Galois descent in Galois theories

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Abstract : inspired by Kummer theory on abelian varieties, we give similar looking descriptions of the Galois groups occuring in the differential Galois theories of Picard-Vessiot, Kolchin and Pillay, and mention some arithmetic applications.

Résumé : guidés par la théorie de Kummer sur les variétés abéliennes et motivés par quelques applications arithmétiques, nous donnons des descriptions d'apparences similaires des groupes de Galois issus des théories de Galois différentielles de Picard-Vessiot, Kolchin et Pillay.

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The topic I had been assigned by the organizers of the Luminy September 09 School was "Algebraic D-groups and non-linear differential Galois theories". The present account is written in an applied maths spirit : how to compute the Galois groups, and what for ? Thus, we start with a motivating question which, in accordance with the theme of the School, comes from diophantine geometry. We then describe the Galois groups of the various theories under study, in terms that bear a strong similarity. Finally, we apply this description to the study of exponentials and logarithms on abelian schemes.

A general argument of Galois descent occurs along the text, hence the title[1] of these notes; its number theoretic prototype, given by Kummer theory, is recalled in an Appendix to the paper.

Although the presentation is sometimes novel, the results described here are not new. For original sources, we refer the reader to [30] for the Picard-Vessiot theory, [28] for Kolchin's and Pillay's theories, and to [1] and [10] for the applications to algebraic independence. Actually, this text may serve as an introduction to the survey [9], which is itself an introduction to the latter papers (and to the descent argument in the non-linear case).

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[1] also borrowed from a set of talks at the Durham July 09 Conference on model theory. I thank the organizers of both Luminy and Durham meetings for having offered me these opportunities to develop this point of view.

1 Ax-Schanuel

1.1 The multiplicative case

The well-known Schanuel conjecture asserts that if $x = \{x_1, ..., x_n\}$ is a "non-degenerate" n-tuple of complex numbers whose image under the standard exponential function exp is denoted by $y = \{y_1 = exp(x_1), ..., y_n = exp(x_n)\}$, then $tr.deg._{\mathbb{Q}}\mathbb{Q}(x, y) \geq n$. The expression *non-degenerate* will occur under several acceptions in these notes. Here, it means that

$$\forall (m_1, .., m_n) \in \mathbb{Z}^n \setminus 0, m_1x_1 + ... + m_nx_n \neq 0,$$

or equivalently, that for any proper algebraic subgroup H of the algebraic torus $G = \mathbb{G}_m^n$, the complex point x of the Lie algebra LG of G does not lie in the Lie algebra LH of H.

In 1970, Ax [2] proved a functional version of the conjecture, which, in an analytic setting, may be phrased as follows. Let

$$x = (x_1, ..., x_n) \in (\mathbb{C}\{\{z_1, .., z_t\}\})^n$$

be a n-tuple of convergent power series in t variables. For each i, $y_i(z) := exp(x_i)$ lies in $\mathbb{C}\{\{z_1, .., z_t\}\}^*$, and we set $y = exp(x) \in (\mathbb{C}\{\{z_1, .., z_t\}\}^*)^n$. Assume that x is *non-degenerate*, in the sense that

$$\forall (m_1, .., m_n) \in \mathbb{Z}^n \setminus 0, m_1x_1 + ... + m_nx_n \notin \mathbb{C},$$

or equivalently, that for any proper algebraic subgroup H of the algebraic torus $G = \mathbb{G}_m^n$ and any constant point $\xi \in LG(\mathbb{C})$,

$$x \notin \xi + LH.$$

Then, $tr.deg.(\mathbb{C}(x, y)/\mathbb{C}) \geq n + \mu$, where μ denotes the rank of the functional jacobian $\frac{Dx}{Dz} \in Mat_{t,n}(\mathbb{C}\{\{z_1, .., z_t\}\})$ of x.

Let $K \simeq \mathbb{C}(z_1, ..., z_\mu)$ be the field generated by the principal variables. In order to check the above lower bound, it suffices to show that $tr.deg.(K(x, y)/K) \geq n$ (and the two statements are actually equivalent). Furthermore, choosing a sufficiently general line in $\mathbb{C}^\mu$, it suffices to check the latter inequality when $\mu = 1$. Using the differential equation satisfied by exp, we can therefore view Ax's theorem as a corollary of the following differential algebraic statement. Let $K = \mathbb{C}(z)^{alg}$ be the algebraic closure of $\mathbb{C}(z)$, endowed with the (unique) extension ∂ of the derivation $\frac{d}{dz}$, and let $(\mathcal{K}, \partial)$ be a differential extension of (K, ∂), with same constant field $\mathcal{K}^\partial = \mathbb{C}$. Let further $(x, y) \in (\mathcal{K} \times \mathcal{K}^*)^n$ satisfy : $\partial y/y = \partial x$ (where derivations are taken coordinate-wise). Assume that for any proper algebraic subgroup H of $G = \mathbb{G}_m^n$, x does not lie in $LH + LG(\mathbb{C})$. Then $deg.tr.K(x, y)/K \geq n$.

1.2 The constant case

Two years later, Ax [3] extended his results to more general algebraic groups (see also [18]). For instance, making the same analysis as above, we may rephrase Theorem 3 of his paper (actually written in a formal setting and under a slightly stronger hypothesis on G) as follows.

Let $K = \mathbb{C}(z)^{alg}, \partial = \frac{d}{dz}$ and $\mathcal{K}$ be as above, and let G be a commutative algebraic group *defined over* $\mathbb{C}$, with no additive quotient. In other words, G is a semi-abelian variety defined over $\mathbb{C}$, or more generally, a quotient of its universal vectorial extension. The Lie algebra LG of G is a vector space over $\mathbb{C}$, so that there is a unique differential operator ∇_{LG} on $LG(\mathcal{K})$, whose solution space is $LG(\mathbb{C})$ (to define $\nabla_{LG}(x)$, choose any basis of LG *over* $\mathbb{C}$, and take the ∂-derivatives of the coordinates of x; the outcome is independent of the chosen basis). The exponential map $exp_G : LG(\mathbb{C}) \to G(\mathbb{C})$ is a morphism of commutative Lie groups, admitting as kernel a discrete subgroup Ω_G of $LG(\mathbb{C})$, and one can consider its inverse ℓn_G as a multivalued function. For any analytic function $x(z)$ with values in $LG(\mathbb{C})$, $y(z) := exp_G(x(z))$ is a well defined analytic function with values in $G(\mathbb{C})$. For any analytic function $y(z)$ with values in $G(\mathbb{C})$, $\nabla_{LG} \circ \ell n_G(y)$ is also well-defined, since Ω_G is killed by ∇_{LG}. Its explicit expression enables us to extend $\nabla_{LG} \circ \ell n_G$ to a group homomorphism $\partial \ell n_G : G(\mathcal{K}) \to LG(\mathcal{K})$. This is the *logarithmic derivative* of $G/\mathbb{C}$, which we describe in a more algebraic way in §2, then for non constant groups in §3 - and again in the above style in §4.1. Notice that when x and y have an analytic meaning, the relation $\partial \ell n_G(y) = \nabla_{LG}(x)$ is equivalent to the existence of a point $\xi \in LG(\mathbb{C})$ such that $y = exp_G(x - \xi)$.

Exactly as in §1.1, Ax's theorem then reads as follows : let $(x, y) \in (LG \times G)(\mathcal{K})$ satisfy $\partial \ell n_G(y) = \nabla_{LG}(x)$, and suppose that x is non-degenerate : for any proper algebraic subgroup $H/\mathbb{C}$ of G, $x \notin LH + LG(\mathbb{C})$. Then, $tr.deg.(K(x,y)/K) \geq dimG$.

In these notes, we will show that differential Galois theories provide proofs of Ax's theorem under the following *restrictions* :

(L) either $y \in G(K)$ (in which case we can apply the linear Picard-Vessiot theory);

(E) or $x \in LG(K)$ (in which case we can apply the non-linear theory of Kolchin).

But the interesting point about these Galois approaches is that in fact, they then provide an *extension* of these results to the case of *non-constant* algebraic groups, where G will only be defined over K. In the second situation, this is made possible by Pillay's generalization of Kolchin's theory (although the initial proof given in [10] uses a different method). See Theorems 4.1 to 4.4 for the outcome in the case of abelian varieties.

1.3 Motivations

The Manin-Mumford conjecture was proved by Raynaud in 1984, and has known since then a remarkable number of interesting new proofs. Based on work of Bombieri, Pila, Wilkie, and Zannier, Pila recently obtained another one [24], where the strategy of [25] is

combined with Ax's theorem[2] on abelian varieties over $\mathbb{C}$. By a general argument (see [11], Thm. 1 and proof of Thm. 4; also [25]), the conjecture reduces to :

Manin-Mumford (key point): *Let $A/\mathbb{Q}^{alg}$ be an abelian variety. An algebraic subvariety $X/\mathbb{Q}^{alg}$ of A passes through finitely many torsion points of A, unless X contains a translate of a non-zero abelian subvariety of A.*

We now sketch Pila's approach : first, as in [25], write $A(\mathbb{C}) = LA(\mathbb{C})/\Omega_A \simeq \mathbb{R}^{2g}/\mathbb{Z}^{2g}$, so that the torsion points become the rational points of the box $[0, 1[^{2g}$ while X pulls back to a complex analytic subvariety $\mathcal{X}$ of LA. By o-minimality, $\mathcal{X}$ meets $<< T^\epsilon$ rational points of denominator $\leq T$, *outside of the real semi-algebraic subvarieties $\mathcal{W}$ of positive dimension it contains.* But back to $A(\mathbb{Q}^{alg})$, any such torsion point generates many others[3] by Galois action, so their orders are bounded.

To conclude, we must control the possible irreducible complex algebraic subvarieties W of positive dimension in $\mathcal{X}$. Assuming that X contains no translate of a non zero abelian subvariety of A, we claim that no such W exists. Assuming otherwise, consider the function field $K = \mathbb{C}(W)$, and let $x \in LA(K)$ be a generic point over $\mathbb{C}$ of W. Since $exp_A(W) \subset X$, the transcendence degree of $y = exp_A(x)$ over $\mathbb{C}$ is $< dim(A)$. Ax's theorem on the constant abelian variety $G = A$ (in its original formulation, or in the above one, using A. Pillay's remark that it suffices to check the claim when W is a curve) then implies that x lies in $\xi + LA'$, for some abelian subvariety A' of A, with $0 \neq A' \subsetneq A$, and some $\xi \in LA(\mathbb{C})$. Set $\eta = exp_A(\xi)$, and notice that $x' = x - \xi \in LA'(K)$ is still a generic point over $\mathbb{C}$ of the irreducible algebraic variety $W' = W - \xi$, which is therefore contained in LA'. The image of W' under $exp_{A'}$ is contained in the intersection X' of $X - \eta$ and A', which is a subvariety of A' containing no translate of a non zero abelian subvariety. Since the inverse image $\mathcal{X}' \subset LA'$ of X' under $exp_{A'}$ contains W', the proof can now be concluded by induction on the dimension of A. We point out that Ax's theorem was here used only in the (E) setting.

