On André's proof of the Siegel-Shidlovsky theorem.

Daniel Bertrand

Colloque de Théorie des Nombres, Maison franco-japonaise et Université Keio, Novembre 1998.

Abstract. In the first four sections of this paper, we describe Yves André's beautiful new proof [2] , [3] of the theorem of Siegel-Shidlovsky on values of E-fonctions. Our last two sections are devoted to the generalized relation of Fuchs on exponents (cf. [4], [5]), which plays the role of a multiplicity estimate in André's method.

Résumé. Récemment, Yves André a obtenu une démonstration du théorème de Siegel et Shidlovsky, comme sous-produit de sa théorie de Gevrey arithmétique. On décrit cette preuve, d'une facture toute nouvelle en transcendance, ainsi que la généralisation au cas irrégulier de la relation de Fuchs, qui joue dans sa méthode le rôle du lemme de Shidlovsky.

1 Introduction.

Let $\overline{\boldsymbol{Q}}$ be an algebraic closure of $\boldsymbol{Q}$. By an E-function, we shall mean in the present note a power series

$$F = \Sigma_{m\geq 0}\frac{a_m}{m!}z^m \in \overline{\boldsymbol{Q}}[[z]]$$

satisfying the following conditions (which, as far as (ii) and (iii) are concerned, are slightly stricter than Siegel's, cf. [3]):

i) F is a solution of a differential equation with coefficients in $\overline{\boldsymbol{Q}}(z)$ (in particular, the a_n's generate a number field); we shall denote by $D_F \in \overline{\boldsymbol{Q}}(z)[d/dz]$ the monic operator of minimal order such that $D_F(F) = 0$, and by n_F the order of D_F.

ii) for each archimedean absolute value $|.|$ on $\overline{\boldsymbol{Q}}$, the sequence $\{|a_m|; m \geq 0\}$ is bounded from above by a geometric progression (in particular, F defines an entire function $F(z)$ for each complex embedding of $\overline{\boldsymbol{Q}}$);

iii) there exists a sequence $\{d_m; m \geq 0\}$ of positive integers, bounded from above by a geometric progression, such that $d_m a_r$ is an algebraic integer for all $0 \leq r \leq m$.

As is well-known (cf. [11]), we then have

Theorem 0 (Siegel-Shidlovsky). *Let $\overline{F} = {}^t(F_1, \ldots, F_n)$ be a vector of E-functions, and let A be an $n \times n$ matrix with coefficients in $\overline{\boldsymbol{Q}}(z)$ such that $\frac{d}{dz}\overline{F} = A\overline{F}$. Fix a complex embedding of $\overline{\boldsymbol{Q}}$, and a point $\alpha \neq 0 \in \overline{\boldsymbol{Q}}$ away from the poles of A. Then,*

$$tr.deg.(\overline{\boldsymbol{Q}}(F_1(\alpha), \ldots, F_n(\alpha))/\overline{\boldsymbol{Q}}) = tr.deg.(\overline{\boldsymbol{Q}}(z)(F_1, \ldots, F_n)/\overline{\boldsymbol{Q}}(z)).$$

The differential assumption on $\overline{F}$ makes Condition (i) above redundant. Note also that the set of poles of A may be strictly smaller than the union of the sets of singularities of the differential operators $D_{F_i}, i = 1, \ldots, n$. But the requirement that α is not a pole of A is crucial: the conclusion of Theorem 0 ceases to hold if $F_1, \ldots, F_n$ is replaced by $(z - \alpha)F_1, \ldots, (z - \alpha)F_n$.

The new proof devised by Y. André of Theorem 0 relies on three ingredients. The most important one, which looks deceptively simple, reads as follows.

