MORSE THEORY FOR SWEEPS.

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Abstract. Morse inequalities can be rephrased as a lower bound on the number of tangencies between a compact immersed submanifold of the Euclidean space and the hyperplanes defined by the levels of a height function. In this paper, we investigate what happens when one deforms such a one-parameter family of hyperplanes among one-parameter families of hypersurfaces (not necessarily the levels of a function). We exhibit a condition on the deformation which ensures that an analogue of the Morse estimate for the number of tangencies between the deformed hyperplanes and the compact immersed submanifold remains valid. This condition can be viewed as a "non-linking" condition between two Legendrian submanifolds in the spherization of the cotangent bundle of the Euclidean space, endowed with its natural contact structure.

1. Introduction

This paper is organized as follows. In this first section, we motivate our definitions and state the main theorem. We discuss its relationship with contact topology. The second section is devoted to the proof.

1.1. **An example.** Consider a smooth affine curve bounding a convex domain in the real projective plane (see fig. 1). Consider the family of lines that goes through a given point. If the point is at infinity, then these lines are the levels of a function defined on the affine plane, and Morse theory gives a lower bound for the number of tangencies between the smooth curve and the lines belonging to this family. On the other hand, if the point is moved inside the convex domain, there is no tangencies between the curve and the lines. This example shows that when one deforms a one-parameter family of hypersurfaces, tangencies of a given compact submanifold with these hypersurfaces may disappear.

1.2. **Morse estimates rephrased.** Consider a smooth compact manifold M of dimension m, and a smooth function $f : M \to \mathbb{R}$. By the Whitney embedding theorem, one can assume that M is a submanifold of $\mathbb{R}^n$, for some $n \in \mathbb{N}$. Furthermore one can assume that the function f is the restriction to M of one of the coordinates, say x_n. A *point* $q \in M$ *is critical for* f iff the "horizontal" hyperplane $\{x \in \mathbb{R}^n | x_n = f(q)\}$ passing through this point is *tangent to* M. In other words, c is a critical value of f iff the hyperplane $\{x \in \mathbb{R}^n | x_n = c\}$ is not transversal to M.

For this specific family of "horizontal" hypersurfaces, Morse theory gives an estimate of the number of non-transversal hypersurfaces. From this point of view, it is natural to ask whether there exists Morse-like estimates for the number of "critical" hypersurfaces for more general one-parameter families of hypersurfaces. In this paper, we give a condition on a family of hypersurfaces which ensures that such a Morse-like estimate exists. This

Date: July 2005.

1991 *Mathematics Subject Classification.* 53D10.

Key words and phrases. contact topology, Morse theory.

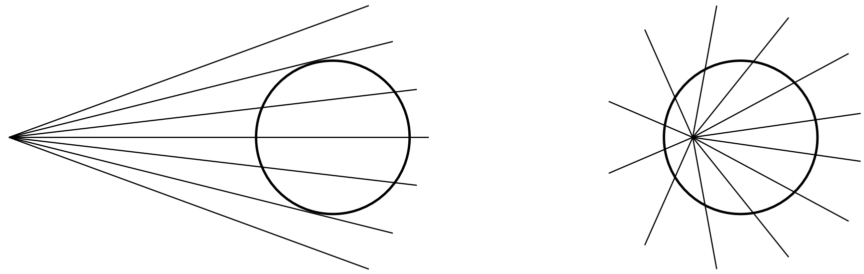

FIGURE 1. A basic example.

condition has a natural interpretation in terms of contact topology. It is in fact a "non-linking" condition between two Legendrian submanifolds which are naturally associated to this situation in the spherization of the cotangent bundle of the Euclidean space $\mathbb{R}^n$, endowed with its natural contact structure (see 1.4 and 1.5).

1.3. **Sweeps.** It this paper, M denotes a smooth, compact, m-dimensional manifold immersed in the Euclidean space $\mathbb{R}^n$. The circle $\mathbb{R}/2\pi\mathbb{Z}$ is denoted by by S^1. For technical reasons, the following definition involves a family of functions. Only the hypersurfaces defined by their 0-levels are relevant.

1.3.1. **Definition.** A *sweep* is a smooth function : $h \ : \mathbb{R}^n \times S^1 \to \mathbb{R}$, viewed as a one-parameter family of smooth functions $h(\cdot, t) \ : \mathbb{R}^n \to \mathbb{R}$, parametrized by $t \in S^1$. For each $t \in S^1$, 0 *must be a regular value* of $h(\cdot, t)$. A sweep is called *standard* if in addition $h(q, 0) > 0$ and $h(q, \pi) < 0$ for all $q \in M$.

