Special values of σ-functions.

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Special numbers and special functions

Frits Beukers 60, Utrecht, July 10-12, 2013

0. References

[J-R] O. Jacquinot, K. Ribet : *Deficient points on extensions of abelian varieties by* $\mathbb{G}_m$; J. Number Th., 25, 1987, 133-151.

[B] : *Special points and Poincaré biextensions* (with an *Appendix* by B. Edixhoven); ArXiv 1104.5178v1

[B-M-P-Z] D. B., D. Masser, A. Pillay, U. Zannier : *Relative Manin-Mumford for semi-abelian schemes*; ArXiv 1307.1008v1

[B-E] D.B, B. Edixhoven: *Pink's conjecture, Poincaré biextensions and generalised Jacobians*; in preparation.

1. Relative Manin-Mumford

Manin-Mumford

F. Beukers-D. Zagier (AA, 1997) : a, b algebraic, $ab \neq 0$. Consider three algebraic solutions $(x, y), xy \neq 0$, to $(W) : ax + by = 1$. Then, the sum of their logarithmic height is ≥ 0.09. "So", at most 2 solutions with $(x, y) \in G_{tor} = (\mu_\infty)^2$, where $G = \mathbb{G}_m^2$.

MM : (Raynaud-Hindry) G_0 *semi-abelian variety over* $\mathbb{C}$, W_0 *closed irred. subvariety of dim.* $d < dimG_0$, *with* $W_0 \cap G_{0,tor}$ *Zariski-dense in* W_0. *Then,* W_0 *is a torsion translate of a strict alg. subgp* H_0.

Equivalently : $S/\mathbb{C}$ irred. alg. var., $G = G_0 \times S$ constant group scheme over S, $s : S \to G$ a section of G/S.

$$G_{tor} = \cup_{\lambda \in S(\mathbb{C})} (G_\lambda)_{tor}.$$

Assume that $S_{tor}^{G,s} := \{\lambda \in S(\mathbb{C}), s(\lambda) \in G_{tor}\}$ is Z-dense in S and that $dim(S) < dim(G/S)$. Then, s factors through a strict subgroup scheme H of G.

Relative Manin-Mumford

RMM : *$S/\mathbb{C}$ irred. alg. var., G/S semi-abelian scheme, not necessarily constant, s a section of G/S.*

$$G_{tor} = \cup_{\lambda \in S(\mathbb{C})} (G_\lambda)_{tor}.$$

Assume that $S^{G,s}_{tor} := \{\lambda \in S(\mathbb{C}), s(\lambda) \in G_{tor}\}$ is Z-dense in S and that $dim(S) < dim(G/S)$. Does then s factor through a strict subgroup scheme H of G ?

Almost a consequence of Pink's general conjecture on mixed Shimura varieties, but not quite. In fact, restricting to a *curve $S/\mathbb{Q}^{alg}$ and to the crucial case: $dim(G/S) = 2$*

- Masser-Zannier : **yes** *if G/S is an abelian scheme.*
- [B], [B-E] : **no** *if $G \in Ext_S(E, \mathbb{G}_m)$*, not isoconstant, where $E = E_0 \times S$ is a constant CM curve, and *s is a* **Ribet section**.
- [B-M-P-Z] : **yes** *in all other cases.*

Unlikely intersections

Bombieri-Masser-Zannier : infinitely many $(x, y) \in \mathbb{G}_m^2$, lying in $(W) : x + y = 1$ for which $\exists r, s \in \mathbb{Z}, x^r y^s = 1$ (likely intersection). But finitely many if we ask for two independent relations (unlikely intersection). More generally (Zilber, BMZ, Pink) :

for G/S, $W \subset G$, of dimension $d < dim(G/S)$, set

$$G^{[>d]} = \cup_{\lambda \in S(\mathbb{C})} \ H_\lambda \ {}_{,\{H_\lambda \subset G_\lambda, codim_{G_\lambda} H_\lambda > d\}} .$$

Then : $W \cap G^{[>d]}$ *Z-dense in* $W \Rightarrow W$ *lies in a strict subgroup scheme of* G/S *?* In particular:

RMM for a curve S : if $dim(G/S) \geq 2$, and $s : S \to G$ is a section of G/S which does not factor through any strict (and flat) subgroup scheme of G/S, does its image $s(S) := W$ contain only finitely many points of $G_{tor} = G^{[>dim(G/S)-1]} \subset G^{[>1]}$?