Theorem 1 (André)**.** *Let F be an E-function, and let $D_F \in \overline{\boldsymbol{Q}}(z)[d/dz]$ be the (monic) differential operator of minimal order such that $D_F(F) = 0$. Then, the differential equation $D_F(y) = 0$ admits a basis of analytic solutions at any point $\alpha \in \mathbf{P}_1(\boldsymbol{C}), \alpha \neq 0, \infty$.*

(Such a point α may well be a singularity of the differential operator D_F and Theorem 1 says in this case that it must be an *apparent* singularity.) Denoting by $n = n_F$ the order of $D = D_F$, and by ord_α the order function on the local ring $\overline{\boldsymbol{Q}}[[z - \alpha]]$, we may then consider the n successive maxima $\{e^\alpha_{n-1}, \ldots, e^\alpha_0\}$ of ord_α on the $\boldsymbol{C}$-vector space of solutions of $Dy = 0$ which are analytic at α, and, inspired by Weierstrass points on curves, define the *defect* of D at α by the formula

$$\delta_\alpha(D) = \Sigma_{i=0,\ldots,n-1}(e^\alpha_i - i).$$

(By Cauchy's theorem, $\delta_\alpha(D) = 0$ if α is an ordinary point of D; it is positive if α is an apparent singularity). With these notations, Theorem 1 implies :

Corollary 1 (André)**.** *If an E-function F belongs to $\boldsymbol{Q}[[z]]$ and if $\alpha \in \mathbf{P}_1(\boldsymbol{Q}), \alpha \neq 0, \infty$, then*

$$\delta_\alpha(D_F) \geq \; n_F \; . \; ord_\alpha(F).$$

The rationality assumptions on α and on F in this corollary are crucial for its proof. As explained at the end of this section, its conclusion encompasses a extrapolation process of an entirely new nature in transcendence theory.

A standard feature in transcendence proofs, the second ingredient is a multiplicity estimate. A convenient way to describe it here consists in setting, for any differential operator $D \in \boldsymbol{C}(z)[d/dz]$, of order n, and any $\alpha \in \mathbf{P}_1(\boldsymbol{C})$:

$$\delta_\alpha(D) = (\Sigma_{i=0,\ldots,n-1}(e_i^\alpha - i)) - \frac{1}{2}\; irr^\alpha(End(D)),$$

where $\{e_0^\alpha, \ldots, e_{n-1}^\alpha\}$ are the exponents of D at α, and $irr^\alpha(End(D))$ is Malgrange's irregularity of $End(\boldsymbol{C}(z)[d/dz]/\boldsymbol{C}(z)[d/dz]D)$ at α. These notions are explicited in Section 5, but for the moment, it suffices to know that the $\delta_\alpha(D)$'s reduce to the defects defined above when α is an ordinary point or an apparent singularity of D, and that they satisfy the following generalization of Fuchs' global relation on exponents :

Theorem 2 (cf. [4], [5] and §6 below)**.** *For any $D \in \boldsymbol{C}(z)[d/dz]$, the (finite) sum of all the non-zero defects of D satisfies:*

$$\Sigma_{\alpha \in \mathbf{P}_1(\boldsymbol{C})} \delta_\alpha(D) = -n(n-1).$$

Since ordinary points and apparent singularities provide non negative contributions to this sum, we derive from Theorems 1 and 2 :

Corollary 2 (André)**.** *For any E-function F and any $\alpha \in \mathbf{P}_1(\overline{\boldsymbol{Q}}), \alpha \neq 0, \infty$,*

$$\delta_\alpha(D_F) \leq \; - \; \delta_0(D_F) \; - \; \delta_\infty(D_F) - n_F(n_F - 1).$$

The third ingredient of André's proof is the transcendence method itsef. Under the assumption that Theorem 1 does not hold, an auxiliary E-function

$F \in \mathbf{Q}[[z]]$ can be built up with a high order T of vanishing at the point $\alpha = 1$. A interesting feature here is that this construction does *not* appeal to Siegel's lemma: just like Mahler's method (cf. [13]), it relies solely on linear algebra. But an even more remarkable feature is that the full construction occurs at the point α, and requires extrapolations neither at 0 (or other points) nor on higher derivatives at α. Rather, the extrapolation is done on the *other solutions* of the differential equation D_F satisfied by the auxiliary function. Indeed, Corollary 1 implies that $\delta_\alpha(D_F) \geq n_F\ T$. But Corollary 2 gives an upper bound of the type $\delta_\alpha(D_F) < n_F\ T$ for T sufficiently large, whence the searched for contradiction.