In his unpublished note [29], Pink extended the conjecture to a relative context, including the following case :

Relative Manin-Mumford (over curves) : *let X be the image of a non-torsion section of an abelian scheme $\mathcal{A}/S$ of relative dimension ≥ 2 over a curve $S/\mathbb{C}$. Assume that X is not contained in a translate of an elliptic subscheme of $\mathcal{A}/S$. Then, X should meet finitely many of the torsion points of the various fibers of $\mathcal{A}/S$.*

So, we here have an abelian variety A over $K = \mathbb{C}(S)$. It need not come from $\mathbb{C}$, but one may hope that again, an Ax-type theorem, now over a non-constant algebraic group, will help. And indeed, in their work on the conjecture, Masser and Zannier do appeal to such an algebraic independence statement, though now in the (L) setting : see [22], p. 493, line 14, for the test case of the square of an elliptic scheme.

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[2] That Ax's earlier version on tori [2] can play a similar role for the multiplicative analogue of Manin-Mumford, had already been observed in [25], Final Remark 2.

[3] i.e. more than T^δ, where $\delta > \epsilon$. The Kummer theory described in the Appendix would similarly yield large Galois orbits for the division points in the Mordell-Lang conjecture.

2 Picard-Vessiot & Kolchin

The differential Galois theories attached to these names (the second one generalizing the first one) concern constant algebraic groups. More precisely, let (K, ∂) be a differential field, with an algebraically closed constant field $K^\partial := C$ of characteristic 0. We fix a differential closure $\hat{K}$ of K; in particular, $\hat{K}^\partial = C$. Let further G be a connected algebraic group, possibly non commutative, defined over C; the Picard-Vessiot theory concerns affine algebraic groups $G \subset GL_{n/C}$. We denote by LG the Lie algebra of G, and set : $G_K = G \otimes_C K$. The main point in the approach of Kolchin and his school (see in particular [19]) is the existence of the logarithmic derivative of G, a canonical differential algebraic map

$$\partial \ell n_G : G \to LG,$$

which, at the level of K-rational points, can be described as follows.

Any point $y \in G(K)$ provides a derivation $\delta_C y$ on the local ring $\mathcal{O}_{G/C, y}$, with values in K, via the formula $\delta_C y(f_C) = \partial(f_C(y))$. By K-linearity, we can extend $\delta_C y$ to a K-linear derivation ∂y on $\mathcal{O}_{G_K, y}$. In preparation for the next section, we repeat the definition of ∂y in the framework of [12], [28]. First, ∂ extends uniquely to a derivation D_∂^0 of the structure sheaf $\mathcal{O}_{G_K}$, killing $\mathcal{O}_{G/C}$. For instance, for G affine, we have $D_\partial^0 = 1 \otimes \partial$ on $K[G] = C[G] \otimes_C K$, while $\partial y = \delta_C y \otimes 1$. For any point $y \in G(K)$, the formulae $D_{\partial, y}^0(f) = (D_\partial^0 f)(y)$ and $(\delta y)(f) = \partial(f(y))$ define two derivations on the local ring $\mathcal{O}_{G_K, y}$, with values in K, which are only C-linear. But since both reduce to ∂ on K, their difference $\delta y - D_{\partial, y}^0$ vanishes on K, hence is K-linear - and clearly coincides with ∂y : $\partial(f(y)) = \partial y(f) + (D_\partial^0 f)(y)$.

Now, such a K-linear derivation $\partial y : \mathcal{O}_{G_K, y} \to K$ can be viewed as an element of the tangent space $T_y G_K$ of G_K at y. Pulling ∂y back to the tangent space LG_K of G_K at the origin by the differential of right translation by y^{-1}, we obtain the *logarithmic derivative* $\partial \ell n_G(y) \in LG(K)$ of y with respect to the standard extension D_∂^0 of ∂. In the affine case $G \subset GL_{n/C}$, this is given by

$$G(K) \ni y \mapsto \partial \ell n_G(y) = (\partial y) y^{-1} \in LG(K),$$

where $\big((\partial y)_{ij}\big) = \big(\partial(y_{ij})\big) \in T_y G \subset T_y(Mat_{n,n}) \simeq Mat_{n,n}$. In particular, $\partial \ell n_{\mathbf{G}_m} y = \frac{\partial y}{y} := \partial \ell n y$, $\partial \ell n_{\mathbf{G}_a} y = \partial y$.

These formulae make sense over any differential extension $(\mathcal{K}, \partial)$ of (K, ∂). For any such $\mathcal{K}$, we write

$$G^\partial(\mathcal{K}) = \{y \in G(\mathcal{K}), \partial \ell n_G(y) = 0\},$$

and point out that in the present constant case, we have $G^\partial(\hat{K}) = G(C)$. We also note that $\partial \ell n_G$ is surjective at the level of $\hat{K}$-rational points, cf. [19], Prop. 6.

Given $a \in LG(K)$, the Picard-Vessiot-Kolchin theory studies the differential extension $K(y)/K$, where y is a solution in $G(\hat{K})$ of

$$\partial \ell n_G(y) = a,$$

and its Galois group

$$\Gamma_a = Aut_\partial(K(y)/K) := \{\sigma \in Aut(K(y)/K), \sigma\partial = \partial\sigma\}.$$

The logarithmic derivative is a cocycle for the adjoint action of G on LG:

$$\partial\ell n_G(uv) = \partial\ell n_G u + u(\partial\ell n_G v)u^{-1},$$

or equivalently : $\partial\ell n(u^{-1}v) = u^{-1}(-\partial\ell nu + \partial\ell nv)u$. Therefore,
  i) two solutions $y, \tilde{y}$ satisfy $y^{-1}\tilde{y} \in G^\partial(\hat{K}) = G(C)$, so, the field $K(y)$ does not depend on the choice of y, and for a given y, the Galois group admits a natural embedding ρ into $G(C) : \forall\sigma \in \Gamma_a, y^{-1}\sigma y = \rho(\sigma) \in G^\partial(\hat{K}) = G(C)$. If we replace y by another solution $yc, c \in G(C)$, then, ρ is changed into $c^{-1}\rho c$.
  ii) Consider the "affine" action of $G(K)$ on $LG(K)$ given by

$$g \bullet a = gag^{-1} + \partial\ell n_G(g).$$

If $\partial\ell ny = a$ and $g \in G(K)$, then, $\tilde{y} = gy$ generates the same extension of K as y, and satisfies $\partial\ell n(\tilde{y}) = g \bullet a$. So, the extension $K(y)/K$, and its Galois group, depend only on the *orbit* $G(K) \bullet a$ of a.

With these points in mind, the main theorem of Kolchin's theory (see [30], §1.4 for the Picard-Vessiot case) can be summarized as follows:

Theorem 2.1. : *i) $Im(\rho) = J_a(C)$, where J_a/C is an algebraic subgroup of G/C;*
  ii) there is a "Galois correspondence" : for instance, $K(y)^{J_a(C)} = K$;
  iii) $tr.deg.(K(y)/K) = dim(J_a)$.

  The Galois correspondence shows that J_a is connected if and only if K is integrally closed in $K(y)$. To fix the ideas, we shall now assume that the base differential field K is algebraically closed, so that all Galois groups over K become connected. We can now describe the Galois goup in a style which will become the leit-motiv of these notes. See [30], I.31.(1) and I.31.(2) for the Picard-Vessiot case, with a weaker assumption on K.

Theorem 2.2. : *assume that K is algebraically closed. Then, J_a is a minimal algebraic subgroup J/C of G/C such that $LJ(K)$ meets the orbit of a under $G(K)$.*

Proof : consider the set $\mathcal{H}$ formed by all the algebraic subgroups H/C of G/C such that $G(K) \bullet a \cap LH \neq \emptyset$. Note that this set is stable under conjugation by $G(C)$, since $c(g \bullet a)c^{-1} = (cg) \bullet a$ for any $c \in G(C)$. We will show that any such H contains a $G(C)$-conjugate of J_a - this actually holds for any K - , and that J_a itself lies in this set.
  i) Let $H \in \mathcal{H}$ and let $g \in G(K)$ such that $g \bullet a := \tilde{a} \in LH(K)$. Since the restriction of $\partial\ell n_G$ to H is $\partial\ell n_H$, which is surjective on $\hat{K}$-points, there is a solution $z \in H(\hat{K})$ of the equation $\partial\ell n_G(z) = g \bullet a$. So, $z = \tilde{y}c = gyc$ with $c \in G(C)$, and the representation $c^{-1}J_a c$ of Γ_a attached to yc lies in $H(C)$.
  ii) Consider the $K(y)$-subvariety yJ_a of G. Its set $yJ_a(K(y))$ of $K(y)$-points is Zariski-dense and stable under Γ_a, and is therefore the set of $K(y)$-points of a K-torsor Z under

6

the algebraic group $J_a \otimes_C K$. Since K is algebraically closed, this torsor is trivial, and there exists $u \in Z(K)$ such that $Z = uJ_{a/K}$. In particular, $y = u\gamma$ for some $\gamma \in J_a(K(y))$, and $a = \partial\ell n_G(u\gamma) = u \bullet \eta$ where $\eta = \partial\ell n_G \gamma \in LJ_a(K(y))$. Therefore, $u^{-1} \bullet a = \eta \in LJ_a$ (which must then be K-rational), and the $G(K)$-orbit of a does meet $LJ_a(K)$.

