How to Generate Mathematical Experimentation, and Does it Provide Mathematical Knowledge?

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Sophisticated observations and controlled experiments are part and parcel of modern science. Mathematics, on the contrary, seems to have configured itself as a discipline by carefully rejecting or at least holding at bay every symptom of empiricism. The qualification of ancient numerical activities as empirical has for a long time meant their disqualification as authentic mathematics.[1] And though mathematics may act as an auxiliary of natural philosophy (or more recently, of natural sciences) and in turn might be stimulated by physical or biological questions, the very idea of cooperation does assume a distinction between two types of disciplines, two ways of dealing with objects and arguments. Following Thomas Kuhn, in particular, science studies have tended to exclude mathematics from the so-called Baconian sciences. "Perhaps, therefore, the cleavage between mathematical and experimental science still remains, rooted in the nature of the human mind"; thus Kuhn concludes his famous 1976 article[2] – in which he describes the birth and development of a cluster of modern sciences, labelled by him "Baconian sciences," as the emancipation from classical sciences, "grouped together" well into the nineteenth century, writes Kuhn, "as mathematics." Anchored as it is in traditions of educational and intellectual systems, this opposition between Baconian and mathematical sciences has developed in more recent authors a variety of harmonics, in the epistemic as well as in the political sphere: certainty, proof, theory, constraints, quantification, abstraction, dogmatism are often associated with the mathematical, possibilities, negociations, things, concreteness, bodies and mediations, with the Baconian.[3]

This very idea of a demarcation line creeps into the work even of those defending an empiricist philosophy of mathematics[4] such as Imre Lakatos, to take just a single, post-foundationalist example. Lakatos contrasts two types of theories[5]: the Euclidean and the quasi-empirical. The differences he detects between them touch on the way truth propagates inside a theory as well as the way a theory develops; truth goes from top to bottom in Euclidean theories, that is from basic, taken-as-true statements like axioms towards the theorems, by *deductive* procedures, while in quasi-empirical theories basic statements are *explained* by the rest of the system. For Lakatos, a Euclidean theory has a three-stage development: first a period of trial and error, then a foundational reorganization, and lastly a steady period of problem-solving; while the development of quasi-empirical theories is described as a dynamic back-and-forth mixture of

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1 Such a characterization can still be found in semi-popular accounts of the development of mathematics. And if it has receded in the last decades it is much more because of a new perception of the theoretical structure of these mathematical texts than because of any recognition of the empirical nature of all mathematics. For the example of Babylonian mathematics, see Hoyrup 1990 (for a geometrical interpretation) and Ritter 1989 (for an algorithmic interpretation).

2 See Kuhn 1976, p. 64 in the 1977 edition.

3 For examples in a variety of areas, see Shapin and Schaffer 1985, Schwartz 1990, Irigaray 1992.

4 On quasi-empiricism, see Pecatte 1998, http://peccatte.karefil.com/Quasi/QuasiEmpirisme.html and the anthology Tymoczko 1998.

5 See Lakatos 1967.

speculation, criticisms, refutation and controversy. Contrary to *idées reçues*, Lakatos, of course, states that most mathematical theories are indeed quasi-empirical and not Euclidean. But if the frontier is thus displaced to the very core of mathematics, and if tests required for a domain to pass through are different from traditional ones, there is still a frontier to define and still mathematical domains to be considered as Euclidean.

However, the fact that many mathematicians share a quasi-empiricist view of their own work is perhaps not foreign to the success among them of Lakatos's *Proofs and Refutations.* Indeed we possess much testimony from professional mathematicians about the importance of experimentation (with numbers, formulas or concepts) in their discoveries. With the banalization of computers, a more radical and amibitious claim has been made in favour of "experimental mathematics" – defined as "the utilization of advanced technology to explore mathematical structures, test conjectures and suggest generalizations."[6] Jonathan Borwein and others have even attempted to theorize the nature of their own involvement with this "experimental mathematics" via a classification of mathematical experiments. Following Peter Medawar's *Advice to a Young Scientist*, they distinguish between four forms and uses of experimentation in mathematics: an Aristotelian (mostly illustrative and restricted to pedagogical purposes), a Kantian (playing with assumptions to generate new mathematics, as in classical non-Euclidean geometries), a Baconian ("the consequence of messing about"), and a Galilean,[7] producing crucial tests and allowing the mathematician "to discriminate between possiblities" – this last being the only one, according to them, "which can make experimental mathematics a truly serious enterprise". This seriousness and the disciplinary vocation of this trend have since 1992 been secured by a professional journal dedicated to it[8] and by a number of achievements, such as the detection of new formulas for π.[9]

This rough survey might suggest that mathematics should be taken into account in the history of experimental practices, but it still confines its place narrowly, and exclusively to a preliminary stage of discovery.[10] I would like to argue here that a richer view of the way experimentation intervenes in mathematics is required, a view which does full justice to the complex procedures and varied roles of mathematical experiments on one side, and, on the other, undermines a too neat distinction between discovery and justification, actions and deductions, the handling of natural and of conceptual objects.[11]

6 In Bailey & Borwein 2001, p. 52.

7 Of course, the use of these adjectives does not mean that the conceptions of experimentations in the corresponding authors fit those of the associated historical figures. It will be clear for Bacon, for instance, in what follows.

8 Called Experimental Mathematics, see http://www.expmath.org/: "It publishes formal results inspired by experimentation, conjectures suggested by experiments, descriptions of algorithms and software for mathematical exploration, surveys of areas of mathematics from the experimental point of view, and general articles of interest to the community."

9 See Bailey & Borwein 2001, section 4. The formula expresses π as the sum, for all positive integers k, of $16^{-k}(4(8k+1)^{-1}-2(8k+4)^{-1}-(8k+5)^{-1}-(8k+6)^{-1})$. This simple formula allows one to compute the n-th digit of π (written in base 2 as a sequence of 0 and 1) directly without computing the previous digits and thus with only a small amount of computer memory.

10 "Experiment has always been, and increasingly is, an important method of mathematical discovery," claims the "statement of philosophy" of the journal Experimental Mathematics."While we value the theorem-proof method of exposition, and do not depart from the established view that a result can only become part of mathematical knowledge once it is supported by a logical proof, we consider it anomalous that an important component of the process of mathematical creation is hidden from public discussion."

We shall here begin the task by exploring two examples. Although separated by two centuries, they do share a number of features. Both are situated inside theoretical mathematics (defined accordingly to the criteria of their respective periods), both display close links with contemporary theories of experimentation in the natural sciences. Both authors, although of different status, belong to central academic milieux, and their views with respect to experimentation are not marginal. In both cases also, we have access to a public version of their practices. My main issue will not be to decide whether or not mathematics is a Baconian science, but, in both cases, to analyze how it is. That is, to explain what sort of experience is involved, for what purpose, at which stages of the practice, and how these experimentations are linked by our authors with the construction of a knowledge which is fully recognized to be mathematical.

I. A Baconian Mathematician: Bernard Frenicle de Bessy

I. 1 La Méthode des Exclusions

Bernard Frenicle de Bessy, probably born during the first decade of the seventeenth century, died in 1674,[12] a respected member of the French Academy of Sciences. A prominent participant in various Parisian scientific circles, he had been an intimate of Marin Mersenne, and in contact with Gilles Personne de Roberval, the Pascal family and later with Edme Mariotte. Frenicle reported for instance to Mersenne in 1634 about the behaviour of bullets thrown from the mast of a ship – a contribution to discussions around Galileo's work – and he showed a life-long interest in astronomical matters. But his fame was due mainly to his talent with numbers: he maintained an active, sometimes heated, correspondence on numerical questions with Pierre Fermat, René Descartes and John Wallis. In the 1640s, Fermat could not find no better way to express the value of his promise to his friend and former colleague Pierre de Carcavi to explain his methods on numbers than by writing: "There is certainly nothing more difficult in the whole of Mathematics and except for M. de Frenicle and perhaps M. Descartes, I doubt that anyone knows its secret."[13] Claude Mylon, too, who took Mersenne's place as coordinator after the latter's death, evoked "his fame as extremely learned in numbers" in a 1657 letter to Christiaan Huygens.[14]

What is left of Frenicle's work has been partially published, at several different times. My primary interest here will be in the *Méthode des exclusions*, published posthumously in 1693 together with one treatise on combinations and another on magic squares. These memoirs were then united with a *Traité des triangles rectangles en nombres* in the fifth volume of the *Mémoires de*

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11 In particular, I shall not play here the well-known game of "Lakatos vs Kuhn." Important elements can be taken from both (and others...), for instance the structuring role of induction in Lakatos or the self-definition of a mathematical community via shared problems or procedures in Kuhn. However, as explained in Goldstein 1995, different levels of mathematical practice and their different (partially independant) dynamics have to be articulated to understand the historical development of a field.

