1 to adjust the constant part A_{n+1} in $e^{F_{n+1}}A_{n+1}$ so that (i) holds. The estimate $(iii)_1$ follows the same lines as Proposition 5.2.2. The estimate $(ii)_1$ comes from the truncation and rest estimates of Exercise 5.2.6 and the fact that $|F_{n+1}\|_{h_{n+1}} = (h_n-h_{n+1})^{-a}\left(O^2(|T_{N_{n+1}}F_n|_{h_{n+1}}, |Y_{n+1}|_{h_{n+1}}) + O(|F|_h, |R_{N_{n+1}}F_n|_{h_{n+1}})\right)$. The factor $e^{-2\pi\delta_{n+1}N_{n+1}}$ corresponds to $|R_{N_{n+1}}F_n|_{h_{n+1}}$ where $\delta_{n+1}=h_{n+1}-h_n$. The factor $(h_n-h_{n+1})^{-a}$ is accounted for in N_{n+1}^V

Case 2 : $\rho_n \notin DS_\alpha^U(N_{n+1})$. Then we let k_{n+1} be as in Lemma 5.2.3. First of all we observe that since $|\rho_n - k_{n+1}\alpha| \leq \frac{1}{N_{n+1}^U}$ and since $\alpha \in DC(\tau,\gamma)$ then $\|\rho_n\| \geq \frac{\gamma}{N_{n+1}^{\tau+1}}$. This provides the bound $\|P_n\| \leq \gamma^{-2}N_{n+1}^{2(\tau+1)}$ for the matrix P_n that diagonalizes A_n. Next we apply the fibered conjugacy $P_nE_{k_{n+1}}(\cdot)P_n^{-1}$ to $e^{F_n(\cdot)}A_n$ and we get after using Lemma 5.2.1 $e^{\bar{F}_n(\cdot)}\bar{A}_n$ with $\rho(\alpha, e^{\bar{F}_n(\cdot)}\bar{A}_n) = \rho(\bar{A}_n) = \rho_n - k_{n+1}\alpha$. Since $\rho_n - k_{n+1}\alpha \in DS_\alpha^U(1000N_{n+1})$, we find ourselves in the context of Case 1 with $\bar{A}_n$ and $\bar{F}_n$ instead of A_n and F_n and $10N_{n+1}$ instead of N_{n+1} for the truncation order. Note that the control on P_n and the fact that $|k_{n+1}| \leq N_{n+1}$ gives that $|\bar{F}_n|_{\bar{h}_n} \leq CN_{n+1}^b e^{4\pi\bar{h}_nN_{n+1}}$ where b is some constant (that depends only on τ) and $e^{4\pi\bar{h}_nN_{n+1}}$ accounts for the norm of $E_{k_{n+1}}$ on the analytic band of width $\bar{h}_n := \frac{h_n}{50}$. Now $(iii)_2$ and $(ii)_2$ follow as in Case 1. Note the factor $N_{n+1}^V e^{4\pi h_{n+1}N_{n+1}}$ that accounts for the small divisor and the size of the conjugacy and the bands width loss, while the factor 10π in the rest term comes from the fact that we truncated at $1000N_{n+1}$ while $\bar{h}_n - h_{n+1} = \bar{h}_n/2 \geq \delta_n/100$. □

Convergence of the scheme and quasi-reducibility.

Recall that h_n be the sequence $h_0=h$ and $h_{n+1}=h_n/100$ for $n\geq 0$. Note that $h_n \to 0$! Note also that $h_{n+1}-h_n=\delta_n=\frac{99}{100^{n+1}}$ and $h_{n+1}=\frac{1}{100^{n+1}}$. Let $N_n = [\frac{1}{50\pi}(\frac{3}{2})^n 100^n]$.

Lemma 5.2.4 *Let $C, V > 0$ and ϵ_n be a sequence such that for every n we have at least one of the following that holds*

$$\epsilon_{n+1} \leq CN_{n+1}^V(\epsilon_n^2 + e^{-2\pi\delta_{n+1}N_{n+1}}\epsilon_n) \tag{5.12}$$
$$\epsilon_{n+1} \leq CN_{n+1}^V e^{4\pi h_{n+1}N_{n+1}}(\epsilon_n^2 + e^{-10\pi\delta_{n+1}N_{n+1}}\epsilon_n). \tag{5.13}$$

Then for any $\eta > 0$, there exists $\epsilon^*(h, C, V) > 0$ such that if $\epsilon_0 < \epsilon^*$ then for every n

$$(\star) \qquad CN_{n+1}^V e^{4\pi h_{n+1}N_{n+1}}\epsilon_n \leq \eta e^{-(\frac{3}{2})^n}.$$

Proof. For any fixed l, if ϵ^* is sufficiently small then $CN_{l+1}^V e^{4\pi h_{l+1}N_{l+1}}\epsilon_l \leq \eta e^{-(\frac{3}{2})^l}$. Then, if l is sufficiently large (depending only on h, C, V), the proof of $(\star)$ for $n > l$ will follow by induction. Indeed after order $n = l$, the factor CN_{n+1}^V does not count much in the inequalities (5.12)–(5.13) and we basically just have to show that if $\epsilon_n \leq e^{-(\frac{3}{2})^n}e^{-4\pi h_{n+1}N_{n+1}}$ then $\epsilon_n^2 + e^{-2\pi\delta_{n+1}N_{n+1}}\epsilon_n \leq e^{-(\frac{3}{2})^{n+1}}e^{-4\pi h_{n+2}N_{n+2}}$ and $e^{4\pi h_{n+1}N_{n+1}}(\epsilon_n^2 + e^{-20\pi\delta_{n+1}N_{n+1}}\epsilon_n) \leq e^{-(\frac{3}{2})^{n+1}}e^{-4\pi h_{n+2}N_{n+2}}$. The latter is a straightforward calculation.

□

As a consequence of Lemma 5.2.4 and Proposition 5.2.4 we obtain quasi-reducibility for cocycles that are defined above Diophantine rotation and that are close to a constant. Namely we say that a cocycle $(\alpha, A(\cdot))$ is C^ω quasi-reducible if for any $\xi > 0$ there exists $h' > 0$ and $B \in C^\omega_{h'}(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ and $\bar{A} \in SL$ such that

$$B(\cdot + \alpha)A(\cdot)B(\cdot)^{-1} = A'(\cdot)$$

with $|A' - \bar{A}|_{h'} \leq \xi$. Note that h' can be very small and B very large!

Corollary 5.2.2 *Fix τ, γ, h and $A_0 \in \mathrm{SL}(2, \mathbb{R})$. There exists $\epsilon := \epsilon(\tau, \gamma, h, \|A_0\|)$ such that if $A \in C^\omega_h(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ satisfies*

- $|A - A_0|_h \leq \epsilon$

then the cocycle (α, A) is C^ω quasi-reducible.

Eliasson's trick on finiteness of the number of resonancies.

Let $N_n = [\frac{1}{50\pi}(\frac{3}{2})^n 100^n]$. Recall that $U \geq \sigma + 1$

Lemma 5.2.5 *If $\alpha \in DC(\tau)$ and $\rho \in DS^U_\alpha(\sigma)$ for some $\sigma > 0$, then the resonance sequence k_n, $n \geq 1$, associated to ρ and $\{N_n\}_{n\geq 1}$ becomes zero after some n.*

Proof. Given any $\kappa > 0$, since $U \geq \sigma + 1$, the fact that $\rho \in DS_\alpha^U(\sigma, \kappa)$ implies that for n large $\|\rho(\alpha, A) - \sum_{i=1}^n k_i\alpha\| \geq \frac{1}{N_{n+1}^U}$. □

Proof of Theorem 5.2.2.