The above proof shows that up to $G(C)$-conjugacy, $\mathcal{H}$ contains a *unique* minimal element (we give a more direct proof of this fact in the commutative case in §3 below). It does not truly provide an algorithm to compute J_a, but it certainly gives upper bounds, which may suffice if one knows in advance enough elements of Γ_a. And of course, a situation where we can conclude (still with K algebraically closed) is given by

Corollary : *Assume that $a \in LG(K)$ is non degenerate : for any proper algebraic subgroup $H \subset G$, the $G(K)$-orbit of a does not meet LH. Then $tr.deg.K(y)/K = dimG$.*

When G is commutative, the action of G on LG reads as : $g \bullet a = a + \partial\ell n_G g$, the set $\partial\ell n_G(G(K))$ is a subgroup of LG, and the theorem determines the Galois group as the smallest algebraic subgroup H of G such that a lies in $\partial\ell n_G(G(K)) + LH$. We derive :

1) - *Kolchin's theorem on* $\mathbf{G}_m^n$: let $y = (y_1, ..., y_n) \in \hat{K}^{*n}$ such that $\partial y_i/y_i = a_i \in K$ for all $i = 1, ..., n$. Assume that $\forall (m_1, .., m_n) \in \mathbb{Z}^n \setminus 0, m_1 a_1 + ... + m_n a_n \notin \partial\ell n(K^*)$. Then, $tr.deg.K(y)/K = n$.

Now, if $tr.deg(K/C) = 1$ and if each a_i is itself of the type ∂x_i for some $x_i \in K$, the condition on the a_i's simply becomes : $\forall (m_1, .., m_n) \in \mathbb{Z}^n \setminus 0, m_1 x_1 + ... + m_n x_n \notin C$. Indeed,

$$\partial\ell n(K^*) \cap \partial(K) = \{0\} , \quad \text{and } K \cap \partial^{-1}(0) = C. \qquad (1)$$

In the setting of §1.2,

> *we have therefore proved Case (E) of Ax's theorem in the case of tori.*

2) - *Ostrowski theorem on* $\mathbb{G}_a^n$: let $x = (x_1, ..., x_n) \in \hat{K}^n$ such that $\partial x_i = b_i \in K$ for all $i = 1, ..., n$. Assume that $\forall (\mu_1, .., \mu_n) \in C^n \setminus 0, \mu_1 b_1 + ... + \mu_n b_n \notin \partial(K)$. Then, $tr.deg.K(x)/K = n$.

Now, if $tr.deg.(K/C) = 1$ and if each b_i is itself of the type $\partial\ell n y_i$ for some $y_i \in K^*$, the condition on the b_i's simply becomes : $\forall (m_1, .., m_n) \in \mathbb{Z}^n \setminus 0, y_1^{m_1}...y_n^{m_n} \notin C^*$, i.e. $m_1 x_1 + ... + m_n x_n \notin C$. Indeed, the natural map

$$\iota : C \otimes_{\mathbb{Z}} \partial\ell n(K^*) \to K \text{ is injective }, Im(\iota) \cap \partial(K) = \{0\} , \text{ and } K^* \cap \partial\ell n^{-1}(0) = C^*. \qquad (2)$$

In the setting of §1.2,

> *we have therefore proved Case (L) of Ax's theorem in the case of tori.*

We refer to [9], §6, for a similar treatment of Cases (E) and (L) of Ax's theorem on general commutative algebraic groups G defined over C. The required analogues $(1^*), (2^*)$ of the displayed Formulae $(1),(2)$ are discussed in §4.1 below.

3 D-groups and Pillay's theory

3.1   General setting

In a series of papers started in 1997 [26], A. Pillay extended Kolchin's theory to the context of algebraic D-groups. These groups, which, after the work of P. Cassidy (see [13], §2), had been considered by Buium [12] with an eye towards the functional analogue of the Manin-Mumford and Mordell-Lang conjectures, are defined over a differential field (K, ∂), and usually not over its field of constants C (but even then, new phenomena can occur, cf. §3.2). For the sake of simplicity, we will restrict the presentation of Pillay's theory to the case of *commutative D-groups*, with an additive notation. Again, we suppose that $K^{\partial} = C$ is algebraically closed of characteristic 0, and we fix a differential closure $\hat{K}$ of K.

So, let G/K be a connected commutative algebraic group over K. We **assume** that ∂ extends to a derivation $D_{\partial} : \mathcal{O}_{G_K} \to \mathcal{O}_{G_K}$ of the structure sheaf $\mathcal{O}_{G_K}$ compatible with its structure of Hopf algebra. (The derivation D_{∂}^0 considered in §2 when G is defined over C did satisfy this property.) We **fix** such an extension D_{∂}, and say that G, tacitly equipped with D_{∂}, *is a D-group*. We can then define the logarithmic derivative of G in exactly the same way as before : for any $y \in G(K)$, we have the two derivations $D_{\partial,y}$, δy of the local ring $\mathcal{O}_{G,y}$, and $\partial \ell n_G(y) \in LG(K)$ is their difference $\delta y - D_{\partial,y}$, pulled back from $T_y G$ to LG via the canonical splitting of the tangent bundle $TG \simeq G \times LG$. Again, this extends over any differential extension $\mathcal{K}$ of K, and we set $G^{\partial}(\mathcal{K}) = \{y \in G(\mathcal{K}), \partial \ell n_G(y) = 0\}$. By [28], 2.5, $\partial \ell n_G$ is surjective at the level of $\hat{K}$-points.

Given $a \in LG(K)$, Pillay's theory studies the differential extension $K(y)/K$, where y is a solution in $G(\hat{K})$ of

$$\partial \ell n_G(y) = a,$$

and its Galois group $\Gamma_a = Aut_{\partial}(K(y)/K)$. To avoid "new constants", we now request that

$$G \text{ is "K-large"} : G^{\partial}(\hat{K}) = G^{\partial}(K);$$

Notice that this hypothesis was automatically satisfied in the case of §2.

The fact that D_{∂} respects the group structure of G is again reflected by a cocycle identity which, in our commutative situation, becomes :

$$\forall u, v \in G, \partial \ell n_G(uv) = \partial \ell n_G u + \partial \ell n_G v.$$

Consequently,

i) two solutions differ by an element of $G^{\partial}(\hat{K}) = G^{\partial}(K)$, hence define the same extension of K, and the Galois group Γ_a comes with a canonical embedding ξ into $G^{\partial}(K)$:

$$\forall \sigma \in \Gamma_a \ , \ \sigma y - y := \xi(\sigma) \in G^{\partial}(\hat{K}) = G^{\partial}(K);$$

ii) the extension $K(y)/K$ and its Galois group depend only on the image of a in the group

$$Coker(\partial \ell n_G, K) := LG(K)/\partial \ell n_G(G(K)).$$

We extract from the main theorem of Pillay's theory (see [28], §3) :

Theorem 3.1. *: assume that the D-group G is K-large. Then :*
    i) $Im(\xi) = N_a^\partial(K)$, where N_a/K is a D-subgroup of G/K;
    ii) there is a "Galois correspondence" : for instance, $K(y)^{N_a^\partial(K)} = K$;
    iii) $tr.deg.(K(y)/K) = dim(N_a)$.

Here, a D-subgroup H of G is an algebraic subgroup of G whose ideal sheaf $\mathcal{I}_H \subset \mathcal{O}_G$ is stable under D_∂, and the D-structure of H is given by the derivation induced by D_∂ on $\mathcal{O}_H$. In particular, $(\partial\ell n_G)_{|H} = \partial\ell n_H$, H is K-large, and $\overline{G} = G/H$ acquires a natural structure of D-group, which is K-large as well. (Notice that the sequence $0 \to H^\partial(\hat{K}) \to G^\partial(\hat{K}) \to \overline{G}^\partial(\hat{K}) \to 0$ is exact, since $\partial\ell n_H$ is surjective on $\hat{K}$-points.) Still assuming K-largeness, we have :

Theorem 3.2. *: the identity component of N_a is the smallest D-subgroup H/K of G/K such that a lies in $LH + \mathbb{Q}.\partial\ell n_G G(K)$.*

Suppose for a moment that K is algebraically closed. Then N_a is connected and $\partial\ell n_G G(K)$ is already a $\mathbb{Q}$-vector space, so the theorem acquires the same form as Theorem 2.2. Here, the commutativity assumption allows to drop any requirement on K.

Proof : as in §2, consider the set $\mathcal{H}$ formed by all the D-groups H/K of G/K such that $a \in LH(K) + \mathbb{Q}.\partial\ell n_G G(K)$. Then,
    i) for any H in $\mathcal{H}$, the equation corresponding to some multiple of a has a solution in $H(\hat{K})$, so the connected component of $Im(\xi)$ lies in H.
    ii) the image $\overline{y}$ of y in $\overline{G} = G/N_a$ is stable under Γ_a, hence K rational. Analysing commutative algebraic groups, we see that $G(K)$ projects onto a subgroup of finite index of $\overline{G}(K)$, so $m\,y = u + \gamma$ for some $m > 0, u \in G(K), \gamma \in N_a(K(y))$, and $m\,a = \partial\ell n_G(u) + \eta$, where $\eta = \partial\ell n_{N_a}(\gamma)$ is a (necessarily K-rational) point of LN_a.

As promised in §2 , we now give a direct proof (independent of Galois theory) that $\mathcal{H}$ admits a unique minimal element. This fact depends crucially on the hypothesis of K-largeness of G, which implies that for any D-subgroup H of G, $LH(K) \cap \partial\ell n_G(G(K)) = \partial\ell n_H(H(K))$, i.e. that the natural map $Coker(\partial\ell n_H, K) \to Coker(\partial\ell n_G, K)$ is injective. The claim then easily follows. When G is not K-large, the snake lemma merely says that the sequence

$$G^\partial(K)/H^\partial(K) \hookrightarrow (G/H)^\partial(K) \to Coker(\partial\ell n_H, K) \to Coker(\partial\ell n_G, K)$$

is exact.

Anyway, the outcome in the K-large case is exactly the same as in the constant one. For instance, assume that a is *non degenerate* : *for any proper algebraic D-subgroup $H \subset G$, $a + \mathbb{Q}.\partial\ell n_G G(K)$ does not meet $LH(K)$.* Then $N_a = G$, i.e. Kolchin-Ostrowski still holds true. In §4, we will find conditions on a ensuring its non-degeneracy, and thereby obtain "non-constant" analogues of the (E) and (L) statements. But more work is required before we get there, because the K-largeness hypothesis is seldom satisfied in these non-constant situations. A good example (and a way to overcome the difficulty) is given by the following case.

3.2 The case of D-modules

Let $B \in \mathfrak{gl}_n(K)$. In §2, we used Kolchin's view-point to describe the Picard-Vessiot extension attached to the homogeneous equation $\partial y = By, y \in \hat{K}^n$. Given $a \in K^n$, we now study the inhomogeneous equation

$$\partial y = By + a.$$

Setting $A = \begin{pmatrix} B & a \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_{n+1}(K)$, and $Y = \begin{pmatrix} y \\ 1 \end{pmatrix} \in \hat{K}^{n+1}$, this equation is essentially the same as $\partial Y = AY$, and can indeed be treated by pure Picard-Vessiot means, as was done in [5], [7]. But we will now describe it from the point of view of Pillay's theory, starting with a vectorial group $G := V \simeq K^n$ over K.

