Counterexamples with semi-abelian varieties.

D. Bertrand (IMJ)

Recent Developments in Model Theory

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1. Motivation : the MM-ML-AO-ZP conjectures

Mordell-Lang : let A be an abelian variety over a number-field k, let Δ be a subgroup of finite rank of $A(k)$, and let X be an irreducible closed subvariety of A of codimension $d \geq 1$. Assume that $X \cap \Delta$ is Zariski dense in X. Then, X is a translate of a proper abelian subvariety of A. Cf. D. Roessler's talk.

Manin-Mumford : restrict to $\mathrm{rank}(\Delta) = 0$, i.e. $\Delta = A_{tor}$, translate $\rightsquigarrow$ a component of an algebraic subgroup.

André-Oort : let $\mathbf{S}$ be a Shimura variety over k, let Δ be a set of special points of $\mathbf{S}$, and let X be an irreducible closed subvariety of $\mathbf{S}$ of codimension $d \geq 1$. Assume that $X \cap \Delta$ is Zariski dense in X. Then, X is a subvariety of Hodge type ($\sim$ a component of a Hecke transform of a proper Shimura subvariety of $\mathbf{S}$). Cf. J. Pila's talk.

Generalizations :

• in MM-ML, replace A by a torus $T = \mathbb{G}_m^r$, or by $A \times T$, or even by an arbitrary

semi-abelian variety $G \in Ext(A, T)$

• *Bombieri-Masser-Zannier* / *Zilber* : unlikely intersections / CIT (cf. J. Kirby's talk) : e.g. in MM, replace G_{tor} by

$$G^{[<d]} = \cup G', \dim G' < d.$$

• *Relative Manin-Mumford* (RMM): replace G/k by

a semi-abelian scheme G/S over a variety S/k.

• *Pink's general conjecture* : in AO, replace $\mathbf{S}$ by

a mixed Shimura variety,

e.g. those parametrizing one-motives (= points on semi-abelian varieties) + unlikely intersections + relative Manin-Mumford...

What is a semi-abelian variety ?

• Chevalley's theorem : any connected commutative algebraic group G with no $\mathbb{G}_a$-subgroup :

$$0 \longrightarrow T \longrightarrow G \stackrel{\pi}{\longrightarrow} A \longrightarrow 0 \ ,$$

Assume $T \simeq \mathbb{G}_m^r$ split. For $r = 1$, add a zero section $\rightsquigarrow$ a line bundle, algebraically equivalent to 0 (Weil-Rosenlicht), so parametrized by

$$q \in Pic^0(A/k) = \hat{A}(k) \simeq Ext_{alg.gr/k}(A, \mathbb{G}_m).$$

• Generalized jacobians : let C/k be a proper smooth curve. Pinch it at two points (q_1, q_2) to get a singular curve C', with normalization $\nu : C \to C'$, hence

$$0 \to \mathbb{G}_m \to Pic^0(C'/k) \to Pic^0(C/k) \to 0$$

Then, $Pic^0(C'/k) = G_q$ for $q = \phi_\Theta(q_1 - q_2) \in \widehat{Pic^0(C/k)}$

The Poincaré bi-extension $\mathcal{P}$

Put all the G_q's together $\rightsquigarrow \mathcal{P} =$ the Poincaré bundle minus zero section, plus a rigidification, cf. drawing on black (or white) board. $\mathcal{P}$ expresses the biduality $\hat{\hat{A}} \simeq A$, via a canonical $\hat{\mathcal{P}}_{p,q} \simeq \mathcal{P}_{q,p}$.

$$\begin{array}{ccc} P & \in & G = G_q \\ \downarrow & & \pi \downarrow \\ p & \in & A \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{P} & \ni & P \\ \downarrow \varpi & & \downarrow \\ \hat{A} \times A & \ni & (q,p) \end{array}$$

$\mathcal{P}|_{q \times A} = G_q$, $\quad \mathcal{P}|_{\hat{A} \times 0} = \mathbb{G}_m \times \hat{A}$ (in particular, "1" $\in \mathcal{P}_{0,0}$).
For $\varphi : A' \to A$ wth transpose $\varphi^* = \hat{\varphi} : \hat{A} \to \hat{A}'$, $\mathcal{P}_{q,\varphi(p')} \simeq \mathcal{P}'_{\hat{\varphi}(q),p'}$

Compare $(\hat{V} \times V) \ni (\lambda, v) \to \lambda(v) =$ "v"$(\lambda) \leftarrow ($"v"$, \lambda) \in \hat{\hat{V}} \times \hat{V}$
For $\varphi : V' \to V$, with transpose $\hat{\varphi}$, $\lambda(\varphi(v')) = \hat{\varphi}(\lambda)(v')$.

