ARRANGEMENTS OF NESTED CURVES

EMMANUEL FERRAND

Abstract. Motivated by Arnold's theory of invariants of plane curves, we introduce the semi-group of equivalence classes of *arrangements of nested curves*. There exists a natural invariant of plane curves without inverse self-tangencies with values in this semi-group. We show that the associated Grothendieck group is $\mathbb{Z} \times \mathbb{Z}$. These two factors correspond to previously known invariants of plane curves without inverse self-tangencies, namely Whitney's index and Arnold's J^- invariant. We show that arrangements of nested curves are *not* classified by their finite type invariants.

1. Introduction

This note is organized as follows. In this introductory section, we recall the basic facts and some previous work on the classification of planar immersions of the circle. This motivate the definition of the semi-group of equivalence classes of *arrangements of nested curves*. In the next section, we compute the Grothendieck group associated to this semi-group. The last section is devoted to the study of finite type invariants of these objects.

Denote by $\mathcal{F}$ the space of all immersions of the oriented circle into the (oriented) plane $\mathbb{R}^2$, and denote by Δ the codimension-one subset of $\mathcal{F}$ whose elements are the immersions with an *inverse self-tangency*, i.e, a non-transverse double point where the velocity vectors are in opposite direction (see fig. 1, center).

Following Arnold [Ar], we would like to find out some informations about the topology of $\mathcal{F} \setminus \Delta$. The generic stratum of Δ (the set of all immersions with exactly one quadratic inverse self-tangency) has a natural coorientation : say that a generic path in $\mathcal{F}$ intersects the generic stratum of Δ *positively* if the number of double points increases (see fig. 1, from left to right).

Arnold proved [Ar] that this coorientation of Δ induces a one-cocycle by duality, and that this cocycle happens to be trivial. Arnold chose a specific primitive of this cocycle, i.e. a locally constant function, called J^-, on $\mathcal{F} \setminus \Delta$, so that it satisfies the additional constraint of being additive with respect to the connected sum.

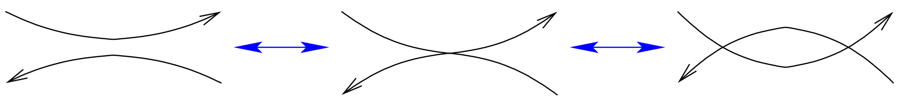

Figure 1. An inverse self-tangency.

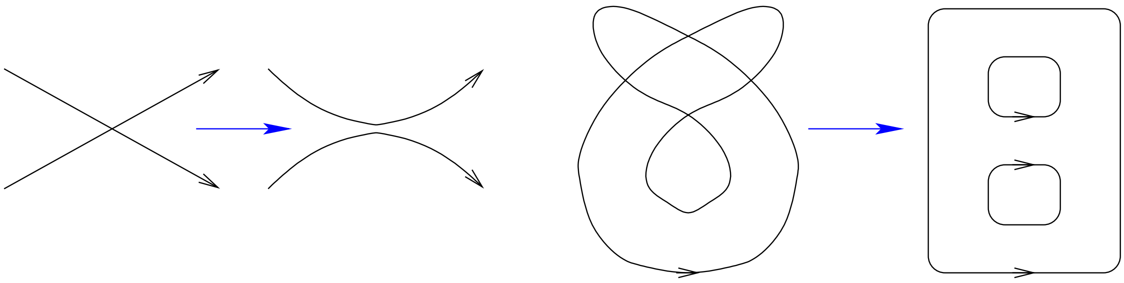

Figure 2. Smoothing of the double points.