Given a basis of V over K, the choice of derivation D_∂ on the affine ring $K[Sym(V^*)] \simeq K[X_1, ..., X_n]$ of V inducing ∂ on K and respecting the group structure of V is tantamount to the choice of a matrix $B \in Mat_{nn}(K)$ such that $D_\partial(X_1, ..., X_n) = (X_1, ..., X_n)B$, i.e. of a D-module structure on V. The associated logarithmic derivative, denoted in this vectorial case[4] by

$$\nabla_V = \partial\ell n_V : V \to LV \simeq V \ ,$$

is then given by

$$V(K) \simeq K^n \ni y \mapsto \partial\ell n_V(y) = \partial y - By \in LV(K) \simeq K^n,$$

where $\partial \ {}^t(y_1, ..., y_n) = \ {}^t(\partial y_1, ..., \partial y_n)$. Then,

$$V^\partial(\hat{K}) = \{y \in \hat{K}^n, \partial y = By\}$$

(which we will abusively denote by V^∂) is the C-space of solutions of a linear equation, so that V is usually not K-large. In order to apply Pillay's theory, we extend K to the Picard-Vessiot extension $\mathbb{K} := K_V := K(V^\partial(\hat{K}))$. By definition, $V_\mathbb{K} = V \otimes_K \mathbb{K}$ is now clearly $\mathbb{K}$-large.

Given a K-rational point $a \in LV \simeq V$, we consider the equation

$$\partial\ell n_V y = a \ , \ i.e. \ \partial y - By = a.$$

By Theorem 3.2 and the $\mathbb{Q}$-divisibility of rational points in vectorial groups, its Galois group $\Gamma_a = Aut_\partial(\mathbb{K}(y)/\mathbb{K})$ is a C-subspace of V^∂ of the form $N_a^\partial = N_a^\partial(\hat{K}) = (N_a)^\partial(\mathbb{K})$, where $N_a \simeq LN_a$ is the smallest $\mathbb{K}$-vector subspace of $V_\mathbb{K}$ stable under $\partial\ell n_V$, such that $a \in \partial\ell n_V(V(\mathbb{K})) + N_a$. Let us try to turn this into a more manageable description.

A first remark is that N_a is actually defined over K. Indeed, let $\mathcal{H}_\mathbb{K}$ be the set of all $\mathbb{K}$-subspaces of $V_\mathbb{K}$ satisfying the property above. Since a is defined over K, the set $\mathcal{H}_\mathbb{K}$ is

---

[4] In this paragraph, we keep to the notation $\partial\ell n_V$ to ease the comparison with the previous cases, but we will state the final result with ∇_V to prepare for the study of the D-module $L\tilde{A}$ in §4.2.

stable under the action of $J(C) := Aut_\partial(\mathbb{K}/K)$, and its unique minimal element N_a too is stable under $J(C)$. By Picard-Vessiot theory, N_a must then be defined over K. Another proof of this fact will be given presently, cf. Proof (i) below.

In these conditions, it is tempting to consider the set $\mathcal{H}_K$ of all K-vector subspaces N of V stable under $\partial\ell n_V$, such that $a \in \partial\ell n_V(V(K)) + N(K)$. Since $\mathcal{H}_K \subset \mathcal{H}_\mathbb{K}$, the Galois group N_a is contained in all its elements. But in general, N_a **will not belong** to $\mathcal{H}_K$, and a priori, one cannot even speak of the smallest element of $\mathcal{H}_K$. See [7] for some counterexamples.

As noticed in [5], [7], there is however a case where N_a does belong to $\mathcal{H}_K$, and therefore becomes its smallest element, viz. when the differential system $\partial y = By$ can be split over K into a direct sum of irreducible systems, i.e. when the D-module structure on V attached to D_∂ is semi-simple over K. Then, any D-submodule H/K of V admits a D-submodule complement over K, and this yields the injectivity of $Coker(\partial\ell n_H, K) \to Coker(\partial\ell n_V, K)$. So we can already say that $\mathcal{H}_K$ has a unique minimal element. Furthermore, as shown in Proof (ii) below, given any quotient $\overline{V} = V/H$ of such a semi-simple V, the natural map

$$Coker(\partial\ell n_{\overline{V}}, K) \to Coker(\partial\ell n_{\overline{V}}, \mathbb{K})$$

is injective. So, for H in $\mathcal{H}_\mathbb{K}$, the image $\overline{a}$ of a in $L\overline{V}(K)$ lies in $\partial\ell n_{\overline{V}}(\overline{V}(\mathbb{K}))$ only if it already lies in $\partial\ell n_{\overline{V}}(\overline{V}(K))$, in which case $a + \partial\ell n_V(V(K))$ meets LH. Consequently, any H in $\mathcal{H}_\mathbb{K}$ contains an element of $\mathcal{H}_K$, the minimal elements of $\mathcal{H}_\mathbb{K}$ and of $\mathcal{H}_K$ coincide, and we derive as in [5], [7] (see also [16], Lemme 2.2.10 and Thm. 2.2.5) :

Theorem 3.3. : *assume that the D-module* $(V/K, \nabla_V)$ *is semi-simple. Then, the Galois group* $Aut_\partial(\mathbb{K}(y)/\mathbb{K})$ *is* N_a^∂, *where* N_a *is the smallest D-submodule* N *of* V *defined over* K *such that* a *lies in* $\nabla_V(V(K)) + N$.

Descent proofs (as promised above) : (i) Firstly, the other proof that under no hypothesis on V, N_a is always defined over K. Consider the tower of extensions, all Picard-Vessiot *over* K in view of the system $\partial Y = AY$ mentioned at the beginning :

$$
\begin{array}{cccccc}
 & \mathbb{K}(y) & & \xi & & \\
 & | & \}N_a^\partial & \hookrightarrow & V^\partial & \\
\Gamma\{ & \mathbb{K} & & \rho & & \\
 & | & \}J & \hookrightarrow & GL(V^\partial) & \\
 & K & & & &
\end{array}
$$

The full Galois group $Aut_\partial(\mathbb{K}(y)/K)$ is of the form $\Gamma(C)$ for some algebraic subgroup Γ/C of GL_{n+1}. Since $\mathbb{K}/K$ is P-V, N_a^∂ is a normal subgroup of Γ, with quotient J naturally acting on V^∂ by a C-rational representation $\rho : J \to GL(V^\partial) \simeq GL_{n/C}$ of the type described in §2. Since N_a is abelian, J also acts on N_a^∂ by conjugation, and a familiar computation from affine geometry shows that the homomorphism ξ commutes with these actions of J :

$$\forall \sigma \in N_a^\partial, \tau \in J \ , \ \xi(\tau\sigma\tau^{-1}) = \rho(\tau)\big(\xi(\sigma)\big).$$

Therefore, ξ identifies N_a^∂ with a J-submodule of V^∂, and by the standard P-V theory of §2, N_a must then be a D-submodule of V defined over K.

The J-equivariance property on which this proof is based can be viewed as a first instance of the arguments of Galois descent alluded to in the introduction of the paper. Indeed, ξ is the restriction to N_a^∂ of a C-rational cocycle $\hat{\xi}_a : \Gamma \to V^\partial$, defined by the same formula $\hat{\xi}_a(\tau) = \tau y - y$, whose class in $H^1(\Gamma, V^\partial)$ depends only on the image $\tilde{a}$ of a in $Coker(\partial \ell n_V, K)$. More precisely, the map

$$\Xi_K : Coker(\partial \ell n_V, K) \to H^1(\Gamma, V^\partial) : \tilde{a} \mapsto \Xi_K(\tilde{a}) = \text{ class of } \hat{\xi}_a$$

is a group embedding. Now, it is a well-known feature of group cohomology that $\hat{\xi}_a$, restricted to any normal subgroup N of Γ, provides a Γ/N-invariant cohomology class in $H^1(N, V^\partial)$, cf. [32], I.2.6.b.

(ii) We now assume that V is semi-simple. Any quotient $\overline{V}$ is then also semi-simple, and we must prove that the kernel of the map $Coker(\partial \ell n_{\overline{V}}, K) \to Coker(\partial \ell n_{\overline{V}}, \mathbb{K})$ vanishes. But via the map Ξ_K attached to $\overline{V}$ and the analogous map $\Xi_{\mathbb{K}}$ at the level of $\mathbb{K}$, this kernel injects into the kernel of the restriction map

$$H^1(\Gamma, \overline{V}^\partial) \to H^1(N_a^\partial, \overline{V}^\partial)^J = Hom_J(N_a^\partial, \overline{V}^\partial).$$

The latter kernel identifies with $H^1(J, \overline{V}^\partial)$ by the inflation map [32], *loc. cit.*. Finally, the faithful representation V^∂ of J is completely reducible, so J is a reductive group and $H^1(J, \overline{V}^\partial) = 0$. (Notice that all cocycles appearing here are continuous for the Zariski topology. Even when $J = PSL_2$ and $C = \mathbb{C}$, I do not know if the abstract cohomology groups of the abstract group $J(C)$ would also vanish.)

In the next section, we show that a similar descent argument applies to various D-groups attached to abelian varieties. Obstructions occur in the case of semi-abelian varieties, and we refer to [1], [10], [9] for examples and counterexamples they lead to.

4   Abelian varieties

From now on, we fix a smooth irreducible curve S over the field $\mathbb{C}$ of complex numbers, and a non-zero vector field $\partial \in H^0(S, TS)$ on S, which we identify with a derivation on the function field $K = \mathbb{C}(S)$, with $K^\partial = \mathbb{C}$. We denote by $\overline{K} \subset \hat{K}$ its algebraic closure. In the sequel, we may tacitly restrict S to a non empty open subset, or more generally to a finite cover, and still denote by S the resulting curve. We consider an abelian variety A/K, extended to an abelian scheme $\mathcal{A}/S$, with relative Lie algebra $L\mathcal{A}/S$. We assume that the largest abelian variety $A_0/\mathbb{C}$, isomorphic over $\overline{K}$ to an abelian subvariety of A, is embedded in A, and call it the constant part, or $\mathbb{C}$-trace, of A.