Lemma 5.2.5 implies that after some n only **Case 1** has to be considered in Proposition 5.2.4. But this actually allows to change the definition of h_n after some large n_0 to $h_{n+1} = h_n(1 - \frac{1}{4^{n+1}})$ and proceed with the proof using only $(ii)_1$ and $(iii)_1$ and obtain convergence of the sequence B_n in $C^\omega_{h_{n_0}/2}$ which gives reducibility. □

Chapitre 6

Homeomorphisms of the Circle

Let $\pi : \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $x \mapsto x[1]$. Given a homeomorphism $f : \mathbb{T}$ we define its lift F to the universal cover $\mathbb{R}$ of $\mathbb{T}$ such as

$$f \circ \pi = \pi \circ F$$

We call $\mathrm{Homeo}_+(\mathbb{T})$ the set of orientation preserving circle homeomorphisms. We have that $f \in \mathrm{Homeo}_+(\mathbb{T})$ if and only if f has a lift F that is strictly increasing and $F(x+1) - F(x) = 1$.

6.1 Rotation number

The following Proposition shows that a circle homeomorphism behaves in average like a rotation.

Proposition 6.1.1 *Let $f \in \mathrm{Homeo}_+(\mathbb{T})$ and F denote its lift to $\mathbb{R}$. Then there exists $\rho(f) \in \mathbb{T}$ such that the following limit holds uniformly in $x \in \mathbb{R}$*

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}[1].$$

The number $\rho(f)$ is called the rotation number of f.

Proof. Two lifts of f differ by an integer, so that the limit of the proposition if it exists is defined modulo the addition of an integer, or equivalently $\rho(f) \in \mathbb{T}$. Fix now a lift F of f.

Independence of x and uniformity : We have that F^n is increasing and $F^n(x)-F^n(x+1)=1$. Thus we can reduce the study of the limit to $x \in [0,1)$. Also, for $x, y \in [0,1)$ we get that $|F^n(x)-F^n(y)|<1$. Consequently,

$$\left|\frac{F^n(x)-x}{n}-\frac{F^n(y)-y}{n}\right|<\frac{2}{n} \tag{6.1}$$

and if the limit exists for one x then the same limit holds uniformly for all $y \in \mathbb{R}$.

Existence : Define $a_n = F^n(0)$. Then, apply (6.1) to $x = a_m$ and $y = 0$ to get that

$$a_m + a_n - 2 \leq a_{m+n} \leq a_m + a_n + 1 \tag{6.2}$$

Now exercise 6.1.1 allows to conclude the proof. □

Exercise 6.1.1 *Show that if a sequence a_n satisfies (6.2), then a_n/n has a finite limit.*

The following shows that the rotation number of a homeomorphism is an invariant of topological conjugacy.

Proposition 6.1.2 *If $h \in \mathrm{Homeo}_+(\mathbb{T})$ then, $\rho(h^{-1}\circ f \circ h) = \rho(f)$*

Proof. Let F and H be lifts of f and h. By Proposition 6.1.1, we can choose F such that $F^n(0) = n\rho(f) + u_n$ with $u_n = o(n)$. Suppose WLOG that $H(0) = 0$. Check that $G = H^{-1}\circ F \circ H$ is a lift of $h^{-1}\circ f \circ h$. Then

$$G^n(0) = H^{-1}\circ F^n \circ H(0) = H^{-1}(n\rho(f)+u_n) = [n\rho(f)+u_n] + H^{-1}(\{n\rho(f)+o(n)\})$$

Therefore $G^n(0)/n \to \rho(f)$ and the proof is completed. □

6.2 Poincaré classification

We say that a point x is periodic for f of period q if $f^q(x) = x$ and there is no other $0 < q' < q$ such that $f^{q'}(x) = x$.

Proposition 6.2.1 *Let $f \in \mathrm{Homeo}_+(\mathbb{T})$. Then, $\rho(f) = \frac{p}{q}$, $p \wedge q = 1$, if and only if f has a periodic orbit of period q. In this case all the periodic orbits of f have the same period q.*

Proof. Let F be a lift of f. We have that $\rho(f) = \frac{p}{q}$ if and only if there exist x_0 such that $F^q(x_0) - x_0 - p = 0$. Indeed, if $F^q(x_0) - x_0 - p = 0$ then $F^{nq}(x_0) - x_0 - np = 0$, which implies that $\rho(f) = \frac{p}{q}$. Conversely, if there is no such x_0 then by by the intermediate value theorem we have that $F^q(x) - x - p$ is either strictly positive or strictly negative for every $x \in \mathbb{T}$. Suppose it is positive, the other case being similar. By compacity, there exists $\epsilon > 0$ such that $F^q(x) - x - p > \epsilon$ for every $x \in \mathbb{T}$. Hence $F^{nq}(x) - x - np > n\epsilon$ for every $x \in \mathbb{T}, n \in \mathbb{N}^*$. Hence $\rho(f) \geq \frac{p}{q} + \epsilon > \frac{p}{q}$.

So, suppose $\rho(f) = \frac{p}{q}$ with $p \wedge q = 1$. Then there exists x_0 such that $f^q(x_0) = x_0$. If there exists x' of period q' then there is p' such that $F^{q'}(x') - x' - p' = 0$, hence $\rho(f) = \frac{p'}{q'}$ hence $q' = lq$ and $p' = lp$. We necessarily have $F^q(x') - x' - p = 0$ otherwise we cannot have $F^{q'}(x') - x' - p' = 0$. We conclude that all the periodic points have period q.

Now, if x_0 is a periodic point for f of period q then there exists p such that $F^q(x_0) - x_0 - p = 0$, hence $\rho(f) = \frac{p}{q}$. If $p = lp'$ and $q = lq'$ then necessarily $F^{q'}(x_0) - x_0 - p' = 0$ otherwise we cannot have $F^q(x_0) - x_0 - p = 0$. By the definition of a period, $q' \geq q$ hence $l = 1$ and $p \wedge q = 1$.

□

Proposition 6.2.2 *The map* $\rho : \mathrm{Homeo}_+(\mathbb{T}) \to \mathbb{T}, f \mapsto \rho(f)$ *is continuous for the* C^0 *topology.*

Proof. We have that $\rho(f) < \frac{p}{q}$ if and only if there exists $\epsilon > 0$ such that

$$F^q(x) - x - p < -\epsilon, \forall x$$

indeed if not, then either there exists x such that $F^q(x) - x - p = 0$ in which case $\rho(f) = p/q$ or for every x, $F^q(x) - x - p > 0$, in which case $\rho(f) \geq p/q$. But $F^q(x) - x - p < -\epsilon, \forall x$ is an open condition in the C^0 topology, i.e. it is verified for g C^0 close to f if it is verified for f. □

The following Proposition due to Poincaré shows that circle homeomorphisms with irrational rotation number are indeed acquainted to rotations, by semi-conjugacy.

We use the trigonometric order on the circle and write $x < y < z$ if according to the trigonometric orientation y falls between x and z.

We recall that a map $h : \mathbb{T} \to \mathbb{T}$ has degree d if it has a lift $H : \mathbb{R} \to \mathbb{R}$ such that $H(x+1) = H(x) + d$. Homeomorphisms in $\mathrm{Homeo}_+(\mathbb{T})$ have degree

1, but there exist continuous maps from $\mathbb{T} \to \mathbb{T}$ of degree one that are not homeomorphisms.

Theorm 6.2.1 *If $f \in \mathrm{Homeo}_+(\mathbb{T})$ is such that $\rho(f) \notin \mathbb{Q}$, then there exists $h : \mathbb{T} \to \mathbb{T}$ of degree one and that preserves the trigonometric ordering of points on the circle and satisfies $h \circ f = R_\alpha \circ h$. We then say that f is semi-conjugated to R_α.*

Exercise 6.2.1 *Show that a continuous $h : \mathbb{T} \to \mathbb{T}$ that preserves the trigonometric ordering of points must have degree* 1.

Proof. Let $\alpha := \rho(f)$. The main ingredient in the proof of Poincaré's Theorem is the fact that the order in the orbit of any point x by f is the same as the order of the sequence $n\alpha$ on the circle.