2 An illustration of the method

As a warm-up, let us show how Theorem 1 immediately implies the following consequence of Theorem 0 . The proof provides a simple illustration of André's extrapolation process.

Corollary 0 (Lindemann-Weierstrass). *Let $\beta_1, \ldots, \beta_n$ be complex algebraic numbers, linearly independent over $\boldsymbol{Q}$. Then, $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\boldsymbol{Q}$.*

Proof : as is well known, it suffices to derive a contradiction from the assumption that

$$\gamma_1 e^{\alpha_1} + \ldots + \gamma_n e^{\alpha_n} = 0,$$

where $\gamma_1, \ldots, \gamma_n$ are non-zero complex algebraic integers, and $\alpha_1, \ldots, \alpha_n$ are distinct complex algebraic integers.

As an auxiliary function, choose (with Lindemann !)

$$F(z) = \Pi_\sigma(\gamma_1^\sigma e^{\alpha_1^\sigma z} + \ldots + \gamma_n^\sigma e^{\alpha_n^\sigma z}),$$

where σ runs through all the complex embeddings of the field generated by the α_i, γ_j's. Then, $F = \Sigma_{m \geq 0} \frac{a_m}{m!} z^m$ with coefficients $a_m \in \boldsymbol{Z}$, and since F is a sum of exponential functions, it defines an E-function; furthermore, the minimal monic differential operator D_F annihilating F has constant coefficients (and positive order, because $F \not\equiv 0$). Now, I claim that

$$\Phi(z) = \frac{F(z)}{1-z} = \Sigma_{m \geq 0} \frac{b_m}{m!} z^m$$

also defines an E-function. Indeed, $F(1) = 0$ by assumption, so that Φ is an entire function of exponential growth; since the coefficients b_m are *rational* numbers, this does imply that for each archimedean absolute value $|.|$ on $\overline{\boldsymbol{Q}}$, the sequence $\{|b_m|; : m \geq 0\}$ grows at most geometrically. Furthemore, the $b_m = \Sigma_{r=0,\ldots,m} \frac{m!}{r!} a_r$ all belong to $\boldsymbol{Z}$, and Condition (iii) is satisfied. Finally, consider the monic differential operator

$$D_\Phi := \frac{1}{z-1} \mathrm{o} D_F \mathrm{o} (z-1) = D_F(\frac{d}{dz} + \frac{1}{z-1}) \in \boldsymbol{Q}(z)[d/dz]$$

(where the third term means that d/dz is replaced by $d/dz + \frac{1}{z-1}$ in the expression of D_F as a polynomial in d/dz). The $\boldsymbol{C}$-linear map $\xi : y \mapsto (z-1)y$ gives a bijection between the solution spaces of the differential equations $D_\Phi y = 0$ and $D_F y = 0$, so that D_Φ is indeed the minimal differential operator annihilating Φ.

Theorem 1 now implies to all the solutions of $D_\Phi y = 0$ are analytic at 1. In view of the bijection ξ, we infer that *all* the solutions of $D_F y =$ vanish at 1. Since D_F has constant coefficients, this contradicts Cauchy's theorem (which here plays the role of Corollary 2) [1].

The latter argument on ξ *is* the proof of the derivation
Theorem 1 $\Rightarrow$ Corollary 1 : under the hypotheses of Corollary 1 , let T be the order of F at α, and let $\Phi(z) = F(z)/(\alpha - z)^T$. Because of the $\boldsymbol{Q}$-rationality assumptions on F and α, one checks as above that Φ defines an E-function, and the map $\xi : y \mapsto (z-\alpha)^T y$ shows that all the solutions of D_F near α are analytic functions, of order $\geq T$. Thus α is an apparent singularity, with minimal exponent $e_0 \geq T$. Now, the exponents $\{e_0, \ldots, e_{n_F - 1}\}$ of D_F at α are distinct, since no logarithmic solution occur. Hence, $e_i \geq T + i$ for all $i = 0, \ldots, n_F - 1$, and $\delta_\alpha(D_F) \geq n_F T$.