Ribet points (work from the 80's by Breen and Ribet)

Analogues of Lagrangian subspaces : apart from $\hat{A} \times 0$ and $0 \times A$, unexpected abelian subvarieties $B \subset \hat{A} \times A$ such that $\mathcal{P}|_B$ is a trivial $\mathbb{G}_m$-torsor can arise. Indeed, for any antisymmetric

$$\varphi : \hat{A} \to A, \hat{\varphi} = -\varphi, \text{with graph } B' = (id, \phi)(\hat{A}),$$

$$\mathcal{P}|_{B'} \simeq (id, \varphi)^* \mathcal{P} \simeq (\hat{\varphi}, id)^* \hat{\mathcal{P}} \simeq (id, \hat{\varphi})^* \mathcal{P} \simeq -\mathcal{P}|_{B'} \in Pic^0(B'),$$

so $\mathcal{P}$, restricted to the graph B of 2φ, admits a canonical section

$$\sigma : B \to \mathcal{P}|_B.$$

For any $q \in \hat{A}$ and antisymmetric φ, the point $R = \sigma(q, 2\varphi(q))$ of $\mathcal{P}_{q \times A} = G_q$, with $\pi(R) = p = 2\varphi(q)$, is called the **Ribet point** *of G_q attached to φ. Ditto for $R' \sim R$ "isogeneous to" R.*

Ribet's initial construction (JNT 25, 1987, 133-151 and 152-161) :
Given an isogeny $\varphi : \hat{A} \longrightarrow A$ and $q \in \hat{A}$, consider the extension

$$G = G_q \in Ext(A, \mathbb{G}_m), \text{ parametrized by } q \in \hat{A}$$

and its pullback

$$\varphi^* G = G'_{\hat{\varphi}(q)} = G' \in Ext(\hat{A}, \mathbb{G}_m) \simeq A, \text{ parametrized by } \hat{\varphi}(q) \in A.$$

$$\begin{array}{ccccccccccc}
 & & & & R_0 & \longrightarrow & q & & & & \\
0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & G' & \stackrel{\pi}{\longrightarrow} & \hat{A} & \longrightarrow & 0 & & [\hat{\varphi}(q) \in A] \\
 & & \| & & \downarrow \varphi & & \downarrow \varphi & & & & \\
0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & G & \stackrel{\pi}{\longrightarrow} & A & \longrightarrow & 0 & & [q \in \hat{A}] \\
 & & & & R_1 & \longrightarrow & \varphi(q) & & & &
\end{array}$$

Choose a point $R_0 \in G'$ above q. Then, $R_1 = \varphi(R_0) \in G$ above $\varphi(q)$. The dual of the one-motive $\{R_0 \in G'_{\hat{\varphi}(q)}\}$ is $\{R_2 \in G_q\}$, with $R_2 = R_0 \in \mathcal{P}$ above $(q, \hat{\varphi}(q))$. The Ribet point is

$$R = R_1 - R_2 \in G_q, \text{ lying above } p = (\varphi - \hat{\varphi})(q).$$

2. Kummer theory

Aim : MM *and reducing* ML *to* M (= Mordell)

For $P \in X(k) \cap \Delta$, study the Galois orbit $\Gamma_k.\frac{1}{\ell}P \subset X(\bar{k}) \cap \Delta$.

$$[k(\tfrac{1}{\ell}P) : k] \leq \sharp G[\ell].$$

Should be large if P is far from a special subvariety of G.
We say that P is non-degenerate if $\mathbb{Z}.P$ is Zariski-dense in G.

- Torus case $G = T = (\mathbb{G}_m)^r$: for $\alpha = (\alpha_1, ..., \alpha_r) \in T(k)$,

$$\alpha \text{ non} - \text{deg.} \;\Leftrightarrow [k(\alpha^{\frac{1}{\ell}} : k] >> \ell^r$$

- Abelian case $G = A$: $r \rightsquigarrow 2g$. Setting $F := k(A[\ell])$,

$$p \text{ non} - \text{deg.} \;\Leftrightarrow [F(\tfrac{1}{\ell}p) : F] >> \ell^{2g}.$$

- $G = G_q \in Ext(A, \mathbb{G}_m), q$ non-tor. : P non-deg. $\Leftrightarrow p = \pi(P)$ non-deg. We expect ℓ^{2g+1}, and do get it, *except* is P is a Ribet point R, in which case we have our **first counter-example** :

$$[F(\tfrac{1}{\ell}R) : F] << \ell^{2g}$$

However, one does not need a big power of ℓ (this is already apparent in Hindry 1988), and indeed, ML (hence MM) holds true for any semi-abelian variety G/k.