Lemma 6.2.1 *For any n, m, n', m' and $x \in \mathbb{T}$, we have that*

$$n\alpha - m < n'\alpha - m', \iff F^n(x) - m < F^{n'}(x) - m'$$

Proof. The proof of the lemma is straight forward from the observation that $\alpha < \frac{m-m'}{n-n'}$ if and only if $F^{n-n'}(y) - y - (m - m') < 0$ for every y. We then apply the latter to $y = F^{n'}(x)$. □

If $\rho(f) \notin \mathbb{Q}$, we have that f has no periodic orbits. In particular we can define $h(f^n(0)) = n\alpha$. Now, for any $z \in \mathbb{T}$ we denote by z^- and z^+ the closets points to the left and to the right of z that are in the closure of the orbit of 0 by f (it is possible that $z^- = z^+ = z$ or not). We construct now an increasing sequence $f^{n_i}(0)$ that converges to z^- as follows : let $n_0 = 0$, then take n_1 to be the first $n \geq n_0$ such that $f^n(0) \in (f^{n_0}(0), z^-)$, then take n_2 to be the first $n \geq n_1$ such that $f^n(0) \in (f^{n_1}(0), z^-)$, etc. Construct similarly the subsequence $f^{m_i}(0)$ that decreases towards z^+.

By the preservation of the order of points between f and the R_α we have that $n_i\alpha$ is an increasing sequence that is bounded by the decreasing sequence $m_i\alpha$. Their limit points $\hat{z}^-$ and $\hat{z}^+$ must coincide, $\hat{z}^- = \hat{z}^+ = \hat{z}$, otherwise we get a contradiction with the definition of z^- and z^+. We then define $h[z^-, z^+] = \hat{z}$. Observe also that for any sequence $f^{p_i}(0)$ that converges to z^- or to z^+ the preservation of order implies that $p_i\alpha$ converges to $\hat{z}$.

We can define in this way h for any z since the intervals $[z^-, z^+]$ (or points in the case $z^- = z^+$) for distinct z are either equal or disjoint. The map h thus defined is continuous since it is continuous on the boundary and inside each $[z^-, z^+]$. It is surjective since it contains the orbit $n\alpha$. Since it preserves the trigonometric ordering on the circle its lift is an increasing map and its degree is equal to one.

□

From his result on semi-conjugacy, Poincaré obtained the following alternative for circle homeomorphisms with irrational rotation number.

Definition 6.2.1 *We say that an interval I is wandering for f if for every $n \in \mathbb{Z}$, $f^n(I) \cap I = \emptyset$.*

Corollary 6.2.1 *For $f \in \mathrm{Homeo}_+(\mathbb{T})$ the following are equivalent*

(i) f does not have a wandering interval

(ii) f is transitive

(iii) f is minimal

(iv) f is conjugated to R_α (i.e. the semi-conjugacy h satisfies $h \in \mathrm{Homeo}_+(\mathbb{T})$).

Proof. We clearly have that $(iii) \implies (ii) \implies (i)$, as well as $(iv) \implies (iii)$. Now, the increasing semi-conjugacy h is not injective if and only if there exists I such that $h(I) = \hat{z}$ (actually I is of the form (z^-, z^+)). Since $h(f^n(I)) = R_\alpha^n h(I) = \hat{z} + n\alpha \neq \hat{z}$ we necessarily have that $h^n(I) \cap I = \emptyset$. Hence $(i) \implies (iv)$ which finishes the proof of the Corollary. □

Chapitre 7

Circle Diffeomorphisms

A natural question, is when can the Poincaré result on semi-conjugacy be improved to insure a conjugacy between $f \in \mathrm{Homeo}_+(\mathbb{T})$ and R_α. Denjoy solved this problem by showing that a sufficient condition is that the homeomrophism f be regular, in particular $f \in \mathrm{Diff}^2(\mathbb{T})$ is sufficient. He also showed that there exist $f \in \mathrm{Diff}^1$ with irrational rotation number α that is not conjugated to R_α.

7.1 The Denjoy Theorem

Theorm 7.1.1 *If $f \in \mathrm{Diff}^2$ and $\rho(f) = \alpha \in \mathbb{R} - \mathbb{Q}$, then f is conjugated to R_α : There exists $h \in \mathrm{Homeo}_+(\mathbb{T})$ such that $h \circ f = R_\alpha \circ h$.*

Proof.

Let q_n be the sequence of best denominators associated to α. Suppose n is even and let $J_n = [0, R_\alpha^{2q_{n-1}}(0)]$. We have $\sum_{l=0}^{q_n-1} |R_\alpha^l(J_n)| \leq 2$. We also have that for every orbit $x_0, x_1, \dots, x_{q_n-1} = x, R_\alpha(x), R_\alpha^2(x), \dots, R_\alpha^{q_n-1}(x)$ it is possible to rearrange the points such that $x_{l_0} \in J_n$, $x_{l_1} \in R_\alpha(J_n), \dots, x_{l_{q_n-1}} \in R_\alpha^{q_n-1}(J_n)$.

By semi-conjugacy the same situation holds for f : If $I_n = [0, f^{2q_{n-1}}(0)]$ then $\sum_{l=0}^{q_n-1} |f^l(I_n)| \leq 2$. Also, every orbit $x_0, x_1, \dots, x_{q_n-1} = x, f(x), f^2(x), \dots, f^{q_n-1}(x)$ can be rearranged such that $x_{l_0} \in I_n$, $x_{l_1} \in f(I_n), \dots, x_{l_{q_n-1}} \in f^{q_n-1}(I_n)$.

Now take any pair of points $(x, y) \in \mathbb{T}$, and let $x_{l_0}, y_{l'_0} \in I_n, x_{l_1}, y_{l'_1} \in f(I_n), \dots, x_{l_{q_n-1}}, y_{l'_{q_n-1}} \in f^{q_n-1}(I_n)$.

We have that

$$\begin{aligned}
|\ln Df^{q_n}(x) - \ln Df^{q_n}(y)| &= \left|\sum_{l=0}^{q_n-1} \ln Df(f^l(x)) - \sum_{l=0}^{q_n-1}\right| \\
&= \left|\sum_{i=0}^{q_n-1} \ln Df(x_{l_i}) - \sum_{l=0}^{q_n-1} Df(y_{l'_i})\right| \\
&\leq \max_{\theta\in\mathbb{T}} |D\ln Df(\theta)| \sum_{i=0}^{q_n-1} |x_{l_i} - y_{l'_i}| \\
&\leq \max_{\theta\in\mathbb{T}} |D\ln Df(\theta)| \sum_{i=0}^{q_n-1} |f^i(I_n)| \\
&\leq C(f)
\end{aligned}$$

Where $C(f) := 2\max_{\theta\in\mathbb{T}} |D\ln Df(\theta)|$.

But since $f^{q_n} \in \mathrm{Diff}^1(\mathbb{T})$, there exists x_0 such that $Df^{q_n}(x_0) = 1$. Taking $y = x_0$ in the inequality above, we hence get that for any $x \in \mathbb{T}$ we have $Df^{q_n}(x) \leq e^{C(f)}$.

The latter inequality implies that f cannot have wandering intervals. Indeed, if I is a wandering interval for f then necessarily $|f^{q_n}(I)| \to 0$ as $n \to \infty$ which clearly contradicts $Df^{q_n}(x) \leq e^{C(f)}$ for every $x \in I$. By Corollary 6.2.1 we conclude that f is conjugated to R_α.

□

7.2 Denjoy counterexamples

Theorm 7.2.1 *For any $\alpha \in \mathbb{R} - \mathbb{Q}$ there exists a C^1 diffeomorphism of the circle f such that $\rho(f) = \alpha$ and f has a wandering interval.*

Proof. Let $\{l_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers such that $\sum_{n\in\mathbb{Z}} l_n = 1$ and $\frac{l_{n+1}}{l_n} \to 1$ as $n \to \infty$.

We take on the circle a collection of disjoint intervals $\{I_n\}_{n\in\mathbb{Z}}$ such that $|I_n| = l_n$ and such that the I_n's are ordered like the orbit of 0 under R_α, that is $I_n < I_j$ is and only if $\{n\alpha\} < \{j\alpha\}$ (for the cyclic ordering on $\mathbb{T}$).

