ON THE DYNAMICAL MANIN-MUMFORD CONJECTURE FOR PLANE POLYNOMIAL MAPS

ROMAIN DUJARDIN, CHARLES FAVRE, AND MATTEO RUGGIERO

Abstract. We prove the dynamical Manin-Mumford conjecture for regular polynomial maps of $\mathbb{A}^2$ and irreducible curves avoiding super-attracting orbits at infinity, over any field of characteristic 0.

Contents

Introduction

The dynamical Manin-Mumford conjecture for polarized endomorphisms of algebraic varieties, first formulated by S.-W. Zhang in two influential papers [32, Conjecture 2.5] and [33, Conjecture 1.2.1], has been a driving force for the development of the field of arithmetic dynamics. It was realized by Ghioca, Tucker and Zhang [19] and Pazuki [26] that the original formulation of the conjecture was too optimistic, and a modified conjecture was proposed in [19] and more recently in [18]. It can be stated as follows: *let $f : X \to X$ be a polarized endomorphism of a smooth projective variety over a field of characteristic zero, and $Z \subset X$ be a subvariety containing a Zariski dense set of preperiodic points. Then either Z is preperiodic or Z is special*, in the sense that it is contained in some subvariety Y that is both f^n- and ψ-invariant, for some $n \geq 1$, where ψ is another polarized endomorphism commuting with f^n, and Z is preperiodic under ψ. Recall that an endomorphism is said to be polarized if there is an ample line bundle $L \to X$ such that $f^*L \simeq L^d$ for some $d \geq 2$. A basic example is that of non-invertible endomorphisms of $\mathbb{P}^k$, for which we can take $L = \mathcal{O}(1)$ and d is the degree of f.

Despite its importance, very few cases of the conjecture have been settled so far. One first case is of course the original Manin-Mumford conjecture, which was solved by Raynaud [27, 28]. Viewed as a dynamical statement, it deals with endomorphisms of

Date: December 27, 2023.

M.R. was partially supported by the ANR grant Fatou ANR-17-CE40-0002-01.

Abelian varieties, and was generalized to related settings, such as commutative algebraic groups (see Hindry [22], and also Lang's classical paper [25]). Uniform versions involving height bounds were subsequently obtained by S.-W. Zhang, David and Philippon, Chambert-Loir and others. We refer to the recent work of Kühne [24] in the semi-abelian case for a latest update, and more references. Closer to algebraic dynamics is the case of polarized endomorphisms of $(\mathbb{P}^1)^k$, which was solved by Ghioca, Nguyen and Ye [17, 16] (see also [18]). Note that such mappings are of product type, that is of the form $f(x_1, \ldots, x_k) = (f_1(x_1), \ldots, f_k(x_k))$, so their dynamical complexity reduces to dimension 1, which is a key step in the proof.

In this paper we will establish the dynamical Manin-Mumford conjecture for a wide class of 2-dimensional examples, whose dynamical behavior is truly higher dimensional.

Note that besides polarized endomorphisms, a partial answer to the conjecture was given in [11] for plane polynomial automorphisms.

Let $\mathbb{k}$ be any field of characteristic zero. If required we fix an algebraic closure $\mathbb{k}^{\mathrm{alg}}$ of $\mathbb{k}$. Let $f\colon \mathbb{A}^2_{\mathbb{k}} \to \mathbb{A}^2_{\mathbb{k}}$ be a polynomial self-map of the affine plane of degree $d \geq 2$, which in coordinates is written as

$$f(z, w) = \big(P(z, w), Q(z, w)\big),$$

with $P, Q \in \mathbb{k}[x, y]$. We say that f is *regular* if it extends to an endomorphism of $\mathbb{P}^2_{\mathbb{k}}$ of degree $d \geq 2$; this means that $P = P_d + \text{l.o.t.}$ and $Q = Q_d + \text{l.o.t.}$, where P_d and Q_d are homogeneous polynomials of degree d without common factors. In particular f induces a rational map $f_\infty := [P_d : Q_d]$ on the line at infinity, which is fixed. Note that any endomorphism of $\mathbb{P}^2_{\mathbb{k}}$ with a totally invariant line is conjugate to a regular polynomial map, and that a generic polynomial map of $\mathbb{A}^2$ whose components are polynomials of degree $\leq d$ is regular.

For regular polynomial maps of $\mathbb{A}^2$, it seems that none of the known obstructions to the dynamical Manin-Mumford conjecture can arise. As said above, according to [18], one obstruction would be the existence of a preperiodic curve C for some endomorphism ψ commuting with f. Such pairs (f, ψ) were classified in [8] over $\mathbb{C}$ (see also [9, 23]), and after a ramified cover they are all induced from a product map or a monomial map on the multiplicative 2-torus. Thanks to [17] it appears that in any such case, C must be also f-preperiodic. Thus we expect that the dynamical Manin-Mumford conjecture holds unconditionally in this case. Our main result confirms this expectation in the vast majority of cases.

Theorem A. *Let f be a regular polynomial endomorphism of $\mathbb{A}^2$ of degree $d \geq 2$, defined over an arbitrary field $\mathbb{k}$ of characteristic 0.*

Let $C \subset \mathbb{A}^2$ be an irreducible algebraic curve containing infinitely many preperiodic points of f, and suppose that the closure of C in $\mathbb{P}^2$ contains a point $p \in L_\infty$ which is not eventually superattracting. Then C is preperiodic under f.

Corollary B. *Under the assumptions of the theorem, if f has no super-attracting points on the line at infinity, then any irreducible algebraic curve $C \subset \mathbb{P}^2$ containing infinitely many periodic points is preperiodic, that is, the dynamical Manin-Mumford conjecture holds for f.*

Our strategy relies on techniques from Arakelov geometry, in particular on the notion of canonical height, which is now classical in arithmetic dynamics, together with a variety of techniques from holomorphic and non-Archimedean dynamics.

We first assume that $\mathbb{k}$ is a number field, in which case we prove a stronger statement (Theorem 3.1). We first recall in Section 1, that there exists a canonical height h_f on $\mathbb{A}^2(\mathbb{k}^{\mathrm{alg}})$, for which preperiodic points are exactly points of zero height. By a theorem of Zhang, all points of C lying at infinity are preperiodic, hence, by replacing C by some iterate we may assume that they are periodic, and, thanks to our assumptions, one of these periodic points is not superattracting for f_∞. Fix such a point $p \in \overline{C} \cap L_\infty$. A second consequence of Zhang's theorem is that or each place v of $\mathbb{k}$, the dynamical Green function $g_{f,v}(z,w) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^n(z,w)\|$ satisfies

$$\int_C g_{f,v} dd^c g_{f,v} = 0. \tag{1}$$

To make sense of this equality at a finite place, we consider the Berkovich analytification of C, in which case $dd^c g_{f,v}$ is the Laplacian of the subharmonic function $g_{f,v}|_C$ in the sense of Thuillier, see [31].

For an appropriate choice of v, we may suppose that the periodic point p is either repelling or parabolic for f_∞. In both cases, we construct in Section 2 a local (super-)stable manifold $W^{\mathrm{ss}}_{\mathrm{loc}}(p)$. When p is repelling and v is Archimedean this is classical. We extend this construction to the non-Archimedean setting, and allow for non-repelling p (see Theorem 2.1). Then, in both (repelling or parabolic) cases we establish graph transform type estimates which will be useful for the local analysis of the Green function at p. When p is parabolic this borrows from the work of Hakim [20].

In Section 3 we combine these estimates with condition (1) to show that, near p, C must locally coincide with $W^{\mathrm{ss}}_{\mathrm{loc}}(p)$, which finally implies that C is periodic. Note that a similar argument appears in the recent work [15].

A key point in this argument is that at the chosen place v, p belongs to the Julia set of f_∞. This is no longer true when f_∞ is superattracting at p, and we are not able to conclude in this case. Still, this situation leads to interesting dynamical considerations and objects, and we plan to come back to this issue in a later work.

Finally, we develop a specialization argument in Section 4 to reduce Theorem A for an arbitrary $\mathbb{k}$ to the case where $\mathbb{k}$ is a number field. So here we rather deal with an algebraic family of endomorphisms of $\mathbb{P}^2$ parameterized by some algebraic variety S. The main issue is to ensure that for such a family, an infinite set of preperiodic points cannot shrink to a finite set too often on S. To do so, one needs to control collisions of periodic points (using the Shub-Sullivan theorem [29] in the spirit of [11]); and the splitting of local preimages of a super-attracting cycle, a phenomenon that was studied in particular by Chio and Roeder [7].

1. Dynamical heights

In this section, we recall some basic facts on canonical heights attached to endomorphisms of the projective plane defined over a number field. Our purpose is to establish Proposition 1.7, which is the key arithmetic geometry input in our main theorem.

Throughout this section, we assume that $\mathbb{k}$ is a number field.

1.1. **Vocabulary of number fields.** We denote by $M_{\mathbb{k}} = \{v\}$ the set of places of $\mathbb{k}$, that is, the set of all multiplicative norms $|\cdot|_v$ on $\mathbb{k}$ that restrict to either the standard euclidean norm $|\cdot|_\infty$, or to a p-adic norm $|\cdot|_p$ on $\mathbb{Q}$ for some prime number $p > 1$. We normalize the p-adic norm by $|p|_p = p^{-1}$. We let $\mathbb{k}_v$ be the completion of $\mathbb{k}$ w.r.t. $|\cdot|_v$, and write $n_v := [\mathbb{k}_v : \mathbb{Q}_v]$. The product formula asserts that for any $a \in \mathbb{k}$, we have

$$\prod_{v \in M_{\mathbb{k}}} |a|_v^{n_v} = 1 \ .$$

The set $M_{\mathbb{k}}$ splits into the finite set $M_{\mathbb{k}}^\infty$ of Archimedean places (whose restriction to $\mathbb{Q}$ is $|\cdot|_\infty$), and the set of finite (or non-Archimedean) places.