Still, the reason for the Galois-degeneracy of Ribet points is worth studying. We will now restrict to an elliptic curve

$$A = E \simeq \hat{E}, \varphi \rightsquigarrow \beta \in \mathcal{O} = End(E), \hat{\varphi} \rightsquigarrow \overline{\beta} = -\beta.$$

$k =$ a number field, with absolute Galois group $\Gamma_k = Gal(\bar{k}/k)$.
$\ell =$ a prime number, larger than a "constant" $c(G, k, P)$ depending only on the indicated data.

$*$ The case of the multiplicative group $\mathbb{G}_m$:

i) $\mathbb{G}_m[\ell] := \mu_\ell = \{\ell\text{-th roots of unity}\} \simeq \mathbb{F}_\ell$, on which Γ_k acts by the cyclotomic character $\chi_\ell : \Gamma_k \to GL_1(\mathbb{F}_\ell) = \mathbb{F}_\ell^* : \gamma.\zeta_\ell = \zeta_\ell^{\chi_\ell(\gamma)}$.

ii) Let $\alpha \in \mathbb{G}_m(k)$. The ℓ-th roots of α provide an "affine" representation of Γ_k :

$$\gamma.\alpha^{\frac{1}{\ell}}/\alpha^{\frac{1}{\ell}} = \xi_\alpha(\gamma) \in \mu_\ell \; ; \; \gamma.(\zeta_\ell^n \alpha^{\frac{m}{\ell}}) = \zeta_\ell^{\chi_\ell(\gamma)n} \xi_\alpha(\gamma)^m \alpha^{\frac{m}{\ell}}$$

The corresponding vectorial representation is given by

$$M_\alpha[\ell] := \{x \in \mathbb{G}_m(\bar{k}), x^\ell \in <\alpha>\}/<\alpha> = \{\zeta_\ell^n \alpha^{\frac{m}{\ell}}, \begin{pmatrix} n \\ m \end{pmatrix} \in \mathbb{F}_\ell^2\};$$

$$0 \to \mu_\ell \to M_\alpha[\ell] \to \mathbb{F}_\ell \to 0 \quad \rightsquigarrow \quad M_\alpha[\ell] \in Ext_{\mathbb{F}_\ell[\Gamma_k]}(\mathbf{1}, \mu_\ell)$$

$$\rho_\alpha(\gamma) = \begin{pmatrix} \chi_\ell(\gamma) & \xi_\alpha(\gamma) \\ 0 & 1 \end{pmatrix} \qquad \begin{array}{c} k(\zeta_\ell, \alpha^{\frac{1}{\ell}}) \\ \uparrow \\ k(\zeta_\ell) \\ \uparrow \\ k \end{array}$$

$\xi_\alpha : Gal(k(\zeta_\ell, \alpha^{\frac{1}{\ell}})/k(\zeta_\ell)) \hookrightarrow \mu_\ell$:

$\chi_\ell : Gal(k(\zeta_\ell)/k) \hookrightarrow \mathbb{F}_\ell^* = GL_1($

$*$ The general case (with $A = E$ for simplicity).

For $P \in G_q(k)$, with $\pi(P) = p \in E(k)$, the picture becomes :

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_\ell & \longrightarrow & G_q[\ell] & \xrightarrow{\pi} & E[\ell] & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \iota & & \\
0 & \longrightarrow & \mu_\ell & \longrightarrow & M_P[\ell] & \xrightarrow{\pi} & M_p[\ell] & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{F}_\ell & = & \mathbb{F}_\ell & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & ,
\end{array}$$

"blending" $M_p[\ell] \in Ext(\mathbf{1}, E[\ell])$ and $G_q[\ell] \in Ext(E[\ell], \mu_\ell)$ (notice $\widehat{G_q[\ell]} \in Ext(\mathbf{1}, \hat{E}[\ell])$. The corresponding Galois representations are :

$$\begin{array}{ccccc} & & F_P & & \\ & & \uparrow & & \\ & & F_{pq} & & \\ & \nearrow & & \nwarrow & \\ F_q & & & & F_p \\ & \nwarrow & & \nearrow & \\ & & F & & \\ & & \uparrow & & \\ & & k & & \end{array} \quad ,$$

$$P \in G(k), \pi(P) = p, G = G_q$$

$$F_P = k(G[\ell], \tfrac{1}{\ell} P)$$

$$F_{pq} = k(G[\ell], \tfrac{1}{\ell} p)$$

$$F_q = k(G[\ell]) = k(E[\ell], \tfrac{1}{\ell} q)$$

$$F_p = k(E[\ell], \tfrac{1}{\ell} p)$$

$$F = k([E[\ell])$$

$$\rho_P(\gamma) = \begin{pmatrix} \chi_\ell(\gamma) & {}^t\xi_q(\gamma) & \tau_P(\gamma) \\ 0 & \rho_E(\gamma) & \xi_p(\gamma) \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tau_P : Gal(F_P/F_{pq}) \hookrightarrow \mu_\ell \simeq \mathbb{F}_\ell$$

$${}^t\xi_q : Gal(F_q/F) \hookrightarrow \hat{E}[\ell] \simeq \mathbb{F}_\ell^2$$

$$\xi_p : Gal(F_p/F) \hookrightarrow E[\ell] \simeq \mathbb{F}_\ell^2$$

$$\rho_E : Gal(F/k) \hookrightarrow GL_2(\mathbb{F}_\ell)$$

Theorem (Jacquinot-Ribet, 1984)