Now we define a diffeomorphism f on $W = \cup_{n\in\mathbb{Z}} I_n$ as follows : $f \in \mathrm{Diff}^\infty(I_n, I_{n+1})$ and f' can be prolongated to the closure $\bar{I}_n = [a_n, b_n]$ so that $f'(a_n) = f'(b_n) = 1$, and $\min f'_{|I_n}, \max f'_{|I_n} \to 1$ as $n \to \infty$. The condition

$\frac{l_{n+1}}{l_n} \to 1$ as $n \to \infty$ insures that such an f can be constructed (of course not unique).

We also construct a map $h : W \to \mathbb{T}$ such that $h(I_n) = R_\alpha^n(0)$. We clearly have $h \circ f = R_\alpha \circ h$ on W. We also have that f and h are monotonous on W. Since both W and the orbit $\{R_\alpha^n(0)\}_{n\in\mathbb{Z}}$ are dense on the circle we can extend f and h to the whole circle getting a semi-conjugacy h, that is not injective such that $h \circ f = R_\alpha \circ h$ on $\mathbb{T}$.

We just have to show that f is of class C^1. Indeed, f is C^1 by construction on W. We need to show that it is also C^1 at the points $x \in \Omega = \mathbb{T} - W$. Since $f' = 1$ on the boundary of W, we finish if we show that $f'(x) = 1$ for every $x \in \Omega$. Equivalently, we must show that $|f([x, z])|/|[x, z]| \to 1$ as $z \to x$. We can assume $x < z$ the other case being similar. If x is a left boundary point of some interval in W then we are done since $f' = 1$ on the boundary of W. If not, then x is accumulated on the right by intervals in W. Hence, we have that

$$\sum_{I_n\in[x,z]} l_n < |[x, z]| < \sum_{I_n\in[x,z]} l_n + \hat{l}_{n_z}$$

where $\hat{l}_{n_z} = 0$ if z does not belong to W and $\hat{l}_{n_z} = |[x, z] \cap I_{n_z}|$ where n_z is such that the interval I_{n_z} contains z. Also

$$\sum_{I_n\in[x,z]} l_{n+1} < |f([x, z])| < \sum_{I_n\in[x,z]} l_{n+1} + \hat{l'}_{n_z}$$

where $\hat{l'}_{n_z} = 0$ or $\hat{l'}_{n_z} = |f([x, z] \cap I_{n_z})|$. As $z \to x$ we have that $n_z \to \infty$ hence in the case $\hat{l}_{n,z} \neq 0$ we have that $\hat{l'}_{n_z}/\hat{l}_{n_z} \sim 1$ as $z \to x$, because $\min f'_{|I_n}, \max f'_{|I_n} \to 1$ as $n \to \infty$. It also holds that $\min_{n:I_n\in[x,z]} \to \infty$ as $z \to x$, hence we have that $l_{n+1}/l_n \to 1$ as $z \to x$ uniformly for all n such that $I_n \in [x, z]$. We conclude that $|f([x, z])|/|[x, z]| \to 1$ as $z \to x$. □

Exercise 7.2.1 *Show how to construct inductively a sequence I_n as in Denjoy counterexample.*

Exercise 7.2.2 *Give an explicit construction of f on W.*

7.3 Non regular conjugacies. The Liouville phenomenon

Denjoy Theorem states that if $f \in \mathrm{Diff}^{\infty}(\mathbb{T})$ has an irrational rotation number α then f is topologically conjugated to the rotation R_α. I n this section we see that circle diffeomorphisms of class C^∞ with a Liouville rotation number α do not have in general a C^1 conjugacy with R_α.

More precisely define

$$\mathcal{A}_\alpha = \mathrm{cl}_\infty\left(\{h \circ R_\alpha \circ h^{-1} : h \in \mathrm{Diff}^\infty(\mathbb{T})\}\right)$$

where cl_∞ stands for closure in the C^∞ topology : diffeomorphisms of $\mathcal{A}_\alpha$ are the diffeomorphisms that can be approached in any topology by diffeomorphisms of the form $h \circ R_\alpha \circ h^{-1}, h \in \mathrm{Diff}^\infty(\mathbb{T})$. Observe that by continuity of the rotation number $\rho(f) = \alpha$ for $f \in \mathcal{A}_\alpha$.

Theorm 7.3.1 *If α is a Liouville number then for f in a G^δ-dense set of $\mathcal{A}_\alpha$ f is not C^1 conjugated to R_α.*

Proof. We will use the following criterion for non existence of a C^1 conjugacy.

Lemma 7.3.1 *If f is such that*

$$\limsup_{n\in\mathbb{Z}} \|Df^n\| = +\infty$$

then f is not C^1 conjugated to R_α.

Proof. The proof of the Lemma is very simple. Assume $f = h\circ R_\alpha \circ h^{-1}$ with h of class C^1. Then $f^n = h \circ R_\alpha^n \circ h^{-1}$ has a bounded derivative, independent of n. This contradicts the limsup of the lemma.

□

Let $A_{n,m} = \{f \in \mathcal{A}_\alpha : \|Df^n\| > m\}$. Clearly $f \in \cap_{m\in\mathbb{N}} \cup_{n\in\mathbb{Z}} A_{n,m}$ satisfies $\limsup_{n\in\mathbb{Z}} \|Df^n\| = +\infty$. Since $A_{n,m}$ is a relatively open set in $\mathcal{A}_\alpha$ for any C^r topology, we just need to show that $A_{n,m}$ is dense in $\mathcal{A}$. For this we fix m and $r \in \mathbb{N}$ and start with $f = h \circ R_\alpha \circ h^{-1}$, and we show that for any $\epsilon > 0$, there exists $h_n \in \mathrm{Diff}^\infty(\mathbb{T})$ such that, if n is large, $f_n := h \circ h_n \circ R_\alpha \circ h_n^{-1} \circ h^{-1}$ satisfies

(i) $\|f_n - f\|_{C^r} \leq \epsilon$

(ii) There exists l_n such that $\|Df^{l_n}\| > m$

Construction of h_n.

We define a sequence $\psi_n \in \mathrm{Diff}^\infty([0,1])$ such that for some $A_n \to \infty$ and $\epsilon_n = \frac{1}{n}$

(1) $\psi_n(x) = x$ for $x \in [0, \frac{1}{20}, [\frac{1}{2}, 1]$
(2) $\psi_n(x) < \epsilon_n x$ for $x \in [\frac{1}{10}, \frac{1}{2} - \frac{1}{10}]$
(3) $\|\psi_n\|_{C^r} < A_n$

Of course A_n depends on r but we drop this dependance in the notations since r is supposed to be fixed.

Exercise 7.3.1 *To construct the sequence ψ_n, start by showing that there exists a sequence $\phi_n \in C^\infty([0,1], \mathbb{R})$ such that $\phi_n > 0$, $\phi_n(x) = 1$ for $x \in [0, \frac{1}{20}, [\frac{1}{2}, 1]$, $\phi_n(x) < \epsilon_n$ for $x \in [\frac{1}{10}, \frac{1}{2} - \frac{1}{10}]$ and $\int_0^1 \phi_n(\theta) d\theta = 1$. Then integrate ϕ_n on $[0,1]$ to get ψ_n.*

Let now q_n be a subsequence of the sequence of best denominators of α such that

$$\|q_n \alpha\| < \frac{1}{q_n^{2n}}$$

and such that $q_n > A_n$.

Next define $H_n \in \mathrm{Diff}^\infty([0, \frac{1}{q_n}])$ by $H_n(x) = \frac{1}{q_n}\psi_n(q_n x)$. We have that $H_n(0) = 0$ and $H_n(\frac{1}{q_n}) = \frac{1}{q_n}$ and $H_n'(0) = H_n'(\frac{1}{q_n}) = 1$ while $H_n^{(r)}(0) = H_n^{(r)}(\frac{1}{q_n}) = 0$, for every $r \geq 2$. Hence we can extend H_n to a strictly increasing diffeomorphism of $\mathbb{R}$ by asking that $H_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ H_n$.