3 From G-functions to E-functions.

Let $\mathcal{L}$ be the formal Laplace transform on the ring $\overline{\boldsymbol{Q}}[[z]]$, sending $F = \Sigma_{m \geq 0} b_m z^m$ to $\mathcal{L}F = \Sigma_{m \geq 0} b_m \frac{m!}{z^{m+1}}$ (when F defines an entire functions of ex-

[1] In [3], André gives yet another proof of Lindemann-Weierstrass, as a direct consequence of Chudnovsky's theorem on G-functions (see Section 3 below), and contrasts it with the proof of Bézivin and Robba [6], which relied on the Borel-Polya-Dwork-Bertrandias rationality criterion.

ponential growth, $\mathcal{L}F$ is the Taylor expansion at 0 of $\int_0^\infty F(u)e^{-zu}du$). A G-function (again in a stricter sense than Siegel's, now because of Condition (i)) may then be defined as a power series $f(z) = \Sigma_{m\geq 0} a_m z^m$ such that $f(z) = \frac{1}{z}\mathcal{L}F(\frac{1}{z})$ for some E-function F, i.e. such that :

i) f is a solution of a differential equation with coefficients in $\overline{\boldsymbol{Q}}(z)$ (the relation between D_F and the annihilator D_f of f in $\overline{\boldsymbol{Q}}(z)[d/dz]$ is described below);

ii) for each archimedean absolute value $|.|$ on $\overline{\boldsymbol{Q}}$, the sequence $\{|a_m|; m \geq 0\}$ is bounded from above by a geometric progression ;

iii) there exists a sequence $\{d_m; m \geq 0\}$ of positive integers, bounded from above by a geometric progression, such that $d_m a_r$ is an algebraic integer for all $0 \leq r \leq m$ (see also [12]).

A fundamental theorem of Chudnovsky [9] asserts that the operator D_f is then an G-operator, i.e. that the resolvent

$$R(z,t) = \Sigma_{m\geq 0} \frac{R_m(t)}{m!}(z-t)^m$$

of the associated system satisfies Galočkin's property : there exists a non-zero polynomial $q(t)$ and a sequence $\{D_m; m \geq 0\}$ of positive integers, bounded from above by a geometric progression, such that for all $r = 0, \ldots, m$, the entries of the matrices $D_m q(t)^m \frac{R_r(t)}{r!}$ are polynomials with algebraic integers as coefficients, all of whose archimedean values grow at most geometrically with m. The first inequality in Theorem IV.5.2 of [1] then implies that D_f satisfies the condition introduced by Bombieri in [7], X, on generic radii of convergence, hence is a fuchsian operator with quasiunipotent local monodromy at each of its singularities, in view of a classical result of Katz. The following consequence will suffice for the proof of Theorem 1 .

Proposition 1 (cf. [2], §3). *Let f be an G-function, and let $D_f \in \overline{\boldsymbol{Q}}(z)[d/dz]$ be the (monic) differential operator of minimal order such that $D_f(f) = 0$. Then, the differential equation $D_f(y) = 0$ admits a regular singularity at 0.*

This result truly pertains to number theory. We now describe André's purely formal derivation

Proposition 1 $\Rightarrow$ Theorem 1 : let F be an E-function, and let $\partial = d/dz$. Then $\mathcal{L}(zF) = -\partial(\mathcal{L}F)$, $\mathcal{L}(\partial(F)) = z\mathcal{L}F - F(0)$, and more generality, for any element Δ, of order δ (in ∂) in the ring $\mathcal{D} = \overline{\boldsymbol{Q}}[z, \partial]$ of differential operators with polynomial coefficients,

$$\mathcal{L}(\Delta(F)) = \mathcal{F}(\Delta)\ (\mathcal{L}F) + p_F(z),$$

where $p_F \in \overline{\boldsymbol{Q}}[z]$ is a polynomial of degree at most $\delta - 1$, and where $\mathcal{F} : z \mapsto\ -\ \partial\ ,\ \partial \mapsto z$ denotes the Fourier transform on the ring $\mathcal{D}$, with inverse $\mathcal{F}^{-1} : z \mapsto \partial\ ,\ \partial \mapsto -z$. Now,