But

2 . What is a Ribet section ?

Relative version of the “deficient points" of [J-R].

Poincaré biextension

Like M. Jourdain with prose, you may have already met Ribet sections when looking at Néron-Tate pairing $< p, q >_L$ for $p, q \in E(\mathbb{Q}^{alg})$, $\alpha \in End(E) = \mathcal{O}$ *totally imaginary* and $p = \alpha q$. $< \alpha x, y >_L = < x, \overline{\alpha} y >_L = - < x, \alpha y >_L$, so $\boxed{< \alpha q, q >_L = 0}$.

This reflects a property of the Poincaré bundle $\mathcal{P}$ on $E \times \hat{E}$. For $q' = \phi_L(q) \in \hat{E}$, -2 $< p, q >_L = \hat{h}_{\mathcal{P}}((p, q'))$. On the graph $B \simeq \hat{E}$ of $f = \alpha \circ \phi_L^{-1} : \hat{E} \to E$ (with $\hat{f} = -f \in End(\hat{E}, E)$), the pull-back $(f, id_{\hat{E}})^* \mathcal{P} \simeq \mathcal{P}_{|B}$ satisfies : $\hat{h}_{\mathcal{P}_{|B}}((f(q'), q')) \equiv 0$, so $\mathcal{P}_{|B} \equiv 0$ (in fact, order 1 or 2 in $Pic_0(B)$). So, $2\mathcal{P}_{|B}$ has a canonical section $s : B \to \mathcal{P}_{|B} \rightsquigarrow$ a point $s_f(q') \in \mathcal{P}$ above $(2f(q'), q') \in E \times \hat{E}$.

Now, $q' \in \hat{E} \simeq Pic_0(E)$ defines $G := G_{q'} = \mathcal{P}^*{}_{|E \times q'} \in Ext(E, \mathbb{G}_m)$: a semi-abelian surface with a distinguished point $s_f(q')$ above the point $p = 2f(q') \in E$: $s_f(q')$ is the Ribet point of $G_{q'}$.

More generally (A. Chambert-Loir, [B-E]) : $S/\mathbb{Q}^{alg}$ a smooth irred. alg. var.; A/S an abelian scheme; $\mathcal{P} = \mathcal{P}^A$ the Poincaré biextension over $A \times \hat{A}$, with its rigifications above $e_S \times \hat{A}$ and above $A \times \hat{e}_S$, and its two families of group laws $+_1$, $+_2$. There are canonical isomorphisms of $\mathbb{G}_m$-torsors (over any base S'/S):

$$\forall y \in \hat{A}, z \in \hat{\hat{A}} \simeq A, \mathcal{P}^{\hat{A}}(y,z) \simeq \mathcal{P}^A(z,y)$$
$$\forall f : B \to C, y \in B, x \in \hat{C}, \mathcal{P}^C(fy,x) \simeq \mathcal{P}^B(y,\hat{f}x), \text{ so}$$
$$\forall f : \hat{A} \to A, y \in \hat{A}, x \in \hat{A}, \tau_{f,x,y} : \mathcal{P}(fy,x) \xrightarrow{\sim} \mathcal{P}(\hat{f}x,y).$$