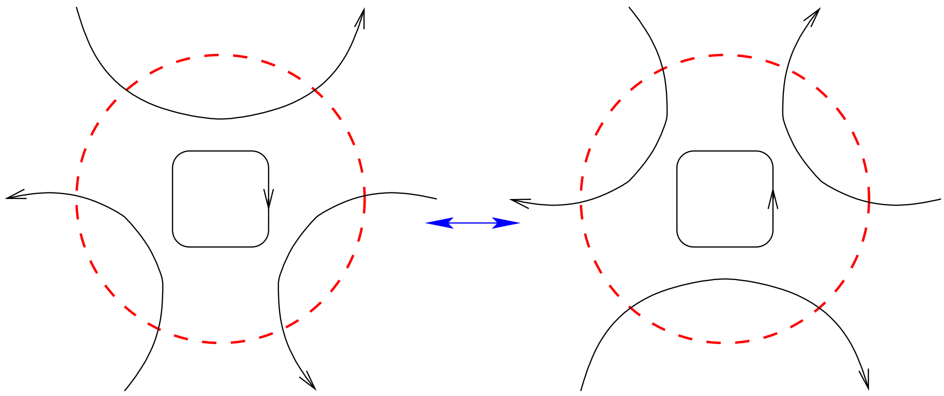

Figure 3. The triangular surgery.

Definition. *An arrangement of nested curves* (or simply *"an arrangement"* in the sequel) is a union of finitely many smooth, oriented, pairwise disjoint Jordan curves in the oriented plane $\mathbb{R}^2$, considered up to ambient isotopies.

By smoothing all its double points like in fig. 2, a generic immersed curve γ gives rise to an arrangement of nested curves $\mathsf{A}(\gamma)$. In [Vi], Viro has found, among other things, a way to compute the invariant $J^-(\gamma)$ is terms of $\mathsf{A}(\gamma)$.

An arrangement A is a combinatorial object. It can be encoded as a rooted tree with signed edges : put one vertex in each connected component of $\mathbb{R}^2 \setminus \mathsf{A}$, and one positive (resp negative) edge between two vertices if the corresponding two regions are separated by one positively (resp. negatively) oriented curve. Our goal is to examine what kind of informations about the classification of curves without inverse self-tangency it is possible to extract from this combinatorial structure.

We introduce now a natural equivalence relation among arrangements.

Definition. A *triangular surgery* on an arrangement consists in cutting off some disk where the arrangement locally looks like fig. 3, left, and pasting there a disk like fig. 3, right.

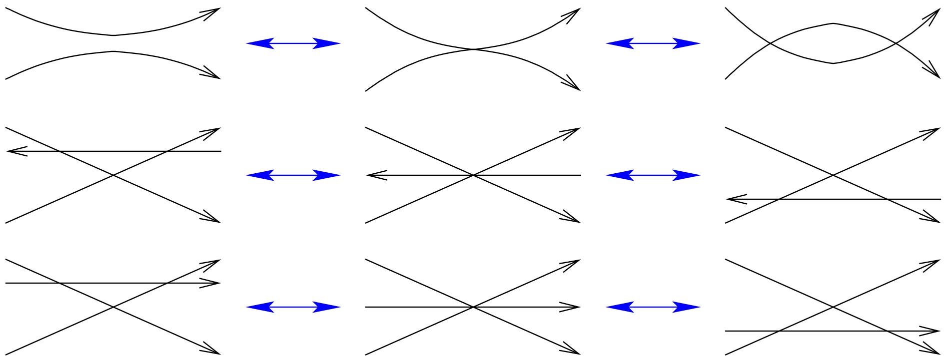

FIGURE 4. A direct self-tangency, a coherent triple point crossing, and a non-coherent triple point corssing.

Definition. Two arrangements are said to be *equivalent* if one can be obtained from the other by means of triangular surgeries and ambient isotopies. The equivalence class of an arrangement A will be denoted by $[\mathsf{A}]$.

Lemma. The map $\gamma \to [\mathsf{A}(\gamma)]$ is an invariant of plane curves without inverse self-tangencies.