Let $G = G_q$ with a non-torsion $q \in \hat{E}(k)$, and let $P \in G(k)$, with a non-torsion $p = \pi(P) \in E(k)$.

i) Assume that p and q are linearly independent over $\mathcal{O}$. Then,

$$Gal\big(F_P/F\big) \simeq \mu_\ell \rtimes \big(\hat{E}[\ell] \times E[\ell]\big).$$

ii) Assume that $q = \beta p$ in $E(k)/E_{tor}$, with $\beta \in \mathcal{O}_{\mathbb{Q}}, \overline{\beta} \neq -\beta$. Then,

$$Gal(F_P/F) \simeq \mu_\ell \rtimes E[\ell].$$

iii) Assume that $q = \beta p$ with $\overline{\beta} = -\beta$. Then,

either $P \not\sim R \Rightarrow Gal(F_P/F) \simeq \mu_\ell \times E[\ell]$.,

or $P \sim R \Rightarrow Gal(F_R/F) \simeq E[\ell]$.

Proof : for X, X' in $\mathfrak{u}_P =$ "*Lie*" $Gal(F_P/F)$, with coefficients $({}^t y, x) \in Im({}^t\xi_q, \xi_p) \subset \hat{E}[\ell] \times E[\ell]$, $t \in \mathbb{F}_\ell \simeq \mu_\ell$,

$$X = \begin{pmatrix} 0 & {}^t y & t \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, X', \text{ we have } [X, X'] = \begin{pmatrix} 0 & 0 & t(X, X') \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{where } t(X, X') = < y|x' > - < y'|x >$$

Polarization $\phi_L = \hat{\phi}_L : \hat{E} \simeq E$ (symmetric) $\rightsquigarrow$ antisymmetric Weil pairing $< | >$ and ^{t}; also, $\hat{\beta} = \overline{\beta}$ (Rosati involution on $End E = \mathcal{O}$).

Since $< | >$ is non-degenerate, this settles Case (i) :

$Gal(F_P/F_{pq}) \simeq Im(\tau_P) = \mathbb{F}_\ell \Rightarrow Gal(F_P/F) =$ Heisenberg group.

Case (ii) , with $q = \beta p$: for any x, x' in $E[\ell]$, occurring in matrices X, X' and such that $< x|x' > \neq 0$, we get :

$$\begin{aligned} t(X, X') &= < \beta x|x' > - < \beta x'|x > = < \beta x|x' > - < x'|\overline{\beta}x > \\ &= < \beta x|x' > + < \overline{\beta}x, x' > = < (\beta + \overline{\beta})x|x' > \neq 0 \end{aligned}$$

since $\beta + \overline{\beta}$ is a non-zero integer. Again, $Im(\tau_P) = \mathbb{F}_\ell$.

Case (iii) : now, $\beta + \bar{\beta} = 0 \Rightarrow Gal(F_P/F)$ is *abelian*. The two cases are distinguished by the possibility to lift $\beta \sim \varphi : \hat{E}[\ell] \to E[\ell]$ to a Γ_k-equivariant self-duality

$$\Phi : \widehat{M_P}[\ell] \simeq M_P[\ell]$$

Then, $P \sim R \Leftrightarrow M_P[\ell]$ is antisymmetrically self-dual $\Leftrightarrow \tau_P = 0$. Holds over any tannakian category, cf. DB. ArXiv 1011.4685.

3. Lehmer's problem on heights

Unlikely intersections (or *Bogomolov*, cf. E. Breuillard's talk) $\rightsquigarrow$ upper bounds for normalized heights $\hat{h}$. Conclude with lower bound of Lehmer type : $\hat{h}(P) >> [\mathbb{Q}(P) : \mathbb{Q}]^{-\frac{1}{?}}$, where "?" should measure **how far** $P \in G(\bar{\mathbb{Q}})$ **is from a special subvariety of** G.
From $G = T$ or A, expect :

$$P \text{ non} - \text{deg.} \Rightarrow \hat{h}(P) >> [\mathbb{Q}(P) : \mathbb{Q}]^{-\frac{1}{dimG}}$$

For $G = G_q \in Ext(A, \mathbb{G}_m)$, q non-tor., $P \sim R$ gives our **second counter-example**. Qualitatively, this is reflected by

Theorem

Let R be a Ribet point in $G_q(k)$. For any place $v \in \mathcal{M}_k$ of k, R lies in the maximal compact subgroup of the topological group $G(k_v)$.

