POSITIVE ISOTOPIES OF LEGENDRIAN SUBMANIFOLDS

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ABSTRACT. We show that there is no positive loop inside the component of a fiber in the space of Legendrian embeddings in the contact manifold ST^*M, provided that the universal cover of M is $\mathbb{R}^n$.

1. INTRODUCTION AND FORMULATION OF THE RESULTS

On the Euclidean unit 2-sphere, the set of points which are at a given distance of the north pole is in general a circle. When the distance is π, this circle becomes trivial: it is reduced to the south pole. Such a focusing phenomenon cannot appear on a surface of constant, non-positive curvature. The image by the exponential map of a circle vectors tangent to the surface at a given point is never reduced to one point.

In this note, we generalize this remark in the context of contact topology. Our motivation comes from the theory of the orderability of the group of contactomorphisms of Eliashberg, Kim and Polterovitch [EKP].

Consider a $2n+1$-dimensional manifold V endowed with a contact structure ξ induced by a globally defined one-form α. Given a compactly supported function $h : V \times [0,T] \to \mathbb{R}$, there exists a unique time-dependent contact vector field X_h such that $\alpha(X_h) = h$ (see, for example, [AG]).

In local Darboux coordinates $q_1, \dots, q_n, p_1, \dots, p_n, u$, one can write $\alpha = du - \sum_{i=1}^n p_i dq_i$. The coordinates of the vector field X_h are then

$$X_h^{q_i} = -\partial_{p_i} h, \quad i = 1, \dots, n$$

$$X_h^{p_i} = \partial_{q_i} h + p_i \partial_u h, \quad i = 1, \dots, n$$

$$X_h^u = h - \sum_{i=1}^n p_i \partial_{p_i} h.$$

These formulas are the contact analogue of the classical Hamilton equations in mechanics, and we will call such a function h a *contact Hamiltonian.*

Example 1. Consider a Riemannian manifold (N, g). Its unit tangent bundle $\pi : S_1N \to N$ has a natural contact one-form: If u is a unit tangent vector, and U a vector tangent to S_1N at u, then

$$\alpha(u) \cdot U = g(u, D\pi(u) \cdot U).$$

The constant contact Hamiltonian $h = 1$ induces the geodesic flow.

Definition 1. *A smooth path $L_t, t \in [0,1]$ in the space of Legendrian embeddings in (V, ξ) is called* positive *if it is induced by a positive contact Hamiltonian.*

Such a path will be also called a *positive isotopy* below. This notion of positivity does not depend on the choice of the contact form α.

Example 2. Any fiber of $\pi : S_1N \to N$ is Legendrian. Moving a fiber by the geodesic flow is a typical example of a positive path.

The space $J^1(N) = T^*N \times \mathbb{R}$ of one-jets of functions on a n-dimensional manifold N has a natural contact one form $\alpha = du - \lambda$, where λ is the Liouville one-form of T^*N and u the $\mathbb{R}$-coordinate. Given a smooth function $f : N \to \mathbb{R}$, its one-jet extension j^1f is a Legendrian submanifold. A path between two functions gives rise to an isotopy of Legendrian embeddings between their one-jets extensions.

Theorem 1. *If N is a closed manifold, there is no closed positive path in the component of the space of Legendrian embeddings in $J^1(N)$ containing the one-jet extensions of functions.*

The Liouville one-from of T^*N induces a contact distribution of the fiber-wise spherization ST^*N. This contact structure is contactomorphic to the one of example 1. Our generalization of the introductory remark is as follows.

Theorem 2. *There is no positive path of Legendrian embeddings between two distinct fibers of $\pi : ST^*N \to N$, provided that the universal cover of N is $\mathbb{R}^n$.*

Definition 2. *A smooth path $L_t = i_t(L_0), t \in [0,1]$ in the space of Legendrian* immersions *in (V, ξ) is called* positive *if, for any $x \in L_0$, $\alpha(\frac{\partial}{\partial t}(i_t(x))) > 0$.*

Remark 1. In the particular case when all $L_t, t \in [0,T]$ are *embedded* Legendrian submanifolds, then one can produce a positive contact Hamiltonian so that we get a positive isotopy like in definition 1.