12 The date comes from his *Inventaire après décès*, recently found by Robert Descimon who kindly communicated this document to me. The usual date given in Frenicle's biographies (1675) is thus in error, as is the frequent confusion, stemming from Condorcet's *Eloges*, between this Frenicle and his brother Nicolas, poet and *Conseiller* at the *Cour des Monnaies*.

13 Fermat, Œuvres, II, p. 248 : *Il n'y a certainement quoi que ce soit dans toutes les Mathématiques plus difficile que ceci, et hors M. de Frenicle et peut-être M. Descartes, je doute que personne en connoisse le secret.*

14 Huygens, *Œuvres*, II, p. 1: *Monsieur Frenicle de Becis, que vous connaissez par reputation pour etre extremement scavant dans les nombres.*

l'académie royale des sciences depuis 1666 jusqu'à 1699, which appeared in 1729 in Paris. These Parisian editions were part of a policy of the Academy of Sciences to bear witness to the fruitful activities of its members through the publication of their writings.

The *Méthode* occupies eighty-five quarto pages in the 1729 edition. Although the fortunes of publication have occasionally isolated it from the other treatises, in particular from the *Traité des triangles rectangles*, indications in the correspondance, the analysis of the various opuscules and a list of writings found in Mersenne's papers convincingly suggest that the *Méthode des exclusions*, probably written around 1640, was intended to head the whole and thus to play a role not dissimilar to that of Descartes' *Discours de la méthode* for its various appendices. In Frenicle's case however, the appended treatises are all devoted to problems concerning numbers: problems on right-angled triangles whose sides are integers; problems on combinations; problems on magic squares; problems on numbers with a given number of divisors or in a fixed relation to the sum of their divisors; problems on figurate numbers, in the Nicomachian tradition. In what follows, I will not discuss the precise relationships between the *Methode* and these other texts.[15] But that there was a relation, and an efficient one, was taken as granted by the contemporaries and immediate successors of Frenicle. For instance, in the preface to the first edition of 1693, the editor, Philippe de la Hire, justifies the publication by claiming that "[i]t was a particular method which [Frenicle] used for the resolution of problems by means of which he easily solved very difficult questions concerning numbers on which Algebra offered often little grasp, which provoked the admiration of the erudite with whom he had intercourse, as can be noted in various places in their works."[16] As will be seen later, these references to "method", "solution of problems" and "algebra" are quite important in situating Frenicle's point of view.

The structure of the book is quite simple. After a short introduction, ten (numbered) rules are explained, often by means of an example, after which ten problems are proposed and discussed at length, all except one being related to right-angled triangles. The introduction justifies this presentation as follows: "Because ordinarily each question is treated diversely according to the different properties which must be used, rules could not be given for all the diverse cases which may be met with. This is why it has been judged appropriate to give examples which will be more useful in making the method comprehensible, after the explanation of some general rules which should be observed to arrive at the solution of the problem."[17]

The ten rules run as follows (I have omitted the numerical illustrations and summed up the statements themselves).

Rule 1. If you generally know what is proposed but not the proposed particular, find, by means of several known particulars, a common rule and then find what is asked by means of it.

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15 Examples of these links are discussed in Goldstein 1995 and 2001. See also Coumet 1968.

16 Frenicle 1693, preface: *C'etoit une méthode particulière dont il se servoit pour la solution des Problemes par le moyen de laquelle il résolvait facilement des questions de nombres tres difficiles sur lesquelles l'Algebre avoit souvent peu de prise, ce qui donnait de l'admiration aux scavants avec qui il avoit commerce, comme on peut le remarquer en plusieurs endroits de leurs ouvrages.*

17 Frenicle 1729, p.3: *Parce qu'ordinairement, chaque question se traite diversement suivant les différentes propriétés dont il se faut servir, il serait impossible de donner des règles pour tous les divers cas qu'on pourrait rencontrer. C'est pourquoy l'on a jugé à propos de donner des exemples qui seront plus utiles pour faire entendre cette méthode, après avoir expliqué quelques règles générales qu'on peut observer pour parvenir à la solution du problème.*

2. If you do not know, even generally, what is proposed, find its properties by systematically constructing similar numbers.
3. In order not to omit any necessary number, establish an order of investigation as simple as possible.
4. To shorten the research, use Exclusion, by eliminating numbers recognized as not useful for the question – in particular, multiples.
5. Exclusion can also be performed by considering the final digits of the numbers.
6. Exclusion can be carried out by means of certain other specific properties of the required object.
7. A second important means of simplifying the search is diminution, which consists in using the smallest possible numbers.
8. In some cases, while looking for a number, you will find that another is required, and then another, etc. This is sometimes useful in impossible problems or at least to eliminate non-useful numbers.
9. If several numbers are required, look for each independantly by means of the preceeding rules and then look to see if their properties are compatible.
10. If, in the course of the research, you find several numbers, observe their properties and try to determine if they are always connected. Also note the exceptions and examine carefully their origins.

During the Renaissance, method had been mostly associated with pedagogical purposes. In his *Thresor de la langue française*, Jean Nicod defined it for instance as a short way to teach or to learn something.[18] The change of emphasis from a way to teaching to a way of doing and finding is a much discussed topic in early-modern scholarship.[19] Frenicle's list of rules, as strange as its mixture of of common sense trivia (1 or 9) and minute technical suggestions (4 or 5) may appear to the actual reader, nonetheless points to a conception of method that is action-oriented and investigatory. Let us focus on one example, the third, to see concretely how the rules operate. What follows simply paraphrases Frenicle's instructions.

I. 2 The method at work: one example

Frenicle's third example deals with the question of deciding if a given integral number is the hypotenuse of a right-angled triangle[20] and of finding the sides of the triangle if such is the case. The number 221 is proposed as the given number. As we do not know for the moment, as Frenicle says, any characteristic properties of hypotenuses which would allow us to answer directly the question, rule 2 should be used. For this, we need to know some property of hypotenuses: the most

18 Nicot 1606, quoted from the 1621 edition, p. 408: "*Methode, Methodus. C'est une briefve façon d'enseigner ou apprendre quelque chose.*"

19 The literature on early-modern method is vast, see for instance Bruyère 1984; Dear 1998; Desan 1987; Di Liscia, Kessler & Methuen 1997; Gilbert 1960; Mac Lean 2002; Vasoli 1968. The change of emphasis in early-modern mathematical texts from a way of teaching to a way of doing and finding, and the place of Frenicle's conception of method among them, is discussed in Goldstein 2008.

20 We remind the reader that the hypotenuse is the longest side, that opposite the right angle. The difficulty in this problem is that the triangles considered here are supposed to have integral sides. For instance, 221 is certainly the hypotenuse of (infinitely) many right-angled triangles, like ($\sqrt{48000}$, $\sqrt{841}$, 221) because of Pythagoras' Theorem: $221^2=48841=48000+841=(\sqrt{48000})^2+(\sqrt{841})^2$, but here the smaller sides would not be integral. An example of a right-angled triangle with integral sides is that with sides 3, 4, 5 (the hypotenuse is 5); we note that Pythagoras' Theorem applies: $3^2+4^2=5^2$.

obvious, for him as probably still for us to-day, is Pythagoras' Theorem, that is, "the square of the hypotenuse is the sum of the squares of the other two sides of the triangle." This means that one can hope to construct (squares of) hypotenuses by adding two squares and by looking to see if the sum is indeed the square of an integer.

According to rule 3 this time, an order must be imposed on this investigation, that is on the construction of relevant cases. Frenicle's choice is to take the first squares in increasing order (1, 4, 9, 16, 25, etc.) and add to each of them those which are smaller (thus the part of the table with 16 as the first entry reads "16+1, 16+4, 16+9" in this order). Contrary to other orders, discussed by Frenicle at this occasion, this one allows the mathematician to extend the table if needed without recalculating it from scratch.

A table (see Table 1) is thus produced, and the cases where the sum of squares provides a square – i.e., the cases when one has effectively got a hypotenuse – are located, for instance by comparison with a previously established table of squares. However, the first 35 computations provide only 2 hypotenuses: the almost trivial 5, the well-known hypotenuse of the triangle with sides 3 and 4, and its no less obvious multiple 10 – a quite disappointing result. Rules 4–6 are thus applied to shorten the construction of examples by predicting and hence eliminating *a priori* useless computation. This new shortened table (see an extract in Table 2) is displayed, going from 100+1 = 101 up to 256+225 = 481, while the cases in the first table which would have survived these exclusions are marked with a cross (see Table 1).

At this point, Frenicle suggests seeking a convenient property of hypotenuses in the tables: "We need to see if the square sums which have been found cannot teach us something … and if we cannot find some [new] property of these hypotenuses,"[21] that is try and apply rule 10.