1.3.2. **Example.** Assume that M is contained in a Euclidean ball which does not contain the origin. Then the family of functions $h(x_1, \ldots, x_n, t) = cost(t) \cdot x_1 + sin(t) \cdot x_2$ defines a sweep, which, in addition, is standard provided that $x_1 > 0$ on M. This is the situation of fig. 1. A deformation of this sweep is shown in fig. 2.

1.3.3. **Definition.** The *enveloppe of a sweep* h is the subset E_h of $\mathbb{R}^n$ defined by the following equations

$$E_h = \{x \in \mathbb{R}^n, \exists t \in S^1 \text{ such that } h(x,t) = 0 \text{ and } \frac{\partial h}{\partial t}(x,t) = 0\}.$$

This is nothing but the classical definition found in textbooks (see for example [DC]) for the enveloppe of a family of submanifolds defined by implicit equations. E_h is in general a singular subset of $\mathbb{R}^n$ (and is not necessarily compact). However, one can prove that, for a generic sweep h, it is a singular hypersurface with a well defined tangent space almost everywhere.

1.3.4. **Definition.** A subspace $H \subset T_x(\mathbb{R}^n)$ is called *tangent* to E_h at some point x if there exist $t \in S^1$ such that $h(x,t) = 0$, $\frac{\partial h}{\partial t}(x,t) = 0$ (so that $x \in E_h$), and if, in addition, the one-form $\frac{\partial h}{\partial x}(x,t)dx$ vanishes on H.

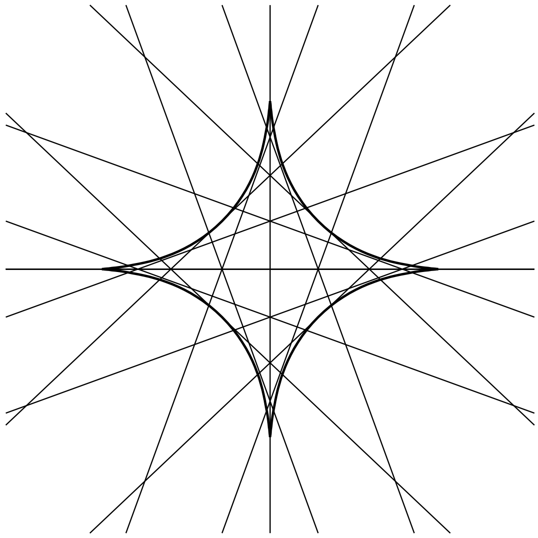

FIGURE 2. The enveloppe of a deformation of the example 1.3.2.

1.3.5. **Examples.** If x belongs to the smooth stratum of of E_h, a subspace H tangent to E_h at x coincides with the usual tangent space. If h is the sweep defined above in 1.3.2, then the envelope E_h is reduced to the codimension 2 subspace $\{x \in \mathbb{R}^n | x_1 = x_2 = 0\}$. Any hyperplane H which contains E is tangent to E in the sense of the above definition.

1.3.6. **Theorem A.** Consider a smooth one parameter family $h^s, s \in [0,1]$ of sweeps, such that h^0 is *standard*, and such that, for any $s \in [0,1]$, the corresponding envelope E_{h^s} is *not* tangent to M. Then the total number (counted with multiplicities) of angles $t \in S^1$ such that M is tangent to the hypersurface $\mathcal{H}^1_t = \{x \in \mathbb{R}^n | h^1(x,t) = 0\}$ is not smaller than twice the sum of the Betti numbers of M (with respect to any field of coefficients).

This theorem is proved in section 2. The classical Morse estimate for functions is recovered as follows. As noticed in 1.2, estimating the number of critical points of some function f is the same as estimating the number of tangencies between M and the "horizontal" hyperplanes $\{x \in \mathbb{R}^n | x_n = c\}, c \in \mathbb{R}$. Denote by c_- (resp. c_+) the minimum (resp. the maximum) of the function x_n on M. On can embed the family of hyperplanes $\{x \in \mathbb{R}^n | x_n = c\}, c \in [c_- - 1, c_+ + 1]$ into a sweep h as follows :

- $h(x,t) = x_n - (c_- - 1 + \frac{2t}{\pi}(c_+ + 2 - c_-)$ for $t \in [0, \frac{\pi}{2}]$
- $h(x, t+\pi) = -h(x,t)$, and
- $h(q,t) < 0$ for all $q \in M$ and all $t \in [\frac{\pi}{2}, \pi]$.