Theorem 3. **i.** *There exist a component of the space of Legendrian embeddings in $J^1(S^1)$ which contains a closed positive path.*

ii. *There exists a closed positive path in the component of the space of Legendrian immersions in $J^1(S^1)$ containing the one-jet extensions of functions.*

iii. *Given any connected surface N, there exists a positive path of Legendrian immersions between any two fibers of $\pi : ST^*N \to N$.*

2. The big Legendrian and the reduced Legendrian

The following constructions are closely related to the notion of *Legendrian cobordism.* See [AG] for details.

2.1. Consider a $2n+1$-dimensional manifold V endowed with a contact structure ξ induced by a globally defined one-form α. Consider a one-parameter family $\phi_t : L_0 \to V, t \in [0, T]$ of Legendrian immersions.

Such a data gives rise to the following Legendrian submanifold in $V \times T^*[0, T]$, endowed with the contact form $\alpha - wdt$, where (t, w) are canonical coordinates of T^*N.

Definition 3. *The* big Legendrian $\mathcal{L}$ *is defined by the formula*

$$\mathcal{L} = \{(\phi_t(x), t, \alpha(\frac{d}{dt}\phi_t(x))) \subset V \times T^*N, x \in L_0\}.$$

Remark 2. When the one-parameter family comes from an isotopy of embedded Legendrian submanifolds induced by a contact Hamiltonian

$$h : V \times [0, T] \to \mathbb{R}, \quad (x, t) \mapsto h(x, t),$$

then the "w" coordinate in the above formula becomes

$$\alpha(\frac{d}{dt}\phi_t(x)) = h(\phi_t(x), t).$$

Remark 3. A path of Legendrian immersions is positive if and only if the associated big Legendrian lies in the $w > 0$ half-space.

2.2. Suppose now that the contact manifold (V, ξ) is the one-jets space $J^1(N) = \mathbb{R} \times T^*N$ of a n-dimensional manifold N, endowed with its natural contact structure, i.e. the hyperplane field induced by the kernel of the one-form $\alpha = du - \sum_{i=1}^n p_i dq_i$, where $q = (q_1, \ldots, q_n)$ are local coordinates on N, $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ the associated canonical coordinates of T^*N, and u the $\mathbb{R}$ coordinate of $J^1(N)$.

The big Legendrian is immersed in $J^1(N) \times T^*[0, T] = J^1(N \times [O, T])$ and gives rise to the following Legendrian submanifold of $J^1([0, T])$.

FIGURE 1. A typical bifurcation diagram of a positive isotopy.

Definition 4. *The* reduced *Legendrian is*

$$\bar{L} = \{(t, w, u) \subset J^1([0, T] \ / \ \exists q \in N, (q, t, 0, w, u) \in \mathcal{L}\}.$$

The reduced Legendrian is a Legendrian tangle embedded in $J^1([O, T])$, provided that $\mathcal{L}$ is in general position.

Definition 5. *An intersection of a Legendrian submanifold $L \subset J^1(N)$ with the subset $\{p = 0\} \subset J^1(N)$ will be called a* critical point *of L. A* critical value *of L is the projection of a critical point to the u axis.*

Example 3. Given a function $f : N \to \mathbb{R}$, the critical points of the Legendrian submanifold $j^1 f$ are in one to one correspondence with the critical points (in the classical sense) of f.

Definition 6. *The projection of the reduced Legendrian $\bar{L}$ to the zero-jets space $J^0([0, T]) = [0, T] \times \mathbb{R}$ is the* bifurcation diagram of the critical values *of the one parameter family of Legendrian immersions.*

Remark 4. In the case of a *positive* path of Legendrian immersions, the smooth branches of this bifurcation diagram have positive slopes.

3. PROOF OF THEOREM 1

In this section, we assume that N is closed, and we consider an isotopy $L_t, t \in [0, T]$ of Legendrian submanifolds of $J^1(N)$ such that $L_0 = L_T = j^1 f$, for some smooth function $f : N \to \mathbb{R}$. We need to show that such a loop cannot be positive.