Hence for $x = y = q \in \hat{A}$ (old q') and using the $+_1$ law :

$$\mathcal{P}((f - \hat{f})q, q) \simeq \mathbb{G}_{m/S},$$

yielding on $\mathcal{P}_{|A \times q} = G_q \in Ext(A, \mathbb{G}_m)$ the Ribet section

$$s_f(q) \in G_q(S) \text{ above } p = (f - \hat{f})(q) \in A(S).$$

Ribet vs RMM

Theorem ([B-E]) : *for* $q \in \hat{A}(S), f \in End_S(\hat{A}, A)$, *let* $s := s_f(q)$ *be the Ribet section of* $G = G_q/S \in Ext_S(A, \mathbb{G}_m)$. *Then:*

i) $s_f(q)$ *is additive in* f, *and depends only on* $f - \hat{f} \in End^{antisym}$;

ii) $s_f(q)$ *projects to* $p := (f - \hat{f})(q) \in A(S)$;

iii) if $q(\lambda) \in \hat{A}_\lambda[n]$, *then* $s(\lambda) = s_f(q(\lambda))$ *is torsion, of order* $|n^2$ *in* $G_\lambda := G_{q(\lambda)}$; *so,* $S^{G,s}_{tor} = S^{\hat{A},q}_{tor} \sim S^{A,p}_{tor}$. *More precisely,*

iv) assume $q(\lambda) = q_\lambda \in \hat{A}_\lambda[n]$; *then,* $\boxed{n.s(\lambda) = e_n(f(q_\lambda), q_\lambda) \in \mu_n}$.

Now, for $f - \hat{f}$ an isogeny, suppose that q factors through no strict sbgp scheme and $S^{\hat{A},q}_{tor}$ is Z-dense in S (exists if $dim S = dim A/S$). Then, ditto for s. So, $s_f(q)$ *contradicts RMM*.

Characterizing Ribet sections via MM ($g = 1$)

From now on, suppose that $A = E$ is an elliptic curve. So $E = E_0 \times S$, where $E_0/\mathbb{Q}^{alg}$ has CM by $\mathcal{O}$. Also, S must be a curve, and we assume wlog that $S = \hat{E}_0$, so "$\lambda = q$". Finally, we can suppose that $\hat{f} \neq f$, i.e. $h = f - \hat{f}$ is an isogeny.

Elementary proof of the theorem soon to come.

But first, an easy consequence of (iii). On $G = G_q \xrightarrow{\pi} E$, the Ribet section $s = s_f(q)$ lifts all torsion values of $p := \pi(s) = h(q)$ to torsion points of G, i.e.: $S_{tor}^{G,s} = S_{tor}^{E,p}$ (which is infinite).
Let now $s' \in G_q(S)$ such that $\pi(s') = p$. *Assume that* $S_{tor}^{G,s'}$ *is infinite*. Then, $s' - s_f(q) = \delta \in \mathbb{G}_m(S)$ and $S_{tor}^{\mathbb{G}_m \times E_0,(\delta,p)}$ is infinite. By Hindry's thm, δ must torsion, so $s' = s_f(q)$ *up to a root of unity*.

Whittaker-Watson proof of the Theorem (in the shape : given G_q and f, there exists a section $\tilde{s}$ enjoying Properties (i) -(iv)), via

3 . Special values of σ-functions

$$\sigma(z) = z \prod_{\omega \in \Omega, \omega \neq 0} (1 - \frac{z}{\omega}) e^{\frac{z}{\omega} + \frac{1}{2} \frac{z}{\omega^2}}$$

$$\zeta = \frac{\sigma'}{\sigma}, \zeta(z+\omega) = \zeta(z) + \eta(\omega)$$

$$\sigma(z+\omega) = \epsilon(\omega)\sigma(z) e^{\eta(\omega)(z + \frac{\omega}{2})}.$$

$$\wp(z) = -\zeta'(z), div(\wp(z) - \wp(v)) = (q) + (-q) - 2(0).$$

Uniformizing G_q

On $E_0/\mathbb{Q}^{alg}$, identified with $\hat{E}_0$, fix a dfk $\underline{\omega} = \frac{dx}{y}$, with period lattice $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, and choose a determination $v = log_E(q) = \int_0^q \underline{\omega}$.