Proof. A generic path in $\mathcal{F} \setminus \Delta$ may feature three different kinds of curves bifurcations : direct self-tangencies, coherent and non-coherent triple point crossings (see fig. 4). $\mathsf{A}(\gamma)$ is not changed (up to isotopy) when γ experiences a direct self-tangency or a non-coherent triple point crossing. A coherent triple point crossing modifies $\mathsf{A}(\gamma)$ by a triangular surgery. □

The degree $i(\gamma)$ of the Gauss map of an immersion γ will be called the *index* of γ. According to Whitney [Wh], this invariant classifies the connected components of $\mathcal{F}$. We now rephrase the construction of Viro [Vi]. Consider an arrangement A. A pair of Jordan curves in A will be called *injective* if one of the curves lies in the interior of the compact domain bounded by the other, and *free* otherwise. A curve of A will be called *positive* if it is oriented as the boundary of the disk it bounds, and *negative* otherwise. The *sign of a pair* of curves is the product of the signs of their components. We denote by $l(\mathsf{A})$ the absolute number of components of A, by $\langle \circ | \mathsf{A} \rangle$ the number of curves in A, but counted with their signs, by $\langle \circ\circ | \mathsf{A} \rangle$ the number of free pairs, counted with their signs, and by $\langle \circledcirc | \mathsf{A} \rangle$ the number of injective pairs, counted with their signs.

Theorem. (Viro, [Vi]) Les γ be a generic planar immersion of the circle.

$$i(\gamma) = \langle \circ | \mathsf{A}(\gamma) \rangle$$
$$J^-(\gamma) = 1 - 2 \cdot \langle \circledcirc | \mathsf{A}(\gamma) \rangle - l = 1 + 2 \cdot \langle \circ\circ | \mathsf{A}(\gamma) \rangle - (i(\gamma))^2 \quad \square$$

When a curve γ experiences an inverse self-tangency during some regular homotopy, $\mathsf{A}(\gamma)$ changes according to the *dangerous* surgery shown in fig. 5. Viro proved the theorem above by analyzing how the right-hand side behaves under triangular and

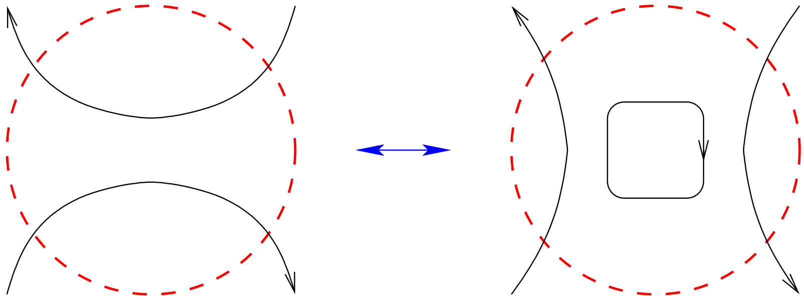

Figure 5. A dangerous surgery

dangerous surgeries (in [Vi], this analysis is carried out in a concise way using using integration with respect to the Euler characteristic).

2. Main theorem

In this section we forget about planar curves, and we consider the classification of arrangements modulo triangular surgeries.

Definition. The index i and the invariant J^- of an arrangement A are respectively $i(\mathsf{A}) = \langle \circ | \mathsf{A} \rangle$ and $J^-(\gamma) = 1 - 2 \cdot \langle \circledcirc | \mathsf{A} \rangle - l = 1 + 2 \cdot \langle \circ \circ | \mathsf{A} \rangle - (i(\mathsf{A}))^2$.

Viro's proof of the theorem above does not use the fact that the arrangement comes from some planar curve γ. This implies that :

Lemma. i and J^- descend to invariants of equivalence classes of arrangements. □

Equip the set of equivalence classes of arrangements with a structure of *commutative semi-group* as follows. The addition, which will be denoted by $\#$, corresponds to the disjoint sum (also denoted by $\#$) of arrangements : $[\mathsf{A}_1] \# [\mathsf{A}_2] = [\mathsf{A}_1 \# \mathsf{A}_2]$, where $\mathsf{A}_1 \# \mathsf{A}_2$ is the arrangement obtained by putting A_1 in one half-plane and A_2 in the other. The identity element is $[\varnothing]$, the class of the empty arrangement. Denote by $\circlearrowleft$ (resp. $\circlearrowright$) the arrangement which consists in exactly one positively (resp. negatively) oriented curve. Denote by A^k the disjoint sum of k copies of A.