The exponential maps of the various fibers of $\mathcal{A}/S$ provide a morphism $exp_A : L\mathcal{A}^{an} \to \mathcal{A}^{an}$ of analytic sheaves over the Riemann surface S^{an}. For a local section $\overline{x}$ of $L\mathcal{A}^{an}$, we denote by $\overline{y} = exp_A(\overline{x})$ its image in $\mathcal{A}^{an}$. We say that

$\overline{x} \in LA$ *is non-degenerate* if for any proper abelian subvariety B of A, $\overline{x}$ does not lie in $LB + LA_0(\mathbb{C})$;

$\overline{y} \in A$ *is non-degenerate* if for any proper abelian subvariety B of A, no non-zero multiple of $\overline{y}$ lies in $B + A_0(\mathbb{C})$. Then:

Theorem 4.1. *: let A/K be an abelian variety, with $\mathbb{C}$-trace A_0, and let $\overline{x} \in LA$, with image $\overline{y} = exp_A(\overline{x}) \in A$. Assume that*

(L) $\overline{y} \in A$ is K-rational and non-degenerate; then, $tr.deg.K(\overline{x})/K = dimA$;

(E) $\overline{x} \in LA$ is K-rational and non-degenerate; then, $tr.deg.K(\overline{y})/K = dimA$.

In case (L), one cannot replace the non-degeneracy hypothesis on $\overline{y}$ by the weaker one on $\overline{x}$, because the periods of exp_A may satisfy non linear algebraic relations[5]. Similarly, under the mere hypothesis that $\overline{x}$ is non-degenerate, the analogue of the Ax-Schanuel theorem (that $tr.deg.(K(\overline{x}, \overline{y})/K) \geq dimA$) would not hold, but one can conjecture that it does as soon as $\overline{y}$ is non-degenerate.

The theorem reflects the existence of large Galois groups attached to $A/K, \overline{x}, \overline{y}$. But before we can speak of Galois groups, we need a D-group.

4.1 From abelian varieties to D-groups

In general, given a commutative algebraic goup G/K, the set of all possible extensions of ∂ to a derivation D_∂ on $\mathcal{O}_G$ is empty, or is an affine space under the space of sections of the tangent bundle TG. It therefore corresponds to a class $\kappa(G/K, \partial)$ in $H^1(G, TG)$ and is non-empty only if this class vanishes. When $G = A$ is proper, $\kappa(A/K, \partial)$ is the image of ∂ under the Kodaira–Spencer map attached to $\mathcal{A}/S$ at the generic point of S, and is known to vanish if and only if $\mathcal{A}/S$ is isoconstant, i.e. $A \simeq A_0$ is isomorphic over $\overline{K}$ to an abelian variety defined over $\mathbb{C}$. So, *a non-isoconstant abelian variety A/K admits no D-group structure.*

To overcome this difficulty (see [12]), we introduce the universal extension $\tilde{A}/K$ of A. This is, in the category of algebraic goups, an extension

$$0 \to W_A \to \tilde{A} \to A \to 0$$

of A by a vectorial group W_A/K, dual to $H^1(A, \mathcal{O}_A)$. In particular, $\tilde{A}$ has dimension $2dimA$, and in fact, its Lie algebra $L\tilde{A}$ is dual to the de Rham cohomology group $H^1_{dR}(A/K)$ of A/K. Now, the latter admits a natural connection (Gauss-Manin), whose dual $\nabla_{L\tilde{A}}$, contracted with ∂, provides a *D-module structure on $L\tilde{A}$*. Finally, $\nabla_{L\tilde{A}}$ can be "integrated" into a *D-group structure on $\tilde{A}$*, which is actually unique. We point out for later use that in the usual identification of a vectorial group with its Lie algebra, *the restrictions of $\nabla_{L\tilde{A}}$ and $\partial\ell n_{\tilde{A}}$ to $W_A \simeq LW_A$ coincide*, so that statements comparing their values often factor through the quotients A, LA. But unless A/K is isoconstant, W_A is not a D-subgroup or

---

[5] This problem can be addressed through the study of the group J appearing below. Applications of the type discussed in [22] may require such sharpenings.

a D-submodule of $\tilde{A}, L\tilde{A}$. Another property to keep in mind in what follows is that any algebraic subgroup of $\tilde{A}$ projecting onto A must fill up $\tilde{A}$; by Poincaré and the fonctoriality of universal extensions, this implies that any connected D-subgroup of $\tilde{A}$ is of the form $\tilde{B} + W$, where B is an abelian subvariety of A and W is a D-submodule of $L\tilde{A}$ contained in W_A.

We now give an analytic description of the logarithmic derivative $\partial\ell n_{\tilde{A}}$ of the D-group structure of $\tilde{A}$, in the style of §1.2. Extend $\tilde{A}/K$ to a group scheme $\tilde{\mathcal{A}}/S$, and consider the exact sequence of analytic sheaves over S^{an} given by the exponential morphism :

$$0 \to \Omega_{\tilde{\mathcal{A}}} \to L\tilde{\mathcal{A}}^{an} \to^{exp_{\tilde{A}}} \tilde{\mathcal{A}}^{an} \to 0.$$

Its kernel $\Omega_{\tilde{\mathcal{A}}}$ is the $\mathbb{Z}_{S^{an}}$- local system of periods of $\tilde{\mathcal{A}}$, and by the analytic definition of the Gauss-Manin connection, these generate over $\mathbb{C}_{S^{an}}$ the space $(L\tilde{A})^{\partial}$ of horizontal sections of the connexion $\nabla_{L\tilde{A}}$. Therefore, given $y \in \tilde{A}(K)$, extended to a section $y(z)$ of $\tilde{\mathcal{A}}/S$, and any local choice $\ell n_{\tilde{A}} y(z)$ of an inverse of y under $exp_{\tilde{A}}$, $\nabla_{L\tilde{A}} \circ \ell n_{\tilde{A}} y(z)$ extends to a well-defined section of $L\tilde{A}$ over S^{an}, actually with moderate growth at infinity, hence over S. Finally, as shown in [10], Appendix H, the resulting point in $L\tilde{A}(K)$ coincides with the logarithmic derivative $\partial\ell n_{\tilde{A}}(y)$ of y. In brief, $\partial\ell n_{\tilde{A}} = \nabla_{L\tilde{A}} \circ \ell n_{\tilde{A}}$, and the analytic relation $y = exp_{\tilde{A}}(x)$ implies

$$\partial\ell n_{\tilde{A}} y = \nabla_{L\tilde{A}} x.$$

Conversely, when x and y have an analytic meaning, this differential relation is equivalent to the existence of a local horizontal section $\xi \in L\tilde{A}, \nabla_{L\tilde{A}}(\xi) = 0$, of $\nabla_{L\tilde{A}}$ such that $y = exp_{\tilde{A}}(x - \xi)$. But contrary to the situation described in §1.2, the point ξ is not necessarily in the constant part $L\tilde{A}_0(\mathbb{C})$; neither is $\eta = exp_{\tilde{A}}(\xi)$, which, in the elliptic case, can be computed in terms of solutions of Picard type of certain Painlevé VI equations.

Theorem 4.2. : let A/K be an abelian variety, with $\mathbb{C}$-trace A_0, and let $x \in L\tilde{A}(\hat{K}), y \in \tilde{A}(\hat{K})$ satisfy the differential relation $\partial\ell n_{\tilde{A}} y = \nabla_{L\tilde{A}} x$. Assume that

(L) = [1], Theorem 3 : $y \in \tilde{A}(K)$, and its image $\overline{y}$ on $A(K)$ is non-degenerate; then $tr.deg.K(x)/K = 2\,dimA$;

(E) = [10], Theorem 1.4 : $x \in L\tilde{A}(K)$, and its image $\overline{x} \in LA(K)$ is non-degenerate; then, $tr.deg.K(y)/K = 2\,dimA$.

The theorem announced at the beginning of the section easily follow, by the functoriality of the exponential morphisms, the fact that any K-rational point of LA, A can be lifted to a K-rational point of $L\tilde{A}, \tilde{A}$, and a dimension count. And the present theorem will clearly follow from Theorems 4.3 and 4.4 below, since its non-degeneracy hypotheses mean, in their notations, that $B = A$, and transcendence degrees, controled by the dimension of Galois groups in view of Theorem 3.1.iii, cannot decrease when we go from $\mathbb{K}$ to K. Finally, we can replace K by its algebraic closure in these statements, so, we henceforth assume that $K = \overline{K}$ is an *algebraically closed field* of transcendence degree 1 over $\mathbb{C}$.

We now collect some facts about the D-groups $\tilde{A}, L\tilde{A}$, in particular with respect to their (non)-K-largeness, referring to [10] and [9] for their proofs. Suffices to say here that

14

those concerning $L\tilde{A}$ are based on deep theorems on variations of Hodge structures, cf. [14], while those concerning $\tilde{A}$ depend on a combination of the latter with similarly deep theorems of model theory, cf. [21], and with Manin's theorem below.

Fact 1 : $(L\tilde{A})^\partial(K) = L\tilde{A}_0(\mathbb{C})$. So, $L\tilde{A}$ is K-large only if A/K is isoconstant. In general, we denote by $\mathbb{K} = K((L\tilde{A})^\partial)$ the extension generated by the solutions of $\nabla_{L\tilde{A}}(\xi) = 0$. This is a Picard-Vessiot extension of K, and we call J its Galois group.

Fact 2 : the D-group $\tilde{A}/K$ is often K-large (for instance, as soon as the Kodaira-Spencer map $\kappa(A/K, \partial)$ has maximal rank), but not always. However, the field $K(\tilde{A}^\partial)$ generated over K by the elements of $\tilde{A}^\partial(\hat{K})$ is always contained in the above Picard-Vessiot extension $\mathbb{K}$ of K. So, viewed over $\mathbb{K}$, both D-groups $L\tilde{A}$ (by definition) and $\tilde{A}$ are $\mathbb{K}$-large.

Fact 3 : $L\tilde{A}$ is a semi-simple D-module over K. So, the (connected) algebraic group J is reductive, and actually, J is even semi-simple ($J = [J, J]$).

Fact 4 : let U_A be the maximal D-subgroup of $L\tilde{A}$ contained in W_A. Then, the quotient $\tilde{A}/U_A$ is K-large.