For such an h, the critical levels of $x_{n|M}$ are in *one-to-two* correspondence with the tangencies counted in theorem A.

1.4. **Reformulation in terms of contact topology.** Although the definitions above (envelope, tangency, ...) allow to prove theorem A without introducing the vocabulary of contact topology (see section 2), they suggest a relationship to the latter. The relevant contact manifold is $ST^*\mathbb{R}^n$, the manifold of *cooriented contact elements* of $\mathbb{R}^n$, endowed with its natural contact structure. As a manifold, it is the spherization of the cotangent bundle. Denote by π the natural projection $\pi : ST^*\mathbb{R}^n \to \mathbb{R}^n$. See for example [AG] for the basic definitions of contact topology.

The spherization of the *conormal bundle* of M is a *Legendrian* submanifold L_M of $ST\mathbb{R}^n$. There is also a natural *Legendrian lift* $\mathcal{E}_h$ of the envelope of a sweep h :

$$\mathcal{E}_h = \{(x, [p]) \in ST^*\mathbb{R}^n, \exists t \in S^1 \text{ such that } h(x,t) = 0, \frac{\partial h}{\partial t}(x,t) = 0, [p] = [\frac{\partial h}{\partial x}(x,t)dx]\}.$$

In the above formula, $[p]$ denotes the class of a non-zero covector p modulo multiplication by *positive* real numbers. One can prove that $\mathcal{E}_h$ is an embedded smooth Legendrian submanifold provided that h is "generic", but this fact is not needed below.

A tangency point between M and the envelope E_h of a sweep h as defined above is nothing but the projection in $\mathbb{R}^n$ of an intersection between $\mathcal{E}_h$ and L_M. Furthermore, the non-transversality of M and some hypersurface $\mathcal{H}$ can be expressed as $L_M \cap L_{\mathcal{H}} \neq \emptyset$. Hence theorem A can be rephrased as follows:

Theorem A'. Consider a smooth one parameter family $h^s, s \in [0,1]$ of sweeps, such that h^0 is standard, and such that, for any $s \in [0,1]$, $\mathcal{E}_{h^s} \cap L_M = \emptyset$. Denote by $\mathcal{H}_t^s$ the hypersurface $\mathcal{H}_t^s = \{x \in \mathbb{R}^n | h^s(x,t) = 0\}$. Then the total number (counted with multiplicities) of angles $t \in S^1$ such that $L_M \cap L_{\mathcal{H}_t^s} \neq \emptyset$ is not smaller than twice the sum of the Betti numbers of M. □

The hypothesis of the theorem is a "non-linking" condition between some Legendrian submanifolds (or varieties), its conclusion is the necessity of some "Legendrian intersections".

1.5. **Another interpretation in terms of Legendrian linking.** From these ingredients, one can cook up another interpretation of the "non-linking" condition in theorem A. Consider the three dimensional manifold $J^1(S^1, \mathbb{R}) = \mathbb{R} \times T^*\mathbb{R}$, endowed with its natural contact one-form $du - \tau dt$, where (τ, t) are canonical coordinates of T^*S^1 and u is the $\mathbb{R}$ coordinate.

To a sweep h, one can associate the following subset of $J^1(S^1, \mathbb{R})$:

$$l_h = \{(u, \tau, t) \in J^1(S^1, \mathbb{R}) | \exists q \in M \text{ such that } \frac{\partial h}{\partial q}(q,t) = 0, u = h(q,t), \tau = \frac{\partial h}{\partial t}(q,t)\}$$

For a generic h, one can prove that l_h is a *smooth, Legendrian link* (but this fact is not needed here). Denote by l_0 the one-jet expansion of the constant 0 function ($l_0 = \{(u, \tau, t) \in J^1(S^1, \mathbb{R}) | u = \tau = 0\}$). The following proposition gives an interpretation of the hypothesis of theorem A in terms of these Legendrian links.

1.5.1. **Proposition.** For any sweep h, the envelope E_h is not tangent to M iff $l_h \cap l_0 = \emptyset$.