Definition. Two arrangements A_1 and A_2 are called *stably equivalent* if there exists $k \in \mathbb{N}$ such that $[\mathsf{A}_1 \# (\circlearrowleft \# \circlearrowright)^k] = [\mathsf{A}_2 \# (\circlearrowleft \# \circlearrowright)^k]$.

Main theorem. The two invariants i and J^- classify equivalence classes of arrangements *up to stabilization*. More precisely if $i(\mathsf{A}_1) = i(\mathsf{A}_2)$ and $J^-(\mathsf{A}_1) = J^-(\mathsf{A}_2)$, then there exists $k \in \mathbb{N}$ such that

$$[\mathsf{A}_1 \# (\circlearrowleft \# \circlearrowright)^k] = [\mathsf{A}_2 \# (\circlearrowleft \# \circlearrowright)^k].$$

Furthermore, for any A, there exists $k \in \mathbb{N}$ such that

$$[\mathsf{A} \# (\circlearrowleft \# \circlearrowright)^k] = [\circlearrowright]^{k + \frac{1 - J^-(A) - i(A)}{2}} \# [\circlearrowleft]^{k + \frac{1 - J^-(A) + i(A)}{2}}.$$

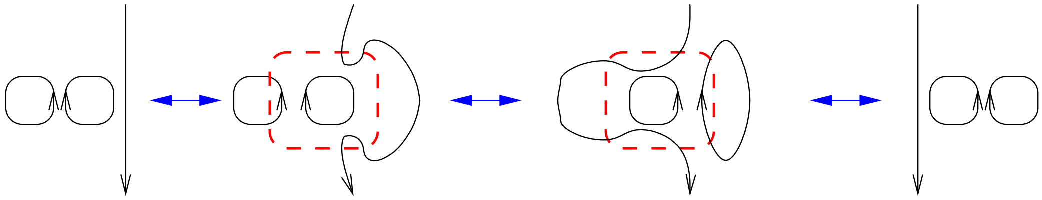

FIGURE 6. A pair $\circlearrowleft \# \circlearrowright$ can travel freely up to triangular surgery

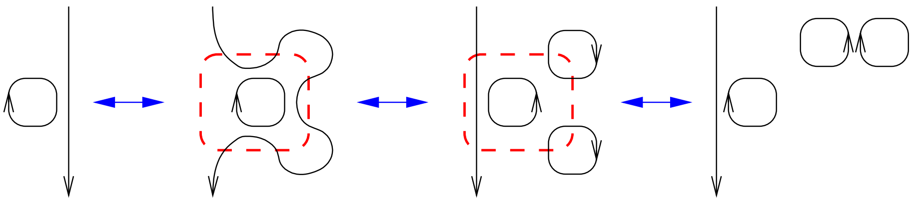

FIGURE 7. An empty curve can travel freely, up to stabilization.

Denote $J^-(\mathsf{A})-1$ by $j(\mathsf{A})$. It follows from the formula $j(\mathsf{A}) = -2\cdot\langle \circledcirc | \mathsf{A}(\gamma)\rangle - l$ that this quantity is, like i, *additive under disjoint union*. Hence we get a semi-group morphism $\mathsf{A} \to (i(\mathsf{A}), j(\mathsf{A}))$ with values in $\mathbb{Z}\times\mathbb{Z}$. The theorem above implies that any semi-group morphism from the set of equivalence classes of arrangements with values in some group G factorizes through $\mathbb{Z}\times\mathbb{Z}$. In other words *the Grothendieck group of our semi-group is isomorphic to* $\mathbb{Z}\times\mathbb{Z}$.