The tables at this stage have furnished, through their squares, four hypotenuses: 5 and 10 (a multiple) in the first table, 13 and 17 in the second.[22] However, "considering [his] first table", and more specifically the third column, where the sums of squares are found, Frenicle "[sees] that it contains these numbers 5, 10, 13, 17, which I have found to be hypotenuses and that they follow each other in the column": that is, those numbers which have been found to be hypotenuses (and whose *squares* are thus somewhere in the third column of the tables) are *themselves* in this third column, and they appear here directly as sums of squares (5=1+4, 10=9+1, 13=9+4, 17=16+1).

This step appears to be a *coup de théâtre*: here the mathematician is left to fall back on his cleverness, acute eye, good fortune or previously acquired knowledge – no strict procedure of dealing with the tables is proposed. It is in particular important to notice that another order in the construction of the tables would not have necessarily provided such directly observable evidence.

In any case, this observation leads to a conditional statement: "It could be," writes Frenicle, "that not only are squares of hypotenuses the sum of two squares, but that hypotenuses themselves are, in the same way, the sum of two squares, which needs to be examined."[23]

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21 Frenicle 1729, p. 21: *Il faut voir si les sommes quarrées qu'on a trouvées ne pourront rien apprendre […] et si on ne pourra point trouver quelque propriété des dites hypotenuses.*

22 For lack of place, Table 2 has not been completely inserted here: we see only the appearance of the hypotenuse 13.

23 Frenicle 1729, p. 22 : *Il se pourrait donc faire que non seulement les quarrez des hypotenuses sont la somme de deux quarrez, mais que les hypotenuses mesmes sont pareillement la somme de deux quarrez, ce qu'il faut examiner.*

†	4.	1	5.
	9.	1	10.
†	9.	4	13.
†	16.	1	17.
	16.	4	20.
†	16.	9	25. ℞ 5.
	25.	1	26.
†	25.	4	29.
	25.	9	34.
†	25.	16	41.
	36.	4	40.
	36.	9	45.
	36.	16	52.
†	36.	25	61.
	49.	1	50.
†	49.	4	53.
	49.	9	58.
†	49.	16	65.
	49.	25	74.
†	49.	36	85.
†	64.	1	65.
	64.	4	68.
†	64.	9	73.
	64.	16	80.
†	64.	25	89.
	64.	36	100. ℞ 10
†	64.	49	113.
	81.	1	82.
†	81.	4	85.
	81.	9	90.
†	81.	16	97.
	81.	25	106.
	81.	36	117.
	81.	49	130.
†	81.	64	145.

Table 1

100.	1	101.
100.	9	109.
100.	49	149.
100.	81	181.
121.	4	125.
121.	16	137.
121.	36	157.
121.	64	185.
121.	100	221.
144.	1	145.
144.	25	169. ℞ 13.
144.	49	193.
144.	121	265.

Table 2

Frenicle then elaborates a new test to check this property. This is an interesting point because, until now, Frenicle's procedure has not challenged recent views that experimentation can only provide hints towards a statement which is then to be proved through a standard demonstration. But now Frenicle proposes looking for a confirmation of his hint not by means of a traditional proof, but by studying *new* occurences of the phenomenon he has detected; more specifically, he wants to determine if the succeeding numbers in the third column are all hypotenuses. For these cases, we do not yet have a table of sums of squares long enough to be checked directly, as was the case for the first examples. Using brute force (my words) to test these numbers – that is, computing their squares and trying to decompose them into a sum of two squares, which more or less comes back to an extension of the table – would be too long (Frenicle's words) and he instead chooses another way. It relies on a problem conveniently treated first in the *Méthode*, and which explains how to find the two small sides of a right-angled triangle if one is given its hypotenuse expressed as a sum of two squares.[24] Frenicle applies the procedure to the next prime number in the column,[25] that is 29 (sum of the squares 25 and 4), finds that it is indeed a hypotenuse[26] and sums up: "The same thing can be examined for the next hypotenuses [...] If taking them in order, without any choice, one finds the same for all, I conclude that this rule is general. [Sums of two squares are hypotenuses]."[27] Just to fix ideas, seventy-three cases are thus discussed.

Frenicle turns then to the inverse problem, that of deciding if all hypotenuses are sums of squares. Again, rule 10 enters into action. Frenicle examines first the case of so-called primitive hypotenuses, that is hypotenuses of triangles whose sides have no common divisors: "I see here [in the table] several numbers which are hypotenuses of very different triangles"[28] – he gives four examples – "and which are sums of squares". Consequently, Frenicle goes on, there is no indication that other prime numbers will behave differently. But "to be more convinced," one might more carefully study exceptions and counter-instances, as rule 10 proposes: that is, check that the prime numbers which are not in the tables of hypotenuses, like 7, 11, etc. are *not* sums of squares.

Counter-instances are also examined inside the tables, for the case of multiples. If it is clear that multiples of hypotenuses are hypotenuses (with the three sides of the triangle multiplied by the same number), one sees in the tables that some of them are not sums of two squares (for instance 15, which is 3 times a hypotenuse and thus the hypotenuse of the triangle of sides 9, 12, 15, is not in our tables). By looking for differences between the two sorts (Frenicle mentions four numbers of the first sort and three of the second), Frenicle finds that in order to be sums of two squares, the multiples of hypotenuses should be either multiples of a hypotenuse by a square or twice a square or divisible only by numbers which are themselves other hypotenuses. Once more, no suggestion paves the way to establishing this kind of difference, but the finding is confirmed by other numbers, either in the tables or outside. The final result of this exploration is thus a criterion describing the type of numbers which are hypotenuses of right-angled triangles with integral sides.[29]

I. 3 Frenicle's experimental practice and its links to mathematical knowledge

Although quite short (the seventh example for example is almost twice as long), this example conveys a feeling of striking contrast with what is considered a standard Euclidean presentation, with axioms and deductive structure. In this respect, it is quite different from Frenicle's *Traité des*

24 If the hypotenuse is given as a sum of two squares of integers, the two other sides are given as the difference of the two squares and as twice the product of the integers themselves.

25 After 17 come 20, 25, 26 but these are multiples of hypotenuses (resps. 5, 5 and 13) and thus obviously hypotenuses themselves (of triangles whose sides are all multiple of triangles already obtained). Thus, testing the property on them woud not constitute an independant test of the conjecture.

26 The two other sides are given by 25-4=21 and 2x5x2=20. One easily checks Pythagoras' relation in this case: $20^2+21^2=29^2$.

27 Frenicle 1729, p. 23: *La mesme chose se pourra examiner aux autres hypotenuses suivantes [...] Si en les prenant de suite, sans aucun choix, on trouve la mesme chose a toutes, je conclus que ladite regle est generale. [Les sommes de deux carrés sont hypotenuses].*

28 Frenicle 1729, p. 23 : *Je voy ici [dans la table] plusieurs nombres qui servent d'hypotenuses à des triangles fort differens [...] et qui sont somme de deux carrés.*

29 This criterion is not very manageable and Frenicle in fact proceeds to find another one: every prime number which is 1 plus a multiple of 4 is an hypotenuse and inversely. The recourse to the various rules is quite analogous.

triangles rectangles en nombres, which touches upon the same topic as the *Méthode* but is written as a list of theorems deduced from axioms. However, it is not the case that the *Méthode* only provides empirical conjectures which would be then seriously proved in the *Traité*,[30] thus separating the roles of empirical practice and mathematical knowledge. Nor is it the case that the corresponding *theory* – that of right-angled triangles with integral sides – could be easily described either as Euclidean or as quasi-empirical in Lakatos's classification. The *Méthode* claims to provide sure statements and results. Let me then summarize its main characteristics from the point of view of experimental practice and analyze how it is supposed to produce mathematical knowledge.

A first important feature already evoked is the embodiment of this practice into a method – here a written *ars inveniendi.* A method is a cognitive tool, it furnishes the concrete procedures from which the result can be derived. With such a tool, the way of discovering new things or new statements is expected to be displayed, and the pedagogical purpose is directed to shaping the mathematician more than to shaping the mathematical text. Different scholars may develop different methods – making them public, in particular by means of writing, is a crucial issue in the social organization of early-modern science, when challenges were being progressively replaced by collaborative work.

Frenicle's method can be described as experimental in several ways. First of all, it relies on methodical induction; it produces facts from a deliberate and ordered observation of events – in the example given above, it produces statements characterizing hypotenuses from the observation of the presence of certain numerical values in tables. These facts are expressible as principles (such as the final criterion obtained, linking hypotenuses and sums of squares), which in turn can be used in practice to solve specific problems. For instance, in our case the number 221 (the specific value proposed in Frenicle's problem 3) appears in table II as a sum of two squares (121 and 100); Frenicle's criterion implies that it is a hypotenuse (of the triangle of sides 21, 220, 221).