By Weil-Rosenlicht–Barsotti, $G_q \xrightarrow{\pi} E$ admits a rational section $\rho : E \dashrightarrow G_q$, with $(\rho) = (-q) - (0)$ (identified with q, up to sign). The 2-cocycle which describes the group law on $\mathbb{G}_m \times E$ is a rat'l function on $E \times E$, expressed by $\frac{\sigma(z+z'+v)\sigma(z)\sigma(z')\sigma(v)}{\sigma(z+z')\sigma(z+v)\sigma(z'+v)}$.
So, the exponential morphism exp_{G_q} on G_q^{an} is represented by

$$LieG_q \ni \begin{pmatrix} t \\ z \end{pmatrix} \mapsto \begin{pmatrix} f_v(z)\ e^t \\ \wp(z) \end{pmatrix} \in G_q$$

where $f_v(z) = \frac{\sigma(v+z)}{\sigma(v)\sigma(z)} e^{-\zeta(v)z} \Rightarrow \frac{df_v}{f_v} = (\zeta(v+z) - \zeta(z) - \zeta(v))dz$
$= \frac{1}{2}\frac{\wp'(z)-\wp'(v)}{\wp(z)-\wp(v)} dz = exp_E^*(\xi_q)$, for the standard logarithmic form ξ_q on E with residue divisor $-1.(0) + 1.(-q)$.

$G_q^{an} \simeq \mathbb{C}^2/\Pi_G$, where $\Pi_G = \mathbb{Z}\varpi_0 \oplus \mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2$ with

$$\varpi_0 = \begin{pmatrix} 2\pi i \\ 0 \end{pmatrix}, \varpi_1 = \begin{pmatrix} \kappa_v(\omega_1) \\ \omega_1 \end{pmatrix}, \varpi_2 = \begin{pmatrix} \kappa_v(\omega_2) \\ \omega_2 \end{pmatrix},$$

where $\kappa_v(\omega_i) = \zeta(v)\omega_i - \eta_i v, \; (i = 1, 2)$ are the basic periods of ξ_q.

Under this description, a section $s \in G/S$ above $p \in E(S)$, and its logarithm $log_G(s)$ above $log_E(p) = u$, are given by

$$s = \begin{pmatrix} \delta_s \\ p = \pi(s) \end{pmatrix}, \; log_G(s) = \begin{pmatrix} -g(u,v) + \zeta(v)u + \ell_s \\ u \end{pmatrix}$$

where $\delta_s := s - \rho(p) \in \mathbb{C}(S)^*$ depends only on s (and on the choice of the section ρ), $\ell_s = log_{\mathbb{G}_m}(\delta_s)$, and

$$g_\lambda(u, v) = log\left(\frac{\sigma_\lambda(u+v)}{\sigma_\lambda(v)\sigma_\lambda(u)}\right).$$

is a "Green function" for the divisor $\Delta - E \times 0 - 0 \times E$ on $E \times E$.

CM relations

Assume now that E_0 has CM by $\mathcal{O}$. Then, $H^1_{dR}(E_0/\mathbb{Q}^{alg})$ splits into two eigenspaces for $\mathcal{O}$, generated by $\frac{dx}{y}$ and by $(x - s_2)\frac{dx}{y}$, where

$$s_2 = \lim_{\varepsilon\to 0}\sum_{0\neq\omega\in\Omega}\frac{1}{\omega^2}\frac{1}{|\omega|^\varepsilon} \in \mathbb{Q}^{alg}.$$

In particular, $\tilde{\eta}(\omega) := \eta(\omega) - s_2\omega, \omega \in \Omega$, satisfies

$$\forall\gamma\in\mathcal{O}, \tilde{\eta}(\gamma\omega) = \overline{\gamma}\,\tilde{\eta}(\omega).$$