When $v \in M_{\mathbb{k}}^\infty$, the algebraic closure $\mathbb{C}_v$ of $\mathbb{k}_v$ is isometric to $\mathbb{C}$. When v is a finite place extending $|\cdot|_p$ on $\mathbb{Q}$, then $|\cdot|_v$ extends canonically to $\mathbb{k}_v^{\mathrm{alg}}$, and its completion $\mathbb{C}_v$ is both complete and algebraically closed.

1.2. **Regular polynomial maps.** Let (z, w) be affine coordinates on the affine plane $\mathbb{A}^2_{\mathbb{k}}$. We also consider homogeneous coordinates $[z_0 : z_1 : z_2]$ on the projective plane $\mathbb{P}^2_{\mathbb{k}}$ and identify the affine plane $\mathbb{A}^2_{\mathbb{k}}$ with the Zariski open set $z_0 \neq 0$ so that $z = z_1/z_0$ and $w = z_2/z_0$. We denote by $L_\infty = \{z_0 = 0\}$ the line at infinity.

Let $f \colon \mathbb{A}^2_{\mathbb{k}} \to \mathbb{A}^2_{\mathbb{k}}$ be any polynomial self-map of the affine plane of degree $d \geq 2$ that extends to an endomorphism of $\mathbb{P}^2_{\mathbb{k}}$. In the coordinates z, w, it is given by

$$f(z, w) = \big(P(z, w), Q(z, w)\big),$$

where $P, Q \in \mathbb{k}[z, w]$ satisfy $\max\{\deg P, \deg Q\} = d$. The fact that f extends to a regular endomorphism $\overline{f} \colon \mathbb{P}^2_{\mathbb{k}} \to \mathbb{P}^2_{\mathbb{k}}$ is equivalent to say that $P = P_d + \text{l.o.t.}$ and $Q = Q_d + \text{l.o.t.}$, where P_d and Q_d are homogeneous polynomials of degree d without common factors.

For $n \in \mathbb{N}$ we write $f^n(z, w) = \big(P_n(z, w), Q_n(z, w)\big)$. The restriction of f_∞ to L_∞ is an endomorphism of $\mathbb{P}^1_{\mathbb{k}}$ given in homogeneous coordinates by

$$f_\infty([z_1 : z_2]) = [P_d(z_1, z_2) : Q_d(z_1, z_2)] \ .$$

1.3. **Green functions.** The next proposition follows from the Nullstellensatz (see e.g. [30, Theorem 3.11]).

Proposition 1.1. *For any $v \in M_{\mathbb{k}}$, there exists a constant $C_v \geq 1$ such that*

$$C_v^{-1} \leq \frac{\max\{1, |P(z, w)|_v, |Q(z, w)|_v\}}{\max\{1, |z|_v, |w|_v\}^d} \leq C_v \tag{2}$$

for all $z, w \in \mathbb{C}_v$. Moreover, for all but finitely many $v \in M_{\mathbb{k}}$ we may take $C_v = 1$.

By the previous proposition, the sequence of functions

$$g_{v,n}(z, w) := \frac{1}{d^n} \log \max\{1, |P_n(z, w)|_v, |Q_n(z, w)|_v\}$$

converges uniformly on $\mathbb{C}_v^2$ to a continuous function g_v, and the next proposition follows.

Proposition 1.2. *For any $v \in M_{\mathbb{k}}$, the function $g_v \colon \mathbb{C}_v^2 \to \mathbb{R}_+$ is continuous, it satisfies the invariance equation $g_v \circ f = d g_v$, and we have*

$$\left|g_v(z,w) - \log\max\{1, |z|_v, |w|_v\}\right| \le \frac{\log C_v}{d-1} \ .$$

The set $\{(z,w) \in \mathbb{C}_v^2,\ g_v(z,w) = 0\}$ coincides with the set of points having bounded orbits.

We shall also consider the global Green function in $\mathbb{C}_v^3$. Write $\widetilde{P}(z_0,z_1,z_2) = z_0^d P(\frac{z_1}{z_0}, \frac{z_2}{z_0})$ and $\widetilde{Q}(z_0,z_1,z_2) = z_0^d Q(\frac{z_1}{z_0}, \frac{z_2}{z_0})$ so that $F(z_0,z_1,z_2) = (z_0^d, \widetilde{P}, \widetilde{Q})$ is a homogenous map of degree d that lifts f to $\mathbb{C}_v^3$. Observe that in homogenous coordinates, (2) can be rewritten as follows:

$$C_v^{-1} \le \frac{\max\{|z_0|^d, |\widetilde{P}(z_0,z_1,z_2)|_v, |\widetilde{Q}(z_0,z_1,z_2)|_v\}}{\max\{|z_0|_v, |z_1|_v, |z_2|_v\}^d} \le C_v$$

so that the next result also holds.

Proposition 1.3. *The function $G_v(z_0,z_1,z_2) = g_v(z_1/z_0, z_2/z_0) + \log|z_0|$ is continuous on $\mathbb{C}_v^3 \setminus \{0\}$, 1-homogeneous (that is, $G_v(\lambda Z) = \log|\lambda| + G_v(Z)$), and satisfies $G_v \circ F = d G_v$. We have*

$$|G_v(z_0,z_1,z_2) - \log\max\{|z_0|_v, |z_1|_v, |z_2|_v\}| \le \frac{\log C_v}{d-1} \ .$$

The set $\{(z_0,z_1,z_2) \in \mathbb{C}_v^3,\ G_v(z_0,z_1,z_2) \le 0\}$ coincides with the set of points having bounded F-orbits.

1.4. **Canonical heights on points.** We refer to [6] for generalities on heights. Consider the line bundle on $\mathcal{O}(1) \to \mathbb{P}^2_{\mathbb{k}}$. The space of sections of this line bundle can be canonically identified with the space of linear forms $a_0 z_0 + a_1 z_1 + a_2 z_2$ with $a_i \in \mathbb{k}$. More precisely, in the trivialization of the bundle over $\{z_i \neq 0\}$, this section is given by $\frac{1}{z_i} a_0 z_0 + a_1 z_1 + a_2 z_2$.

For any $v \in M_{\mathbb{k}}$, we consider the line bundle $\mathcal{O}(1) \to \mathbb{P}^2_{\mathbb{C}_v}$ and endow it with the metrization $|\cdot|_v$ induced by G_v, in the sense that any section $\sigma = a_0 z_0 + a_1 z_1 + a_2 z_2$ as above

$$|\sigma|_v([z_0 : z_1 : z_2]) = |a_0 z_0 + a_1 z_1 + a_2 z_2| e^{-G_v(z_0,z_1,z_2)} \tag{3}$$

(this expression is well-defined thanks to the homogeneity property of G_v). Let us now explain how this collection of metrizations defines a function on the set of algebraic points in $\mathbb{P}^2$ as well as on the set of all algebraic curves in $\mathbb{P}^2_{\mathbb{k}}$ defined by an equation with coefficients in $\mathbb{k}^{\mathrm{alg}}$.

For any $p \in (\mathbb{k}^{\mathrm{alg}})^3 \setminus \{0\}$, we set

$$h_f(p) := \frac{1}{N(p)} \sum_{p' \in \mathrm{Gal}\cdot p} \left(\sum_{v \in M_{\mathbb{k}}} n_v G_v(p') \right)$$

where Gal denotes the absolute Galois group of $\mathbb{k}^{\mathrm{alg}}$ over $\mathbb{k}$, and $N(x)$ is the cardinality of the set $\mathrm{Gal}\cdot x \subset (\mathbb{k}^{\mathrm{alg}})^3$.

The product formula entails that $h_f(z_0,z_1,z_2) = h_f(\lambda z_0, \lambda z_1, \lambda z_2)$ for any $\lambda \in \mathbb{k}^{\mathrm{alg}}$ so that we have a well-defined function $h_f \colon \mathbb{P}^2(\mathbb{k}^{\mathrm{alg}}) \to \mathbb{R}$.

Proposition 1.4. *The function h_f takes non-negative values, and satisfies $h_f \circ f = d h_f$. The set $\{h_f = 0\}$ coincides with the set of preperiodic points of f. Furthermore, for any $(z, w) \in (\mathbb{k}^{\mathrm{alg}})^2$ we have*

$$h_f(z, w) := \frac{1}{N(z, w)} \sum_{(z', w') \in \mathrm{Gal} \cdot (z,w)} \left(\sum_{v \in M_{\mathbb{k}}} n_v g_v(z', w') \right) .$$

As above Gal denotes the absolute Galois group of $\mathbb{k}^{\mathrm{alg}}$ over $\mathbb{k}$, and $N(z, w)$ is the cardinality of the set $\mathrm{Gal} \cdot (z, w) \subset (\mathbb{k}^{\mathrm{alg}})^2$. The proof follows directly from Northcott's theorem, see [4, Corollary 1.1.1].

1.5. **Analytification of affine curves.** Let C be any irreducible algebraic curve in $\mathbb{A}^2_{\mathbb{k}}$ defined by an equation $\{R = 0\}$ with $R \in \mathbb{k}[z, w]$. Denote by $\overline{C}$ the Zariski closure of C in $\mathbb{P}^2$.

Fix any place $v \in M_{\mathbb{k}}$. We denote by C^{an}_v the analytification in the sense of Berkovich of C over $\mathbb{C}_v$. This is a connected, locally connected and locally compact space. When $|\cdot|_v$ is Archimedean, hence $\mathbb{C}_v$ is isometric to $\mathbb{C}$, C^{an}_v is the complex analytic subspace (possibly with singularities) defined as usual by the vanishing of R in $\mathbb{C}^2_v$. When $|\cdot|_v$ is non-Archimedean, then C^{an}_v is defined as the set of multiplicative semi-norms on the ring $\mathbb{C}_v[z, w]/(R)$ whose restriction to the base field equals $|\cdot|_v$. A point is said to be rigid when the semi-norm has non-trivial kernel.

One can also define the analytification of $\overline{C}$ by considering suitable affine coordinates in $\mathbb{P}^2$ and patching the previous construction in a natural way, see [3, §3.4]. Observe that $\overline{C}^{\mathrm{an}}_v \setminus C^{\mathrm{an}}_v$ consists of rigid points.