Authors of the time opposed such induction to two other ways to look for truths: axiomatic deduction usually associated with Euclidean mathematics[31] on one hand, and ordinary induction, the "groping," the extrapolation *à tatons* from arbitrary mixtures of singular cases, on the other hand. The stakes are especially high for problems concerning numbers, because "arithmetical questions can sometimes be better found by a hard-working man who will examine stubbornly the sequence of numbers, than by the skill of the best mind," as Descartes dismissively writes to Mersenne (Descartes, OC, II, p.91): on many occasions, Frenicle and the other arithmeticians are obliged to exhibit solutions with 8 or 9 digits to show that they can obtain them by a "sure rule" and not haphazardly.[32]

A second important point is that this induction operates through actions on mediating devices. The vocabulary stresses the activity of the mathematician, whose senses, and not only mind, are called upon: "I see", "I display," "I look, " "I take," "I mark," "I experiment" ("je voy, je dispose, je regarde, je prends, je marque, j'éprouve," etc.), writes Frenicle in various places in the *Méthode*: the first person singular dominates the text. It is noticeable that the use of these verbs reconfigures common categorizations, between standardized mathematical procedures (like

30 For instance, the criterion given in our example 3 is not included. See Goldstein 1995 for more information on the statements in the *Traité* and their links with the *Méthode.*

31 But not only with them: several domains are handled *more geometrico,* as in Spinoza's Ethics. See also Schüling 1969 and Dear 1995.

32 Examples are given in Goldstein 1995 and 2001.

adding or dividing), intellectual processes (inferring) and physical actions (seeing, marking). And instruments, such as tables and lists, are created and used to help perform these actions. As we have seen briefly in Example 3, tables are not simple registers of direct computations for instance. Their construction and display are carefully planned to favour the search; multiple lists and tables (of examples, of counter-examples, of exceptions, etc.) are generated in the process. The dealing of, on, and around tables also has the ability to concretely embody words which are currently used in mathematical texts to represent mental operations, like "I consider" or "I seek".

A third and last point I would like to underline is the way conviction is obtained: the variety of the objects put forward when a theory or a statement is under test, the regularity of the procedures and, in some other examples, the size of the solutions, are the three warrants of truth. We have already met the first while discussing Example 3 of the *Méthode*, and alluded to the third above. The second occurs in the sixth example: Frenicle observes there that 7, the difference of the two smallest sides of right-angled triangles 5, 12, 13 and 8, 15, 17, is also the difference both of 1 and 8 and of 2 and 9; in each case, of a square and twice a square. He decides that this strange connection between two phenomena should be explored by means of the smallest numbers attached to these situations. As seen above, hypotenuses of triangles are sums of squares and Frenicle thus attempts to tie the roots of these squares to the roots of the squares and double squares appearing in the second representation of 7. This attempt is immediately embodied in a new table and three means of relating all these numbers suggested and checked, after 7, on 17. Frenicle comments on the success of his final strategy in these terms: "The third way is more regular and the triangles can be found in an entirely analogous manner, it is thus the true method of finding triangles. Each couple is used independantly."[33] If the regularity is not directly inscribed in the first table, the mathematician's work will make it visible. The variety of the tested objects protects from bias, the regularity of the procedure guaranties that all cases are exhausted. Frenicle, in the course of establishing his theorems, never uses the word "*prouve*" (proves). He speaks of assurance ("*pour s'assurer davantage de cette règle*") or of confirmation ("*ce qui me confirme davantage en cette opinion*") as far as the conclusion is concerned, and of experimenting ("*éprouver*") as far as the procedure itself is concerned. Typically, the conviction is established with three or four cases; Frenicle deals with the problem of representativity in two ways: either the cases are "very different" or they are consecutive.[34] The results are then confirmed by testing the following entries in the available tables or a "large" number.[35] To this, of course, a decisive test should be added: a numerical solution of the problem itself should be obtained which can be simply checked. In our Example 3, for instance, one can check that the proposed number 221 is the hypotenuse of the triangle which Frenicle's method has indeed provided (21, 220, 221). In

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33 Frenicle 1729: La troisième voie est plus régulière et les triangles se trouvent par des façons entièrement semblables, aussi est-ce la vraie méthode de trouver les triangles [à partir des autres nombres]. On se sert de chaque couple à part.

34 See for instance Frenicle 1729, p. 23 : *les uns ne peuvent pas avoir plutost cette propriété que les autres puisqu'elle se trouve en plusieurs triangles fort différents* (these cannot have this property more than the others because one can find it in several very different triangles) or : *si en les prenant de suite et sans aucun choix, on trouve la mesme chose à toutes, je conclus que ladite règle est générale* (if taking them in sequence and without any selection, one finds the same thing for all, I conclude that the rule is general).

35 Frenicle 1729 p. 33 : *pour s'assurer davantage de cette règle, on prendra quelque grand nombre, comme le produit du cube de 5 par le quarré de 13 et par 17* (to be more sure of this rule, one will take some large number, such as the product of the cube of 5 by the square of 13 and by 17).

Frenicle's own words, already mentioned: "This research is mainly useful for possible questions, using for most of them no proof other than construction."[36] Two systems of conviction are thus articulated: one for the intermediate properties, theorems and truths, reminiscent of those in experimental sciences; one reduced to verification for the numerical problem itself.

This issue leads us back to that of mathematical knowledge. The aim of Frenicle's book is to solve concrete problems, to handle questions "diversely according to the different properties which ought to be employed," as stated at the beginning of the treatise. The emphasis is thus put on particulars and on effective solutions, even if the method and its rules promise a general path to these particulars. Indeed, method often delineates a specific body of problems which can be reached and explored. Descartes's emphasis on algebra for instance leads him to a concordant classification of geometrical questions to be handled.[37] Frenicle does not act differently in his *Méthode* when he explains that his main aim is to solve *possible* questions dealing with numbers. The point of this apparently quasi-universal category is in fact to avoid "impossible" problems: Fermat, in particular, had drawn the attention of his contemporaries to negative statements (like the very famous "Last Theorem of Fermat": no sum of two cubes, two fourth powers, etc. can be a cube, a fourth power, etc.), for which he had developed a privileged method, that of infinite descent.[38] Such statements require another kind of answer than an explicit number, and, except in some easy cases, are out of reach of the effective procedures developped by Frenicle. Problems within *his* reach should have a standard form, whose traces for example can be followed in the correspondence or in books like the edition, commented and translated, of the *Arithmetic* of Diophantus, by Gaspard Bachet de Méziriac in 1621; they ask for an explicit solution, sometimes with specific numerical values in the data – this is the case in Diophantus as in our Example 3 – sometimes not. Numerical answers to these concrete questions are thus considered as correct and ordinary solutions. In the last example – to find a right-angled triangle in which the hypotenuse and the difference between the smallest side and the others are squares of integers – the size of the proposed solution (473304, 2276953, 2325625) in itself cannot but convince the reader of the use of a method to discover it. But in two cases (7 and 8), no numerical answer is provided. In the example 7 (to find a triangle such that the hypotenuse and the sum of the other sides are squares), the size of the possible triangles reached by the tables through the various exclusions does not go further than 6-digit numbers and does not give any correct solution. In such cases, Frenicle considers as a result a statement like "no solution exists less than a given number," thus adapting the nature of the answer to his method of investigation and to the size of his tables.

Moreover, several statements isolated as "theorems" or sometimes "truths" are disseminated in the text, as we have already seen in Example 3 above. Such statements do not occur at the end of the example, but normally in the middle; they are the product of the investigation of the tables and provide the basis for deriving the numerical solutions, if possible. Like the tables or the problems, they sometimes constitute a kind of investment, reappearing in several occasions. However, they are not collected and reorganized here in order to provide a supply of truths about

[36] Frenicle 1729, p. 10 : *Cette recherche ne sert principalement qu'aux questions possibles, ne se servant pour la plupart d'autre démonstration que de la construction.*

[37] See Bos 1996 and 2003.

[38] On negative statements and proofs by contradiction in early-modern mathematics, see Mancosu 1991 and 1996. Fermat's programme is discussed in Goldstein 1995 (see also http://www.institut.math.jussieu.fr/~cgolds/ STPFermat-GOLDSTEIN.pdf.)

numbers; the emphasis on such truths is not for themselves, but only insofar as they are required to concretely solve the various problems.

Mathematical practice appears here so close to an *ars* that mathematical knowledge cannot be detached from mathematical know-how; the *Méthode* also models how to deal with errors of various kinds, how to train the attention of the mathematician, how to choose between possible ways of ordering tables or determining counter-instances. The experience then *is* knowledge, and is supposed to be transmitted as such – tables circulate from examples to examples, or from books to other books, as do tested statements and numerical values.