3.1. According to Chekanov's theorem [Ch], there exists $p \in \mathbb{N}$ and a smooth function $F : N \times [0, T] \times \mathbb{R}^{2p} \to \mathbb{R}$ such that

- outside of a compact, F if equal to a fixed quadratic form of signature (m, m) in the $\mathbb{R}^{2m}$ variable,
- the big Legendrian $\mathcal{L}$ associated with such an isotpy is given, using the notations of definition 4, by

$$\mathcal{L} = \{(q, t, p, w, u), \text{ such that } \exists v \in \mathbb{R}^{2p} \text{ such that}$$
$$\partial_v F(q, t, v) = 0, p = \partial_q F(q, t, v), w = \partial_t F(q, t, v), u = F(q, t, v)\}.$$

Remark 5. Hence in this case the bifurcation diagram of $L_t, t \in [0, T]$ is nothing but the bifurcation diagram of the critical values of the family of functions $F(t, ., .) : N \times \mathbb{R}^{2n} \to \mathbb{R}, t \in [O, T]$.

3.2. Selecting and following the critical values. A reference for the construction described below is, for example, [Vi].

Denote by F_t^a the set $\{F(t, ., .) \leq a\}$. Due to the fact that F is quadratic outside of a compact set, $H_*(F_t^a, F_t^{-C})$ is independent of C for C sufficiently big. In addition $H_*(F_t^C, F_t^{-C})$ is isomorphic to $H_*(N)$ (all homology groups are taken with $\mathbb{Z}_2$ coefficients).

For $k \in \{1, \ldots, \dim H * (N)\}$, denote by

$$c_{t,k} = inf\{b \mid \dim i_* H_*(F_t^b) \geq k\},$$

where i is induced by the inclusion $F_t^a \hookrightarrow F_t^C$. It is a critical value of $F(t, ., .)$, and *a continuous function of the parameter* t. If the path is in general position[1], then these continuous functions are *piecewise smooth*. If in addition the path is *positive*, then these piecewise smooth functions are *increasing* with respect to t.

3.3. Since the positivity of a path is an open condition, one can assume that the path is in general position. In particular one can assume that F_0 is a Morse function with a finite number of distinct critical values. Among these critical values, $dim(H_*(N))$ of them are selected a time $t = 0$ by the above construction. From each of these selected values, a continuous piecewise smooth increasing graph is starting. On can follow a given selected critical value not only for $t \in [0, T]$, but also for any iterate of the loop. But this would need an arbitrary large number of critical values at $t = 0$. This contradiction ends the proof of theorem 1

[1] i.e. the family F_t intersects transversally the complement of the set Morse functions with distinct critical values

3.4. Remark. The same proof would work for a weaker notion of positivity: To get a bifurcation diagram with positive slope as above it is sufficient that $h(t,.)$ is positive near the the submanifold $\{p=0\} \subset J^1(N) \times [0,T]$.

4. Proof of theorem 2

We need to prove that there is no positive path between two distinct fibers of $\pi : ST^*\mathbb{R}^n \to \mathbb{R}^n$. The relationship with theorem 1 comes from the following classical construction.

4.1. Isomorphism with jet spaces. The space $ST^*\mathbb{R}^n$ of cooriented contact elements of $\mathbb{R}^n$ and $J^1(S^{n-1})$, both endowed with their natural contact structures, are contactomorphic (see, for example, [FP]).

A fiber $\pi^{-1}(x)$ corresponds to the one-jet extension $j^1 f_x$ of the function f_x induced on S^{n-1} by the scalar product with x

4.2. Consider two distinct points $x \neq y$ in $\mathbb{R}^n$, and assume that there exists a positive path between $j^1 f_x$ and $j^1 f_y$. One can assume that this path is in general position. At time $t=0$ (resp. $t=T$), there are exactly two distinct selected critical values $\pm||x||$ (resp. $\pm||y||$) . But since $x \neq y$, it is impossible to connect $\pm||x||$ and $\pm||y||$ by continuous piecewise smooth increasing graphs of functions. This contradiction ends the proof the theorem 2

5. Proof of theorem 3

5.1. A positive loop. We begin by the description of a positive loop in the space of Legendrian embeddings in $J^1(S^1)$. Due to theorem 1, this cannot happen in the component of the zero section $j^1 0$.