Suppose first v is Archimedean. The metrization of $\mathcal{O}(1)$ defined by (3) induces a measure $\mu_{C,v}$ on $\overline{C}^{\mathrm{an}}_v$ which is locally defined by $\mu_{C,v} := \Delta \log |\sigma|_v$ where σ is a local section of $\mathcal{O}(1)$. The plurisubharmonicity of G_v ensures that $\mu_{C,v}$ is a positive measure. The Lelong-Poincaré formula implies that the mass of $\mu_{C,v}$ is equal to $\deg(C)$, and we have $\mu_{C,v} = \Delta(g_v|_{C^{\mathrm{an}}_v})$ on C^{an}_v. Observe that since G_v is continuous, $\mu_{C,v}$ gives no mass to points.

The construction is completely analogous in the non-Archimedean case. We again obtain a positive measure $\mu_{C,v}$ on $\overline{C}^{\mathrm{an}}_v$ of total mass $\deg(C)$ which is given in C^{an}_v by $\mu_{C,v} = \Delta(g_v|_{C^{\mathrm{an}}_v})$ where Δ is the Laplace operator defined by Thuillier [31]. Observe that the continuity of the metrization implies that $\mu_{C,v}$ does not charge any rigid point (but it may still charge some non-rigid point in C^{an}_v). We refer to [6, §1.3] for the details of the constructions.

1.6. **Canonical heights on curves.** Let C be any irreducible algebraic curve in $\mathbb{A}^2_{\mathbb{k}}$ as in the previous section. We now define the canonical height of the curve $\overline{C}$ following the recipe given in [6, §3.1.2], taking z_0 as a section of $\mathcal{O}(1)$ (note that this section vanishes exactly along the line at infinity). We obtain:

$$h_f(\overline{C}) := \sum_{p \in \overline{C} \cap L_\infty} (\overline{C}, L_\infty)_p \times h_f(p) + \sum_{v \in M_{\mathbb{k}}} \int_{C^{\mathrm{an}}_v} g_v d\mu_v \ . \tag{4}$$

Note that $h_f(\overline{C}) \geq 0$ because the canonical height is non-negative on closed points, and the Green functions g_v are also non-negative.

Define the essential minimum of h_f by the following formula:

$$\mathrm{essmin}_C(h_f) := \sup_{F \text{ finite} \subset \overline{C}(\mathbb{k}^{\mathrm{alg}})} \inf_{\overline{C}(\mathbb{k}^{\mathrm{alg}}) \setminus F} h_f \ .$$

Theorem 1.5 (Zhang's inequality [32, Theorem 1.10])**.** *We have*

$$2\,\mathrm{essmin}_C(h_f) \geq \frac{h_f(\overline{C})}{\deg(C)} \geq \mathrm{essmin}_C(h_f) + \inf_{p \in \overline{C}(\mathbb{k}^{\mathrm{alg}})} h_f(p) \ .$$

Since $h_f(x) = 0$ if and only if x is preperiodic, we obtain:

Corollary 1.6. *An irreducible algebraic curve C containing infinitely many f-preperiodic points satisfies $h_f(\overline{C}) = 0$.*

1.7. **A first characterization of special curves.**

Proposition 1.7. *Suppose that $C \subset \mathbb{A}^2_{\mathbb{k}}$ is an irreducible algebraic curve containing a sequence of distinct points $p_n \in C(\mathbb{k}^{\mathrm{alg}})$ such that $h_f(p_n) \to 0$.*

Then all points in $\overline{C} \cap L_\infty$ are preperiodic for $\overline{f}$, and for any $v \in M_{\mathbb{k}}$ the function $g_v|_{C_v^{\mathrm{an}}}$ is harmonic on $\{g_v > 0\}$.

Proof. Note that $h_f(\overline{C}) \geq 0$. By Theorem 1.5, our assumption implies that $\mathrm{essmin}_C(h_f) \leq 0$, therefore $h_f(\overline{C}) = 0$. Then the result follows from (4) and the fact that a point p is f-preperiodic if and only if $h_f(p) = 0$. □

2. Super-stable manifolds and local estimates

2.1. **Construction of super-stable manifolds.** In this section we work under the following hypothesis: $(\mathbb{k}, |\cdot|)$ is a complete metrized field of characteristic 0 (which may be either Archimedean or non-Archimedean).

Theorem 2.1. *Suppose $f\colon (\mathbb{A}^2_{\mathbb{k}}, 0) \to (\mathbb{A}^2_{\mathbb{k}}, 0)$ is a germ of analytic map fixing the origin of the form*

$$f(x,y) = \Big(\lambda x + \mu y + g(x,y), y^d(1 + h(x,y))\Big) \ , \tag{5}$$

where $d \geq 2$, $\lambda \neq 0$, $\mu \in \mathbb{k}$, $h(0,0) = 0$, and $g(x,y) = \mathcal{O}(|(x,y)|^2)$. Then there exists a unique smooth analytic curve which is transverse to $\{y = 0\}$ and f-invariant.

We shall call this curve the *local super-stable manifold* of the origin, and denote it by $W^{\mathrm{ss}}_{\mathrm{loc}}(0)$. After a linear change of coordinates of the form $(x,y) \mapsto (x + \frac{\mu}{\lambda} y, y)$, we may and will assume from now on that $\mu = 0$. Expressing the invariant curve as a graph of the form $x = \varphi(y)$ and making a change of coordinates of the form $(x,y) \mapsto (x - \varphi(y), y)$, f takes the form

$$f(x,y) = (\lambda x + x\tilde{g}(x,y), y^d(1 + \tilde{h}(x,y))). \tag{6}$$

It follows that f is analytically conjugate to $y \mapsto y^d$ on $W^{\mathrm{ss}}_{\mathrm{loc}}(0)$, hence the terminology.

The result is classical when $\Bbbk$ is Archimedean and/or f is locally invertible (see e.g. [21, Appendix]). For convenience we include a proof that works simultaneously in the Archimedean and non-Archimedean settings, and is adapted to $\{y = 0\}$ being superattracting.

Proof. As explained above, we look for an analytic map $y \mapsto \varphi(y)$, with $\varphi(0) = 0$ such that the change of coordinates $\Phi(x, y) = (x + \varphi(y), y)$ satisfies

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x\tilde{g}(x, y), y^d(1 + \tilde{h}(x, y)))$$

with $\tilde{g}, \tilde{h}$ analytic and vanishing at 0. This property is equivalent to the identities:

$$\begin{cases} \lambda\varphi(y) + g(\varphi(y), y) = \varphi(y^d(1 + \tilde{h}(0, y))) \\ \tilde{h}(0, y) = h(\varphi(y), y) \end{cases}$$

so that we aim at finding some analytic function φ satisfying

$$\lambda\varphi(y) + g(\varphi(y), y) = \varphi(y^d(1 + h(\varphi(y), y))) \ .$$

For any $r > 0$, let us introduce the Banach space $\mathcal{B}_r$ that consist of those power series $\varphi(y) := \sum_{j\geq 1} a_j y^j$ which are convergent in the disk of radius r, and satisfy $\|\varphi\|_r := \sup_{|y|<r} |\varphi(y)| < \infty$. Note that in the non-Archimedean case, we have $\|\varphi\|_r := \sup_j |a_j| r^j$.

For any $\varphi \in \mathcal{B}_r$, we set

$$T\varphi(y) := \frac{1}{\lambda}\Big(\varphi(y^d(1 + h(\varphi(y), y))) - g(\varphi(y), y)\Big) \ .$$

We claim that for $r > 0$ and $\rho > 0$ sufficiently small, T is a well-defined strictly contracting map on $B(0, \rho) \subset \mathcal{B}_r$. Then, applying the Banach fixed point theorem implies the existence of the desired φ.

First, we may suppose that g is analytic in the polydisk of radius r, and since g vanishes up to order 2 at the origin, we have

$$|g(x, y)| \leq C \max\{|x|, |y|\}^2$$

for some $C > 0$ and all $|x|, |y| < r$. Similarly, we may suppose that h is analytic in the polydisk of radius $r > 0$, and that $|h(x, y)| \leq \frac{1}{2}$ for all $|x|, |y| < r$.

Pick any $\varphi(y) = \sum_j a_j y^j \in B(0, \rho) \subset \mathcal{B}_r$. Reduce $r > 0$ if necessary so that $\frac{3}{2}r^d < r$. Then $\tilde{\varphi} : y \mapsto \varphi(y^d(1 + h(\varphi(y), y)))$ is well-defined and analytic on the disk of radius r.

In the non-Archimedean case, $|1 + h(\varphi(y), y))| = 1$, so that one has

$$\|\tilde{\varphi}(y)\|_r = \sup_{j\geq 1} |a_j| r^{dj} \leq r \sup_{j\geq 1} |a_j| r^{dj-1} \leq r|\varphi|_r \ .$$

In the Archimedean case, the Schwarz lemma yields $|\varphi(y)| \leq \frac{|\varphi|_r}{r}|y|$ for all $|y| < r$, hence $\|\tilde{\varphi}\|_r \leq \frac{3}{2}r^{d-1}|\varphi|_r$. Note that we also have:

$$\|g(\varphi(y), y)\|_r \leq C \max\{\|\varphi\|_r, r\}^2$$

so for $\varphi \in B(0, \rho)$ we deduce

$$|T\varphi|_r \leq \frac{1}{\lambda}\left(\frac{3}{2}r^{d-1}\rho + C \max\{\rho, r\}^2\right).$$

By choosing $\rho = r$ and then r small enough, this estimate shows that $T\varphi$ is well-defined on the ball $B(0,r) \subset \mathcal{B}_r$ and $T(B(0,r)) \subset B(0,r)$.