I. 4 Baconian mathematics

Such a conception of mathematical knowledge may appear idiosyncratic and doomed. Moritz Cantor writes, in the second volume of his *Vorlesungen über Geschichte der Mathematik*: "If this treatise is called Méthode …, it is a somehow deceptive title […] Other research from Frenicle must have been known privately, because from the works published by the Academy, one cannot explain the high esteem of Fermat in particular for Frenicle."[39] and this dismissive reaction has been shared by most historians (and some mathematicians of Frenicle's time). However, the social pratice of mathematics in Frenicle's direct environment provides a friendlier contextualization for the *Méthode* and the conception of mathematical knowledge it relies upon. Arithmeticians transmitted challenges through an intense correspondence, challenges where the question was supposed to test at the same time the skill of who launched the challenge and the skills of his correspondents. Those appearing in Mersenne's, Fermat's or Descartes's letters are exactly of the same type (and sometimes exactly the same) as those presented as examples in the *Méthode*. Solutions were in general only numerical, while their size or number witnessed that they had been obtained through a powerful method, not by groping. Frenicle's method was then even able to counter the claims of the (algebraically oriented) Analysts of his time: while some of them, proud of the efficiency of their new techniques, would try and dismiss arithmetical problems as lacking of universality or arithmeticians as lacking of wit, Frenicle could intervene sucessfully by challenging them with problems they could not always solve.[40]

Frenicle's experimental practice, on the other hand, is not idiosyncratic either. First of all, it presents striking similarities with the conception of experience promoted in Francis Bacon's *Novum Organum*. The aim of Bacon's science is knowledge of things in the world, a knowledge necessary for the operationality of human actions: the emphasis is thus put on the *particularia*, particular things: "So that we lead men to the particular things themselves, their series and their orders."[41] The privileged way to do this is through a methodical induction – a procedure toward truth that Bacon explicitely opposed to two other possibilities: the a-priori axioms and deductive reasoning from them of the scholastic tradition and the common, ordinary induction, a wild

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39 Cantor 1880-1908, vol. 2, p. 714: *Wenn eine Abhandlung die Überschrift Méthode pour trouver la solution des problèmes par exclusion trägt, so ist dieses ein einigermassen irreführender Titel. […] Andere Untersuchungen Frénicle's müssen unter der Hand bekannt gewesen sein, denn aus diesen von der Akademie veröffentichen Arbeiten ist die grosse Hochschätzung, welche Fermat insbesondere Frenicle widmete, in keiner Weise zu erklären.*

40 See Goldstein 2001 for a detailed example of such challenges.

41 Bacon, *Novum Organum*, I, aphorism XXXVI: *Homines at ipsas particularia et eorum series et ordines adducamus.*

extrapolation from arbitrary mixtures of singular cases. Frenicle inscribes himself perfectly in this project of a third way in mathematics and we have seen it was essential for him to distance himself from any hit-and-miss practices. The Baconian method is an *ars inveniendi*, an art of discovery, which should provide an assured way to the (secret) truths of the world. More than a technique, it is a propeudeutic and an education, an education of the whole self, in particular of the senses. Men should learn familiarity with things,[42] in Jeffrey Barnouw words, an "exigent, deliberating contact with reality." The will in Frenicle's method to discuss errors (that is, false interpretation of some observations) and the way to correct them, is a good illustration of this insistance on a progressive education.

The Baconian method proceeds in two steps, the first one towards the axioms, the second towards the practice.[43] Experimentation is mainly concerned with the first part: experiments produce facts from a deliberate and especially ordered observation of events and these facts are expressed in principles and axioms (first movement); then, the principles are applied to practice (second movement). Frenicle's method follows eactly the same pattern, its methodical induction providing general statements, true principles, which are then used in the resolution of the problems.

Interestingly consonant with Frenicle's activities, Bacon's accent is not so much on instruments in the ordinary sense of the word, objects of copper, glass or wood, but on those graphico-mental instruments, tables. Experiments should be written, trusted to paper. An ordered construction of a succession of tables (one for the positive instances, one for the negative, one for the comparison of their appearances) is a privileged technique in the application of Bacon's method. Specific techniques of repetition and prolongation of already-known cases preserve the role of individual sagacity, to which is devolved the crucial task of discovering the relevant analogies which will allow one to proceed to the principles. As we have seen, tables and lists of counter-examples also play a decisive role in Frenicle's method, while specific signs drawn in tables allow one to identify positive and negative instances of the phenomenon under study.

A last commun feature, of course, is the role of exclusions. Bacon urges one, as Frenicle does in his rule 10, to be particularly careful of negative instances, of exclusions of nature, of what is not the case. "Foundations for a true induction lay in these exclusions, but it will be completed only through an affirmation."[44] Through these controlled rejections in the realm of the nature of things and the correction of errors in the realm of interpretation, the truth, at the end of an ordered, progressive process, will be discovered and usable, Bacon writes. And Frenicle does.

Bacon's works were thoroughly discussed in Mersenne's circles, and although I have no trace of a direct and conscious borrowing from Frenicle's part, the parallelism of his treatise with Bacon's prescriptions might be followed quite far. But the Method also displays common features with other experimental practices at close range: the idea of probable truth was at the center of important discussions by Mersenne and Gassendi and both share with Mariotte a theory of phenomena as rationalized, mediated sense-data. All of them – interpreting here Bacon as well – rely on comparisons and computations on such phenomena to reach, by methodic induction, their principles and properties. According to Corinne Massignat, they also agree to take

42 Bacon, *Novum Organum* I, XXXVI: *cum rebus ipsis consuescere.*
43 Bacon, *Novum Organum*, I, CIII: *ascendendo primo ad Axiomata, descendendo ad Opera.*
44 Cf. Bacon, *Novum Organum* II, XIX.

perception into account, but only when it is transformed into something collectively debatable and accessible: phenomenon in this circle designates this "objectivation of the sensible data by the work of reason…The essential function of reason is to measure, confront, compare appearances in order to establish their laws and properties." Frenicle's method perfectly illustrates how the rough and sudden perception of a value in a table can be transformed through a systematic procedure of reason into what can be fairly described as a mathematical phenomenon.[45] In this perspective, numbers can be explored through the same procedures as falling bodies and their properties shaped as much as discovered through similar procedures of experimentation.

II. Charles Hermite, the Mathematician as Naturalist

The second case I would like to discuss carries us two centuries ahead and is centered around the work of Charles Hermite. Despite his unusually delayed career,[46] Hermite occupied a central position on the mathematical scene during the second half of the nineteenth century: professor at the Ecole polytechnique and at the Sorbonne, member of the Academy of the Sciences (and its president in 1890), Hermite also acted as a hub for mathematical connections through his extensive international correspondence, with mathematicians as different as Angelo Genocchi, Gösta Mittag-Leffler, Leopold Kronecker or James Sylvester.

His contributions to mathematics are numerous and various, from partial differential equations to number theory, from complex analytic functions to mechanics and he gave his name to several important concepts, like Hermitian forms. "You have never stopped cultivating the highest parts of the mathematical sciences, those where pure number reigns: analysis, algebra, arithmetic,"[47]summarized Henri Poincaré, one of Hermite's heirs and proteges, during Hermite's Jubilee.

II. 1 Numbers and nature

During the nineteenth century, however, the very concept of number was deeply rethought. While views on the relation between geometry and the physical space were hotly debated, number and its corresponding fields of research appeared to many as independant of the real world. Gauss wrote to Bessel in 1830 that "we must humbly grant that, while number is merely a product of our mind, space also possesses a reality outside our mind, and that we cannot entirely prescribe its laws a priori."[48] This position would survive: for Dedekind or Cantor, for instance, fifty years later,

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[45] Massignat 1998, résumé: *Il faut prendre en compte la perception mais en la transformant en un objet susceptible d'une adhésion commune. Une telle objectivation des données sensibles ne peut que résulter d'un travail de la raison.… La fonction essentielle de la raison devient alors la mesure, la confrontation, la comparaison des apparences afin d'établir lois et propriétés entre ces apparences.*

[46] Hermite (1822-1901) entered the Ecole polytechnique as was expected in the 1840s of a promising French mathematician, but left after a year, partly for health reasons, and devoted himself entirely to mathematics with the support of his family during the next decades. He obtained his first position in the 1860s.