Lemma 1. A pair $\circlearrowleft \# \circlearrowright$ can travel freely inside an arrangement, up to triangular surgery.

Proof. This lemma is illustrated an proved by fig. 6. □

Lemma 2. A curve bounding a disk can go through another curve, up to the birth or the death of a pair $\circlearrowleft \# \circlearrowright$.

Proof. This lemma is illustrated an proved by fig. 7. Observe that in one way, a pair $\circlearrowleft \# \circlearrowright$ is created. In the other way a pair is consumed. □

Proof of the main theorem. Consider an arrangement A and one curve in A which bounds a disk. By lemma 2, one can move this curve anywhere in the complement of the arrangement, provided that we have enough pairs $\circlearrowleft \# \circlearrowright$ at the right places to make it go through the other curves. On the other hand, by lemma 1, for all $k\in\mathbb{N}$, $\mathsf{A}\,\#\,(\circlearrowleft \# \circlearrowright)^k$ is equivalent to any union of A and k copies of $\circlearrowleft \# \circlearrowright$, with the k pairs located anywhere in $\mathbb{R}^2\setminus\mathsf{A}$. So, up to stabilization, one can assume that the pairs needed to move the curve are present.

Hence there exists $k, k^+, k^- \in \mathbb{N}$ such that $[\mathsf{A}\,\#\,(\circlearrowleft \# \circlearrowright)^k] = [\circlearrowleft^{k^+} \;\#\; \circlearrowright^{k^-}]$. By Viro's formula, $i(\mathsf{A}) = k^+ - k^-$ and $J^-(\mathsf{A}) - 2k = 1-(k^+ + k^-)$. Note that i and J^- classify classes of the form $[\circlearrowleft^{k^+} \;\#\; \circlearrowright^{k^-}]$. Due to the additivity of i and $J^- - 1$,

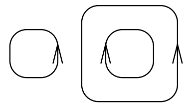

FIGURE 8. No triangular surgery is possible on this arrangement.

one has that

$$[\mathsf{A} \# (\circlearrowleft \# \circlearrowright)^k] = [\circlearrowleft]^{k+\frac{1-J^-(A)-i(A)}{2}} \# [\circlearrowleft]^{k+\frac{1-J^-(A)+i(A)}{2}}.$$

Consider now two arrangements A_1 and A_2. For $i \in \{1,2\}$, find some integers $k_i, k_i^+, , k_i^-$ such that

$$[\mathsf{A}_i \# (\circlearrowleft \# \circlearrowright)^{k_i}] = [\circlearrowleft^{k_i^+} \ \# \ \circlearrowright^{k_i^-}].$$

Suppose for instance that $k_1 \leq k_2$. Then, according to the above discussion,

$$[\mathsf{A}_1 \# (\circlearrowleft \# \circlearrowright)^{k_2}] = [\mathsf{A}_2 \# (\circlearrowleft \# \circlearrowright)^{k_2}]. \qquad \square$$

3. Finite type invariants

The classification of arrangements up to triangular surgery seems to be rather complicated. The theorem above imply that we already know all *additive* invariants. On the other hand i and J^- do not classify arrangements (stabilization is really needed) : consider an arrangement A such that for any connected component C of $\mathbb{R}^2 \setminus \mathsf{A}$, the oriented curve(s) of A bounding C are oriented as the boundary of C, for some orientation of C. *It is impossible to perform any triangular surgery* on such a arrangement since the local model for the surgery is not oriented as a boundary. Such an A is, up to planar isotopy, the single element of $[\mathsf{A}]$. An example is given by the arrangement of Fig. 8, for which $i = 1$ and $J^- = 0$. Its class becomes equivalent to $[\circlearrowleft]$, but after one stabilization by a pair $\circlearrowleft \# \circlearrowright$.

Remark. Denote by P_- a negative injective pair. One can check that

$$[\circlearrowleft \# \circlearrowright \ \# \mathsf{P}_-] = [\circlearrowleft \# \circlearrowright].$$

Hence $[\mathsf{A} \# \mathsf{P}_-]$ is stably equivalent to $[\mathsf{A}]$, for any arrangement A.