Since $H_n(x+1) = H_n(x) + 1$, H_n defines a lift of a circle diffeomorphism $h_n \in \mathrm{Diff}^\infty(\mathbb{T})$. The following properties of h_n are inherited from (1)–(3) of ψ_n.

(h1) $h_n(x) = x$ for $x \in [\frac{1}{2q_n}, \frac{1}{q_n}]$
(h2) $h_n(x) < \epsilon_n x$ for $x \in [\frac{1}{10q_n}, (\frac{1}{2} - \frac{1}{10})\frac{1}{q_n}]$
(h3) $\|h_n\|_{C^r} < q_n^r A_n$

The important fact is that $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$, from where we conclude that (using $(h3)$ and the Fa De Bruno formula for differentiating compositions of maps)

$$\begin{aligned}
\|h_n \circ R_\alpha \circ h_n^{-1} - R_\alpha\|_{C^r} &\leq \|h_n \circ R_\alpha \circ h_n^{-1} - h_n \circ R_{\frac{p_n}{q_n}} \circ h_n^{-1} + R_{\frac{p_n}{q_n}} - R_\alpha\|_{C^r} \\
&\leq \|h_n\|_{C^r}^{r^2} |\alpha - \frac{p_n}{q_n}| \\
&\leq \frac{1}{q_n^n}
\end{aligned}$$

if n is sufficiently large. Hence $\|f_n - f\|_{C^r} \leq \epsilon$.

Now, let l_n be such that $R_\alpha^{l_n}([\frac{1}{2q_n}, \frac{3}{4q_n}]) \subset [\frac{1}{10q_n}, (\frac{1}{2} - \frac{1}{10})\frac{1}{q_n}]$ (this is possible by minimality of R_α).

Finally, let $J_n := [\frac{1}{2q_n}, \frac{3}{4q_n}]$ and $I_n := h \circ h_n(J_n) = h(J_n)$, by $(h1)$. We have that

$$\begin{aligned}
f_n^{l_n}(I_n) &= h \circ h_n \circ R_\alpha^{l_n} \circ h_n^{-1} \circ h^{-1}(I_n) \\
&= h \circ h_n \circ R_\alpha^{l_n}(J_n)
\end{aligned}$$

Because $R_\alpha^{l_n}(J_n) \subset [\frac{1}{10q_n}, (\frac{1}{2} - \frac{1}{10})\frac{1}{q_n}]$ we get from $(h2)$ that $|h_n \circ R_\alpha^{l_n}(J_n)| \leq \epsilon_n J_n$, hence since $I_n = h(J_n)$

$$|h \circ h_n \circ R_\alpha^{l_n}(J_n)| \leq \|h\|_{C^1} \epsilon_n |J_n| \leq \|h\|_{C^1} \|h^{-1}\|_{C^1} \epsilon_n |I_n|$$

In conclusion, $|f_n^{l_n}(I_n)|/|I_n| \to 0$ as $n \to \infty$, which implies that $\|Df_n^{-l_n}\| \to \infty$ as $n \to \infty$ and completes the proof of (ii) and hence of Theorem 7.3.1.

□

The result we have proved has a general implication on the set of all circle diffeomorphisms having rotation number α. Define this set as

$$\mathcal{F}_\alpha := \{f \in \mathrm{Diff}^\infty(\mathbb{T}) : \rho(f) = \alpha\}$$

A difficult result by Yoccoz shows the following

Theorm 7.3.2 *For any irrational α, the set $\mathcal{A}_\alpha$ is dense in $\mathcal{F}_\alpha$*

Then our result gives the following general Corrolary

Corollary 7.3.1 *If $\alpha \in \mathbb{R} - \mathbb{Q}$ is Liouville then for f in a G^δ-dense set of $\mathcal{H}_\alpha$, f is not C^1 conjugated to R_α.*

Chapitre 8

KAM with parameter exclusion. Smooth conjugacies in the Arnol'd family.

8.1 The Arnol'd family

For $|\epsilon| < \frac{1}{2\pi}$, define the following family of circle diffeomorphisms

$$f_{t,\epsilon}(\theta) = \theta + \varphi(t) + \epsilon \Delta f(\theta)$$

Where $\Delta f \in C^\infty(\mathbb{T}, \mathbb{R})$ is such that for every $\theta \in \mathbb{T}$, $f'(\theta) > -2\pi$ We denote by $I_0 = [0,1]$ and by $lip(I_0)$ the Lipschitz norm of function on I_0.

Assume that φ is Lipschitz and that there there exists M such that

$$2\|\varphi\|_{lip(I_0)} \leq M, \quad \inf_{t \in I_0} \varphi'(t) \geq \frac{2}{M} \tag{8.1}$$

When $\varphi(t) = t$ and $\Delta f(\theta) = \sin(2\pi\theta)$, the family is called the Arnol'd family.

For $t \in I_0$, we say that $f_{t,\epsilon}$ is linearizable if there exists $h \in \mathrm{Diff}^\infty(\mathbb{T})$ such that $h \circ f_{t,\epsilon} \circ h^{-1} = R_{\rho(f_{t,\epsilon})}$. Our goal in this section is to show that as $\epsilon \to 0$, most of the members of the family $f_{t,\epsilon}$ become linearizable. This result should be put into contrast with the result on Liouville phenomenon, that indicates to us that the diffeomorphisms of the family $f_{t,\epsilon}$ that have a Liouville rotation number will in general be non linearizable. The diffeomorphisms $f_{t,\epsilon}$ that have a rational rotation number are also non linearizable in general and they form an open dense set in the parameter space $t \in I_0$ (see exercise 8.1.1).

The abundance in measure of the linearizable diffeomorphisms in $f_{t,\epsilon}$ comes from the fact that the set of Diophantine numbers has full measure.

Theorm 8.1.1 *For any $\eta > 0$, there exists ϵ_0 such that if $\epsilon < \epsilon_0$ then the set of $t \in [0,1]$ such that $f_{t,\epsilon}$ is linearizable has Lebesgue measure larger than $1-\eta$.*

In this statement ϵ_0 depends on η and M and Δf. We will see later that there exists $r_0 \in \mathbb{N}$ universal (for example $r_0 = 100$ is sufficient but one could do much better!!) such that the dependence of ϵ_0 on Δf is only through $\|\Delta f\|_{r_0}$.

Exercise 8.1.1 It is not difficult to show using Poincaré classification that for an open and dense set of t $\rho(f_{t,\epsilon})$ is rational and $f_{t,\epsilon}$ is not linearizable.

The proof of Theorem 8.1.1 in the case $\Delta f(\theta) = \sin(2\pi\theta)$ goes back to Arnold (1970). It was originally based on a quadratic scheme with parameter exclusion inspired by the work of Kolmogorov on Hamiltonian systems. The inductive conjugacy scheme in this context can be carried out in the real analytic category since the function $\sin(2\pi\theta)$ is real analytic. The scheme is then similar to the one we presented in the proof of the Poincaré-Siegel Theorem with a succession of band width losses due to the loss of regularity in resolving the linearized equation.

In the general case of a smooth function Δf the proof of Theorem 8.1.1 is mainly due to Moser and the proof we present here is inspired from his work.

Let

$$f_t(\theta) = R_{\varphi(t)}(\theta) + \Delta f_t(\theta)$$

with $\Delta f_t \in C_0^{lip,\infty}(I \times \mathbb{T}, \mathbb{R})$, that we simply write as

$$f(\theta) = R_\varphi(\theta) + \Delta f(\theta)$$

because we will always assume the functions we use are parametrized by t.

For I a collection of intervals of I_0, we denote $C^{lip,\infty}(I, \mathbb{T}, \mathbb{R})$ the set of families of smooth maps in the $\mathbb{T}$ variable and Lipschitz in the parameter $t \in I$. We denote $C_0^{lip,\infty}(I, \mathbb{T}, \mathbb{R})$ the subset of maps $f \in C^{lip,\infty}(I, \mathbb{T}, \mathbb{R})$ such that if we write $f_t(z) = (f_t^1(z), f_t^2(z)) \in \mathbb{T}^d \times \mathbb{T}$, then $\int_{\mathbb{T}} f_t^2(z)dz = 0$ for $t \in I$.