In order to prove that T is strictly contracting, observe that we can write

$$g(x,y) - g(x',y) = (x-x')\hat{g}(x,x',y)$$

where $\hat{g}(x,x',y)$ is again analytic in the polydisk of radius r, and

$$|\hat{g}(x,x',y)| \leq C' \max\{|x|,|x'|,|y|\}$$

for some constant $C' > 0$. For any pair of analytic functions $\varphi_1, \varphi_2 \in B(0,r) \subset \mathcal{B}_r$ we infer:

$$|g(\varphi_1(y),y) - g(\varphi_2(y),y)| \leq C' \left\|\varphi_1 - \varphi_2\right\|_r \max\{\left\|\varphi_1\right\|_r, \left\|\varphi_2\right\|_r, r\}$$

hence

$$\left\|T\varphi_1 - T\varphi_2\right\|_r \leq \frac{1}{\lambda}(3r^d + C'r)\ \left\|\varphi_1 - \varphi_2\right\|_r \ .$$

Again, by choosing r sufficiently small, we obtain that T is strictly contracting and we are done. ☐

2.2. **The rescaling argument in the repelling case.** We work in $\mathbb{A}^2_{\Bbbk}$ where $\Bbbk$ is an arbitrary complete metrized field of characteristic 0. We start with a preparation lemma.

Lemma 2.2. *Suppose f is an analytic map of the form*

$$f(x,y) = \Big(\lambda x + x g(x,y), y^d(1 + h(x,y))\Big) \tag{7}$$

where $|\lambda| > 1$, $d \geq 2$ and $g(0) = h(0) = 0$.

Then there exists an analytic change of coordinates Φ such that

$$\Phi^{-1} \circ f \circ \Phi(x,y) = (\lambda x(1 + xy\tilde{g}(x,y)), y^d(1 + x\tilde{h}(x,y)))$$

for some analytic functions $\tilde{g}, \tilde{h}$.

Recall that the form (7) is what is obtained from (5) after conjugating to get $\mu = 0$ and declaring that the stable manifold of Theorem 2.1 is $\{x = 0\}$.

Proof. By Böttcher's theorem (see [2, Chapter 4] for the non-Archimedean case) applied to $y \mapsto f(0,y)$ we may suppose that x divides h. Similarly, since $|\lambda| > 1$, by we may linearize $x \mapsto f(x,0)$, and suppose that f is of the form $f(x,y) = (\lambda x(1 + g_1(y)) + x^2 y h_1(x,y), y^d(1 + \mathcal{O}(x)))$ for some analytic functions g_1, h_1 with $g_1(0) = 0$.

We claim that there exists $\Phi(x,y) = (x(1 + \varphi(y)), y)$ with $\varphi(0) = 0$ such that

$$\Phi^{-1} \circ f \circ \Phi(x,y) = (\lambda x + x^2 y h_2(x,y), y^d(1 + \mathcal{O}(x)))$$

for some analytic function h_2. Indeed, φ is then characterized by the equation

$$\lambda x(1 + \varphi(y))(1 + g_1(y)) + \mathcal{O}(x^2) = (\lambda x + \mathcal{O}(x^2))(1 + \varphi(y^d))$$

that is, $(1 + \varphi(y))(1 + g_1(y)) = (1 + \varphi(y^d))$, a solution of which is given by the infinite product

$$1 + \varphi(y) = \prod_{k=0}^{\infty} \left(1 + g_1\left(y^{d^k}\right)\right)^{-1}$$

and we are done. ☐

The next result is similar to [11, Lemma 4.2].

Proposition 2.3. *Suppose f is an analytic map of the form*

$$f(x,y) = (\lambda x(1+xyg(x,y)), y^d(1+xh(x,y))) \tag{8}$$

where $|\lambda|>1$, $d\geq 2$ and g,h are analytic functions.

Then $f^n(\frac{x}{\lambda^n},y)\to(x,0)$ when $n\to\infty$, uniformly on a polydisk of sufficiently small radius centered at the origin.

Proof. Fix $0<r\leq 1/4$ small enough so that g,h are both analytic on the polydisk of radius r and $|g(x,y)|,|h(x,y)|\leq 1$ for $|x|,|y|<r$. Let us first show that if $|x|\leq\frac{r}{2|\lambda|^n}$ and $|y|\leq r$, then the n first iterates of (x,y) remain in $\mathbb{D}_r^2$. We argue by induction. So assume that $(x_0,y_0)\in\mathbb{D}_{r|\lambda|^{-n}/2}\times\mathbb{D}_r$, put $f^j(x_0,y_0)=(x_j,y_j)$, let $k\leq n$ and assume that $(x_j,y_j)\in\mathbb{D}_r^2$ for $j\leq k-1$. Observe that for $j\leq k-1$, $|y_{j+1}|\leq 2|y_j|^d$ from which it follows that

$$|y_k|\leq\left(2^{1/(d-1)}|y_0|\right)^{d^k}\leq\left(2^{1/(d-1)}r\right)^{d^k}\leq r.$$

Observe that the first inequality together with $r\leq 1/4$ also yield $|y_j|\leq 2^{-d^j}$. For the first coordinate, we use recursively the relation $x_{j+1}=\lambda x_j(1+x_jy_jg(x_j,y_j))$ to get

$$|x_k|\leq|\lambda|^k|x_0|\prod_{j=0}^{k-1}(1+|x_j||y_j|)\leq\frac{r}{2}|\lambda|^{k-n}\prod_{j=0}^{k-1}\left(1+2^{-d^j}\right)\leq r|\lambda|^{k-n}, \tag{9}$$

where the last inequality follows from

$$\prod_{j=0}^{k-1}\left(1+2^{-d^j}\right)\leq\prod_{j=0}^{k-1}\left(1+2^{-2^j}\right)=\frac{2^{2^k}-1}{2^{2^k-1}}<2,$$

in which the middle equality is easily obtained by induction.

Now take $(x,y)\in\mathbb{D}_{r/2}$, and consider $f^n(\frac{x}{\lambda^n},y)$. Denote as before $(x_0,y_0)=(\frac{x}{\lambda^n},y)$ and $(x_j,y_j)=f^j(x_0,y_0)$. The first part of the proof shows that (x_j,y_j) is well-defined for all $j\leq n$, and that $y_n\to 0$. Now we have

$$x_n=\lambda^n x_0\prod_{j=0}^{n-1}(1+x_jy_jh(x_j,y_j))=x\prod_{j=0}^{n-1}(1+x_jy_jh(x_j,y_j)).$$

The inequality $\left|\prod(1+z_j)-1\right|\leq\exp\left(\sum|z_j|\right)-1$ shows that to establish the convergence $x_n\to x$ it is enough to show that $\sum_{j=0}^{n-1}|x_jy_jh(x_j,y_j)|$ tends to 0. But by (9), $|x_j|\leq r|\lambda|^{j-n}$, thus

$$\sum_{j=0}^{n-1}|x_jy_jh(x_j,y_j)|\leq\sum_{j=0}^{n-1}r|\lambda|^{j-n}2^{-d^j}\leq r|\lambda|^{-n}\sum_{j=0}^{\infty}|\lambda|^j2^{-d^j},$$

and we are done. □

2.3. **Graph transform for $\lambda = 1$.** In this paragraph we assume that $\Bbbk = \mathbb{C}$ and f is of the form

$$f(x,y) = \Big(x + g(x,y), y^d(1 + h(x,y))\Big) \ , \tag{10}$$

with $g(x,y) = \mathcal{O}(|x,y|^2)$, $h(0,0) = 0$, and $g(x,0) = cx^{k+1} + \mathcal{O}(x^{k+2})$ for some $k \geq 1$ and $c \neq 0$. Observe that $f|_{\{y=0\}}$ has a parabolic point at the origin with k attracting and k repelling petals (see e.g. [1, §6.5]). An *attracting petal* is a f-invariant (connected and simply-connected) open subset U containing 0 in its boundary, and such that, for all $z \in U$, $f^n(z) \to 0$ tangentially to some real direction (in our case, to the normalized k-th roots of $-c$). A *repelling petal* is an attracting petal for f^{-1} (they are tangent to the normalized k-th roots of c). One can chose the k attracting petals and k repelling petals so that their union fills up a punctured neighborhood of the origin.

A vertical graph V in a domain of the form $\Omega \times \mathbb{D}_\rho$ is a submanifold of the form $V := \{(\varphi(y), y),\ y \in \mathbb{D}_\rho\}$ for some holomorphic function $\varphi : \mathbb{D}_\rho \to \Omega$. In the next theorem we consider pull backs of such graphs in $\mathbb{D}_r^2$ in the graph transform sense, that is, when pulling back some vertical graph under f, we keep only the component of $f^{-1}(V) \cap \mathbb{D}_r^2$ containing $f^{-1}(V \cap \{y = 0\})$. Abusing notation we simply denote it by $f^{-1}(V)$.

Theorem 2.4. *Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be of the form* (10) *with $h(0) = 0$, $g(x,0) \not\equiv 0$ and $g(x,y) = \mathcal{O}(|(x,y)|^2)$. Let U be any repelling petal of $f|_{\{y=0\}}$ and consider a germ V of analytic curve transverse to $\{y = 0\}$, intersecting U.*

Then there exists $r > 0$ depending only on f, such that for large enough n, the analytic sets $f^{-n}(V)$ are vertical graphs in $\mathbb{D}_r^2$ converging to $W^{\rm ss}_{\rm loc}(0)$ in the C^1 topology.

Lemma 2.5. *Suppose f is a holomorphic map of the form* (10) *as in Theorem 2.4. Then there exist an integer $k \geq 1$, and an analytic change of coordinates Φ such that*

$$\Phi^{-1} \circ f \circ \Phi(x,y) = (x + x^{k+1} + x^{2k+1}\tilde{g}(x,y), y^d(1 + x\tilde{h}(x,y)))$$

for some analytic functions $\tilde{g}, \tilde{h}$.

Proof. The proof is essentially contained in [20, Proposition 2.3]. We provide the details for the sake of completeness. By Theorem 2.1 and by applying the Böttcher theorem to $y \mapsto f(0,y)$ we may assume that both g and h are divisible by x, so that we may write

$$f(x,y) = \Big(x + xg(x,y), y^d(1 + xh(x,y))\Big) \ , \tag{11}$$

with $g(0,0) = 0$.

By a local change of coordinates involving only x, we can arrange so that $f(x,0) = (x + x^{k+1} + \mathcal{O}(x^{2k+1}), 0)$. Expand the first coordinate of f in power series of x as follows:

$$x \circ f(x,y) = x(1 + g_0(y)) + x^{k+1}(1 + g_k(y)) + \sum_{j \neq 0,k}^{\infty} x^{j+1} g_j(y),$$

with $g_j(0) = 0$ for $0 \leq j \leq 2k - 1$.