Take $\epsilon > 0$ and consider a Legendrian submanifold L embedded in the half-space $\{p > 2\epsilon\} \subset J^1(S^1)$. Consider the contact flow $\phi_t : (q,p,u) \to (q-t, p, u-t\epsilon), t \in \mathbb{R}$. The corresponding contact Hamiltonian $h(q,p,t) = -\epsilon + p$ is positive near $\phi_t(L)$, for all $t \in \mathbb{R}$. Hence the same motion of L can be induced by an everywhere positive Hamiltonian.

On the other hand, one can go from $\phi_{2\pi}(L)$ back to L just by increasing the u coordinate, which is also a positive path. This proves the first statement of theorem 3.

5.2. Take $L \subset p > 2\epsilon$ as above, but assume in addition that L is homotopic to $j^1 0$ through Legendrian immersions. The $0-$jet projection of such an L is depicted in fig. 5.1.

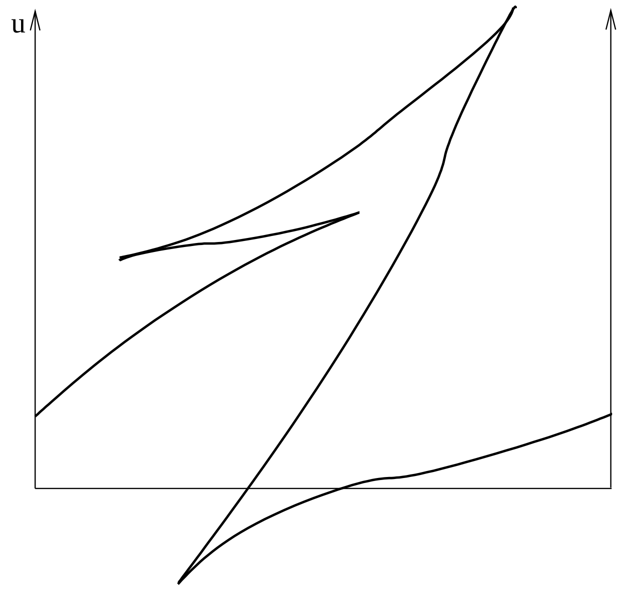

FIGURE 2. The projection of an $L \subset \{p > 2\epsilon\}$.

Step 1. This homotopy can be transformed into a positive path of Legendrian immersions between $j^1 0$ and a translate L' of L, by combining it with an upwards translation with respect to the u coordinate.
Step 2. Then, using the flow ϕ_t (defined in 5.1) for $t \in [0, 2k\pi]$ with k big enough, one can reach reach another translate L'' of L, on which the u coordinate will be arbitrarily low.
Step 3. A path from L to $j^1 0$ can be modified into a positive path between L'' and j^0 using upward translation again.

This proves the second statement of theorem 3.

5.3. The proof of the third statement is very similar. Consider two points x and y on the surface N, a path from x to y, and an open neighborhood U of this path, diffeomorphic to $\mathbb{R}^2$. Since ST^*U is contactomorphic to $ST^*\mathbb{R}^2$, it is enough to consider the particular case $N = \mathbb{R}^2$.

The case when $x = y$ follows from the preceding statement via the contactomorphism used in 4.1.

Suppose now that $x \neq y$. One can assume that x is the origin of $\mathbb{R}^2$. We need to find a positive path of Legendrian immersions in $J^1(S^1)$ between $j^1 0$ and $j^1 f_y$, using the notations of 4.1.

To achieve this, it is enough to construct a positive path of Legendrian immersions between $j^1 0$ and a translate of $j^1 0$ that would be entirely below $j^1 f_y$ with respect to the u coordinate.

This can be done as in 5.2, just by decreasing even more the u coordinate in step 2. This finishes the proof of theorem 3

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