Proof. An intersection of l_h and l_0 is a point $(u, \tau, t) \in J^1(S^1, \mathbb{R})$ such that there exists $q \in M$ such that $0 = u = h(q,t)$ and $0 = \tau = \frac{\partial h}{\partial t}(q,t)$. This means precisely that q belongs to $M \cap E_h$. □

1.5.2. Consider a one-parameter family of sweeps h^s, $s \in [0,1]$ which satisfy the hypothesis of theorem A. The corresponding one-parameter family of Legendrian objects l_h^s, $s \in [0,1]$ does *not* realize a *Legendrian isotopy* or even a *Legendrian homotopy*, but *Legendrian cobordism* between l_{h^0} and l_{h^1} . On the other hand, for each $s \in [0,1]$, l_{h^s} is given by a *generating family.* See [AG] for an introduction to Legendrian cobordism, and [EG] for a detailed survey of generating families in symplectic aand contact topology.

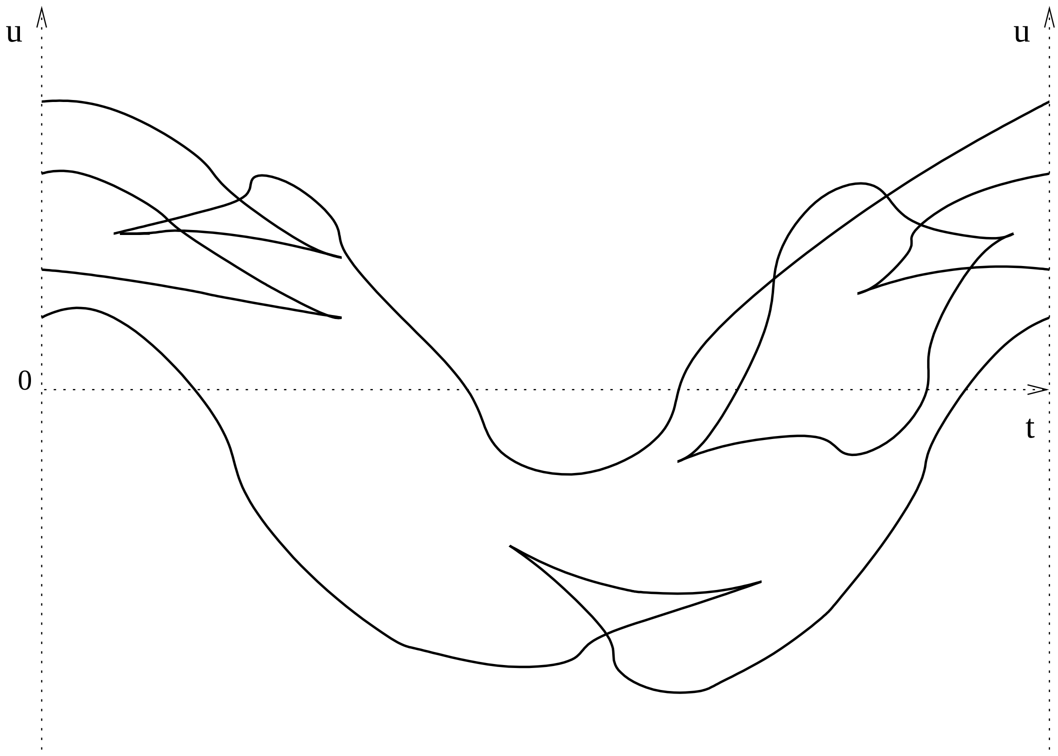

FIGURE 3. The front of l_h, for some typical standard sweep h.

In this picture, the conclusion of theorem A is that, the number of intersections (counted with multiplicities) of *front* of l_h^s, i.e., its projection to the 0-jets space $\mathbb{R}\times S^1$ by forgetting τ, with the zero-section $\{(u,t)\in\mathbb{R}\times S^1|u=0\}$ of the $0-$jets space cannot be smaller than twice the sum of the Betti numbers of M.

The front of a typical *standard* sweep is shown in fig 3. When h is standard, one can guess why the front of l_h must intersect $\{(u,t)\in\mathbb{R}\times S^1|u=0\}$ by the following informal argument : One can associate to each homology class of M a continuous path in the front, by a minimax argument. Since $h(\cdot,0)>0$ and $h(\cdot,\pi)<0$, these continuous paths must intersect the axis $u=0$ in $]0,\pi[$.

2. Proof of theorem A.

2.0.3. Denote by $H_*(M)$ the homology groups of M with coefficients in some field, and by dim $H_*(M)$ the sum of the corresponding Betti numbers. Consider a one-parameter family of sweeps $h^s, s\in[0,1]$ satisfying the hypothesis of theorem A. We use the notation $\mathcal{H}_t^s$ to denote the hypersurface $\{x\in\mathbb{R}^n|h^s(x,t)=0\}$. Consider the following subset Γ^s of $M\times S^1$:

$$\Gamma^s=\{(q,t)\in M\times S^1, h^s(q,t)=0\}.$$

Theorem A is implied by the following 4 lemmas.