Set $\boxed{\theta(z) = \sigma(z)e^{-\frac{1}{2}s_2z^2}}$, $\theta(z+\omega) = \epsilon(\omega)\theta(z)e^{\tilde{\eta}(\omega)(z+\frac{\omega}{2})}$, so $\tilde{\zeta}(z)$ $= \frac{\theta'}{\theta} = \zeta(z) - s_2z)$. Then, for any $\gamma \in \mathcal{O}$, with $N(\gamma) = \gamma\overline{\gamma}$:

$$\left(\frac{\theta(\gamma z)}{\theta(z)^{N(\gamma)}}\right)^2 = \gamma^2 \prod_{e\in E[\gamma], e\neq 0}(\wp(z) - \wp(e)).$$

NB : the RHS is a square as soon as $E[\gamma] \cap E[2] = \{0\}$ or $E[2]$.

The Ribet section, analytically

Reverting the roles of p and q, assume that $q = \alpha p$, where $\alpha = \beta - \overline{\beta}$ for some $\beta \in \mathcal{O} \setminus \mathbb{Z}$. Then, $(2, \alpha)$ and $(2, \alpha + 1)$ are either (2) or (1). Furthermore, $N(\alpha + 1) - N(\alpha) - 1 = \alpha + \overline{\alpha} = 0$. So, choosing as we may $v = \alpha u$:

$$\tilde{\delta}(p) := \frac{\sigma(u+v)}{\sigma(v)\sigma(u)} e^{-s_2 uv} = \frac{\theta((\alpha+1)u)}{\theta(\alpha u)\theta(u)}$$

lies is $\mathbb{C}(\wp(u), \wp'(u))$, hence in $\mathbb{C}(S)^*$, and defines a section

$$\tilde{s}_\beta(p) = (\tilde{\delta}(p), p) \in G_q(S).$$

Proposition : *for* $G = G_q$, *the section* $\tilde{s} = \tilde{s}_\beta(p) \in G(S)$ *satisfies*
ii) $\pi(\tilde{s}) = p$ *where* $q = (\beta - \overline{\beta})(p)$;
iv) if $p(\lambda) \in E[n]$, *then* $\boxed{n.\tilde{s}(\lambda) = e_n(\beta p_\lambda, p_\lambda)}$. So, $S_{tor}^{G,\tilde{s}} = S_{tor}^{E,p}$.

Proof of (iv)

[NB: set $p' = Np$, $N = \alpha\overline{\alpha}$. Then, $s' := -Ns \in G_q(S)$ projects to $p' = (\beta - \overline{\beta})q$ and still satisfies $S_{tor}^{G,s'} = S_{tor}^{E,p}$ infinite. So, by the MM charact'n of Ribet sections, $s' = s_\beta(q)$ up to a root of unity.]

Proof : $log_G^{(1)}(\tilde{s}) = -g(u,v) + \zeta(v)u + log(\tilde{\delta}) = \zeta(v)u - s_2 uv$, so for $p \in E[n]$, $u = \frac{1}{n}\omega, v = \frac{1}{n}\alpha\omega$ (and $\alpha = \beta - \overline{\beta}$),

$$log_G^{(1)}(\tilde{s}) - \frac{1}{n}\kappa_v(\omega) = \frac{1}{n^2}\eta(\omega)\alpha\omega - \frac{1}{n^2}s_2\alpha\omega^2 = \frac{1}{n^2}\alpha\tilde{\eta}(\omega)\omega.$$

Legendre $\Rightarrow 2\pi i = \tilde{\eta}_2\omega_1 - \tilde{\eta}_1\omega_2 = (\overline{\tau} - \tau)\tilde{\eta}_1\omega_1 \Rightarrow \alpha\tilde{\eta}(\omega)\omega \in 2\pi i\mathbb{Z}$, so $\exists\varpi \in \Pi_G$ such that $log_G(\tilde{s}) - \frac{1}{n}\varpi \in \frac{1}{n^2}\mathbb{Z}\varpi_0$, and $\tilde{s} \in G_q[n^2]$.