We will also need the analogues of the displayed formula (1) and its sharper version (2) given at the end of §2 in the case of tori. These are provided by sharper and sharper versions of Manin's kernel theorem, whose simplest case reads as follows : $\partial\ell n_{\tilde{A}}(\tilde{A}(K)) \cap \nabla_{L\tilde{A}}(L\tilde{A}(K)) = \nabla_{L\tilde{A}}(W_A(K))$, and more precisely :

$$y \in \tilde{A}(K), \partial\ell n_{\tilde{A}}(y) \in \nabla_{L\tilde{A}}(L\tilde{A}(K)) \Rightarrow \text{a multiple of } \overline{y} \text{ lies in } A_0(\mathbb{C}). \qquad (1')$$

(By "a multiple", we mean a multiple by a *non-zero* integer; this convention will be used throughout the rest of the text.) This was extended by Chai to quotients of $\tilde{A}$ by D-subgroups of $\tilde{A}$, as follows : let H be a D-subgroup of $\tilde{A}$, with image $\overline{H}$ in A; then

$$y \in \tilde{A}(K), \partial\ell n_{\tilde{A}}(y) \in \nabla_{L\tilde{A}}(L\tilde{A}(K)) + LH \Rightarrow \text{a multiple of } \overline{y} \text{ lies in } A_0(\mathbb{C}) + \overline{H}, \quad (1^*)$$

and further extended to quotients of $L\tilde{A}$ by D-submodules, as follows : assume that A/K is a *simple* abelian variety, and let N be any proper D-submodule of $L\tilde{A}$; then

$$y \in \tilde{A}(K), \partial\ell n_{\tilde{A}}(y) \in \nabla_{L\tilde{A}}(L\tilde{A}(K)) + N \Rightarrow \text{a multiple of } \overline{y} \text{ lies in } A_0(\mathbb{C}) \quad (2').$$

See [8], §3, for a proof of (2'), based on André's normality theorem [1], Thm. 2, from which yet another sharpening can be deduced, as follows. Recall the definition of non-degeneracy of $\overline{y} \in A$ given at the beginning of the section, let A/K be any abelian variety, and let N be any proper D-submodule of $L\tilde{A}$; then

$$y \in \tilde{A}(K), \partial\ell n_{\tilde{A}}(y) \in \nabla_{L\tilde{A}}(L\tilde{A}(K)) + N \Rightarrow \overline{y} \text{ is degenerate in } A. \qquad (2^*)$$

Indeed, the semi-simplicity of $L\tilde{A}$ allows us to speak of the smallest D-submodule N_0 satisfying this property. By the normality theorem, there exists a D-subgroup H of $\tilde{A}$ such that $N_0 = LH$, and since N is proper, $\overline{H} \neq A$. The conclusion now follows from (1^*).

The reader will notice that Formulae (1) and (2) of §2 on a torus $\mathbb{G}_m^n$ are the exact analogues of the abelian Formulae (1^*) and (2^*).

4.2 Abelian logarithms

Let A/K be an abelian variety, with $\mathbb{C}$-trace A_0, and let $y \in \tilde{A}(K)$ with image $\overline{y} \in A(K)$. Recall the notation $\mathbb{K} = K((L\tilde{A})^\partial)$ from Fact 1, the assumption that K is algebraically closed and the convention that a multiple means a multiple by a non-zero integer.

We consider the differential system in the unknown $x \in L\tilde{A}(\hat{K})$:

$$\nabla_{L\tilde{A}} x = a \text{ where } a = \partial \ell n_{\tilde{A}} y \in L\tilde{A}(K). \qquad (*).$$

Its Galois group over $\mathbb{K}$ is well-defined, since the D-module $L\tilde{A}$ is $\mathbb{K}$-large, and is the normal subgroup N_a^∂ of Γ in the diagram of §3.2, which here reads :

$$
\begin{array}{ccccc}
\mathbb{K}(x) & & \xi & & \\
| & \}N_a^\partial & \hookrightarrow & (L\tilde{A})^\partial & \\
\Gamma\{ \quad \mathbb{K} & & \rho & & \\
| & \}J & \hookrightarrow & GL((L\tilde{A})^\partial) & \\
K & , & & &
\end{array}
$$

where $\xi(\sigma) = \sigma(x) - x$, and J acts $\mathbb{C}$-rationally on $(L\tilde{A})^\partial$.

Theorem 4.3. : *let $y \in \tilde{A}(K)$, and let B be the smallest abelian subvariety of A such that some multiple of $\overline{y}$ lies in $B + A_0(\mathbb{C})$. Then, the Galois group $Aut_\partial(\mathbb{K}(x)/\mathbb{K})$ of $(*)$ over $\mathbb{K}$ is equal to $(L\tilde{B})^\partial$, and in particular, $tr.deg.(\mathbb{K}(x)/\mathbb{K}) = 2dimB$.*

Proof : by Fact 3 and Theorem 3.3, this Galois group is N_a^∂, where N_a is the smallest D-submodule N/K of $L\tilde{A}$ such that $a := \partial \ell n_{\tilde{A}}(y) \in N + \nabla_{L\tilde{A}}(L\tilde{A}(K))$.

Fix a lift $y' \in \tilde{B}(K)$ of $\overline{y}$. Then, $u = y - y'$ lies in $W_A(K)$, where $\nabla_{L\tilde{A}}$ and $\partial \ell n_{\tilde{A}}$ coincide. Therefore, $\partial \ell n_{\tilde{A}}(y) = -\partial \ell n_{\tilde{B}}(y') + \nabla_{L\tilde{A}}(u) \in L\tilde{B} + \nabla_{L\tilde{A}}(L\tilde{A}(K))$, and N_a is contained in $L\tilde{B}$.

To prove the converse inclusion, we set $x' = x - u$, and note that by Fact 1, the Galois group of $(*)$ over $\mathbb{K}$ is the same as that of $\nabla_{L\tilde{B}} x' = a'$ where $a' = \partial \ell n_{\tilde{B}} y' \in L\tilde{B}(K)$. Furthermore, the Picard-Vessiot extensions $\mathbb{K}$ and $\mathbb{K}_B(x')$ of $\mathbb{K}_B = K((L\tilde{B})^\partial)$ are linearly disjoint, since a normal subgroup of the reductive group J cannot admit a non-zero vectorial quotient. So N_a is in fact the smallest D-submodule N/K of $L\tilde{B}$ such that $\partial \ell n_{\tilde{B}}(y') \in N + \nabla_{L\tilde{B}}(L\tilde{B}(K))$. Since the image $\overline{y}'$ of y' in the abelian variety B is by definition non-degenerate, the strong form (2^*) of Manin–Chai, applied to $\tilde{B}$, now implies that N_a fills up $L\tilde{B}$.

4.3 Abelian exponentials

Let A/K be an abelian variety, with $\mathbb{C}$-trace A_0. Recall the notation $\mathbb{K} = K((L\tilde{A})^\partial)$, and let now $x \in L\tilde{A}(K)$ with image $\overline{x} \in LA(K)$. We consider the differential system in the unknown $y \in \tilde{A}(\hat{K})$:

$$\partial \ell n_{\tilde{A}} y = a \text{ where } a = \nabla_{L\tilde{A}} x \in L\tilde{A}(K). \qquad (**).$$

Its Galois group over $\mathbb{K}$ is well-defined, since the D-group $\tilde{A}$ is $\mathbb{K}$-large, and is represented by the top level of the tower of extensions :

$$
\begin{array}{ccccc}
\mathbb{K}(y) & & \xi & & \\
| & \}H_a^\partial & \hookrightarrow & & \tilde{A}^\partial \\
\text{no } \Gamma \ ! \quad \mathbb{K} & & \rho & & \\
| & \}J(C) & \to & Aut(\tilde{A}^\partial) \\
K & & , & &
\end{array}
$$

where $\xi(\sigma) = \sigma(y) - y$. Note that $J(C)$ acts as an abstract group on $\tilde{A}^\partial(\mathbb{K}) = \tilde{A}^\partial(\hat{K})$, which we abbreviate as $\tilde{A}^\partial$.

Theorem 4.4. : *let $x \in L\tilde{A}(K)$, and let B be the smallest abelian subvariety of A such that $\overline{x}$ lies in $LB + LA_0(\mathbb{C})$. Then, the Galois group $Aut_\partial(\mathbb{K}(y)/\mathbb{K})$ of $(**)$ over $\mathbb{K}$ is equal to $\tilde{B}^\partial$, and in particular, $tr.deg.(\mathbb{K}(y)/\mathbb{K}) = 2dimB$.*

Proof : by Theorem 3.2, this Galois group is H_a^∂, where H_a is the smallest D-subgroup $H/\mathbb{K}$ of $\tilde{A}$ such that $a := \nabla_{L\tilde{A}}(x) \in LH + \partial\ell n_{\tilde{A}}(A(\mathbb{K}))$. Lifting $\overline{x}$ to a point x' of $L\tilde{B}(K)$, we see by the same argument as above that $H_a \subset \tilde{B}$. To prove the reverse inclusion, we suppose for a contradiction that $G := \tilde{B}/H_a \neq 0$.

We first prove that H_a is automatically defined over K. Here, we cannot appeal to the cohomological argument of Proof (i) of §3.2, because the full extension $\mathbb{K}(y)/K$ is not "normal" in any reasonable sense, so that in the diagram above, there is no natural action of $J(C)$ on H_a^∂ by conjugation. Instead, we use the rigidity of abelian varieties : the projection A'' of H_a on A is necessarily defined over K, and H_a is isogenous to $\tilde{A}'' \times W''$, where $W'' \subset W_A$ is a D-submodule of $L\tilde{A}$ over $\mathbb{K}$. Now, W'' is generated over $\mathbb{K}$ by W''^∂, which is stable under the action of $J(C)$ by the minimality of H_a. So, W'' is indeed defined over K, and H_a as well.

We now consider the D-quotient $G = \tilde{B}/H_a$ over K, which, up to an isogeny, can also be viewed as a quotient of $\tilde{A}$, and denote by ξ, η the images of x, y in G. Being stable under $Aut_\partial(\mathbb{K}(y)/\mathbb{K})$, the point η is $\mathbb{K}$-rational, and the class of $\nabla_{LG}(\xi) = \partial\ell n_G(\eta) \in LG(K)$ in $Coker(\partial\ell n_G, K)$ becomes trivial in $Coker(\partial\ell n_G, \mathbb{K})$. Going to a minimal non trivial D-quotient $\overline{G}$ of G over K, we will presently check that the natural map

$$
Coker(\partial\ell n_{\overline{G}}, K) \to Coker(\partial\ell n_{\overline{G}}, \mathbb{K})
$$

is injective. Consequently, η lifts to a point $y' \in \tilde{B}(K)$ such that for some proper D-subgroup H' of $\tilde{B}$ over K,

$$
\partial\ell n_{\tilde{B}}(y') \in \nabla_{L\tilde{B}}(x') + LH' \subset \nabla_{L\tilde{B}}(L\tilde{B}(K)) + LH'.
$$

From version (1^*) of Manin-Chai, we deduce that $\partial\ell n_{\tilde{B}}(y')$ lies in $\nabla_{L\tilde{B}}(W_B(K)) + LH'$, hence $\nabla_{L\tilde{B}}(x')$ as well. In view of Facts 1 and 3, this contradicts the non-degeneracy of $\overline{x}$ in LB.