$$\mathcal{L}(z^\delta \Delta(F) = (-\partial)^\delta \mathcal{L}(\Delta(F)) = (-\partial)^\delta \mathcal{F}(\Delta)\ (\mathcal{L}F)\ ,$$

so that $\mathcal{F}^{-1}(\nabla)(F) = 0$, for any $\nabla \in \mathcal{D}$ annihilating $\mathcal{L}F$.

Set $f(z) = \frac{1}{z}(\mathcal{L}F)(\frac{1}{z})$, and let ∇ be any non-zero $\overline{\boldsymbol{Q}}[z]$-multiple, lying in the ring $\mathcal{D}$, of the annihilator $D_{\frac{1}{z}f(\frac{1}{z})}$ of $\frac{1}{z}f(\frac{1}{z})$. Then, $\nabla(\mathcal{L}F) = 0$. On the other hand, ∇ is equivalent to the pull-back under the inversion $z \mapsto \frac{1}{z}$ of the annihilator D_f of f. Since f is a G-function, we thus deduce from Proposition that ∇ admits a regular singularity at ∞. In particular, writing δ (resp. ν) for the degree in z (resp. the order in ∂) of $\nabla = \Sigma_{i=O,\ldots,\nu} C_i(z)\ \partial^i$, the standard condition of Fuchs at ∞ reads :

$$deg(C_i) - i \leq deg(C_\nu) - \nu.$$

Therefore, the only polynomial C_i of degree δ is C_ν, and $\mathcal{F}^{-1}(\nabla)$ is a differential operator (of order δ in ∂), whose highest coefficient is a *monomial* in z (of degree ν). Consequently, $\mathcal{F}^{-1}(\nabla)$ has *no* singularity outside $\{0\ ,\ \infty\}$.

Finally, the annihilator D_F of F is a right divisor of $\mathcal{F}^{-1}(\nabla)$ in the ring $\overline{\boldsymbol{Q}}(z)[\partial]$. Since all solutions of $D_F(y) = 0$ are solutions of $\mathcal{F}^{-1}(\nabla)(y) = 0$, this implies that D_F has at worst *apparent* singularities outside $\{0\ ,\ \infty\}$, as claimed by Theorem 1.

4 Siegel-Shidlovsky

We can now turn to the promised proof ([3], §2) of Theorem 0 , which, thanks to standard reductions (cf [11], Lemma 5.3, and Siegel's trick), is equivalent to the following assertion : assume that the E-functions $F_1, \ldots, F_n$ are linearly independent over $\overline{\boldsymbol{Q}}(z)$, and let K be a number field containing their Taylor coefficients at 0 and the coefficients of the rational functions entries of the matrix A; then the dimension ν of the K-vector space generated by their values (relatively to the given complex embeding of $\overline{\boldsymbol{Q}}$) at $\alpha = 1$ satisfies

$$\nu \geq n/\kappa,$$

where $\kappa = [K : \boldsymbol{Q}]$.

By definition, ν is the dimension of the smallest K-subspace W of K^n such that $W(\boldsymbol{C}) \ni \overline{F}(1)$. View K^n as the space of initial condition of the differential system $d\overline{Y}/dz = A\overline{Y}$ at its ordinary point $\alpha = 1$, and let $\overline{Y}_1, ..., \overline{Y}_\nu$ be a basis of its solutions in $(K[[z-1]])^n$ whose values at $\alpha = 1$ generate W. In particular, $\overline{F}$ belongs to the $\boldsymbol{C}$-vector space these generate in $(\boldsymbol{C}[[z-1]])^n$.