Contrasts with Kronecker's theorem : let $\alpha \in \mathbb{G}_m(k)$ such that for any place $v \in \mathcal{M}_k$, α lies in the maximal compact subgroup of the topological group $\mathbb{G}_m(k_v) = (k_v)^*$. Then, α is a root of unity.

Normalized heights and Lehmer bounds.

$[n]_G$-equivariant compactification $\bar{G}$ of G, $D \in Pic(\bar{G}/k)$ s.t.

$$[n]^* D \sim n^\kappa D \Rightarrow \hat{h}_D(nP) = n^\kappa \hat{h}_D(P) \Rightarrow \hat{h}_D(\tfrac{1}{\ell}P) = \tfrac{1}{\ell^\kappa}\hat{h}_D(P)$$

So, the following bounds would be best possible :

- Torus case $G = T$: for $P = (\alpha_1, ..., \alpha_r) \in T(\bar{\mathbb{Q}})$: $\hat{h}$ linear

$$P \text{ non}-\text{deg.} \Rightarrow \hat{h}(P) >> [\mathbb{Q}(P) : \mathbb{Q}]^{-\frac{1}{r}} \text{ ?}$$

- Abelian case $G = A$: $r \rightsquigarrow g = \dim A$: $\hat{h}$ quadratic

$$P \text{ non}-\text{deg.} \Rightarrow \hat{h}(P) >> [\mathbb{Q}(P) : \mathbb{Q}]^{-\frac{1}{g}} \text{ ?}$$

- $G = G_q \in Ext(A, \mathbb{G}_m), q$ non-tor. : $\hat{h} = \hat{h}_A \circ \pi + \hat{h}_{lin}$ so

$$\hat{h}(\tfrac{1}{\ell}P) = \tfrac{1}{\ell^2}\hat{h}_A(p) + \tfrac{1}{\ell}\hat{h}_{lin}(P)$$

True, $\frac{1}{\ell} >> (\ell^{2g})^{-\frac{1}{g+1}}$, *but* $\hat{h}_{lin}(R) = 0$, and $\frac{1}{\ell^2} << (\ell^{2g+1})^{-\frac{1}{g+1}}$.

Proof (of Thm. $\sim h_{lin}(R) = 0$, cf. DB, Duke MJ 1995.)
By the product formula, the (absolute, logarithmic) normalized height on $\mathbb{G}_m(k)$ is

$$\hat{h}(\alpha) = \Sigma_{v \in \mathcal{M}_k} \frac{[k_v : \mathbb{Q}_p]}{[k:\mathbb{Q}]} \; |\ell og(|\alpha|_v)|.$$

For $G = G_q$ and $v \in \mathcal{M}_k$, there is a unique extension of $\ell og|.|_v$ to $\lambda_v = \lambda_v^{(q)} : G(k_v) \to \mathbb{R}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k_v^* & \longrightarrow & G(k_v) & \xrightarrow{\pi} & A(k_v) & \longrightarrow & 0 \\ & & \downarrow \ell og|.|_v & & \downarrow (\lambda_v, \pi) & & \| & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} \times A(k_v) & \xrightarrow{\pi} & A(k_v) & \longrightarrow & 0 \end{array}$$

Then, $ker(\lambda_v) =$ maximal compact subgroup of $G(k_v)$, while

$$\hat{h}_{lin}(P) = \Sigma_{v \in \mathcal{M}_k} \frac{[k_v : \mathbb{Q}_p]}{[k:\mathbb{Q}]} \; |\lambda_v(P)|$$

Similar extension of $\ell og|.|_v$ to $\mathbb{L}_v : \mathcal{P}(k_v) \to \mathbb{R}$. One then checks that $\mathbb{L}_v(R) = 0$, while $\mathbb{L}_v|_{G_q} = \lambda_v^{(q)}$.

4. Lindemann-Weierstrass

Aim : determine "algebraic locus" in Pila-Wilkie thanks to lower bounds, à la Ax-Schanuel, for functional transcendence degrees ($\Rightarrow$ MM and RMM via Zannier's strategy).

$S =$ (pointed) algebraic curve over $\mathbb{C}$ $\rightsquigarrow$ $K = \mathbb{C}(S)$.
A/S abelian scheme, $LA/S =$ relative tangent bundle.
$A_0 = K/\mathbb{C}$-trace of $A =$ maximal constant part.
Exponential morphism over S^{an}:

$$0 \longrightarrow \Pi_A \longrightarrow LA^{an} \xrightarrow{exp_A} A^{an} \longrightarrow 0 \ .$$

For $u \in LA(K)$, extended to a section of LA/S, set $p = exp_A(u)$ $\in A(S^{an})$. Then, $tr.deg.K(p)/K)$ should measure **how far u is from a "special" Lie subalgebra of** LA, modulo $\Pi_A +$ *constants*.