17

To prove the required injection of cokernels, we follow the principle of Proof (ii) of §3.2. But first, we note that since H_a does not project onto B by assumption, the quotient $\overline{G}$ is necessarily a quotient of $\tilde{B}/U_B$ (in the notations of Fact 4), and is therefore K-large. So, we must in fact show that given two points $\overline{\alpha} \in L\overline{G}(K), \overline{\eta} \in \overline{G}(\mathbb{K})$ such that $\overline{\alpha} = \partial\ell n_{\overline{G}}(\overline{\eta})$, then $\overline{\eta}$ is automatically defined over K. By Fact 2, the abstract group $J(C) = Aut_\partial(\mathbb{K}/K)$ acts on $\tilde{B}^\partial = \tilde{B}^\partial(\mathbb{K})$, inducing a trivial action on $\overline{G}^\partial(\mathbb{K}) = \overline{G}^\partial(K)$ by K-largeness. Therefore, the cocycle

$$\hat{\xi} : J(C) \to \overline{G}^\partial(K) : \tau \mapsto \hat{\xi}(\tau) := \tau(\overline{\eta}) - \overline{\eta}$$

is a homomorphism of abstract groups. Since by Fact 3, every element of $J(C)$ is a commutator, while $\overline{G}^\partial(K)$ is commutative, $\hat{\xi}$ is trivial. In other words, $\overline{\eta}$ is fixed by $Aut_\partial(\mathbb{K}/K)$, and is therefore K-rational. (Another proof, suggested by A. Pillay, will be found in [9], Remarque 7).

5 APPENDIX

Since $\tilde{A}$ disappears from the statements of this appendix, we will now call y instead of $\overline{y}$ the points of A. We retain our convention that a multiple of y means a multiple by a non-zero integer.

5.1 Kummer theory

Let K be a *number field*, with algebraic closure $\overline{K}$, and let A be an abelian variety over K. Going to a finite extension if necessary, we assume that all the endomorphisms of $A/\overline{K}$ are defined over K and set

$$End(A/K) = End(A/\overline{K}) := \mathcal{O}.$$

Let y be a point in $A(K)$. Since there is no "constant part" anymore, we say that y *is non-degenerate on A* if for any proper abelian subvariety B of A, no multiple of y lies in B, i.e. if the group $\mathbb{Z}.y$ generated by y is Zariski dense in A, or equivalently, if the annihilator $Ann_\mathcal{O}(y)$ of y in $\mathcal{O}$ is reduced to $\{0\}$.

Following the elliptic work of Bashmakov and Tate-Coates from the 70's (see [20], V, §5), K. Ribet devised in [31] a general method to bound from below the degree of the division points of y of prime order. His method readily extends to all orders (see [6]), and yields :

Theorem 5.1. *: let y be a non-degenerate point in $A(K)$. There exists a real number $c = c(A, K, y) > 0$ such that for all $n > 0$,*

$$[K(\frac{1}{n}y) : K] \geq cn^{2dimA}.$$

Set $dimA := g$, and write $[n] \in \mathcal{O}$ for the multiplication by n on A, with kernel $A[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$ in $A(\overline{K})$. Since A is usually not "K-large" for $[n]$, we introduce a field K_∞, analogous to the previously defined field $\mathbb{K}$, which takes into account all torsion points of A :

$$K_\infty = K(A_{tor}), \quad \text{where } A_{tor} = \cup_{n>0} A[n].$$

For each ℓ in the set $\mathcal{P}$ of prime numbers, we set

$$K_{y,(\ell)} = \cup_m K_\infty(\frac{1}{\ell^m} y), \quad K_{y,\infty} = \cup_n K_\infty(\frac{1}{n} y),$$

and define the Tate modules $T_\infty(A) := \varprojlim_n A[n] = \Pi_{\ell \in \mathcal{P}} T_\ell(A)$ in the usual way.

Given a positive integer n, we have the tower of Galois extensions of K, whose Galois groups are indicated in the diagram :

$$
\begin{array}{ccccc}
K_\infty(\frac{1}{n}y) & & \xi_y & & \\
| & \}N & \hookrightarrow & A[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g} \\
\Gamma\{ \quad K_\infty & & \rho & & \\
| & \}J & \hookrightarrow & GL(T_\infty(A)) \simeq GL_{2g}(\hat{\mathbb{Z}}) \\
K & & & &
\end{array}
$$

where $N \ni \sigma \mapsto \xi_y(\sigma) = \sigma(\frac{1}{n}y) - \frac{1}{n}y$ is a group embedding. Theorem 5.1 immediately follows from :

Theorem 5.2. : *let y be a K-rational point of A, and let B be the smallest abelian subvariety of A containing a multiple of y. Then, $Gal(K_{y,\infty}/K_\infty)$ is isomorphic to an open subgroup of $T_\infty(B)$.*

Since all torsion points are defined over K_∞, we can assume that y itself lies in B; a map analogous to ξ_y then identifies this Galois group to a subgroup of $T_\infty(B)$. By Nakayama, Theorem 5.2 then amounts to the following claims :

   i) for almost all $\ell \in \mathcal{P}$, $Gal(K_\infty(\frac{1}{\ell}y)/K_\infty) \simeq B[\ell] \simeq (\mathbb{Z}/\ell\mathbb{Z})^{2dimB}$.
   ii) for all $\ell \in \mathcal{P}$, $Gal(K_{y,(\ell)}/K_\infty)$ is an open subgroup of $T_\ell(B) \simeq \mathbb{Z}_\ell^{2dimB}$;

5.2   Ribet's method.

The proof "follows" that of Theorems 3.3, 4.3, 4.4. For the sake of simplicity, we present it under the assumption that y is non-degenerate, i.e. $B = A$, and refer to [31] for the general case (see also [17], [6]). We fix a prime number ℓ, and first treat the mod ℓ claim.

1. *Galois theoretic step* (= analogue of "Proof (i)" of §3.2).

By the same computation from affine geometry, $Im(\xi_y) \simeq N$ is a J-submodule of $A[\ell]$. In these conditons, there exists $\ell_1 = \ell_1(A, K)$ with the following property : assume that $N \neq A[\ell]$ and that $\ell > \ell_1$; then, there exists $\alpha \in \mathcal{O}$, $\alpha \notin \ell\mathcal{O}$, such that $\alpha.y$ is divisible by ℓ in $A(K_\infty)$.

Indeed, Faltings's theorem on the finiteness of isomorphism classes of abelian varieties over K in a given isogeny class implies that for ℓ large enough, N is the intersection with $A[\ell]$ of the kernel of an endomorphism α of A. Alternatively, one can say, as in [31], that :

- $A[\ell]$ is a semi-simple J-module, so there exists $\alpha_\ell \in End_J(A[\ell]), \alpha_\ell \neq 0$, killing N.

- $End_J(A[\ell]) \simeq End(A) \otimes \mathbf{F}_\ell$ (Tate conjecture), so α_ℓ is induced by some $\alpha \in \mathcal{O}, \alpha \notin \ell\mathcal{O}$. Finally,

- $\xi_{\alpha.y} = \alpha\xi_y$, so, $\frac{1}{\ell}\alpha.y$ is fixed by N, i.e. the class of $\alpha.y$ in $Coker([\ell], A(K_\infty))$ vanishes, where we set, for any extension K' of K:

$$Coker([\ell], A(K')) := A(K')/\ell A(K').$$

Remark 1 : in the function field case, and even if A is simple and non isoconstant, the hypothesis that $(End(L\tilde{A}))^{\nabla_{L\tilde{A}}} = End(A) \otimes \mathbb{C}$, which would be a functional version of Tate's conjecture, *does not hold* in general. See [14] §4.4, and a counterexample in [15].

2. *Galois descent* (= analogue of "Proof (ii)" of §3.2)

There exists $\ell_2(A, K)$ such that if $\ell > \ell_2$, and if a point $y' \in A(K)$ is divisible by ℓ in $A(K_\infty)$, then, y' is already divisible by ℓ in $A(K)$, i.e. the kernel Ker of the natural map

$$Coker([\ell], A(K)) \to Coker([\ell], A(K_\infty))$$

vanishes.

Indeed, the exact sequence of inflation-restriction here gives:

$$
\begin{array}{ccccc}
Ker & \to & A(K)/\ell.A(K) & \to & A(K_\infty)/\ell.A(K_\infty) \\
\downarrow & & \downarrow \hat{\xi} & & \downarrow \xi \\
H^1(J, A[\ell]) & \to & H^1(\Gamma, A[\ell]) & \to & Hom_J(N, A[\ell]) ,
\end{array}
$$

while Serre's results on homotheties [33], Thm. 2, and Sah's lemma imply that $H^1(J, A[\ell]) = 0$ for ℓ large enough (depending only A and K).

3. *Diophantine step*

By Step 2, $\alpha.y$ is divisible by ℓ in $A(K)$. A contradiction to the conclusion of Step 1, or to the non-degeneracy of y, now follows from the existence of $\ell_0 = \ell_0(A, K, y)$ such that if $\ell > \ell_0$, and if $\alpha.y \in \ell.A(K)$, then there exists $\alpha' \in \mathcal{O}$ such that $(\alpha - \ell\alpha').y = 0$.

Indeed, this in turn follows from the Mordell-Weil theorem, which implies that $\mathcal{O}.y$ has finite index in its divisible hull in $A(K)$ (see [20] for an effective version). Since Manin's kernel theorem is based on a similar fact, this last step can perhaps be considered as an analogue of Formulae (1), (2), (1*), (2*) of the text.

Remark 2 : even when y is "indivisible in $A(K)$" in the sense of [17], Lemme I, the constant $\ell_0(A, K, y)$ arising in this last step *cannot be made independent of y* in general. Elliptic curves A with complex multiplications by non-principal orders $\mathcal{O}$ already provide counterexamples. Consequently, the constant c occurring in Theorem 5.1 will in general

depend on the non-degenerate point y, even if one insists that $\mathcal{O}.y$ be maximal among the $\mathcal{O}$-orbits of points of $A(K)$.