Let now T be a sufficiently large parameter. By linear algebra, there exist a non zero linear form $\tilde{P} = (P_1, \ldots, P_n) \in ((K[z])^n)^*$, whose entries are polynomials of degrees at most $T\frac{\nu}{n}$ such that

$$\forall\ j = 1, \ldots, \nu\ :\ ord_1 \tilde{P}.\overline{Y}_j \geq T.$$

This implies that $\tilde{P}.\overline{F} = \Sigma_{i=1,\ldots,n} P_i F_i$ has order at least T at $z = 1$. As auxiliary function, we now choose

$$F = \Pi_\sigma \tilde{P}^\sigma.\overline{F}^\sigma,$$

where σ runs through all complex embedings of K. Then, F is an E-function, which belongs to $\boldsymbol{Q}[[z]]$, and which still vanish to an order $\geq T$ at 1, so that the annihilator D_F of F in $\boldsymbol{Q}(z)d/dz$ satisfies

$$\delta_1(D_F) \geq n_F T\ ,$$

in view of Corollary 1.

Following a method introduced by Chudnovsky [8] for Fuchsian operators (see [4] in case of irregular singularities), we now bound *from below* the defects of D_F at 0 and at ∞. By differential Galois theory, a basis of solutions of $D_F(y) = 0$ is given by functions of the form $\Pi_\sigma \tilde{P}^\sigma.\overline{Y}_\sigma$, where each $\overline{Y}_\sigma$ runs through a subset of the space the solutions of the differential system $d\overline{Y}/dz = A^\sigma \overline{Y}$. Therefore, n_F is bounded in terms of n and κ, and there exists positive integers $c_0(A, \kappa), c_\infty(A, \kappa)$, depending only on A and κ, such that

$$\delta_0(D_F) \geq\ -c_0(A, \kappa)\ ,\ \delta_\infty(D_F) \geq\ -n_F[K : \boldsymbol{Q}]\sup_{i=1,\ldots,n}\deg(P_i)\ -\ c_\infty(A, \kappa).$$

Corollary 2 now implies :

$$\delta_1(D_F) \leq n_F \frac{\kappa\nu}{n} T + c_0 + c_\infty.$$

Hence

$$1 \leq \frac{\kappa\nu}{n} + \frac{c_O + c_\infty}{n_F T},$$

and the proof of our assertion follows on taking T sufficiently large with respect to A and K.
[To justify the sketch of proof at the end of §1 , recall that the effect of Siegel's trick (cf. [11], pp. 217 and 231) is to replace n (resp. ν) by the values at sufficiently large integers of the Hilbert function of $\mathbf{P}_{n-1}$ (resp. of a proper subvariety of $\mathbf{P}_{n-1}$), so that $\frac{\kappa\nu}{n}$ can be made smaller than 1.]

This concludes our report on André's proof. Note that Siegel-Shidlovsky is only one of the corollaries of his arithmetic Gevrey theory. For other applications (q-analogues, Euler-type series, ...) and for the theory itself, please consult the original papers [2], [3].

5 Exponents.

In this section and the last one, we give the proof of Theorem 2. This corresponds to a joint work with G. Laumon, which was summarized (under a slightly different viewpoint) in [5], and which sharpens a previous joint work with F. Beukers [4].

Since exponents are a local notion, we first define them over the ring $F[d/dz]$ of differential operators with coefficients in the fraction field $F = C((z))$ of the local ring $A = C[[z]]$ of formal power series with coefficients in an algebraically closed field C of characteristic zero. We denote by $v = ord_0$ the extension to the algebraic closure $\overline{F}$ of the standard valuation on F, and we set $\theta = zd/dz$. Any element D of $F[d/dz]$, of order n, admits a non zero left A-multiple $\tilde{D} \in A[d/dz]$ such that

$$\tilde{D} = b_n\theta^n + b_{n-1}\theta^{n-1} + \ldots + b_0, \inf_{i=0,\ldots,n} v(b_i) = 0,$$

and up to a constant multiple, the indicial polynomial $P = P_D$ of D is given by

$$P_D(X) = b_n(0)X^n + \ldots + b_0(0).$$

We say that D admits a regular singularity (or : is Fuchsian) at 0 if its degree is n, i.e. if $v(b_n) = 0$. The *exponents* of D at 0 are then the n roots $e_0(D), \ldots, e_{n-1}(D)$ of the indicial equation $P(X) = 0$. In case 0 is ordinary, $P(X) = X(X-1)\ldots(X-(n-1))$, while as explained above, $e_i(D) \geq i$ for all $i = 0\ldots, n-1$, if 0 is an apparent singularity; in both of these cases, the exponents provide all the values assumed by v on the set of solutions of $Dy = 0$ in $A[[z]]$.