For $f \in C_0^{lip,\infty}(I_0, \mathbb{T}, \mathbb{R})$, we use the notation $\|f\|_{lip(I),r} = \max_{|\iota| \leq r} \mathrm{Lip}(f^{(\iota)})$ where $f^{(\iota)}$ is the derivative of f of order ι and where $\mathrm{Lip}(f)$ is the maximum of the supnorm of f and its Lipschitz constant.

Theorem 8.1.1 follows from the following more general statement.

Theorm 8.1.2 *There exists r_0, such that for any φ that satisfies (8.1), for any $\eta > 0$ there exists $\epsilon(\eta, M)$, such that if $\|\Delta f\|_{lip(I),r_0} < \epsilon$ then the set of $t \in I_0$ such that f_t is linearizable has measure greater than $1 - \eta$.*

The proof of Theorem 8.1.2 is based on an inductive procedure where at each step we solve the linearized system that corresponds to the conjugacy equation of $R_\varphi + \widetilde{\Delta f}$ to a rotation. This is possible for the parameters t that satisfy some arithmetic condition that corresponds to the induction step we are at. The solution of the linearized equation yields a new family (on the restricted space of parameters) $R_{\tilde{\varphi}} + \widetilde{\Delta f}$ where the nonlinear term $\widetilde{\Delta f}$ satisfies quadratic estimates compared to Δf.

As the induction proceeds we have to exclude more parameters that fail to satisfy the arithmetic condition but the scheme converges quickly (quadratiqually) so the measure of the excluded parameters at each step becomes smaller and smaller leaving a large set of parameters at the end of the induction for which the maps are linearized.

8.2 Solving the linearized system. The inductive step

For $N \in \mathbb{N}$, define

$$\mathcal{D}(N) = \{\alpha \in I_0 \ / \ |1 - e^{i2\pi k\alpha}| \geq N^{-3}, \quad \forall 0 < |k| \leq N\}.$$

Proposition 8.2.1 *There exists $\sigma > 0$ such that If $N \in \mathbb{N}$ and I is a collection of intervals such that $I \subset \{t \in I_0 \ / \ \varphi(t) \in \mathcal{D}(N)\}$, then there exist $\tilde{\varphi} \in \mathrm{Lip}(I, \mathbb{R})$ and $h, \widetilde{\Delta f} \in C_0^{lip,\infty}(I, \mathbb{T}, \mathbb{R})$ such that if we write $H = \mathrm{Id} + \mathrm{h}$ we have that*

$$H \circ f = (R_{\tilde{\varphi}} + \widetilde{\Delta f}) \circ H \tag{8.2}$$

with for all $r' > r \geq 0$

$$\Delta S \leq C_0 N^{\sigma} \Delta_0$$
$$\|h\|_{lip(I),r+1} \leq C_r S N^{\sigma} \Delta_r$$
$$\widetilde{\Delta}_r \leq C_r S N^{\sigma} \Delta_0 \Delta_r + C_{r,r'} N^{\sigma+r-r'} \Delta_{r'}$$

where :

$$S = \|\varphi\|_{lip(I)}$$
$$\Delta S = \|\varphi - \tilde{\varphi}\|_{lip(I)}$$
$$\Delta_r = \|\Delta f\|_{lip(I),r}$$
$$\widetilde{\Delta}_r = \|\widetilde{\Delta f}\|_{lip(I),r}$$

Proof.

Lemma 8.2.1 *There exists $\sigma > 0$ such that if $v \in C_0^{lip,\infty}(I \times \mathbb{T}, \mathbb{R})$, is a trigonometric polynomial with degree $N \in \mathbb{N}$ and I is an interval such that $I \subset \{t \in I_0 \ / \ \varphi(t) \in \mathcal{D}(N)\}$, then there exists $h \in C^{lip,\infty}(I \times \mathbb{T}, \mathbb{R})$ such that :*

$$v = h - h \circ R_{\varphi}$$
$$\|h\|_{lip(I),r+1} \leq C_r \|\varphi\|_{lip(I)} N^{\sigma} \|v\|_{lip(I),r}$$

The proof of Lemma 8.2.1 is done using Fourier expansions and is similar to the proof of existence of solutions to the linearized cohomological equation above α. We leave its proof as an exercise.

We show how the lemma implies Proposition 8.2.1.

For $f \in C_0^{lip,\infty}(I \times \mathbb{T}, \mathbb{R})$ we write f as a Fourier series with Lipschitz coefficients $f_{k,t}$: $f = \sum_{k \in \mathbb{Z}} f_{k,t} e(k\theta)$. Then we define $T_N f + R_N f := f$ with

$$T_N f := \sum_{|k|<N} f_{k,t} e(k\theta)$$
$$R_N f := \sum_{|k|\geq N} f_{k,t} e(k\theta)$$

Observe that for any $r' \geq r, r, r' \in \mathbb{N}$ the following *truncation estimates* hold :

$$\|T_N f\|_{lip,r'} \leq C_{r,r'} N^{2+r'-r} \|f\|_{lip,r}$$
$$\|R_N f\|_{lip,r} \leq C_{r,r'} N^{2+r-r'} \|f\|_{lip,r'}$$

with $C_{r,r'}$ constants that depend on r and r'.

Exercise 8.2.1 Prove the truncation estimates using Fourier series

Since $\Delta f \in C_0^{lip,\infty}(I \times \mathbb{T}, \mathbb{R})$ we can apply Lemma 8.2.1 and get h such that (after replacing σ by $\sigma + 2$)

$$T_N \Delta f = h - h \circ R_\varphi$$
$$\|h\|_{lip(I),r+1} \leq C_r \|\varphi\|_{lip(I)} N^\sigma \|\Delta f\|_{lip,r}$$

We see now that

$$\begin{aligned}\widetilde{\Delta f} \circ (\mathrm{Id} + \mathrm{h}) &= \varphi - \tilde{\varphi} + \Delta f + h \circ (R_\varphi + \Delta f) - h \\ &= \varphi - \tilde{\varphi} + R_N \Delta f + h \circ (R_\varphi + \Delta f) - h \circ R_\varphi\end{aligned}$$

Thus

$$\widetilde{\Delta f} = \varphi - \tilde{\varphi} + (R_N \Delta f + h \circ (R_\varphi + \Delta f) - h \circ R_\varphi) \circ (\mathrm{Id} + \mathrm{h})^{-1}$$

We choose $\tilde{\varphi}$ such that $\widetilde{\Delta f}$ has zero average, that is

$$\varphi - \tilde{\varphi} = \int_{\mathbb{T}} (R_N \Delta f + h \circ (R_\varphi + \Delta f) - h \circ R_\alpha) \circ (\mathrm{Id} + \mathrm{h})^{-1}(\theta) \mathrm{d}\theta$$

Observe that since we chose h to get rid of the "big" part $T_N \Delta f$ of Δf, the remaining part $(R_N \Delta f + h \circ (R_\varphi + \Delta f) - h \circ R_\varphi) \circ (\mathrm{Id} + \mathrm{h})^{-1}$ contains a "rest" term $R_N \Delta f$ and a "quadratic" term $h \circ (R_\varphi + \Delta f) - h \circ R_\varphi$ that is of order $Dh \cdot \Delta f$ which up to some loss in derivatives is a $\mathcal{O}^2(\Delta f)$. Indeed, standard estimations on composition and inverse of maps (we will omit the proofs) now give that $\tilde{\varphi} - \varphi$, and $\widetilde{\Delta f}$ satisfy the conclusion of Proposition 8.2.1.