[47] Hermite 1893, p. 6: *Vous n'avez cessé de cultiver les parties les plus élevées de la Science mathématique, celles où règne le nombre pur, l'Analyse, l'Algèbre et l'Arithmétique.*

[48] Gauss to Bessel, April 9, 1880: *Wir müssen in Demuth zugeben, dass, wenn die Zahl bloss unseres Geistes Product ist, der Raum auch ausser unsern Geiste Realität hat, der wir apriori ihre Gesetze nicht vollständig vorschreiben kann.* Quoted here from Boniface 2007, p. 329.

number was a "free creation of the human mind." Many mathematicians, confronted by the challenge to intuition brought on by the development of Euclidean geometries or teratological functions, were ready to consider numbers as the single sure foundation of truth – a position reinforced by the vigorous efforts of analysts from Cauchy to Weierstrass to Cantor to rigorously define the basic objects of mathematics: functions, integrals, real numbers, etc. "I actually believe," wrote the most extreme promoter of this arithmetizing trend, Leopold Kronecker, "that one day we will succeed in 'arithmetizing' the complete content of all these mathematical disciplines, i. e. found them exclusively on the notion of number taken in the strictest sense, by peeling away the modifications and extensions of this notion, most of which have been inspired by applications to geometry and mechanics."[49]

This context and the emphasis on pure and abstract mathematics in Poincaré's homage made all the more striking what Hermite wrote on March 2, 1876 to the German mathematician Leo Königsberger.

"I think that we should say of mathematics that they are an observational science. I reject as totally wrong the idea that mathematicians are the creators of their science, and nothing seems to me more opposed to the truth and to the reality of matters than what M. Poinsot says, in these words: 'Computation is an instrument which does not produce anything in itself and which in some way only gives back the ideas entrusted to it'... [My note written in protest] would give at the same time the reasons by which I think I can defend the proposition I previously stated that mathematics, and especially abstract analysis, is the product, the result, of observation and not an arbitrary creation of our mind. The feeling you expressed in this passage of your last letter where you said to me: 'the more I think about these matters, the more I recognize that mathematics is an experimental science, as much as the other sciences', and in this other passage: 'It seems to me that the main task, now, just as in descriptive natural history, consists in accumulating as much material as possible and in discovering principles by classifying and describing this material', this feeling, I say, is also mine, and in this simple and precise form you have summarized with respect to mathematics the deep and intimate conviction of my entire life as a mathematician. I thus believe that even the most abstract analysis is for the most part an observational science, I completely assimilate the complex of concepts, known and to be known in this domain of analysis, to those of the natural sciences – the concepts of analysis having their proper individuality, their figure if I dare say, and their multiple correlations, to the same degree as animals and plants."[50]

This letter displays Hermite as a strong opponent to Dedekind's famous claim on the freedom of the mathematician.[51] But it also displays a particular reason for this opposition; the mathematician cannot be free, because even the most abstract domains in mathematics, the

49 Extract from the 1886 Festschrift to Eduard Zeller, quoted from Petri & Schappacher 2007, p. 355: *Und ich glaube auch, dass es dereinst gelingen wird, den gesammten Inhalt aller dieser mathematischen Disciplinen zu "arithmetisieren," d.h. einzig und allein auf den im engsten Sinne genommenen Zahlbegriff zu gründen, also die Modificationen und Eweiterungen dieses Begriffs wieder abzustreifen, welche zumeist durch die Anwendungen auf die Geometrie und Mechanik veranlasst worden sind.* I am not pretending here that all the authors I have mentioned agreed with each other: on the contrary, several concurrent brands of arithmetization and more generally of foundationalism were available in the last decades of the 19th century. On arithmetization and the foundational development in analysis, see in particular Jahnke & Otte 1981, Epple 1999, Boniface 2007 and Petri & Schappacher 2007.

domains "of pure number" in Poincaré's terms, can be completely assimilated to zoology and botany; the notions of mathematics are not free creations, they are natural objects, to be discovered through observation.

It is noticeable that Hermite's position was not idiosyncratic. Besides Königsberger himself, as the letter indicates, several of Hermite's closest correspondents expressed similar views. For instance, Thomas Stieltjes, a Dutch mathematician who obtained a professorship at the University of Toulouse, wrote to Hermite on May 3, 1894: "With respect to fractions P'/P, P"/P, I have to admit that I did not pretend to clarify such a difficult subject by thought and imagination alone. I will proceed as natural scientists do, calling on observation for help."[52]

To which Hermite happily replied ten days later: "I feel so glad to know that you are of such good composition that you are transforming yourself into a naturalist to observe the phenomena of the arithmetical world. Your doctrine is mine; I believe that numbers and the functions of analysis are not the arbitrary product of our mind; I think that they exist outside us with the same character of necessity as the things of objective reality and that we meet or discover them and study them like the physicists, the chemists and the zoologists."[53]

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50 This manuscript is kept in the Handschriftabteilung of the Staatsbibliothek in Berlin: *Je crois qu'on doit dire des mathématiques, qu'elles sont une science d'observation. Je repousse comme de toute fausseté que les géomètres soient les créateurs de leur science, et rien ne me semble plus contraire à la vérité et à la réalité des choses, que ce que dit M. Poinsot, dans les termes suivants: « Le calcul est un instrument qui ne produit rien par lui-même et qui ne rend en quelque sorte que les idées qu'on lui confie.....»*
[Ma note écrite en protestation] donnerait en même temps les raisons dont je crois pouvoir appuyer la proposition précédemment énoncée que les mathématiques, et tout spécialement l'Analyse abstraite, sont le produit, sont le résultat de l'observation, et non une création arbitraire de notre esprit. Le sentiment exprimé dans ce passage de votre dernière lettre où vous me dites: «plus je réfléchis sur toutes ces choses, plus je reconnais que les mathématiques forment une science expérimentale, aussi bien que toutes les autres sciences», et dans cet autre passage: «Il me semble que la tâche principale, actuellement, de même que pour l'histoire naturelle descriptive, consiste à amasser le plus possible de matériaux, et à découvrir des principes en classant et décrivant ces matériaux», ce sentiment, dis-je, est aussi le mien, et sous une forme simple et précise vous avez résumé à l'égard des mathématiques l'intime et profonde conviction de toute ma vie de géomètre. Je crois donc que l'Analyse la plus abstraite est en grande partie une science d'observation, j'assimile absolument le complexe des notions connues et à connaître dans ce domaine de l'analyse, à celles des sciences naturelles, les notions de l'analyse ayant leur individualité propre, leur figure si je puis dire, et leurs corrélations multipliées, au même degré que les animaux et les plantes. The French word, "*géomètre*", was used very generally to denote any mathematician, and has not the restricted meaning of "a specialist of geometry". Hermite indeed expressed all his life his lack of taste for geometry, in particular descriptive and projective geometry. I insist on this point, for it is not a question here of opposing rigorous analysts and intuitive, naturalistic, geometers.

51 More generally, Hermite's position with respect to the various aspects of arithmetization and foundations is analyzed in Goldstein 2008.

52 Hermite & Stieltjes, tome 2, p. 397: *A l'égard des fractions P'/P, P"/P, je vous avouerai que je n'ai point la prétention d'éclaircir un sujet aussi difficile par la réflexion et par l'imagination seules. Je procéderai comme les naturalistes, en appelant au secours l'observation.*

53 Hermite & Stieltjes, tome 2, p. 398: *Je me sens tout joyeux de vous savoir en si bonne disposition que vous vous transformez en naturaliste pour observer les phénomènes du monde arithmétique. Votre doctrine est la mienne, je crois que les nombres et les fonctions de l'Analyse ne sont pas le produit arbitraire de notre esprit; je pense qu'ils existent en dehors de nous avec le même caractère de nécessité que les choses de la réalité objective, et que nous les rencontrons ou les découvrons, et les étudions, comme les physiciens, les chimistes et les zoologistes.*

II. 2 Observation in mathematics

To detect how such observational practice was really put to work, however, is more delicate.[54] In the two-page note alluded to in the letter to Königsberger, entitled "On observation in mathematics,"[55] Hermite illustrates his views with several examples. Most refer to a mainly heuristic use of observation, like this: "Jacobi asked of observation that it reveal the law of representation of integers by a sum of cubes, while having a clever computer construct the tables which have been published on this question in Crelle's Journal."[56] The word "reveal" emphasizes Hermite's realist view of mathematical results and the expression "law of representation" of course resonates with the usual "laws of nature." The example also suggests an interesting division of labour accompanying the use of tables in detecting mathematical phenomena – calculators being used like the various laboratory assistants of other sciences.[57] Still, the role of observation is here confined to the formulation of conjectures; whatever the epistemological view of the author, the practice he advocates seems rather banal. But Hermite adds: "The preceding results are not sufficient to give a complete idea of the role we can attribute to observation."[58]

The note only illustrates briefly how the idea of a proof itself may sometimes be derived from an empirical study. A more thorough exploration of Hermite's other texts and letters, however, shows more subtle interventions. Observation is sometimes called upon *a posteriori* in order to deepen the understanding of a mathematical phenomenon and to check its validity and its limits. For instance, in 1894, Hermite comments on some features of a result of Stieltjes: "The presence of the factor 1/p is favourable for approximation; but before going further, it would appear to me necessary to test this formula by performing numerical applications,"[59] or again: "I clamour for a function escaping from your condition and in which df/dx and df/dy are always continuous and finite inside the domain D. My request has its origin and cause in my tendancy to deduce analytical notions from the observation of analytical facts, as I believe that observation is as fruitful a source of invention in the world of subjective realities as it is in the domain of sensible realities."[60]

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54 See Echeverria 1992 and 1996 for examples of the use of tables in the establishment of number-theoretical results in the 19th century.