Due to the smoothing procedure, an arrangement is of the form $\mathsf{A}(\gamma)$ for some connected curve γ if and only if, for any component C of $\mathbb{R}^2 \setminus \mathsf{A}$ which has a non-connected boundary, the orientation of ∂C inherited from A does *not* come from an orientation of C.

There exists two connected curves γ_1 and γ_2 such that $\mathsf{A}(\gamma_1)$ and $\mathsf{A}(\gamma_2)$ are stably equivalent, but not equivalent : Take $\gamma_1 =\circlearrowleft$, and γ_2 the curve of fig. 9.

Question. Denote by $\bar{\gamma}$ the curve obtained from a curve γ by changing the orientation. Observe that $i(\bar{\gamma}) = -i(\gamma)$ and that $J^-(\bar{\gamma}) = J^-(\gamma)$. Suppose that γ is the curve of fig. 10. This curve is of index 0, hence, by the main theorem, $\mathsf{A}(\gamma)$ and $\mathsf{A}(\bar{\gamma})$ are stably equivalent. *Is it true that these two arrangements are not equivalent ?*

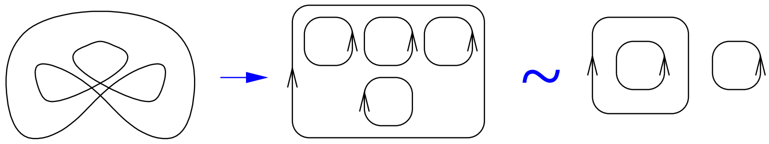

FIGURE 9. A curve with $i = 1$ and $J^- = 0$ whose arrangement is not equivalent to $\circlearrowleft$.

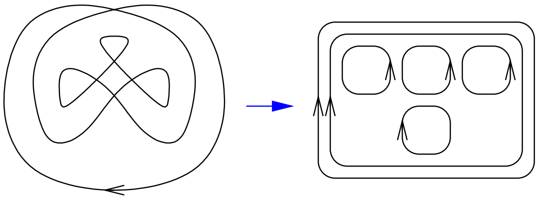

FIGURE 10. Is the arrangement of γ equivalent to that of $\bar{\gamma}$?

The *dangerous surgery* of fig. 5 plays the role of an "unknoting operation" :

Lemma. An arrangement A of positive (resp. vanishing, negative) index $i(\mathsf{A})$ is equivalent to $\circlearrowleft^{i(\mathsf{A})}$ (resp. $\circlearrowleft \# \circlearrowright$, $\circlearrowright^{-i(\mathsf{A})}$) modulo dangerous surgeries.

Proof. A triangular surgery can be realized by two dangerous surgeries. On the other hand, any non-empty arrangement A is equivalent to $\mathsf{A} \# \circlearrowleft \# \circlearrowright$ after one dangerous surgery. Hence the lemma follows from the main theorem. $\square$

This unknoting operation suggests a natural notion of finite type invariant. A dangerous surgery is said to be *positive* if it creates a new little circle locally (As shown in fig. 5, from left to right), and *negative* otherwise. Observe that J^- jumps by -2 during a positive dangerous surgery (of course i is preserved).

A $n-$*singular arrangement* is an arrangement A together with the additional data of n disjoint segments whose interiors are embedded in $\mathbb{R}^2 \setminus \mathsf{A}$ and whose endpoints are located on A, and such that the orientations of A at the endpoints of each segment are compatible, so that one can perform a positive dangerous surgery on A in a neighborhood of each segment. A *resolution* of an $n-$singular arrangement is the arrangement obtained after dangerous surgeries have been made near *some* of these segments (there are 2^n possible resolutions). The sign of such a resolution is minus one to the number of segments near which *no* surgeries have been done. The signed sum over all resolutions of some $n-$singular arrangement gives an element in the complex vector space F spanned by all the equivalence classes of arrangements. Denote by F_n the subspace generated by such sums for all $n-$singular arrangements. An invariant is said to be *of order* n if its linear extension to F vanishes on F_{n+1}. Remark that the disjoint sum $\#$ induces a product on F, which inherits a structure of commutative and associative algebra.