We claim that for all $n \leq 2k - 1$, we can conjugate f by a germ of invertible holomorphic map such that $g_j \equiv 0$ for every $j \leq n$. Applied to $n = 2k - 1$, this claim implies the proposition.

For $n = 0$ this is done by a change of coordinates of the form $\Phi_0(x, y) := (x(1 + \varphi_0(y)), y)$ such that $(1 + g_0)(1 + \varphi_0) = 1 + \varphi_0(y^d)$. This equation can be solved exactly as in Lemma 2.2.

Now assume that $n > 0$, and that the result has been achieved up to $j = n - 1$. Put $\Phi_n(x, y) = \big(x(1 + \varphi_n(y)x^n), y\big)$, so that $\Phi_n^{-1}(x, y) = \big(x(1 - \varphi_n(y)x^n + \mathcal{O}(x^{n+1})), y\big)$. Depending on the position of n and k, we obtain:

$$x \circ (\Phi_n^{-1} \circ f \circ \Phi_n) = \begin{cases} x + x^{n+1}\big(\varphi_n(y) + g_n(y) - \varphi_n(y^d)\big) + \mathcal{O}(x^{n+2}) & \text{if } n < k, \\ x + x^{n+1}\big(\varphi_k(y) + 1 + g_n(y) - \varphi_n(y^d)\big) + \mathcal{O}(x^{n+2}) & \text{if } n = k, \\ x + x^{k+1} + x^{n+1}\big(\varphi_n(y) + g_n(y) - \varphi_n(y^d)\big) + \mathcal{O}(x^{n+2}) & \text{if } n > k. \end{cases}$$

Therefore, to render the term in x^{n+1} constant, it is enough to solve the equation $-g_n(y) = \varphi_n(y) - \varphi_n(y^d)$ which can be done by setting $\varphi_n(y) = -\sum_{m=0}^{\infty} g_n\big(y^{d^m}\big)$. □

Proof of Theorem 2.4. By the previous lemma, we may suppose that

$$f(x, y) = \Big(x + x^{k+1} + x^{2k+1}g(x, y), y^d(1 + xh(x, y))\Big)$$

with g and h holomorphic near the origin. Note that $f|_{y=0}$ has a repelling petal along the positive real axis. Fix $r > 0$ such that f is holomorphic and injective on a neighborhood of $\overline{\mathbb{D}}_r^2$. Reducing and rotating the petal if necessary, we may assume that

$$U := \Big\{x : \ \arg(x) \in \Big(-\frac{\pi}{4k}, \frac{\pi}{4k}\Big), \ |x| < r\Big\}.$$

The holomorphic map $z = \big(kx^k\big)^{-1}$ is univalent on $U \times \mathbb{D}_r$, and takes its values in

$$\Omega_R := \Big\{z : \ \arg(z) \in \Big(-\frac{\pi}{4}, \frac{\pi}{4}\Big), \ |z| > R\Big\},$$

where $R = (kr^k)^{-1}$. The expression of f in the coordinates (z, y) is of the form

$$f(z, y) = \left(z - 1 + \frac{1}{z}a(z, y), y^d\left(1 + \frac{1}{z^{1/k}}b(z, y)\right)\right). \tag{12}$$

so that f is now defined in $\Omega_R \times \mathbb{D}_r$. Fix $M > 0$ such that

$$|a|, \ |b|, \left|\frac{\partial a}{\partial z}\right|, \left|\frac{\partial a}{\partial y}\right|, \ \left|\frac{\partial b}{\partial z}\right|, \ \left|\frac{\partial b}{\partial y}\right| \le M \text{ on } \Omega_R \times \mathbb{D}_r, \tag{13}$$

and reduce r if necessary so that $r < \frac{1}{10d}$ and $MR^{-1/k} \le \frac{1}{100}$.

For any $\rho < r$ and $\sigma > 0$ we let

$$\mathcal{G}(\rho, \sigma) = \{\varphi \colon \mathbb{D}_\rho \to \Omega_R \text{ holomorphic s.t. } \sup_{\mathbb{D}_\rho} |\varphi'| \le \sigma\}\ .$$

Lemma 2.6. *Suppose that $\sigma\rho < \frac{1}{100d}$. For any vertical graph Γ determined by $\varphi \in \mathcal{G}(\rho, \sigma)$, then $f^{-1}\Gamma$ is a vertical graph determined by a function $\psi \in \mathcal{G}(\rho_1, \frac{1}{10})$ where $\rho_1 = \min\big((\rho/2)^{1/d}, r\big)$ and $\mathrm{Re}(\psi) \ge \mathrm{Re}(\varphi(0)) + 9/10$.*

Assuming this result for the moment, let us conclude the proof of the theorem. Let V be any germ of curve intersecting transversally $\{y = 0\}$ at $(x_0, 0) \in U$. Then V is a graph of slope σ over some disk $\mathbb{D}_\rho$ in the second coordinate, for some $\rho > 0$. Reduce ρ by trimming V if necessary so that $\sigma\rho < 1/100d$. Since $r/10 < 1/100d$, the previous lemma implies that we can define inductively a sequence of vertical graphs $V = V_0$, $V_n := f^{-1}V_{n-1}$, where V_n is defined by a holomorphic function $z = \varphi_n(y)$ of uniformly bounded slope over $\mathbb{D}_{\rho_n}$, and furthermore $\rho_n = r$ for large enough n. Moreover, we have $\mathrm{Re}(\varphi_n) \geq \mathrm{Re}(\varphi(0)) + 9n/10$, hence, coming back to the (x, y) coordinates, we see that $V_n = \{x = (k\varphi_n(y))^{-1/k}\}$ converges in the C^1-topology to the curve $W^{\mathrm{ss}}_{\mathrm{loc}}(0) = \{x = 0\}$. □

Proof of Lemma 2.6. Let Γ be a vertical graph of equation $z = \varphi(y)$ in the (z, y) coordinates, with $\varphi \in \mathcal{G}(\rho, \sigma)$. Then the equation of $f^{-1}\Gamma$ is given by $z = \ell(z, y)$, where

$$\ell(z, y) = \varphi\left(y^d\left(1 + \frac{1}{z^{1/k}}b(z, y)\right)\right) + 1 - \frac{1}{z}a(z, y).$$

Fix $y_0 \in \mathbb{D}_{\rho_1}$. We show that the equation $z = \ell(z, y_0)$ admits a unique solution $z \in \Omega_R$. First, observe that by the estimates (13) on $|a|$ and $|b|$, for $z \in \Omega_R$ we have

$$\begin{aligned} |\ell(z, y_0) - (\varphi(0) + 1)| &\leq \left|\varphi\left(y_0^d\left(1 + \frac{1}{z^{1/k}}b(z, y)\right)\right) - \varphi(0)\right| + \frac{M}{R} \\ &\leq 2\rho_1^d\sigma + \frac{1}{100} \leq \rho\sigma + \frac{1}{100} \leq \frac{1}{10}\ , \end{aligned} \tag{14}$$

hence $\ell(\cdot, y_0)$ maps Ω_R to itself. Next, we see that

$$\begin{aligned} \left|\frac{\partial \ell}{\partial z}(z, y_0)\right| &\leq \left|y_0^d\left(\frac{1}{z^{1/k}}\frac{\partial b}{\partial z} - \frac{1}{kz^{1/k+1}}b\right)\right| \cdot \left|\varphi'\left(y_0^d\left(1 + \frac{1}{z^{1/k}}b\right)\right)\right| + \left|\frac{a}{z^2} - \frac{1}{z}\frac{\partial a}{\partial z}\right| \\ &\leq \rho_1^d\frac{2M}{R^{1/k}}\sigma + \frac{2M}{R} \leq \frac{4}{100}\rho\sigma + \frac{2}{100} \leq \frac{1}{10}\ , \end{aligned}$$

so $\ell(\cdot, y_0)$ is a contraction and the equation $z = \ell(z, y_0)$ has a unique solution. This means that $f^{-1}\Gamma$ is a vertical graph over $\mathbb{D}_{\rho_1}$ determined by a holomorphic function ψ satisfying $\psi(y) = \ell(\psi(y), y)$. The slope of this graph can be estimated as above:

$$|\psi'(y)| \leq \left|\frac{\partial\ell/\partial y}{1 - \partial\ell/\partial z}\right| \leq \frac{10}{9}\ \left(\sigma\left(d\rho^{d-1}\left(1 + \frac{M}{R^{1/k}}\right) + \rho^d\frac{M}{R^{1/k}}\right) + \frac{M}{R}\right) \leq \frac{1}{10}.$$

Finally, the estimate $\mathrm{Re}(\psi) \geq \mathrm{Re}(\varphi(0)) + 9/10$ follows from (14) and we are done. □

3. Proof of Theorem A when $\Bbbk$ is a number field

Here we establish the following more precise form of our main theorem, in the number field case.

Theorem 3.1. *Let $\Bbbk$ be a number field and f be a regular polynomial map of $\mathbb{A}^2_{\Bbbk}$. Denote by h_f the induced canonical height.*

Suppose that $C \subset \mathbb{A}^2_{\Bbbk}$ is an irreducible algebraic curve containing a sequence of distinct points $p_n \in C(\Bbbk^{\mathrm{alg}})$ such that $h_f(p_n) \to 0$. If there exists a point of $\overline{C} \cap L_\infty$ which is not eventually superattracting, then C is preperiodic.

Let f_∞ be the restriction to the line at infinity L_∞ of the extension of f to $\mathbb{P}^2$. By Proposition 1.7, all points in $\overline{C} \cap L_\infty$ are preperiodic, so we may replace f by f^N and C by $f^N(C)$, to assume that $\overline{C} \cap L_\infty$ contains a fixed point p which is not super-attracting.

Let $\lambda = f'_\infty(p)$ be the multiplier of p along L_∞. Then one of the two following mutually disjoint cases occur:

(a) either λ is a root of unity;
(b) or there is a place $v \in M_{\mathbb{k}}$ such that $|\lambda|_v > 1$.