2.0.4. **Lemma.** Under the hypothesis of theorem A, for all $s \in [0,1]$, Γ^s is a smooth hypersurface, diffeomorphic to Γ^0.

Proof. Pick some $(q,t) \in \Gamma^s$, and denote by $q = (q_1, \dots, q_m)$ some coordinate system for M near q. Assume that both $\frac{\partial h^s}{\partial q}$ and $\frac{\partial h^s}{\partial t}$ vanish at (q,t). On one hand, $\frac{\partial h^s}{\partial q} = 0$ means that the smooth hypersurface $\mathcal{H}_t^s$ is tangent to M at q. On the other hand $\frac{\partial h^s}{\partial t} = 0$ means that q belongs to the envelope E^s. Altogether, this means that E^s is tangent to M at x, which is forbidden by our hypothesis. Hence the differential of h^s is non-singular near any $(q,t) \in \Gamma^s$. This imply that for any $s \in [0,1]$, Γ^s is a smooth, compact hypersurface in $M \times S^1$. $\square$

2.0.5. **Lemma.** The tangencies between $\mathcal{H}_t^s$ and M we are looking for are in one-to-one correspondence with the critical values of the natural projection $t\ : \Gamma^s \to S^1$.

Proof. We are looking for the t's such that there exists $q \in M$ such that $\mathcal{H}_t^s$ is tangent to M at q. Still denote by $q = (q_1, \dots, q_m)$ some coordinate system on M near q. The tangent space to Γ^s at (q,t) is defined by $\frac{\partial h^s}{\partial q}(q,t)dq + \frac{\partial h^s}{\partial t}(q,t)dt = 0$. On one hand, a tangency of $\mathcal{H}_t^s$ and M at q means that $\frac{\partial h^s}{\partial q}(q,t) = 0$. Hence the equation of the tangent space of Γ^s at (q,t) reduces to $\frac{\partial h^s}{\partial t}(q,t)dt = 0$. On the other hand, since q does not belong to the envelope E^s, $\frac{\partial h^s}{\partial t}(q,t) \neq 0$. This means that $dt_{|\Gamma^s} = 0$ at (q,t). $\square$

2.1. **Lemma.** The closed one-form dt is *exact* on Γ^s.

Proof. Since Γ^s is isotopic to Γ^0 in $M \times S^1$, it is enough to check it at $s = 0$. On can view t as a function from Γ^0 to S^1. Since the sweep h^0 is standard, the pre-image of 0 by this function t is empty. Hence the restriction of t to Γ^0 can be seen as a function with values in $]0, 2\pi[$. $\square$

2.2. **Lemma.** $\dim H_*(\Gamma^0) \geq \dim H_*(M)$.

Proof. Γ^0 is made of two components. One of them, which will be denoted by Γ, is contained in $t^{-1}(]0,\pi[)$. The other is contained in $t^{-1}(]\pi, 2\pi[)$. Denote by $(M\times]0,\pi[)^{\geq 0}$ the subset $h^{-1}([0,\infty[) \cap (M\times]0,\pi[)$, so that $\Gamma = \partial(M\times]0,\pi[)^{\geq 0}$. Consider the long exact sequence of the pair $((M\times]0,\pi[)^{\geq 0}, \Gamma)$:

$$\dots \to H_{*+1}((M\times]0,\pi[)^{\geq 0}, \Gamma) \to H_*(\Gamma) \to H_*((M\times]0,\pi[)^{\geq 0}) \to H_*((M\times]0,\pi[)^{\geq 0}, \Gamma) \to \dots$$

The map $H_*((M\times]0,\pi[)^{\geq 0}) \to H_*((M\times]0,\pi[)^{\geq 0}, \Gamma)$ is zero, hence $H_*(\Gamma)$ injects in $H_*((M\times]0,\pi[)^{\geq 0})$. On the other hand $\dim H_*((M\times]0,\pi[)^{\geq 0}) \geq \dim H_*(M)$, since the composition of the two injections $M \times \{0\} \to (M\times]0,\pi[)^{\geq 0} \to M\times]0,\pi[$ induces an isomorphism in homology. $\square$

The conclusion of theorem A is obtained by applying the classical Morse inequalities to a primitive of $dt|_{\Gamma^1}$. $\square$

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