More precisely (at least up to a sign),
$e_n(\beta p, p) = exp(n(\eta(\beta\frac{\omega}{n})\frac{\omega}{n} - \eta(\frac{\omega}{n})\beta\frac{\omega}{n})) = exp(\frac{1}{n}(\overline{\beta} - \beta)\tilde{\eta}(\omega)\omega)$, so

$$p \in E[n] \Rightarrow n.\tilde{s}_\beta(p) = e_n(\beta p, p) \in \mu_n.$$

4. Back to relative Manin-Mumford

No other obstruction.

Theorem ([B-M-P-Z]) *Let E/S be an elliptic scheme over a curve $S/\mathbb{Q}^{alg}$, and let G/S be an extension of E/S by $\mathbb{G}_{m/S}$. Let further $s : S \to G$ be a section of G/S, with image $W = s(S)$.*

(A) *Assume that $W \cap G_{tor}(\simeq S^{G,s}_{tor})$ is infinite. Then,*
i) either s is a Ribet section;
ii) or s factors through a strict subgroup scheme of G/S.

(B) *More precisely, $W \cap G_{tor}$ is infinite if and only if s is a Ribet section, or a torsion section, or a non isoconstant section of a strict subgroup scheme of G/S.*

The proof uses Zannier's strategy, in the version developped by Masser-Zannier for the case of abelian surfaces : lower bounds for Galois orbits, Pila-Wilkie upper bounds, reducing the problem to algebraic independence of functional logarithms. This brings us to another of Frits's territories.

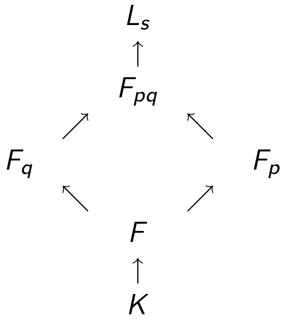
Differential Galois theory

Lemma : *With $S/\mathbb{C}$, let G/S be an extension by $\mathbb{G}_m$ of an elliptic scheme E/S, parametrized by a section q of $\hat{E}/S$, and let G_0 be the constant part of G. Let further s be a section of G/S, with projection $p = \pi \circ s$ to E/S, and let F_{pq} be the field of generalized periods of $\{E, p, q\}$.*

(A) *Assume that $log_G(s)$ is algebraic over F_{pq}. Then, there exists a constant section $s_0 \in G_0(\mathbb{C})$ such that*

i) either $s - s_0$ is a Ribet section;
ii) or $s - s_0$ factors through a strict subgroup scheme of G/S.

(B) *More precisely, $log_G(s)$ is algebraic over F_{pq} if and only if there exists a constant section $s_0 \in G_0(\mathbb{C})$ such that $s - s_0$ is a Ribet section, or a torsion section, or factors through a strict subgroup scheme of G/S projecting onto E/S.*



$$L_s = F_{pq}(log_G(s)) = F_{pq}\big(\ell_s - g(u,v)\big)$$

$$F_{pq} = F_p.F_q$$

$$F_q = F_G = F(v,\zeta(v)) \qquad F_p = F(u,\zeta(u))$$

$$F = F_E = K(\omega_1,\omega_2,\eta_1,\eta_2)$$

$$K = \mathbb{C}(S)$$

$$\rho_{G,s}(\gamma) = \begin{pmatrix} 1 & {}^t\xi_q(\gamma) & \tau_s(\gamma) \\ 0 & \rho_E(\gamma) & \xi_p(\gamma) \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{array}{c} \tau_s : Gal_\partial(L_s/F_{pq}) \hookrightarrow \mathbb{C} \\ {}^t\xi_q : Gal_\partial(F_q/F) \hookrightarrow \mathbb{C}^2 \simeq \hat{\mathcal{V}} \\ \xi_p : Gal_\partial(F_p/F) \hookrightarrow \mathbb{C}^2 \simeq \mathcal{V} \\ \rho_E : Gal_\partial(F/K) \hookrightarrow SL_2(\mathbb{C}) \end{array}$$

Then, $\tau_s = 0 \Leftrightarrow \exists s_0 \in G_0(\mathbb{C}), s - s_0$ is Ribet or factors.