The general definition of exponenents is based on a splitting of D which often requires an extension of scalars to $\overline{F}$, i.e. to $C((t))$ with $t^N = z$ for some integer N (one can in fact take $N = ppcm(1, \ldots, n)$), and the introduction of 'Puiseux polynomials in $\frac{1}{z}$ with no constant term', i.e. elements of $\frac{1}{t}C[\frac{1}{t}] \subset \overline{F}$. Since $z\frac{d}{dz} = \frac{1}{N}t\frac{d}{dt}$, the degree of the indicial equation P_D is invariant under such extensions, and the definition of exponents in the fuchsian case can be extended to the full ring $\overline{F}[\theta]$ in a uniform way. For any $D = D(\theta) \in \overline{F}[\theta]$ and any $\omega \in \overline{F}$, set

$$D^\omega := D(\theta + \omega) = e^{-\int \omega \frac{dz}{z}} \mathrm{o} D \mathrm{o} e^{\int \omega \frac{dz}{z}} \in \overline{F}[\theta].$$

The classical theorem of Poincaré-Hukuhara-Turritin-Levelt then asserts :

Proposition 2 (cf. [14]). *for any monic $D \in F[\theta]$, there exists a unique set $\{\omega_1, \ldots, \omega_s\}$ of distinct Puiseux polynomials in $\frac{1}{z}$ with no constant terms and a unique set $\{D_1, \ldots, D_s\}$ of monic fuchsian operators in $\overline{F}[\theta]$ such that D is the least common left multiple of $\{D_1^{-\omega_1}, \ldots, D_s^{-\omega_s}\}$ in the ring $\overline{F}[\theta]$.*

Note in particular that $\Sigma_{j=1,\ldots,s} ord(D_j) = ord(D)$. If $n_j = ord(D_j), j = 1, \ldots, s$, we now *define* the exponents of D at 0 as the collection

$$\{e_i(D_j); i = 0, \ldots, n_j - 1; j = 1, \ldots, s\}$$

of all the exponents of all the D_j's. By the unicity of the maximal right fuchsian factor of an operator ([14], 2.4.2), these may also be described as the roots of the indicial polynomials of the operators $D^{\omega_j}; j = 1, \ldots, s$. Thus, D always has $n = ord(D)$ exponents. We also write

$$irr^0(EndD) := \; - \; \Sigma_{1 \leq j \neq k \leq s} n_j n_k v(\omega_j - \omega_k) = -2 \; \Sigma_{1 \leq k < j \leq s} n_j n_k v(\omega_j - \omega_k).$$

For our needs, the latter expression, which vanishes if D is fuchsian (or, more generally, if and only if the decomposition above involves a single ω_j) can be taken as the definition of the irregularity of $End(F[d/dz]/F[d/dz]D)$ at 0..

The following effective version of the Hensel-Sibuya lemma (cf. [14]) is the key to the proof of Theorem 2.

Lemma 1. *Let Θ_1, Θ_2 be fuchsian operators of order n_1, n_2. For any not zero $\omega \in \frac{1}{z}C[\frac{1}{z}]$, the exponents of $D = \Theta_2\Theta_1^{-\omega}$ consist of the $n_1 + n_2$ numbers : $\{e_i(\Theta_1), i = 0, \ldots, n_1 - 1 \; ; \; e_i(\Theta_2) - n_1 v(\omega) \; , i = 0, \ldots, n_2 - 1\}$.*