55 "*Sur l'observation en mathématiques*," Hermite 1866.

56 Hermite 1866: *Jacobi a demandé à l'observation de révéler la loi de la représentation des nombres par une somme de cubes en faisant construire par un calculateur habile, les Tables qui ont été publiées sur cette question dans le Journal de Crelle.* Jacobi's corresponding publication is the article [Jacobi 1851]. The question is a particular cae of the so-called "Waring problem," which establishes bounds for the number of powers of a certain degree required to represent every integer. In particular, every integer can be written as a sum of four squares or less, or (it is the case studied by Jacobi), every integer can be written as a sum of 9 cubes (or less). For example, $5818=17^3+7^3+7^3+4^3+4^3+4^3+3^3$. Jacobi also studies the number of such representations. The "able calculator" is the famous Johann Martin Zacharias Dase (or Dahse) (1824-1861) who also worked for Gauss. He is well-known for his computation of the first 200 digits of π in 1844.

57 The comparison could be pushed further. As shown in Maarten Bullynck's thesis, Bullynck 2006, the fabrication of tables could mobilize several types of persons, including amateurs willing to contribute to the development of science, as in the natural sciences. This sociological aspect (already present in Mersenne's circle, see Goldstein 2005) will be explored in more detail elsewhere.

58 Hermite 1866: *Les résultats qui précèdent… ne suffisent point à donner l'idée complète du rôle qu'on peut attribuer à l'observation.*

59 Hermite & Stieljtes, vol. 2, p. 377: *La présence du facteur 1/p est favorable à l'approximation; mais avant d'aller plus loin, il me paraîtrait nécessaire d'expérimenter cette formule, en faisant des applications numériques.*

Observation thus operates at different stages and on different types of objects. It operates before the definitive statement of a result, on numerical data gathered in tables, of course, but also on complicated constructions of functions or of series in order to adjust the correct assumption. It also operates afterwards; for instance to verify a specific phenomenon or to link it wih others (varying properties and hypotheses) and to test the outcome – a step perceived as necessary to describe in the most precise way how mathematical objects, once detected, concretely behave and how they are related to each other. Hermite pushes this naturalization of mathematical objects quite far: they are things to be discovered – like new species – and like species, they have to be classified. "[Science] is above all a classification, a way of bringing closer facts that appearences have separated, although they were linked by some natural and hidden family relationship. Science, in other terms, is a system of relations,"[61] writes Poincaré too in *La valeur de la science.* Classifying means in particular discovering key "characters," which will help to establish proper links between otherwise heteroclite entities.

II. 3 Classifying roots of equations

An example of the issues involved can be followed on one aspect of Hermite's work concerning the roots of algebraic equations. When he investigated the topic in the 1840s, Hermite was confronted by a mixture of partial results, involving quadratic forms, approximations, and continued fractions. But their links had been properly clarified only in the simplest case, that of quadratic roots.[62] A quadratic equation $ax^2+bx+c=0$ where a, b, and c are integers ($a\neq 0$) has in general two (real or complex) roots, given by the well-known expressions

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

and

$$\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

For instance, in the case of the equation $x^2-3=0$, the two roots are real numbers, $a=\sqrt{3}$ and $b=-\sqrt{3}$. Real numbers can be expanded into continuous fractions,[63] a representation which also

60 Hermite & Stieltjes, vol. 1, p. 331-332: *Je réclame à cor et à cri une fonction échappant à votre condition et dans laquelle d f /d x et d f /d y soient toujours continues et finies à l'intérieur d['un domaine] D. Ma requête a pour origine et pour cause ma tendance à faire résulter les notions analytiques de l'observation des faits de l'analyse, croyant que l'observation est la source féconde de l'invention dans le monde des réalités subjectives, tout comme dans le domaine des réalités sensibles.*

61 Poincaré 1906, p. 122: *[La science], c'est avant tout une classification, une façon de rapprocher des faits que les apparences séparaient, bien qu'ils fussent liés par quelque parenté naturelle et cachée. La science, en d'autres termes, est un système de relations.*

62 This case is one of the examples mentioned by him in the note already alluded to.

63 See for instance Davenport 1984, chapter IV. To obtain the development of a real number x, one notices that there exists a unique integer x_0 such that $x-x_0$ is less than 1. The inverse of $x-x_0$ is thus equal to an integer x_1 plus a number again less than 1; after reiteration, one obtains a (perhaps infinite) expression of x in terms of the integers x_0, x_1, x_2, etc. See the example in the text.

provides good approximations by rational numbers. To follow our example, the root $a = \sqrt{3}$ can be developed as:

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}$$

We see that the integers intervening in the development, after the first step, are alternately 1 and 2. Such a periodicity is the key to a characterization proved by Lagrange around 1770: quadratic irrational numbers (that is, roots of a quadratic equation) have periodic continued fractions after a certain step. This property is linked to another result, the so-called reduction of the quadratic form $f_\Delta = (x - \alpha y)^2 + \Delta(x - \beta y)^2$ where Δ is a real positive number.

To reduce such a form means to find new variables (X, Y) (where X and Y are linear combinations of x and y with integral coefficients[64]), such that the form in the new variables X and Y deduced from the initial form, $\tilde{f}_\Delta(X,Y) = AX^2 + BXY + CY^2$ satisfies $A < C$ and $-A < B \leq A$, or $C = A$ and $0 \leq B \leq A$. A form whose coefficients verify these inequalities is said to be reduced. Thus among all the forms equivalent to the initial form – that is, deducible from it by such a change of variables – reduced forms are simple in the sense that their coefficients are small; if the forms are assumed to have integral coefficients, there is indeed a finite number of reduced forms in each class of equivalent forms. Moreover, the minimum value of the reduced form for integral values of the variables X and Y is A and it is also the minimum value of the initial form, and of any form of the class.

The connection with the preceding result on continued fractions is that the sequence of reduced forms associated with the various f_Δ when Δ varies is obtained by a periodic computation; this computation involves the continued fractions linked to a and b.[65]

Léon Charve, a student of Hermite, summarized this situation at the beginning of his own thesis in 1880: "One knows that if one expands a quadratic irrational as a continued fraction, the computation is periodic. This periodicity constitutes a very remarkable property of the roots of quadratic equations, and can even serve as a definition of these irrationals. But the theory of continued fractions is closely linked to the theory of binary quadratic forms... One is thus led to wonder if some type of approximation of the quantities would not give an analogous periodicity for irrationals of higher degree than the second. It is the consideration of quadratic forms which leads to this extension of the theory of continued fractions and yields these new methods of approximation."[66]

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64 We assume that this change of variables is invertible, meaning that it is possible to get back to the initial variables x and y by another linear transformation with integral coeeificents.

65 The details of this computation can be found in Davenport 1984.

66 Charve 1880, p. 36-37: *On sait que, si l'on développe en fraction continue une irrationelle du second degré, le calcul est périodique. Cette périodicité constitue une propriété très remarquable des racines des équations du second degré, et elle peut même servir de définition à ces irrationnelles. Or la théorie des fractions continues est liée étroitement à la théoriedes formes quadratiques binaires...On est alors conduit à se demander si quelque mode d'approximation des quantités ne donnerait pas une périodicité analogue pour les irrationnelles d'un degré supérieur au second. C'est la considération des formes quadratiques qui conduit à cette extension de la théorie des fractions continues, et donne ces nouvelles méthodes d'approximation.*

This is not the place for discussing in detail the heuristic of this generalization, but it is relevant to understand its rationale; the periodicity here is the characteristic feature which provides the classification and understanding of the roots of algebraic equations. But its traces in the quadratic case – the obvious periodicity displayed in the expansion into a continued fraction of the algebraic roots – are no longer available in the case of higher degrees. Continued fractions can no longer be the key to the general situation. It is Hermite's idea to locate the phenomena associated with continuous fractions (rational approximation of the roots and periodicity) in another object, which will display the phenomena for all degrees. He found them by putting centre stage not the roots themselves, but quadratic forms constructed by means of these roots. In one of his first number-theoretical papers, published in 1850, Hermite shows how to use forms to approximate several irrational numbers at the same time, and he also sketches their connection with periodicity. In the cubic case, $x^3 - 2 = 0$, for instance, which has one real root $\alpha = \sqrt[3]{2}$ and two complex conjugates roots β and γ, Hermite introduces the quadratic forms:

$$F_{0,\Delta} = (x + y\sqrt[3]{2} + z\sqrt[3]{4})^2 + 2\Delta(x + \lambda y\sqrt[3]{2} + \lambda^2 z\sqrt[3]{4})(x + \lambda^2 y\sqrt[3]{2} + \lambda z\sqrt[3]{4})$$

where λ is a cubic root of unity. The study of their successive reductions when Δ varies reveals that the reduced forms can be obtained again by a periodic procedure, as the same three changes of variables can be used to reduce the forms.[67]