The ℓ-adic claim can be treated along similar lines[6], as follows. Firstly, by [33], Thm. 1, we may assume, after a finite extension of K, that the fields of definition $K_{(p)}, p \in \mathcal{P}$, of the p-primary parts of A_{tor} are linearly disjoint over K. Since $\mathbb{Z}_\ell$ cannot be a quotient of a p-adic Lie group for $p \neq \ell$, it then suffices to prove Claim (ii) with K_∞ replaced by $K_{(\ell)}$ and $K_{y,(\ell)}$ by $K_{(\ell),y} := \cup_m K_{(\ell)}(\frac{1}{\ell^m}y)$. So, we have a continuous map $\xi_{(\ell),y}$ analogous to ξ_y, and must show that $\xi_{(\ell),y}$ sends $N_{(\ell)} := Gal(K_{(\ell),y}/K_{(\ell)})$ into an open subgroup of $T_\ell(A)$. Set $J_{(\ell)} = Gal(K_{(\ell)}/K)$, and denote by $\Xi_\ell : A(K) \otimes \mathbb{Z}_\ell \to Hom(N_{(\ell)}, T_\ell(A))$ the $\mathbb{Z}_\ell$-linear extension of the map $y \mapsto \Xi_\ell(y) = \xi_{(\ell),y}$.

1. *Galois theoretic step*

$N_{(\ell)}$ is as usual a $J_{(\ell)}$-submodule of $T_\ell(A)$, and is closed. Assuming for a contradiction that it is not open, we deduce from the semi-simplicity of the representation $T_\ell(A) \otimes \mathbb{Q}_\ell$ and from Tate's conjecture the existence of a non-zero element $\alpha \in \mathcal{O} \otimes \mathbb{Z}_\ell$, such that $\Xi_\ell(\alpha.y) = 0$.

2. *Galois descent*

By the inflation-restriction sequence, the kernel of Ξ_ℓ injects into $H^1(J_{(\ell)}, T_\ell(A))$. Bogomolov's theorem on homotheties [11] or, more directly, an earlier result of Serre on the vanishing of $H^1(J_{(\ell)}, T_\ell(A) \otimes \mathbb{Q}_\ell)$ ensure that the latter group is finite. Replacing α by some multiple, we deduce that $\alpha.y = 0$ in $A(K) \otimes \mathbb{Z}_\ell$.

3. *Diophantine step*

Choosing a basis of $\mathcal{O}$ over $\mathbb{Z}$, and a basis of $A(K)$ over $\mathbb{Z}$ modulo torsion, we deduce from the latter conclusion that y is linearly dependent over $\mathcal{O}$, contrary to our assumption that y is non-degenerate in $A(K)$.

5.3   Back to function fields

Let us come back to the situation of §4, with a base field $K = \mathbb{C}(S)$ and an abelian variety A over K, with $\mathbb{C}$-trace A_0. All the notions introduced in the present Appendix remain meaningful, and one can ask if a suitable version of Theorem 5.2 still holds true in this functional context.

This is indeed the case. Although this probably follows from Theorem 5.2 itself and a specialization argument, we here want to indicate a more natural method, where the searched-for algebraic statement on *classical* Galois groups (of finite extensions of $\mathbb{C}(S)$) is *directly deduced* from the purely transcendental Theorem 4.3. on *differential* Galois groups (of Picard-Vessiot extensions of $\mathbb{C}(S)$).

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[6] See [6], Theorem 2, for a very brief sketch. As pointed out to me by M. Bays, the argument is described in more detail in [4].

Since $\tilde{A}$ will reappear, we start with a point $\overline{y}$ in $A(K)$, denote by B the smallest abelian variety such that a non trivial multiple of $\overline{y}$ lies in $B + A_0(\mathbb{C})$, assume without loss of generality that $\overline{y}$ itself lies in B, and choose an arbitrary lift y of $\overline{y}$ to $\tilde{B}(K)$. Fix a point $s_0 \in S(\mathbb{C})$, and consider the image Π of $\pi_1(S, s_0)$ in the monodromy representation attached to the local system formed by the various "logarithms" of the multiples $my, m \in \mathbb{Z}$, of $y \in \tilde{\mathcal{B}}(S)$. More precisely, let $x = \ell n_{\tilde{\mathcal{B}}}(y) \in L\tilde{B}$ be a local determination of a logarithm of y in a neighbourhood of s_0. For any $\gamma \in \pi_1(S, s_0)$, analytic continuation along γ provides an element of the differential Galois group

$$\Gamma(\mathbb{C}) = Aut_\partial(\mathbb{K}(x)/K), \text{ with } \Gamma \subset GL_{2g+1}, \text{ and } \Pi \subset \Gamma(\mathbb{Z}) \subset GL_{2g+1}(\mathbb{Z}),$$

sending x to $\gamma.x = x + \hat{\xi}(\gamma)$, where $\hat{\xi}(\gamma)$ lies in the subgroup $\Omega_{\tilde{\mathcal{B}}}$ of periods of $(L\tilde{B})^\partial$. Notice that $\hat{\xi}$ is only a cocyle, so that $\hat{\xi}(\pi_1(S, s_0)) := \Omega_y$ is in general not a group.

Let now n be a positive integer, and consider the n-th division point $\frac{1}{n}y = exp_{\tilde{A}}(\frac{1}{n}x)$ of y in $\tilde{B}(\overline{K})$. Since $exp_{\tilde{A}}$ is a S^{an}-morphism, the action of $\pi_1(S, s_0)$ on $\frac{1}{n}y$ is given by

$$\gamma.(\frac{1}{n}y) = exp_{\tilde{A}}(\gamma.(\frac{1}{n}x)) = exp_{\tilde{A}}(\frac{1}{n}x + \frac{1}{n}\hat{\xi}(\gamma)) = \frac{1}{n}y + exp_{\tilde{B}}(\frac{1}{n}\omega_\gamma),$$

where $\omega_\gamma = \hat{\xi}(\gamma) \in \Omega_y$. In particular the number of conjugates of $\frac{1}{n}y$ over K, i.e. the degree of $K(\frac{1}{n}y)$ over K, is equal to the number $\delta_y(n)$ of distinct classes in $\frac{1}{n}\Omega_y$ modulo the kernel $\Omega_{\tilde{\mathcal{B}}}$ of $exp_{\tilde{\mathcal{B}}}$. And since $\tilde{B}$ is a vectorial extension, the fields of definition of $\frac{1}{n}y$ and of its image $\frac{1}{n}\overline{y}$ in B coincide, so this is also the degree of $K(\frac{1}{n}\overline{y})$ over K.

Now come the main points :

(i) since $\nabla_{L\tilde{A}}$ is a fuchsian connexion, Π is a Zariski-dense subgroup of the algebraic group Γ. In particular, Ω_y is Zariski dense in the Galois group (N_a^∂), which, as we know by Theorem 4.3, coincides with $(L\tilde{B})^\partial$.

(ii) Assume for a moment that A is defined over $\mathbb{C}$, i.e. that $A = A_0$, in which case all the torsion points of A are defined over $\mathbb{C}$, and $K_\infty = K$. Then, the periods in $\Omega_{\tilde{A}}$ are constant (so, $K = \mathbb{K}$ as well), $\hat{\xi} = \xi$ is a group morphism, and Ω_y is a subgroup of the discrete subgroup $\Omega_{\tilde{\mathcal{B}}}$ of the vectorial group $(L\tilde{B})^\partial$. Being Zariski-dense in the latter vectorial group, Ω_y must be of *finite index*, say ν_y, in $\Omega_{\tilde{\mathcal{B}}}$. In particular, the degree $\delta_y(n) = [K(\frac{1}{n}\overline{y}) : K]$ is bounded from below by cn^{2dimB}, where $c = \frac{1}{\nu_y}$, and we derive, more precisely :

Theorem 5.3. *let A_0 be an abelian variety defined over $\mathbb{C}$, let $K = \mathbb{C}(S)$, let y be a point in $A_0(K)$, and let B be the smallest abelian subvariety of A_0 such that some multiple of y lies in $B + A_0(\mathbb{C})$. There exists an integer $\nu = \nu(A_0, K, y)$ such that for any $n > 0$, the Galois group $Gal(K(\frac{1}{n}y))/K)$ is isomorphic to a subgroup of $B[n]$ of bounded index, equal to 1 as soon as n is prime to ν.*

(iii) When A is not defined over $\mathbb{C}$, the discrete monodromy group Π will in general not even meet N_a^∂ outside of 0, and the above argument does not apply. However, finitely generated subgroups of $GL_m(\mathbb{Z})$ such as Π often satisfy the *strong approximation property* with respect to their Zariski closure G in GL_m, in the sense that their closure $\hat{\Pi}$ in the profinite group $GL_n(\hat{\mathbb{Z}})$ is then open in $G(\hat{\mathbb{Z}})$. This holds true when G is a semi-simple connected and simply connected group, as shown by the theorems of Matthews-Vaserstein-Weisfeiler and of Nori, which play a role in [33]. It clearly fails for tori, but as pointed out to me by Y. Benoist, Nori's Theorem 5.3 in [23] shows that this is in a sense the only obstruction : by Fact 3 of §4, this theorem applies to our group $G = \Gamma$, whose radical N_a^∂ is unipotent. We deduce that for all prime numbers ℓ, the image of Π in $\Gamma(\mathbb{Z}/\ell\mathbb{Z})$ has bounded index. In particular[7], the image of $\frac{1}{\ell}\Omega_y$ in $\frac{1}{\ell}\Omega_{\tilde{\mathcal{B}}}/\Omega_{\tilde{\mathcal{B}}} \simeq B[\ell]$ fills up this group for ℓ sufficiently large.

We will come back to this in a later article, but already mention the following consequence of the discussion above, where we set $K_n = K(A[n])$:

Theorem 5.4. : *let A be an abelian variety over $K = \mathbb{C}(S)$, with $\mathbb{C}$-trace A_0, let y be a point in $A(K)$, and let B be the smallest abelian subvariety of A such that some multiple of y lies in $B + A_0(\mathbb{C})$. There exists an integer $\nu = \nu(A, K, y)$ such that for any $n > 0$, the Galois group $Gal(K_n(\frac{1}{n}y))/K_n)$ contains a subgroup of $B[n]$ of bounded index, and coincides with $B[n]$ as soon as n is prime to ν.*

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[7] Actually, this consequence follows directly from Nori's Theorem 5.1 in [23], according to which for almost all ℓ's, the image of Π in the group $\Gamma(\mathbb{F}_\ell)$ contains the subgroup $\Gamma(\mathbb{F}_\ell)^+$ generated by its elements of order ℓ, hence the full subgroup $N_a^\partial(\mathbb{F}_\ell) \simeq \mathbb{F}_\ell^{2dimB}$.

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