Theorem (DB - A. Pillay, 2008)

Let $u \in LA(K)$ be non-degenerate, i.e. s.t. for any proper abelian subvariety H of A, $u \notin LH(K) + LA_0(\mathbb{C})$, and let $p = exp_A(u)$. Then, $tr.deg.(K(p)/K) = dim(A)$

Semi-constant semi-abelian varieties

$\pi : G/S \to A/S$ semi-abelian scheme of constant toric rank.
LG/S = relative tangent bundle, exponential morphism over S^{an}:
But now, the $K/\mathbb{C}$-trace G_0 and $K/\mathbb{C}$-image G^0 can be very different. Let $G^{sc} := \pi^{-1}(A_0)$ be the "semi-constant part" of G.
Then, $LG \ni U \mapsto P = exp_G(U) \in G$ satisfies :

- if G^{sc} is defined over $\mathbb{C}$ ($\Leftrightarrow G^{sc} = G_0$), we still have :

$$U \in LG(K) \ \mathrm{non-deg.} \ \Rightarrow tr.deg.K(P)/K = dimG.$$

- while $G_0 \subsetneq G^{sc}$ provides our **third counter-example** :

- $E = E_0 \times S$ constant elliptic scheme, $q \in \hat{E}(K)$ non-constant,
$G = G_q \in Ext_S(E, \mathbb{G}_m)$; then, $G_0 = \mathbb{G}_m$;
- $U \in LG(K)$, $0 \neq \pi(U) = u \in LE_0(\mathbb{C})$; then, U is non-deg., but

$$tr.deg.(K(P)/K) = 1 < 2.$$

In the fully constant case (Ax) : $q \in \hat{E}_0(\mathbb{C})$, $G_0 = G$, so

$$U \text{ non} - \text{deg.} \Rightarrow U \notin L\mathbb{G}_m(K) + LG_0(\mathbb{C}) \Rightarrow \pi(U) \notin LE_0(\mathbb{C})$$

No semi-constant nor Ribet-type degeneracy to be considered.

In general, set $q = exp_{\hat{E}}(v), v \in L\hat{E} \simeq LE, G = G_q$. If E has CM, then $E = E_0 \times S$, so for q non-constant, $U \in LG(K)$ gives a Ribet point $exp_G(U) = R$ only if $v = \beta u \in LE(K) \setminus LE_0(\mathbb{C})$. Then, $q \notin \hat{E}(K)$, G is transcendental over K and we will be forced to consider a general Schanuel-André problem (still open).

The results above come from the study of the (model-theoretic) Manin kernels $A^\sharp, G^\sharp =$ DAG groups. In fact,

$$G^{sc} = G_0 \Rightarrow \bar{K}(G^\sharp) = \bar{K} \Rightarrow tr.deg.K(G^\sharp, P)/K(G^\sharp)) = dimG.$$

Otherwise, $\bar{K} \subsetneq \bar{K}(G^\sharp)$; determining when

$$tr.deg.K(G^\sharp, P)/K(G^\sharp, p)) = 1 \text{ or } 0$$

is still an open question in this case (counter-ex. included).

5. Relative Manin-Mumford

CIT : $X \subset G$ irreducible of codimension d, with $X \cap G^{[<d]}$ Zariski dense in $X \Rightarrow X \subset$ strict algebraic subgroup of G ?

RMM : let $S/\mathbb{C}$ be an irreducible variety, let G/S be a semi-abelian scheme, and let $G_{tor} = \cup_{s \in S(\mathbb{C})}(G_s)_{tor}$. Let $X \subset G$, irreducible and of codimension ≥ 1, such that $X \cap G_{tor}$ is Zariski-dense in X.Then, $X \subset$ strict subgroup scheme of G/S ?

In particular, if S is a curve, and $P : S \to G$ is a section of G/S which does not factor through any proper closed subgroup scheme of G/S, then its image $P(S) := X$ should contain only finitely many points of G_{tor}. Our **fourth counter-example** will concern a semi-abelian scheme G/S, and I hasten to say that

$*$ RMM should hold true for any abelian scheme;
$*$ apart from this counter-example and its isogeny class, RMM does hold true for any semi-abelian surface scheme over a curve S/k

Cf. current work of D. Masser, U. Zannier, A. Pillay, D.B.