In the remainder of this section we split the proof of the theorem according to these two cases.

3.1. **When λ is a root of unity.** In this situation we iterate f further so that $\lambda = 1$, and work over the complex numbers. Since p belongs to the Julia set $J(f_\infty)$, which is a perfect set, the union of attracting and repelling petals cover a punctured neighborhood of p, and the attracting petals of p are contained in the Fatou set, we see that there is a repelling periodic point q of f_∞ contained in some local repelling petal of p. Then the local (super-)stable manifold of q is a disk transverse to L_∞ at q, and by Theorem 2.4, the local truncated pull-backs $f^{-n}(W^{\mathrm{ss}}_{\mathrm{loc}}(q))$ under f^n converge to $W^{\mathrm{ss}}_{\mathrm{loc}}(p)$ when $n \to \infty$.

Assume by way of contradiction that $\overline{C}$ does not locally coincide with $W^{\mathrm{ss}}_{\mathrm{loc}}(p)$. We claim that C intersects $f^{-n}(W^{\mathrm{ss}}_{\mathrm{loc}}(q))$ in $\mathbb{C}^2$ close to p for large n. Indeed, locally near p, $\overline{C} \cap W^{\mathrm{ss}}_{\mathrm{loc}}(p) = \{p\}$, so by the persistence of proper intersections, $\overline{C}$ intersects $f^{-n}(W^{\mathrm{ss}}_{\mathrm{loc}}(q))$ close to p for large n. But $p \notin f^{-n}(W^{\mathrm{ss}}_{\mathrm{loc}}(q)) \cap L_\infty$, so these intersection points lie in $\mathbb{C}^2$, as claimed. If now Δ is a small disk in C containing one of these intersection points, then by the Inclination Lemma, the derivative of f^n in the direction of Δ tends to infinity, thus $(f^n|_\Delta)$ is not a normal family. On the other hand, by Proposition 1.7 $g|_\Delta$ is harmonic, which implies that $(f^n|_\Delta)$ is normal (see [13, Prop 5.10]). This contradiction shows that $\overline{C}$ locally coincides with $W^{\mathrm{ss}}_{\mathrm{loc}}(p)$ near p, so it is fixed under f, and by irreducibility this property propagates to the whole of C. This completes the proof in this case. □

Remark 3.2. This argument works essentially the same when $|\lambda|_v > 1$ at some Archimedean place, and may help understand the non-Archimedean argument below.

3.2. **When $|\lambda|_v > 1$.** We may assume that $p = [0 : 0 : 1]$, and by Lemma 2.2 find a local analytic isomorphism $(x, y) \mapsto \psi(x, y) = [z_0(x, y) : z_1(x, y) : 1]$ such that $\psi(0) = p$, $z_0(x, 0) = 0$ so that $\{y = 0\}$ corresponds to the line at infinity[1], and write

$$\widetilde{f} := \psi^{-1} \circ f \circ \psi : (x, y) \longmapsto \Big(\lambda x\big(1 + xyg(x, y)\big), y^d\big(1 + xh(x, y)\big)\Big) .$$

If $\overline{C}$ has an analytic branch at p which coincides with $\{x = 0\}$ then C is fixed as above, and we are done. Otherwise, we may find a Puiseux parameterization of a branch of $\overline{C}$ at p in the (x, y) coordinates of the form $\Gamma(t) = (t^q, \gamma(t))$, where γ is analytic and defined in a small disk $\mathbb{D}_\delta$, and $q \in \mathbb{N}^*$. We seek a contradiction.

We lift ψ to $\mathbb{A}^3_{\mathbb{k}}$, and set $\Psi(x, y) = (z_0(x, y), z_1(x, y), 1)$. As in §1.3, we lift f to a homogeneous polynomial map $F \colon \mathbb{A}^3_{\mathbb{k}} \to \mathbb{A}^3_{\mathbb{k}}$, of the form $F(z_0, z_1, z_2) = (z_0^d, \widetilde{P}, \widetilde{Q})$. Since $\Psi \circ \psi^{-1}$ is a local section of the projection $\mathbb{A}^3 \setminus \{0\} \to \mathbb{P}^2$, $\Psi \circ \widetilde{f}$ must be a multiple of $F \circ \Psi$. From the expression of F we obtain

$$\Psi \circ \widetilde{f} = \frac{F \circ \Psi}{\widetilde{Q} \circ \Psi} .$$

[1]Beware that coordinates are swapped here : $\{z_0 = 0\}$ corresponds to $\{y = 0\}$.

To simplify notation, we write $F^n(z_0, z_1, z_2) = (z_0^{d^n}, \widetilde{P}_n, \widetilde{Q}_n)$. We consider the 1-homogeneous Green function $G_f\colon \mathbb{A}_v^{3,\mathrm{an}} \to \mathbb{R}$ of Proposition 1.3; it satisfies $G_f \circ F = dG_f$ and $g_f(z_1, z_2) = G_f(1, z_1, z_2)$.

Observe that $h := G_f \circ \Psi \circ \Gamma$ is a continuous function on $\mathbb{D}_\delta$. Since $h(t) = g_f \circ \psi \circ \Gamma(t) + \log|z_0 \circ \Gamma(t)|$, by Proposition 1.7, h is harmonic on $\{t \neq 0\}$. Since it is continuous at 0, it is also harmonic on $\mathbb{D}_\delta$, see, e.g., [12, Lemma 3.7]. Write

$$\begin{aligned} d^{nq}\, h(t) &= G_f \circ F^{nq} \circ \Psi \circ \Gamma(t) \\ &= G_f \circ \Psi \circ \widetilde{f}^{nq} \circ \Gamma(t) + \log\left|\widetilde{Q}_{nq} \circ \Psi \circ \Gamma(t)\right| . \end{aligned}$$

By Proposition 2.3, $\widetilde{f}^n\big(\frac{x}{\lambda^n}, y\big) \to (x, 0)$ uniformly in a neighborhood of the origin, hence

$$G_f \circ \Psi \circ \widetilde{f}^{nq} \circ \Gamma\big(\tfrac{t}{\lambda^{n/q}}\big) \to G_f\big(0, z_1(t^q, 0), 1\big)$$

as $n \to \infty$. On the other hand, since $d^{nq}\, h\big(\frac{t}{\lambda^{n/q}}\big) - \log\left|\widetilde{Q}_{nq} \circ \Psi \circ \Gamma\big(\frac{t}{\lambda^{n/q}}\big)\right|$ is a sequence of harmonic functions, it follows that $t \mapsto G_f(0, z_1(t^q, 0), 1)$ is harmonic as well.

Now, observe that the restriction of f to the line at infinity is $f_\infty[z_1 : z_2] = \big[\widetilde{P}(0, z_1, z_2) : \widetilde{Q}(0, z_1, z_2)\big]$, so that $G_f(0, z_1, z_2)$ is the global Green function of f_∞. The equilibrium measure of f_∞ is the probability measure on the analytification of L_∞ defined by $\mu_{f_\infty} := \Delta G_f(0, z_1, 1)$ in the chart $z_2 \neq 0$. Its support is the Julia set of f_∞ (see [2, Theorem 13.39]), and it contains all repelling (rigid) fixed points, see [2, Theorem 8.7]. Therefore, $z_1 \mapsto G_f(0, z_1, 1)$ cannot be harmonic near 0, hence the function $t \mapsto G_f(0, z_1(t^q, 0), 1)$ cannot be harmonic either. This contradiction concludes the proof. □

Remark 3.3. Under the assumptions of Theorem 3.1, the proof shows that the preperiod k of F and the period of $f^k(C)$ are exactly the same as that of any of its non-superattracting points at infinity.

4. Proof of Theorem A for arbitrary $\Bbbk$

In this section we use a specialization argument to deal with maps defined over arbitrary fields. It shares some arguments with [11, §5] (see also [5, §7]). Nevertheless, new ideas are needed to deal with preperiodic points instead of periodic ones.

We are in the setting of Theorem A, so we assume that f is a regular polynomial map of $\mathbb{A}^2$ of degree $d \geq 2$ defined over a field $\Bbbk$ of characteristic 0, and C is a curve containing an infinite set $\mathcal{P} = \{p_n,\ n \geq 0\}$ of preperiodic points and whose closure $\overline{C} \cap L_\infty$ contains at least one point which is not eventually super-attracting.

By enlarging $\Bbbk$ if necessary we may assume that it contains the algebraic closure $\mathbb{Q}^{\mathrm{alg}}$ of its prime field. Let R be the sub-$\mathbb{Q}^{\mathrm{alg}}$-algebra of $\Bbbk$ generated by all coefficients defining f and C. Its fraction field K is finitely generated over $\mathbb{Q}^{\mathrm{alg}}$. Let $S = \operatorname{Spec} R$. This is an affine variety defined over $\mathbb{Q}^{\mathrm{alg}}$, and elements of R can be seen as regular functions on S.

Inverting some elements of R if necessary, we may suppose that C is flat over S, and f extends as a morphism $f\colon \mathbb{P}^2_S \to \mathbb{P}^2_S$. We let $\pi\colon \mathbb{P}^2_S \to S$ be the canonical projection, and write $\mathbb{P}^2_s = \pi^{-1}(s)$. We also let $\mathbb{A}^2_s := \mathbb{A}^2_S \cap \pi^{-1}(s)$.

For each (scheme theoretic) point $s \in S$, we write $C_s = C \cap \mathbb{A}^2_s$ and let $\overline{C}_s$ be the closure of C_s in $\mathbb{P}^2_s$. The flatness of the morphism $C \to S$ implies $\overline{C}_s$ (hence C_s) to be a curve. Similarly, we let $f_s \colon \mathbb{P}^2_s \to \mathbb{P}^2_s$ be the induced map on the fibers: this is an endomorphism of degree d.

We also denote by $p_{n,s} \in \mathbb{A}^2_s$ the specialization of p_n. Note that p_n is defined over some finite extension of K which depends on n.

The first result does not use the assumption that our infinite set of preperiodic points lies on a curve.