Example. The index i is of order zero, J^- is of order one.

Theorem. Any finite type invariant is a function of i and J^-.

Our situation is very close to that of Legendrian knots, described in [FT] (see the comments below), and the proof given here is just an adaptation of that of the theorem 4.5 of [FT].

Proof. We will show that, for any $n \in \mathbb{N}$, an invariant of order n cannot distinguish two arrangements which are stably equivalent. Suppose that A_1 and A_2 are stably equivalent. Pick some k such that $[\mathsf{A}_1 \# (\circlearrowleft \# \circlearrowright)^k] = [\mathsf{A}_2 \# (\circlearrowleft \# \circlearrowright)^k]$. Consider

$$[\mathsf{A}_1] \# ([\varnothing] - ([\varnothing] - [\circlearrowleft \# \circlearrowright])^n)^k = [\mathsf{A}_1 \# (\circlearrowleft \# \circlearrowright)^k] \# R,$$

where R is some expression independent of A_1. This is equal to $[\mathsf{A}_2 \# (\circlearrowleft \# \circlearrowright)^k] \# R$, and hence to $[\mathsf{A}_2] \# ([\varnothing] - ([\varnothing] - [\circlearrowleft \# \circlearrowright])^n)^k$. On the other hand, for $i \in \{1, 2\}$,

$$[\mathsf{A}_i] \# ([\varnothing] - ([\varnothing] - [\circlearrowleft \# \circlearrowright])^n)^k = [\mathsf{A}_i] + [\mathsf{A}_i] \# ([\varnothing] - [\circlearrowleft \# \circlearrowright])^n \# S_i,$$

for some expression S_i. Now observe that for any arrangement A,

$$[\mathsf{A} \# (\circlearrowleft \# \circlearrowright)] - [\mathsf{A}] \in F_1, \text{ and, more generally, } [\mathsf{A}] \# ([\circlearrowleft \# \circlearrowright] - [\varnothing])^n \in F_n.$$

This imply that $[\mathsf{A}_i] = [\mathsf{A}_i] \# ([\varnothing] - ([\varnothing] - [\circlearrowleft \# \circlearrowright])^n)^k$ modulo F_n, and hence A_1 and A_2 are equal modulo F_n. $\square$

Remark. In the search of invariants of arrangements beyond i and J^-, it is natural to look for finite type ones. The theorem above says that this search is bound to fail. In this sense it is a negative result. On the other hand, one may notice that although it is not known whether finite type invariant classify knots, there exists a number of topological or combinatorial contexts where one can define finite type invariants in a natural way and where these invariants *do not form a complete set of invariants*. The most famous example is probably the case of *Legendrian knots* in $\mathbb{R}^3$ with its standard contact structure ζ. It was shown in [FT] that finite type invariants of Legendrian isotopy in $(\mathbb{R}^3, \zeta)$ are in one to one correspondence with those of *regular isotopy*. Pairs of non-equivalent Legendrian knots which are regular isotopic were exhibited by various methods, none of them elementary (see, for example [Ch]). The reason why finite type invariants fail to distinguish Legendrian knots is precisely a stabilization trick, similar to what happen in the case of arrangements. On can view the classification of arrangements as an elementary "toy model" for this problematics.

Acknowledgments. This note is under the strong influence of the papers [Vi] and [FT]. I also thank F. Aicardi for her comments.

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E. F. : Institut Fourier, CNRS UMR 5582, 100 rue des Maths, Domaine Universitaire, BP 74, 38402 Saint Martin d'Hères cedex, France
E-mail address: emmanuel.ferrand@ujf-grenoble.fr