Theorem

Let E_0 be a CM elliptic curve, and let $\varphi : \hat{E}_0 \to E_0$ be an anti-symmetric isogeny. There exists an open subset S of $\hat{E}_0$ with the following property. Let $E = E_0 \times S$, let $G/S \in Ext_S(E, \mathbb{G}_{m/S})$ be the restriction to S of the universal semi-abelian scheme $\mathcal{P}/\hat{E}_0$, and let $R : S \to G$ be the universal Ribet section

$$S \ni s = q \mapsto R(q) = \sigma(q, 2\varphi(q)) \in G_s = G_q.$$

Then, $X = R(S)$ satisfies :

i) $\pi(X)^{Zar} = E$, so X lies in no strict subgroup scheme of G/S;
ii) for any $s = q \in S$ such that $\pi(R(s)) := p(s) = 2\varphi(q) \in (E_s)_{tor} \simeq (E_0)_{tor}$, $R(s)$ is a torsion point of the fiber $G_s = G_q$ of G/S.

The proof uses the construction of σ via Cartier duality (cf. DB, ArXiv 1104.5178v1). Viewing G/S as a generalized jacobian, B. Edixhoven has shown that $s \in \hat{E}_0[\ell] \Rightarrow R(s) \in G_q[\ell^2]$.

Corollary

For any section $P : S \to G$ such that $p := \pi \circ P = 2\varphi$, but not isogeneous to R, the curve $X = P(S)$ meets finitely many points of G_{tor} (i.e. satisfies RMM).

Proof : set $P - R = f : S \to \mathbb{G}_m$. Then, $(f, 2\varphi)(S)$ gives a curve in the (constant, split) semi-abelian variety $\mathbb{G}_m \times E_0$. Apply old MM !

• For the other cases, compute the "algebraic locus" in Pila-Wilkie via *logarithms of K-rational points*. (Recall that $K = \mathbb{C}(S)$.)

The Lie algebras of the universal vectorial extensions $\tilde{E} \simeq \hat{\tilde{E}}, \tilde{G}$ of $E, G = G_q$, carry canonical connections. Lift the K-rational points q, p, P to $\tilde{q}, \tilde{p} \in L\tilde{E}(K), \tilde{P} \in \tilde{G}(K)$. Let

$$\tilde{v} = log_{\tilde{E}} \tilde{q}, \tilde{u} = log_{\tilde{E}} \tilde{p}, \tilde{U} = log_{\tilde{G}} \tilde{P},$$

and set $\mathbb{C} \otimes \Pi_E = (L\tilde{E})^{\partial}, F = K((L\tilde{E})^{\partial})$.

Theorem

Let $G = G_q$ with a non-constant $q \in \hat{E}(K)$, let $P \in G(K)$, with a non-torsion $p = \pi(P) \in E(K)$, and set $F_P = F(\tilde{u}, \tilde{v}, \tilde{U})$.

i) If p and q are linearly independent over $\mathcal{O}$ mod. $E_0(\mathbb{C})$, then,

$$deg.tr.(F_P/F) = 5$$

ii) Assume that $q = \beta p$ in $E(K)/E_0(\mathbb{C})$, with $\beta \in \mathcal{O}, \overline{\beta} \neq -\beta$, or that $q = \beta p + p_0$ with $\overline{\beta} = -\beta$ and a non-torsion $p_0 \in E_0(\mathbb{C})$, or that $p \in E_0(\mathbb{C})$. Then,

$$tr.deg.(F_P/F) = 3$$

iii) Assume that $q = \beta p$ in $E(K)$, with $\overline{\beta} = -\beta$. Then,

either $P \not\sim R \Rightarrow tr.deg.(F_P/F) = 3$,

or $P \sim R \Rightarrow tr.deg.(F_P/F) = 2$.

Cf. DB, Newton 2006 & [BMPZ], 2011. Implies RMM for $P \not\sim R$.

$G \in Ext(E, \mathbb{G}_m)$ leads to an exact sequence of $\mathbb{Z}$-local systems

$$0 \to \Pi_{\mathbb{G}_m} = \mathbb{Z} \to \Pi_G \to \Pi_E \to 0,$$

equivalently, to a representation $\rho_G : \pi_1(S, s_0) \to GL_{2g+1=3}(\mathbb{Z})$, $\rho_G \in Ext_{\pi_1}(\rho_E, \mathbf{1})$, and $\Pi_P := \{log_G(\mathbb{Z}.P)\} \rightsquigarrow \rho_P \in Ext_{\pi_1}(\mathbf{1}, \rho_G)$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pi_G & \xrightarrow{\pi} & \Pi_E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pi_P & \xrightarrow{\pi} & \Pi_p & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z} & = & \mathbb{Z} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & ,
\end{array}$$