Proposition 4.1. *Let as above $f \colon \mathbb{P}^2_S \to \mathbb{P}^2_S$ be a family of endomorphisms over an affine variety defined over $\mathbb{Q}^{\mathrm{alg}}$, and let $\mathcal{P} = \{p_n,\ n \geq 0\}$ be an infinite family of preperiodic points. Then there exists a non-empty Zariski open and dense subset $U \subset S$ such that for any $s \in U \cap S(\mathbb{Q}^{\mathrm{alg}})$, $\mathcal{P}_s = \{p_{n,s},\ n \geq 0\} \subset \mathbb{A}^2_s$ is infinite.*

Before starting the proof, let us fix some additional notation. For each $n \geq 0$, we denote by k_n the preperiod of p_n, so that $q_n := f^{k_n}(p_n)$ is the first periodic point in the orbit of p_n. We let ℓ_n be the (primitive) period of q_n.

Proof. We may suppose that there exists a parameter $s_0 \in S(\mathbb{Q}^{\mathrm{alg}})$ such that $\mathcal{P}_{s_0}$ is finite (otherwise we take $U = S$ and the proof is complete).

Lemma 4.2. *The family of periodic points (q_n) is finite.*

Proof. We follow the arguments of [11, §5]. Set $\mathcal{Q} = \{q_n,\ n \geq 0\}$. For each $\ell \geq 1$, we consider the subvariety Per_ℓ of $\mathbb{P}^2_S$ defined by the equation $f^\ell(z) = z$. Since $\mathbb{P}^2_S \to S$ is proper, the structure map $\mathrm{Per}_\ell \to S$ is also proper.

Let $\mathcal{Q}_\ell$ be the union of the irreducible components of Per_ℓ containing a point of $\mathcal{P}$. Its underlying set is the Zariski closure of $\mathcal{P} \cap \mathrm{Per}_\ell$, hence $\mathcal{Q}_\ell \to S$ is proper. Observe that for $x \in \mathcal{Q}_{\ell,s}$, the multiplicity of x as a point of $\mathcal{Q}_{\ell,s}$ equals its multiplicity as a fixed point of f^ℓ_s. By Nakayama's lemma and the properness of $\mathcal{Q}_\ell$ over S, the function

$$s \longmapsto \sum_{x \in \mathcal{Q}_{\ell,s}} \mathrm{mult}_x(\mathcal{Q}_{\ell,s}) \tag{15}$$

is upper semi-continuous for the Zariski topology, hence

$$\sum_{q \in \mathcal{Q}_\ell} \mathrm{mult}_q(\mathcal{Q}_\ell) \leq \sum_{x \in \mathcal{Q}_{\ell,s_0}} \mathrm{mult}_x(\mathcal{Q}_{\ell,s_0}), \tag{16}$$

where the left hand side is the value of (15) at the generic point. By assumption $\mathcal{P}_{s_0}$ is a finite set, hence so does $\mathcal{Q}_{s_0} = \{q_{1,s_0}, \ldots, q_{r,s_0}\}$ and by the Shub-Sullivan theorem [29], there exists a uniform bound $C > 0$ such that for every j, and for any ℓ, we have

$$\mathrm{mult}_{q_{j,s_0}}(\mathcal{Q}_{\ell,s_0}) \leq \mathrm{mult}_{q_{j,s_0}}(\mathrm{Per}_{\ell,s_0}) \leq C\ .$$

It then follows from (16) that

$$\#\mathcal{Q}_\ell \leq \sum_{q \in \mathcal{Q}_\ell} \mathrm{mult}_q(\mathcal{Q}_\ell) \leq rC$$

hence $\bigcup_\ell \mathcal{Q}_\ell$ is finite, as was to be shown. □

By the previous lemma, replacing f by some iterate f^N we may assume that all periodic points q_n are fixed. Since $\mathcal{P}$ is infinite, one of these fixed points, say q_1, admits infinitely many preimages in $\mathcal{P}$. We may denote $q = q_1$ and suppose $\mathcal{P}$ is made of an infinite set of preimages of q, that is, (after possible reordering of $\mathcal{P}$) for any $p_n \in \mathcal{P}$ there is a minimal $k_n \geq 0$ such that $f^{k_n}(p_n) = q$, and that $k_{n+1} > k_n$. We may adjoin to R the coordinates of q so that $q \in \mathbb{A}^2(R)$, i.e., for any $s \in S$, q_s is a single point (to say it differently, we replace S by a branched cover of a Zariski open dense subset of S).

Let $d(s)$ be the local degree of f_s at q_s, which is upper semicontinuous for the Zariski topology. Since f_s is a finite map of degree d^2, $d(s) \leq d^2$ for every s. Thus there is an analytic hypersurface H such that $d(s) = d_{\min}$ is constant for $s \in S \setminus H$.

We claim that $\mathcal{P}_{s_1}$ is infinite for any $s_1 \in S \setminus H$. We argue again in the complex analytic category fixing an embedding $\mathbb{Q}^{\mathrm{alg}}$ into $\mathbb{C}$. Observe that for any point $s \in S(\mathbb{C})$, $p_{n,s}$ is a finite set included in the fiber $\mathbb{A}^{2,\mathrm{an}}_s \simeq \mathbb{C}^2$ (not necessarily reduced to a single point since p_n lies in a finite extension of R).

Fix an analytic neighborhood V of q_{s_1} in $\mathbb{P}^{2,\mathrm{an}}_{s_1}(\mathbb{C})$ such that $f_{s_1}^{-1}(q_{s_1}) \cap \overline{V} = \{q_{s_1}\}$. Since $d(s)$ is locally constant near s_1, there is an analytic neighborhood W of s_1 in $S^{\mathrm{an}}(\mathbb{C})$ such that for $s \in W$,

$$f_s^{-1}(q_s) \cap V = \{q_s\} \tag{17}$$

Choose any $n > m$, and suppose by contradiction that $p_{m,s_1} = p_{n,s_1}$. Since $k_n - 1 \geq k_m$, we have

$$f_{s_1}^{k_n-1}(p_{n,s_1}) = f_{s_1}^{k_n-1}(p_{m,s_1}) = q_{s_1}.$$

Thus, for s close to s_1, the finite set $f_{s_1}^{k_n-1}(p_{n,s_1})$ is contained in V, hence by (17), $f_s^{k_n-1}(p_{n,s}) = q_s$, and by analytic continuation this property holds throughout S, which contradicts the definition of k_n.

This shows that the $p_{n,s}$ are all distinct for all $s \in S \setminus H$, and concludes the proof of Proposition 4.1. □

Proposition 4.3. *Let $f\colon \mathbb{A}^2_{\Bbbk} \to \mathbb{A}^2_{\Bbbk}$ be a regular polynomial map and $C \subset \mathbb{A}^2_{\Bbbk}$ be an algebraic curve containing infinitely many preperiodic points. Then every point of $\overline{C} \cap L_\infty$ is preperiodic under $f|_{L_\infty}$.*

Proof. We keep the same formalism and notation as above, so that f is viewed as a family over S. Write $\overline{C} \cap L_\infty = \{c_1, \dots c_r\}$ and without loss of generality enlarge R so that the points at infinity c_i have their coordinates in R. Fix $i \in \{1, \dots, r\}$ for the remainder of the proof and consider $c = c_i$. By Proposition 4.1, there is a Zariski open subset U such that for any $s \in U \cap S(\mathbb{Q}^{\mathrm{alg}})$, f_s admits infinitely many preperiodic points on C_s. Therefore, by Proposition 1.7, for every such s, c_s is preperiodic. Fix $s_0 \in U \cap S(\mathbb{Q}^{\mathrm{alg}})$, then c_{s_0} eventually falls on a periodic point q_{s_0}. Replacing f by f^N and C by $f^N(C)$ for some N, we may assume that q_{s_0} is fixed and $c_{s_0} = q_{s_0}$. Enlarging R again if necessary we way assume that q_{s_0} is the specialization at $s = s_0$ of a fixed point $q \in \mathbb{P}^2(R)$ of f.

Our purpose is to show that $c = q$. To simplify notation we write $\hat{f} = f|_{L_\infty}$. Note that the multiplier $\mu := \hat{f}'_{s_0}(q_{s_0})$ belongs to $\mathbb{Q}^{\mathrm{alg}}$. It follows from Kronecker's theorem that either μ is a root of unity or there is a place v on $\mathbb{Q}^{\mathrm{alg}}$ such that $|\mu|_v < 1$.

Case 1. μ is not a root of unity.

Fix a place v on $\mathbb{Q}^{\mathrm{alg}}$ such that $|\mu|_v < 1$, and consider the completion $\mathbb{C}_v$ of $(\mathbb{Q}^{\mathrm{alg}}, |\cdot|_v)$. We then argue in the analytic topology in the Berkovich analytification of $\mathbb{P}^2_{\mathbb{C}_v}$ and $S_{\mathbb{C}_v}$.

Fix a neighborhood W of s_0 in $S^{\mathrm{an}}_{\mathbb{C}_v}$ such that for $s \in W$, q_s is attracting, and a neighborhood V of q_s in L_∞ independent of $s \in W$ such that $\hat{f}_s(V) \subset V$ and for any $z \in V$, $f_s^n(z)$ converges to q_s as $n \to \infty$. Reducing W if necessary we may assume that for any $s \in W$, c_s belongs to V. For $s \in W \cap S(\mathbb{Q}^{\mathrm{alg}})$, c_s is preperiodic and converges to q_s, so it is preperiodic to q_s, that is, there exists a minimal $k = k(s)$ such that $f_s^{k(s)}(c_s) = q_s$. Now we use an argument similar to that of Proposition 4.1: let $H \subset S$ be a hypersurface such that the local degree of f_s at q_s is locally minimal outside H and fix $s_1 \in W \setminus H$. Then, there is a neighborhood W_1 of s_1 in $W \setminus H$ and a neighborhood $V_1 \subset V$ of q_1 such that for any $s \in W_1$, $f_s(V_1) \subset V_1$ and $f_s^{-1}(q_s) \cap V_1 = \{q_s\}$. From this it follows that the only point eventually falling onto q_s in V_1 is q_s itself. Therefore if $s \in W_1$ is so close to s_1 that $c_s \in V_1$, we infer that $c_s = q_s$, and finally $c = q$ by analytic continuation.