□

8.3 The KAM scheme

Lemma 8.3.1 *Let $M > 0$. There exists $N_0(M)$ such that if $N > N_0$ and $\tilde{N} = N^{3/2}$ and if I is an interval of size $1 \geq |I| \geq 1/(2MN^2)$ and if $M^{-1} < \varphi'(t) < M$ for every $t \in I$, then there exists a union of disjoint intervals $\mathcal{U} = \{\tilde{I}_j\}$ such that $\varphi(\tilde{I}_j) \in \mathcal{D}(\tilde{N})$ and $\tilde{I}_j \subset I$ and $|\tilde{I}_j| \geq 1/(2M\tilde{N}^2)$ and $\sum |\tilde{I}_j| \geq (1 - 2dM^2\tilde{N}^{-1})|I|$.*

Proof. We just observe that the set of $t_k \in I$ such that $1+e^{i2\pi\varphi(t)}=0$ with $k \leq \tilde{N}$ consists of at most $d([M\tilde{N}^2|I|]+2)$ points separated one from the other by at least $1/(M\tilde{N}^2)$. Excluding from I the intervals $[t_k - M/\tilde{N}^3, t_k + M/\tilde{N}^3]$ leaves us with a collection of intervals of size greater than $1/(2M\tilde{N}^2)$ of total length $|I| - d([M\tilde{N}^2|I|]+2)M/\tilde{N}^3 \geq (1-2dM^2\tilde{N}^{-1})|I|$. □

Recall that

$$\|\varphi\|_{lip(I_0)} \leq \frac{M}{2}, \quad \inf_{t\in I_0} \varphi'(t) \geq \frac{2}{M} \tag{8.3}$$

Let $N_0 \geq N_0(M)$ of Lemma 8.3.1 and define for $n \geq 1$, $N_n = N_{n-1}^{\frac{3}{2}}$.

Observe that Lemma 8.3.1 implies that if $\mathcal{A}_n$ is a collection of intervals of sizes greater than $1/(2MN_n^2)$ and φ_n are functions satisfying (8.3) on $\mathcal{A}_n$ with M instead of $2M$ then there exists $\mathcal{A}_{n+1}$ that is a collection of intervals with sizes greater than $1/(2MN_{n+1}^2)$ such that $\varphi_n(\mathcal{A}_{n+1}) \subset \mathcal{D}(N_{n+1})$ and $\lambda(\mathcal{A}_{n+1}) \geq (1-2dM^2N_{n+1}^{-1})\lambda(\mathcal{A}_n)$.

We now describe the inductive scheme that we obtain from an iterative application of Proposition 8.2.1. We start with $f = R_\varphi + \Delta f$. At step n we have $f_n = R_{\varphi_n} + \Delta f_n$, defined for $t \in \mathcal{A}_n$, with $\mathcal{A}_{-1} = [0,1]$. We denote $\epsilon_{n,r} = \|\Delta f_n\|_{lip(\mathcal{A}_n),r}$. We obtain h_n and φ_{n+1} a defined on $\mathcal{A}_{n+1}$ such that

$$H_n f_n H_n^{-1} = R_{\varphi_{n+1}} + \Delta f_{n+1}$$

with $\Delta f_{n+1} \in C_0^{lip,\infty}(\mathcal{A}_{n+1}, \mathbb{T}, \mathbb{R})$, and if we denote $\xi_{n,r} = \|h_n\|_{lip(\mathcal{A}_{n+1}),r+1}$ and $\nu_n = \|\varphi_{n+1} - \varphi_n\|_{lip(\mathcal{A}_{n+1})}$ we have from Proposition 8.2.1 that

$$\xi_{n,r} \leq C_r \gamma_n N_n^\sigma \epsilon_{n,r} \tag{8.4}$$

$$\nu_n \leq \epsilon_{n,0} \tag{8.5}$$

$$\epsilon_{n+1,r} \leq C_r \gamma_n N_n^\sigma \epsilon_{n,0}\epsilon_{n,r} + C_{r,r'}\gamma_n N_n^{\sigma+r-r'}\epsilon_{n,r'} \tag{8.6}$$

with $\gamma_n = (1+S_n+\epsilon_{n,0})^\sigma$, where

$$S_n := \|\varphi_n\|_{lip(\mathcal{A}_n)}$$

If during the induction we can insure that $\sum \epsilon_{n,0} < 1/(100M)$ we can conclude from (8.5) and the definition of M that for all n, φ_n satisfies on $\mathcal{A}_n$ the inductive condition

$$\|\varphi_n\|_{lip(\mathcal{A}_n)} \leq 2M, \quad \inf_{t\in\mathcal{A}_n} \varphi_n'(t) \geq \frac{1}{2M} \tag{C1}$$

and Lemma 8.3.1 will insure that $\mathcal{A}_{n+1}$ is well defined and $\lambda(\mathcal{A}_{n+1}) \geq (1 - 2M^2 N_{n+1}^{-1})\lambda(\mathcal{A}_n)$. To be able to apply the inductive procedure we also have to check that H_n is indeed invertible which is insured if during the induction we have

$$\xi_{n,0} < \frac{1}{2}. \tag{C2}$$

We call the latter two conditions the inductive conditions.

We need to prove now that the scheme (8.4)–(8.6) converges provided an adequate control on $\epsilon_{0,0}$ and $\epsilon_{r_0,0}$ for a sufficiently large r_0 is.

Lemma 8.3.2 *Let $\alpha = 4\sigma + 2$, $\beta = 2\sigma + 1$, and $r_0 = [8\sigma + 5]$. If $S_n, \xi_{n,r}, \epsilon_{n,r}$ satisfy (8.4)–(8.6), there exists $\bar{N}_0(\sigma)$ such that if $N_0 = \bar{N}_0 M$ and*

$$\epsilon_{0,0} \leq N_0^{-\alpha}, \quad \epsilon_{0,r_0} \leq N_0^{\beta}$$

then for any n the inductive conditions (C1) and (C2) are satisfied and in fact $\epsilon_{n,0} \leq N_n^{-\alpha}$, $\xi_{n,0} \leq N_n^{-\sigma}$, and for any $s \in \mathbb{N}$, there exists $\bar{C}_r$ such that $\max(\epsilon_{n,s}, \xi_{n,s}) \leq \bar{C}_s N_n^{-1}$.

Proof. We first prove by induction that for every n, $\epsilon_{n,0} \leq N_n^{-\alpha}$ and $\epsilon_{n,r_0} \leq N_n^{\beta}$, provided $\bar{N}_0(\sigma)$ is chosen sufficiently large.

Assuming the latter holds for every $i \leq n$, the inductive hypothesis (C1) and (C2) can be checked up to n immediately from (8.4) and (8.5). Now, (8.6) applied with $r = 0$ and $r' = r_0$ yields

$$\begin{aligned}\epsilon_{n+1,0} &\leq C_0 N_n^{\sigma}(2 + M)^{\sigma} N_n^{-2\alpha} + C_{0,r_0} N_n^{\sigma - r_0} N_n^{\beta} \\ &\leq N_{n+1}^{-\alpha}\end{aligned}$$

provided $\bar{N}_0(\sigma)$ is sufficiently large.

On the other hand, applying (8.6) with $r' = r = r_0$ yields

$$\begin{aligned}\epsilon_{n+1,r_0} &\leq C_{r_0} N_n^{\sigma}(1 + M)^{\sigma} N_n^{-\alpha} N_n^{\beta} + C_{r_0,r_0}(1 + M)^{\sigma} N_n^{\sigma} N_n^{\beta} \\ &\leq N_{n+1}^{\beta}\end{aligned}$$

provided $\bar{N}_0(\sigma)$ is sufficiently large.

To prove the bound on $\epsilon_{n,s}$ we start by proving that for any s, there exist $\tilde{C}_s$ and n_s such that for $n \geq n_s$ we have that $\epsilon_{n,s} \leq \tilde{C}_s N_n^{\beta}$. Let indeed n_s be such that $N_{n_s}^{-1/10}(1 + M)^{\sigma}(C_s + C_{s,s}) < 1$. Let $\tilde{C}_s$ be such that $\epsilon_{n_s,s} \leq \tilde{C}_s N_{n_s}^{\beta}$.