As in [J-R], the proof of $\Rightarrow$ uses representation theory. Conversely, the formulae for $\tilde{\delta} = \frac{\theta((\alpha+1)u)}{\theta(\alpha u)\theta(u)} \in \mathbb{C}(\wp(u), \wp'(u))$ and for $log_G(\tilde{s})$ prove $\Leftarrow$ both in the present differential context and in the Kummer theoretical one of [J-R].

5. Pink's conjecture on mixed Shimura varieties

Ribet curves are special

André-Oort : let X be a Shimura variety, let X_{sp} be its set of special points, and let W be an irreducible closed subvariety of X. Assume that $W \cap X_{sp}$ is Zariski dense in W. Then, W is a special subvariety ($\sim$ a component of a Hecke transform of a proper Shimura subvariety of X).

Pink's conjecture in "amplitude 0" : $X \rightsquigarrow$ a mixed Shimura variety.

Example 1 (Y. André) : $X = E =$ universal elliptic curve over a modular curve Y;

Example 2 : $X = \mathcal{P} =$ the Poincaré bi-extension of $E \times_Y \hat{E}$. Over a CM point $y_0 \sim E_0$, the special subvariety $\mathcal{P}_0 := \mathcal{P}_{|y_0}$ can be viewed as the "universal" extension $\mathcal{G}_0$ of $(E_0)_{\hat{E}_0}$ by $\mathbb{G}_m$, over its parameter space $\hat{E}_0$ (i.e. $S = \hat{E}_0, q = q_{id} : \hat{E}_0 \to (\hat{E}_0)_{\hat{E}_0}$).

In this context, the counterexample to RMM turns into a

Pro-example : *given* $f \in \mathcal{O}$, *let* $W = s_f(q)(\hat{E}_0)$ *be the image of the Ribet section in* $\mathcal{G}_0 = \mathcal{P}_0$. *Then,* W *passes through a Zariski-dense set of special points of the mixed Shimura variety* $\mathcal{P}_0$ *- and is indeed a special curve of* $\mathcal{P}_0$.

In higher dimension, see [B-E].

Back to $g = 1$, we have more generally :

Theorem : *Let* W *be an irred. closed curve in* $\mathcal{P}$, *such that* $W \cap \mathcal{P}_{sp}$ *is infinite. Then,* W *is a special curve of* $\mathcal{P}$.

Proof : André's thm for E/Y and [B-M-P-Z], using $\mathcal{P}_{0,sp} \subset \mathcal{G}_{0,tor}$.

Pink's general conjecture

For X a mixed Shimura variety, set

$$X^{[>d]} = \cup Y_{,codim Y > d, Y \ special} \cdot$$

Then : $W \subset X, dim W = d$ *and* $W \cap X^{[>d]}$ *Zariski-dense in* $W \Rightarrow W$ *lies in a strict special subvariety of* X *?*

• some results for W a curve in $X = \mathcal{P}$ (because $\mathcal{P}_{0,sp} \subsetneq \mathcal{G}_{0,tor}$!);
• Pila-Tsimerman, Klingler-Ullmo-Yafaev : for a pure X, uniformized by $\pi : \mathfrak{X} \to X$, the *weakly* special subvarieties in W are the images of the maximal algebraic subvarieties in $\pi^{-1}(W)$.
• ditto for $\mathbb{C}^3 \times \mathfrak{H} \to \mathcal{P}$ (in progress), cf. $log_G(s_\alpha(p))$, and the role of the constant part G_0 in the Lemma.

Et pour finir :

Joyeux anniversaire,

Frits !