The construction of quadratic forms in the same mould for other algebraic roots provides the same result. Hermite comments: "Perhaps we will succeed in deducing from [the remarkable circumstances which are provided by the reduction of forms whose coefficients depend on roots of algebraic equations with integral coefficients] a complete system of characters for each species of these kinds of quantities, analogous for instance to those given by the theory of continued fractions for the roots of quadratic equations. We cannot in any event have too many elements concurring in throwing some light on this infinite variety of algebraic irrationals, among which the symbols for the roots represent merely the smallest part.... What an immense task it is for number theory and integral calculus to penetrate into the nature of such a multitude of entities created by reason, classifying them into mutually irreducible groups, to constitute them all individually through characteristic and elementary definitions."[68]

As in Frenicle's case, the aim of science is here to understand each mathematical concept or object in its individuality. But since the individuality is now that of a biological species, it is associated with a set of characters which can be detected and checked (here computed). Observation allows the scientist to recognize them and in turn, these characters hopefully provide a procedure to classify any individual – here any algebraic number.[69]

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67 The details of this procedure are given in Goldstein 2007.

68 Hermite 1850: *Peut-être parviendra-t-on à déduire de là [des circonstances remarquables auxquelles donne lieu la réduction des formes dont les coefficients dépendent des racines d'équations algébriques à coefficients entiers] un système complet de caractères pour chaque espèce de ce genre de quantités, analogue par exemple à ceux que donne la théorie des fractions continues pour les racines des équations du second degré. On ne peut du moins faire concourir trop d'éléments pour jeter quelque lumière sur cette variété infinie des irrationnelles algébriques, dont les symboles d'extraction des racines ne nous représentent que la plus faible partie.... Quelle tâche immense pour la théorie des nombres et le calcul intégral de pénétrer au milieu d'une telle multiplicité d'êtres de raisons en les classant par groupes irréductibles entre eux, de les constituer tous individuellement, par des définitions caractéristiques et élémentaires.*

69 Hermite's programme, from his own point of view, did not succeed. The computations involved became quickly much too complicated to be effectively done and provide convincing evidence.

II. 4 Chevreul's experimental method

Such a view of scientific procedures and scientific purposes was not proper to mathematicians and Hermite's biography provides some hints on his involvement with natural scientists. One significant case is the famous chemist Eugène Michel Chevreul, to a text of whom Hermite's note, mentioned earlier, was appended. Like Hermite, Chevreul participated in the Société philomathique and was a member of the Academy of Sciences. Besides his works on stearine and on optical properties of colours, Chevreul had a keen interest in the philosophy of science. For him, to found organic chemistry was first to determine "immediate principles": they were compounds more or less naturally constituted, as he said, "by the action of life"; then organic bodies were to be classified by species, varieties, genus, etc., through a thorough analysis of their properties.[70] Chevreul also promoted the idea of an *a posteriori* experimental method, in terms remarkably consonant with Hermite's practice: "In the case of a simple observation or of what is called an experiment, the a posteriori experimental method intervenes with the express provision that the scientist institute experiments proper for proving the truth of a reasoning performed with the intention of reducing to its immediate cause the phenomenon given by the observation or by the simple experiment. [...]A deep knowledge of concrete things imperatively requires that the study be comparative while examining the property, fact or abstraction that one studies in a series of concrete objects presenting the same property, fact, or abstraction. [...]The natural and truly scientific method [for providing classifications] relies on the mutual relations between species. Thus this aim explains the importance attached to the preeminent quality of what attributes are chosen as characters."[71]

Going from concrete to abstract (and back again) was also a leitmotiv for Hermite. Imagination and observation should first concentrate their efforts on concrete instances, then identify an immediate 'principle', that is, an object encapsulating that which will allow the scientist to apprehend the phenomenon under study: in Hermite's case, for instance, the forms constructed from the roots of algebraic equations. In one of the first publications in which he used such forms, Hermite had already indicated that "it is in some very elementary properties of quadratic forms with an arbitrary number of variables that I found the principles of analysis which I request permission to discuss with you."[72] But then, a posteriori, the mathematician should compare, relate and link, in order to 'reduce to its immediate cause' the phenomenon: we have seen Hermite

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70 Analogous features can of course be found in zoologists or botanists. But it is particularly revealing of this French mid-century milieu to find such resonant descriptions among chemists like Chevreul and mathematicians like Hermite.

71 Chevreul 1866: *Dans le cas d'une simple observation ou de ce qu'on appelle une expérience, la méthode a posteriori expérimentale intervient à la condition expresse que le savant institue des expériences propres à prouver la vérité d'un raisonnement fait avec l'intention de ramener à sa cause immédiate le phénomène donné par l'observation ou par la simple expérience... La connaissance approfondie des choses concrètes exige impérieusement que l'étude soit comparative, en examinant la propriété, le fait, l'abstraction que l'on étudie dans une série d'objets concrets qui présentent la même propriété, le même fait, la même abstraction.... La méthode naturelle, toute scientifique, [de faire des classifications] repose sur les rapports mutuels des espèces. Aussi le but qu'elle se propose explique-t-il l'importance qu'elle attache à la prééminence des attributs choisis comme caractères.*

72 Hermite 1850/1905-1917, vol. 1, p. 101: *C'est dans quelques propriétés très élémentaires des formes quadratiques, à un nombre quelconque de variables, que j'ai rencontré les principes d'Analyse dont je vous demande la permission de vous entretenir.*

requiring from Stieltjes a specific ad-hoc example in order to test a posteriori if the formulation of a result properly delineated its range of validity and above all, the reasons for its validity.

For Hermite, experimenting is not only heuristically useful – either to discover the objets themselves or the idea of a proof – it is necessary in order to validate mathematical knowledge. Paradoxical as it may appear to our post-foundationalist generation, a single proof is not enough for Hermite. First of all, the proof should correspond to the principles of a situation, and, in some cases, Hermite expresses himself in favour of a multiplicity of proofs, as necessary to illuminate all the links and relations involved and thus establish true mathematical knowledge.

Experimentation operates here at several levels: it helps to provide the analytical elements which will act as possible characters, to control and understand results, to find the principle of a proof. As concepts are no more invented by the mathematician than organisms are by the naturalist, the aim of science is to investigate properties of these objects (which can and should be discovered through procedures adapted to their nature), and to classify them. Authentic mathematical knowledge cannot put these procedures aside, in the same way a heuristic aid can be dismissed once found: Hermite's conception of mathematical knowledge indeed *requires* such a practice.

MATHEMATICAL EXPERIMENTATIONS

What do these two examples teach us? First of all that important mathematical milieux were at various times interested in mathematical experiment. Specific mathematical practices were closely related to some of those advocated for contemporary natural sciences: terminology and actions both underline this proximity.[73] This global experimental context is essential to throw light upon key features of mathematical experimentation and, above all, to historicize both the experimental procedures and the links between mathematics and the sciences.[74] These practices of experimentation operated in mathematics at a variety of levels and not only as a kind of refined heuristics: they were sollicited in order to identify interesting mathematical objects or properties, to sketch a proof, to adjust the conditions of validity of a statement. Moreover, these reflections went with reconfigurations of mathematical knowledge itself : in our two examples, the nature of experimentation reshaped what would constitute a statement or a valuable mathematical topic.

But there is an even more puzzling issue to explore, an issue more specific to the history of mathematics: these activities have not been organized nor transmitted as a recognized whole, neither in mathematics itself nor through the links usually constructed by historians and philosophers. There is – and my brief survey at the beginning testifies to it – no centenary tradition of mathematical experiment. The practices described here are to be detected in the shadow of other histories, that of foundations or of algebrization, often in a negative way. Their memories have been erased, or perhaps more accurately, not retained as valuable contributions to the shaping of mathematics. Adopting here a quite Baconian outlook, we may thus wonder if it is not

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[73] For instance, the recurrent use of "method" by Frenicle, the emphasis on "observation" and "classification" by Hermite.

[74] How these internal experimental features informed or contradicted the use of mathematics as a tool box for modeling natural sciences, the relations between what are often coined mathematical and empirical data, is one larger issue raised by this enquiry.

by following their traces that we might understand how mathematics and sciences have been for so long joined and yet separate.

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