We show by induction that $\epsilon_{n,s} \le \tilde{C}_s N_n^{\beta}$ for every $n \ge n_s$. Assume the latter true up to n and apply (8.6) with $r = r' = s$ to get

$$\begin{aligned}\epsilon_{n+1,s} &\le C_s N_n^{\sigma}(1+M)^{\sigma} N_n^{-\alpha}\epsilon_{n,s} + C_{s,s}(1+M)^{\sigma} N_n^{\sigma}\epsilon_{n,s}\\ &\le N_n^{\sigma+1/10}\epsilon_{n,s}\\ &\le \tilde{C}_s N_n^{\sigma+1/10+\beta} \le \tilde{C}_s N_{n+1}^{\beta}.\end{aligned}$$

We will now bootstrap on our estimates as follows. Let $s'(s) = s + [\sigma + \beta + \frac{3}{2}(\sigma+1)] + 1$, and define $\tilde{n}_s \ge \max(n_s, n_{s'})$ such that $N_{\tilde{n}_s}^{-1/10}(1+M)^{\sigma}(C_s + \tilde{C}_{s'} C_{s,s'}) < 1$.

. Let $\bar{C}_s$ be (large !) such that $\epsilon_{\tilde{n}_s,s} \le \bar{C}_s N_{\tilde{n}_s}^{-\sigma-1}$. We will show by induction that for any $n \ge \tilde{n}_s$ we have that $\epsilon_{n,s} \le \bar{C}_s N_n^{-\sigma-1}$. Indeed, apply (8.6) with $r = s$ $r' = s'$ to get

$$\begin{aligned}\epsilon_{n+1,s} &\le \bar{C}_s C_s N_n^{\sigma}(1+M)^{\sigma} N_n^{-\alpha} N_n^{-\sigma-1} + C_{s,s'}\tilde{C}_{s'}(1+M)^{\sigma} N_n^{\beta} N_n^{\sigma+s-s'}\\ &\le \bar{C}_s N_{n+1}^{-\sigma-1}\end{aligned}$$

since $\tilde{n}_s$ was chosen sufficiently large.

Finally, (8.4) yields that for $n \ge \tilde{n}_s$, $\xi_{n,s} \le C'_s N_n^{-1}$.

□

8.4 Proof of the KAM Theorem

We can now finish the proof of the KAM Theorem 8.1.2. The sets $\mathcal{A}_n$ are decreasing and we let $\mathcal{A}_\infty = \liminf \mathcal{A}_n$. Note that the norm $\|\cdot\|_{lip,r}$ norms are well defined on $\mathcal{A}_\infty$ for functions that are in $C_0^{lip,\infty}(\mathcal{A}_n, \mathbb{T}, \mathbb{R})$ for every n. The result of Lemma 8.3.2 implies that

$$\lambda(\mathcal{A}_\infty) \ge \Pi(1 - 2M^2 N_{n+1}^{-1}) \ge 1 - \eta$$

if $N_0 \ge N_0(\eta)$. On $\mathcal{A}_\infty$, φ_n converges in the Lipschitz norm to some $\varphi_\infty \in \mathrm{Lip}(\mathcal{A}_\infty, \mathbb{R})$, and the maps $H_n \circ \ldots \circ H_1, H_n^{-1} \circ \ldots \circ H_1^{-1}$ converge in the $C^{lip(\mathcal{A}_\infty),\infty}$ norm to some G, G^{-1} such that on $\mathcal{A}_\infty$ it holds that $G \circ (R_\varphi + \Delta f) \circ G^{-1} = f_{\varphi_\infty}$.

□

8.5 Liouville phenomenon. Non smooth conjugacies

Exercise 8.5.1 For $0 < \epsilon < \frac{1}{2\pi}$, define the following family of circle diffeomorphisms for $t \in [0,1]$:

$$f_{t,\epsilon}(\theta) = \theta + t + \epsilon \sin(2\pi\theta)$$

In all the sequel $0 < \epsilon < \frac{1}{2\pi}$ will be fixed. For a map f and a point x we denote by $\omega(x, f)$ the omega limit set of the orbit of x under f, that is the accumulation points of $f^n(x)$, $n \in \mathbb{N}$.

The goal of this exercise is to show that there exists $t \in [0,1]$ such that the rotation number $\rho(f_{t,\epsilon}) \in \mathbb{R} - \mathbb{Q}$ and such that the conjugacy of $f_{t,\epsilon}$ to the rotation $R_\rho(f_{t,\epsilon})$ is not Lipschitz.

a) Compute $\rho(f_{0,\epsilon})$ and $\rho(f_{1,\epsilon})$ and show that $\rho(f_{\frac{1}{2},\epsilon}) \neq 0$. Deduce that $[0,1] \to \mathbb{T} : t \mapsto \rho(f_{t,\epsilon})$ is surjective.

b) For every $\frac{p}{q} \in [0,1]$, $p \wedge q = 1$, show that the set of $t \in [0,1]$ such that $\rho(f_{t,\epsilon}) = \frac{p}{q}$ is a non empty interval that will be denoted $[t^-_{\frac{p}{q}}, t^+_{\frac{p}{q}}]$. (*Hint* : Use that the map $\theta \mapsto f_{t,\epsilon}(\theta)$ is analytic in θ). Show that $f_{t^+_{\frac{p}{q}}}$ has finitely many periodic orbits of period q and that if $x_1 < x_2 < \ldots < x_N$ denote the periodic points in cyclic ordering on the circle then $f^q_{t^+_{\frac{p}{q}}}[x_i, x_{i+1}] = [x_i, x_{i+1}]$ and every point $x \in (x_i, x_{i+1}]$ satisfies $\omega(x, f^q_{t^+_{\frac{p}{q}}}) = x_{i+1}$. What can you say about the α-limit set of x ?

c) Deduce that for any $\eta > 0$ there exists a set E that is a collection of open intervals on the circle and an integer l such that $f^l_{t^+_{\frac{p}{q}}}(E)$ is strictly included in E^c and $|E| > 1 - \eta$ (E^c is the complementary in $\mathbb{T}$ of a set E while $|E|$ denotes its Lebesgue measure).

d) We now start an inductive construction of a parameter t_∞ as follows. Let $\eta_n := \frac{1}{n+1}$. Take $\frac{p_1}{q_1} \in [0,1]$ and consider $f_{t^+_{\frac{p_1}{q_1}}}$ to which we apply c) and get E_1 and l_1 corresponding to η_1.

Show that if $\frac{p_2}{q_2} > \frac{p_1}{q_1}$ is sufficiently close to $\frac{p_1}{q_1}$ then

- $\frac{p_2}{q_2} - \frac{p_1}{q_1} < \frac{1}{q_1^{100}}$

– $f^{l_1}_{t^+_{\frac{p_2}{q_2}}}(E_1)$ is strictly included in E_1^c.

Show that there also exists a set E_2 that is a collection of open intervals on the circle and an integer l_2 such that $f^{l_2}_{t^+_{\frac{p_2}{q_2}}}(E)$ is strictly included in E_2^c and $|E_2| > 1 - \eta_2$.

Continue the induction and show that $t^+_{\frac{p_n}{q_n}} \to t_\infty$ such that $\rho(f_{t_\infty}) \in \mathbb{R} - \mathbb{Q}$ and that the conjugating map h such that $h \circ f_{t_\infty} = R_{\rho(f_{t_\infty})} \circ h$ satisfies $|h(E_n)| \leq |E_n^c|$ for every n.

e) ÊDeduce that h is not Lipschitz. What can you say more about h ? What can you say about the set of parameters t such that $\rho(f_{t,\epsilon}) \in \mathbb{R} - \mathbb{Q}$ and such that the conjugacy of $f_{t,\epsilon}$ to the rotation $R_\rho(f_{t,\epsilon})$ is not Lipschitz ? Can you say it is G^δ dense inside the set of of parameters t such that $\rho(f_{t,\epsilon}) \in \mathbb{R} - \mathbb{Q}$?

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