Proof : for such D, one has $s = 2$ with $\omega_1 = \omega, \omega_2 = 0$. Using the second description of exponents given above, we get :

$$P_{D^\omega}(X) = P_{\Theta_1}(X) \; , P_D(X) = P_{\Theta_2}(X + n_1 v(\omega)),$$

and the lemma follows.
[For instance, the exponents of $D = \theta(\theta - \omega)$ are $\{0, -v(\omega)\}$: here, $D_1 = \theta$ since $D^\omega = (\theta + \omega)\theta$, hence $e_0(D_1) = 0$, and $D_2 = \theta - b$ for some $b \in A$ with $b(0) = -v(\omega)$ (indeed, $Dy = 0$ admits a solution in F with order $v(\omega)$ at 0), hence $e_0(D_2) = -v(\omega)$.]

6 The global relation.

To complete the derivation : Proposition 2 + Lemma 1 $\Rightarrow$ Theorem 2, some more local analysis is needed. Let $D = \theta^n + b_{n-1}\theta^{n-1} + \ldots + b_0 \in C((z))[\theta]$. Performing an eventual ramified covering, we deduce from Proposition 2 a (non canonical) decomposition

$$D = D'^{-\omega_s}_s \mathrm{o} D'^{-\omega_{s-1}}_{s-1} \mathrm{o} \ldots \mathrm{o} D'^{-\omega_1}_1 \in C((z^{\frac{1}{N}}))[\theta],$$

where each

$$D'_j = \theta^{n_j} + b_{j,n_j-1}\theta^{n_j-1} + \ldots + b_{j,0} \; , \; (j = 1, \ldots, s)$$

is $C((z^{\frac{1}{N}}))$-equivalent to D_j, hence fuchsian at 0. Looking at the trace of its indicial equation, we get $\Sigma_{i=0,\ldots,n_j-1} e_i(D'_j) = -Res_{z=0} b_{j,n_j-1}\frac{dz}{z}$, and we derive from the relation $b_{n-1} = \Sigma_{j=1,\ldots,s}(b_{j,n_j-1} - n_j\omega_j)$:

$$\Sigma_{j=1,\ldots,s}\Sigma_{i=0,\ldots,n_j-1} e_i(D'_j) = -Res_{z=0} b_{n-1}\frac{dz}{z}.$$

On the other hand, iterating Lemmma 1, we get :

$$\forall j = 1, \ldots, s, \forall i = 0, \ldots, n_j - 1 : e_i(D'_j) = e_i(D_j) + \Sigma_{1 \leq k \leq j-1} n_k v(\omega_j - \omega_k),$$

hence on adding up and recalling the definition of $irr^0(End(D))$:

$$(\Sigma_{j=1,\ldots,s}\Sigma_{i=0,\ldots,n_j-1}e_i(D)) - \frac{1}{2}irr^0(End(D)) = -Res_{z=0}b_{n-1}\frac{dz}{z}.$$

Let now $D = (d/dx)^n + a_{n-1}(d/dx)^{n-1} + \ldots + a_0 \in C(x)[d/dx]$. Localizing at a point $\alpha \in \mathbf{P}_1(C), \alpha \neq \infty$ with local parameter $z = x - \alpha$, we write : $z^n D = \theta^n + b_{n-1}\theta^{n-1} + \ldots + b_0 \in C((z))[\theta]$, with $b_{n-1} = za_{n-1} - \frac{n(n-1)}{2}$, and the local formula above yields :

$$\delta_\alpha(D) = -Res_\alpha a_{n-1}dx.$$

Localizing at $\alpha = \infty$ with local parameter $z = \frac{1}{x}$, we write $(-1)^n x^n D = \theta^n + b_{n-1}\theta^{n-1} + \ldots + b_0$, with $-b_{n-1} = xa_{n-1} - \frac{n(n-1)}{2}$, and the local defect becomes

$$\delta_\infty(D) = -Res_\infty a_{n-1}dx - n(n-1).$$

The residue formula completes the proof of Theorem 2.

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Daniel BERTRAND
Institut de Mathématiques de Jussieu
Case 247
4, place Jussieu
75252 PARIS Cédex 05 (France)

E-mail: `bertrand@math.jussieu.fr`