$\Pi_p = \{log_E(\mathbb{Z}.p)\}, \Pi_G \simeq \hat{\Pi}_q$. In Picard-Vessiot terms (= proof !) :

$$\begin{array}{ccccc} & & F_P & & \\ & & \uparrow & & \\ & & F_{pq} & & \\ & \nearrow & & \nwarrow & \\ F_q & & & & F_p \\ & \nwarrow & & \nearrow & \\ & & F & & \\ & & \uparrow & & \\ & & K & & \end{array} \quad ,$$

$$P \in G(k), \pi(P) = p, G = G_q$$
$$\tilde{v} = log_{\tilde{E}}\tilde{q}, \tilde{u} = log_{\tilde{E}}\tilde{p}, \tilde{U} = log_{\tilde{G}}\tilde{P}$$

$$F_P = k(\tilde{u}, \tilde{v}, \tilde{U})$$
$$F_{pq} = F(\tilde{u}, \tilde{v})$$
$$F_q = K((L\tilde{G})^{\partial}) = F(\tilde{v})$$
$$F_p = F(\tilde{u})$$

$$F = K((L\tilde{E})^{\partial})$$

$$\rho_P(\gamma) = \begin{pmatrix} 1 & {}^t\xi_q(\gamma) & \tau_P(\gamma) \\ 0 & \rho_E(\gamma) & \xi_p(\gamma) \\ 0 & 0 & 1 \end{pmatrix} \quad ,$$

$$\tau_P : Gal_{\partial}(F_P/F_{pq}) \hookrightarrow \mathbb{C}$$
$${}^t\xi_q : Gal_{\partial}(F_q/F) \hookrightarrow \mathbb{C}^2 \simeq \widehat{(L\tilde{E})}^{\partial}$$
$$\xi_p : Gal_{\partial}(F_p/F) \hookrightarrow \mathbb{C}^2 \simeq (L\tilde{E})^{\partial}$$
$$\rho_E : Gal_{\partial}(F/K) \hookrightarrow SL_2(\mathbb{C})$$

6. Pink's general conjecture

In "amplitude" 0, R. Pink's general conjecture reads : let $\mathbf{S}$ be a mixed Shimura variety over $\mathbb{C}$, let Δ be a set of special points of $\mathbf{S}$, and let X be an irreducible closed subvariety of $\mathbf{S}$ of codimension $d \geq 1$. Assume that $X \cap \Delta$ is Zariski dense in X. Then, X is a special subvariety ($=$ of Hodge type) of $\mathbf{S}$.

In this context, the counterexample to RMM turns into a :
Pro-example : given a totally imaginary quadratic integer β, and $g \geq 1$, there is a mixed Shimura variety $S(\beta)$ with a natural embedding $i : X \to S(\beta)$ of the image $X = R(S)$ of the Ribet section, such that

Theorem

The algebraic subvariety $i(X)$ of the mixed Shimura variety $S(\beta)$ passes through a Zariski-dense set of special points of $S(\beta)$ - and is indeed a special subvariety of $S(\beta)$.

Construction of $S(\beta)$ (cf. ArXiv 1104.5178) :

Start with a component S_0 of the pure Shimura variety parametrizing ppav's (A, ϕ) of dimension g with $\beta \in End(A)$ (if $g = 1$, S_0 is a point $\{E_0\}$). Let $\mathcal{A}$ be the universal abelian scheme over S_0. Then, $S_1 = \hat{\mathcal{A}} \times_{S_0} \mathcal{A}$ is a mixed Shimura variety parametrizing (q, p)'s on $\hat{A} \times A, \{A\} \in S_0$. Set $S_1(\beta) = \{A, q, p = 2\beta\phi(q)\}$, and let $\varpi : \mathcal{P} \to S_1$ be the Poincaré bi-extension, viewed as the universal extension $\mathcal{G}$ of $\mathcal{A}$ by $\mathbb{G}_m$, over its parameter space $\hat{\mathcal{A}}$. Then,

$$S(\beta) = \varpi^{-1}(S_1(\beta))$$

is a mixed Shimura variety parametrizing points P on fibers G_q of $\mathcal{G}$ such that $\pi(P) = 2\beta\phi(q)$. In particular, for $g = 1$, $X = R(S)$ has a canonical embedding i into $S(\beta)$ above the injection $S \hookrightarrow \hat{E}_0$.

Now ,

• a special point of $S(\beta)$ represents a couple $(P \in G_q)$ such that the underlying A has CM, G_q is an isotrivial extension, and P is a torsion point on G_q. For $s = q \in (\hat{E}_0)_{tor}$, these conditions are all satisfied by $R(q)$, so $i(X)$ contains infinitely many special points.

• the same study as in the two "multiple choice" theorems shows that the generic Mumford-Tate groups of the special subvarieties of $S(\beta)$ are characterized by the condition $\xi = 0$ or $\tau = 0$. So, $i(X)$ is a subvariety of Hodge type of $S(\beta)$.

Conclusion : the problem comes from the existence of a two-step filtration in the unipotent radical of the generic Mumford-Tate group of the mixed Shimura varieties parametrizing one-motives. No such phenomenon will occur for the study of abelian schemes.

Et pour finir :

Joyeux anniversaire,

Anand !