Case 2. μ is a root of unity.

To deal with this case we embed $\mathbb{Q}^{\mathrm{alg}}$ into $\mathbb{C}$, and work at the complex place. Recall that a holomorphic family $(g_\lambda)_{\lambda\in\Lambda}$ of rational maps on $\mathbb{P}^1(\mathbb{C})$, parameterized by a connected complex manifold is trivial if any two members are conjugate by a Möbius transformation, depending holomorphically on Λ. If c is persistently preperiodic we are done, so assume that c is not persistently preperiodic.

Under our assumptions, there is a dense set $S(\mathbb{Q}^{\mathrm{alg}})$ of parameters such that c_s is preperiodic, but c is not persistently preperiodic. Thus by Chio-Roeder [7, Theorem 2.7] (see also [10, Theorem 4]) every such parameter belongs to the bifurcation locus of the marked family (f_∞, c) (note that c is *not* a critical point here). As a consequence, the bifurcation locus of the family is equal to $S^{\mathrm{an}}_{\mathbb{C}}$. (Note that it is enough to work in a neighborhood of s_0, away from possible singularities of $S^{\mathrm{an}}_{\mathbb{C}}$.)

A first possibility is that the family $(\hat{f}_s)_{s\in S^{\mathrm{an}}_{\mathbb{C}}}$ is non-trivial. Then Gauthier [14, Theorem A] implies that $J(\hat{f}_s) = L_\infty$ for all s. But since $\hat{f}_{s_0}$ has a rationally indifferent fixed point, it admits an attracting petal and $J(\hat{f}_{s_0}) \neq L_\infty$. This contradiction shows that the family $(\hat{f}_s)_{s\in S^{\mathrm{an}}_{\mathbb{C}}}$ is trivial.

Now the situation is that there is a holomorphic family φ_s of Möbius transformations such that $\varphi_s \hat{f}_s \varphi_s^{-1} = g$ is a fixed rational map g on $\mathbb{P}^1$ with a rationally indifferent fixed point at 0. After this conjugacy, the marked family $(\hat{f}, c)$ becomes $(g, \varphi(c))$. Since c coincides with q at s_0 and by assumption c is not persistently preperiodic, there is an open set Ω of parameters such that for $s \in \Omega$, $\varphi_s(c_s)$ belongs to some attracting petal associated to 0 for g. This contradicts the fact that $\varphi_s(c_s)$ must be preperiodic for a dense set of parameters, and the proof is complete. □

Conclusion of the proof of Theorem A. By Proposition 4.3, any point at infinity of C is preperiodic, and, by assumption, one of these points, say c, is preperiodic to a non-superattracting periodic point p. Replace f by f^N and C by $f^N(C)$ for some N, so that $c = p$ is fixed. By Proposition 4.1, there is a non-trivial Zariski open subset $U \subset S$

such that for every $s \in U \cap S(\mathbb{Q}^{\text{alg}})$, C_s contains infinitely many preperiodic points. Then, since p_s is fixed and not superattracting for f_s, Theorem 3.1 asserts that C_s is preperiodic, and more precisely fixed, under f_s (see Remark 3.3). The density of $U \cap S(\mathbb{Q}^{\text{alg}})$ in S (for the Zariski or analytic topology) then implies that $f(C) = C$, and the proof is complete. □

References

[1] Alan F. Beardon. *Iteration of rational functions*, volume 132 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Complex analytic dynamical systems. 11

[2] Robert L. Benedetto. *Dynamics in one non-archimedean variable*, volume 198 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019. 9, 15

[3] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990. 6

[4] Gregory S. Call and Joseph H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.*, 89(2):163–205, 1993. 6

[5] Serge Cantat and Romain Dujardin. Finite orbits for large groups of automorphisms of projective surfaces. *Compositio Math.*, 160:120–175, 2024. 15

[6] Antoine Chambert-Loir. Heights and measures on analytic spaces. A survey of recent results, and some remarks. In *Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume II*, volume 384 of *London Math. Soc. Lecture Note Ser.*, pages 1–50. Cambridge Univ. Press, Cambridge, 2011. 5, 6

[7] Ivan Chio and Roland K. W. Roeder. Chromatic zeros on hierarchical lattices and equidistribution on parameter space. *Ann. Inst. Henri Poincaré D*, 8(4):491–536, 2021. 3, 18

[8] Tien-Cuong Dinh. Sur les endomorphismes polynomiaux permutables de $\mathbb{C}^2$. *Ann. Inst. Fourier (Grenoble)*, 51(2):431–459, 2001. 2

[9] Tien-Cuong Dinh and Nessim Sibony. Sur les endomorphismes holomorphes permutables de $\mathbb{P}^k$. *Math. Ann.*, 324(1):33–70, 2002. 2

[10] Romain Dujardin and Charles Favre. Distribution of rational maps with a preperiodic critical point. *Amer. J. Math.*, 130(4):979–1032, 2008. 18

[11] Romain Dujardin and Charles Favre. The dynamical Manin-Mumford problem for plane polynomial automorphisms. *J. Eur. Math. Soc. (JEMS)*, 19(11):3421–3465, 2017. 2, 3, 10, 15, 16

[12] Charles Favre and Thomas Gauthier. Classification of special curves in the space of cubic polynomials. *Int. Math. Res. Not. IMRN*, (2):362–411, 2018. 15

[13] John Erik Fornaess and Nessim Sibony. Complex dynamics in higher dimension. II. In *Modern methods in complex analysis (Princeton, NJ, 1992)*, volume 137 of *Ann. of Math. Stud.*, pages 135–182. Princeton Univ. Press, Princeton, NJ, 1995. 14

[14] Thomas Gauthier. Dynamical pairs with an absolutely continuous bifurcation measure. *Ann. Fac. Sci. Toulouse Math. (6)*, 32(2):203–230, 2023. 18

[15] Thomas Gauthier, Johan Taflin, and Gabriel Vigny. Sparsity of postcritically finite maps of $\mathbb{P}^k$ and beyond: A complex analytic approach, 2023. 3

[16] Dragos Ghioca, Khoa D. Nguyen, and Hexi Ye. The dynamical Manin-Mumford conjecture and the dynamical Bogomolov conjecture for endomorphisms of $(\mathbb{P}^1)^n$. *Compos. Math.*, 154(7):1441–1472, 2018. 2

[17] Dragos Ghioca, Khoa D. Nguyen, and Hexi Ye. The dynamical Manin-Mumford conjecture and the dynamical Bogomolov conjecture for split rational maps. *J. Eur. Math. Soc. (JEMS)*, 21(5):1571–1594, 2019. 2

[18] Dragos Ghioca and Thomas J. Tucker. A reformulation of the dynamical Manin-Mumford conjecture. *Bull. Aust. Math. Soc.*, 103(1):154–161, 2021. 1, 2

[19] Dragos Ghioca, Thomas J. Tucker, and Shouwu Zhang. Towards a dynamical Manin-Mumford conjecture. *Int. Math. Res. Not. IMRN*, (22):5109–5122, 2011. 1

[20] Monique Hakim. Attracting domains for semi-attractive transformations of $\mathbb{C}^p$. *Publ. Mat.*, 38(2):479–499, 1994. 3, 11
[21] Michael Herman and Jean-Christophe Yoccoz. Generalizations of some theorems of small divisors to non-Archimedean fields. In *Geometric dynamics (Rio de Janeiro, 1981)*, volume 1007 of *Lecture Notes in Math.*, pages 408–447. Springer, Berlin, 1983. 8
[22] Marc Hindry. Autour d'une conjecture de Serge Lang. *Invent. Math.*, 94(3):575–603, 1988. 2
[23] Lucas Kaufmann. Commuting pairs of endomorphisms of $\mathbb{P}^2$. *Ergodic Theory Dynam. Systems*, 38(3):1025–1047, 2018. 2
[24] Lars Kühne. Points of small height on semiabelian varieties. *J. Eur. Math. Soc. (JEMS)*, 24(6):2077–2131, 2022. 2
[25] Serge Lang. Division points on curves. *Ann. Mat. Pura Appl. (4)*, 70:229–234, 1965. 2
[26] Fabien Pazuki. Zhang's conjecture and squares of abelian surfaces. *C. R. Math. Acad. Sci. Paris*, 348(9-10):483–486, 2010. 1
[27] Michel Raynaud. Courbes sur une variété abélienne et points de torsion. *Invent. Math.*, 71(1):207–233, 1983. 1
[28] Michel Raynaud. Sous-variétés d'une variété abélienne et points de torsion. In *Arithmetic and geometry, Vol. I*, volume 35 of *Progr. Math.*, pages 327–352. Birkhäuser Boston, Boston, MA, 1983. 1
[29] Michael Shub and Dennis Sullivan. A remark on the Lefschetz fixed point formula for differentiable maps. *Topology*, 13:189–191, 1974. 3, 16
[30] Joseph H. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer, New York, 2007. 4
[31] Amaury Thuillier. *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov.* Theses, Université Rennes 1, October 2005. 3, 6
[32] Shou-Wu Zhang. Small points and adelic metrics. *J. Algebraic Geom.*, 4(2):281–300, 1995. 1, 7
[33] Shou-Wu Zhang. Distributions in algebraic dynamics. In *Surveys in differential geometry. Vol. X*, volume 10 of *Surv. Differ. Geom.*, pages 381–430. Int. Press, Somerville, MA, 2006. 1

Sorbonne Université, Laboratoire de probabilités, statistique et modélisation, UMR 8001, 4 place Jussieu, 75005 Paris, France

Email address: romain.dujardin@sorbonne-universite.fr

CNRS - Centre de Mathématiques Laurent Schwartz, École Polytechnique, 91128 Palaiseau Cedex, France

Email address: charles.favre@polytechnique.edu

Université de Paris and Sorbonne Université, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, F-75006 Paris, France

Email address: matteo.ruggiero@imj-prg.fr