

Local dynamics of non-invertible maps near normal surface singularities

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We study the problem of finding algebraically stable models for non-invertible holomorphic fixed point germs $f : (X, x_0) \rightarrow (X, x_0)$, where X is a complex surface having x_0 as a normal singularity. We prove that as long as x_0 is not a cusp singularity of X , then it is possible to find arbitrarily high modifications $\pi : X_\pi \rightarrow (X, x_0)$ such that the dynamics of f (or more precisely of f^N for N big enough) on X_π is algebraically stable. This result is proved by understanding the dynamics induced by f on a space of valuations associated to X ; in fact, we are able to give a strong classification of all the possible dynamical behaviors of f on this valuation space. We also deduce a precise description of the behavior of the sequence of attraction rates for the iterates of f . Finally, we prove that in this setting the first dynamical degree is always a quadratic integer.

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Introduction

Let (X, x_0) be an irreducible germ of a complex surface X at a point $x_0 \in X$. In this article, we study the local dynamics of dominant non-invertible germs $f : (X, x_0) \rightarrow (X, x_0)$ of holomorphic maps that have x_0 as a fixed point. We are specifically interested in the case when x_0 is a singular point of the surface,

though in this case we will always assume X is normal at x_0 . Normality in particular implies that x_0 is an isolated singularity.

Understanding local dynamics around fixed points is often essential to understanding the behavior of global holomorphic dynamical systems, so a great deal of research has been devoted to the topic, with most of it focusing on the smooth case of holomorphic germs $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$. In dimension $n = 1$, classical arguments have given an essentially complete understanding of local dynamics; in particular, for *non-invertible* germs $f : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$, it is always possible to make a local holomorphic change of coordinates so that f takes the simple form $f(z) = z^c$ for some integer $c > 1$ [14, 51]. An analogous statement holds if we replace \mathbb{C} by any field of characteristic 0. In positive characteristic normal forms are more involved (see [59]), but still computable. The situation in dimension $n \geq 2$ is far more complicated. A non-invertible *planar* holomorphic germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ comes in two flavors: either the differential df_0 of f at 0 is nilpotent, in which case 0 is said to be a *superattracting* fixed point of f , or else df_0 has exactly one non-zero eigenvalue, and 0 is said to be *semi-superattracting*. In these cases, one might hope to analyze the local dynamics as was done in dimension 1, by finding suitable local coordinates in which f has a simple form, called a *normal form*, making it easier to compute and study the iterates of f . Work in this direction for *planar* semi-superattracting germs has been done in [57], but in the superattracting case this strategy has been successfully carried out for only special classes of germs, see e.g. [23, 1] for results in dimension 2, and [58, 13] for results in higher dimensions. Unfortunately, it seems as if a complete and useful description of normal forms for general superattracting germs is infeasible (see [38]), owing largely to the richness of analytic geometry in dimensions 2 or higher, a central obstruction being that it is unclear how to deal with the growth of the geometric complexity of the critical set of the iterates of f .

Recently, another approach to studying the dynamics of planar non-invertible germs $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ has been successfully employed. In this approach, one investigates the dynamics of f on birationally equivalent models of $(\mathbb{C}^2, 0)$. Given a modification $\pi : X_\pi \rightarrow (\mathbb{C}^2, 0)$, i.e., a proper holomorphic map which is an isomorphism over $\mathbb{C}^2 \setminus \{0\}$, one can lift f to a meromorphic map $f_\pi : X_\pi \dashrightarrow X_\pi$ and study the dynamics of f_π on the exceptional set $\pi^{-1}(0)$ of π . Understanding the orbits of the *exceptional divisors* of π , that is, the divisors of X_π supported in $\pi^{-1}(0)$, gives very concrete information about the local dynamics of f , see [29, 35, 57, 34]. Unfortunately, for any given modification π , the action of f_π on exceptional divisors can be difficult to control due to the presence of indeterminacy points of f_π . One can imagine, for instance, a situation where f_π^n contracts an exceptional prime divisor E of π to an indeterminacy point of f_π for infinitely many n . If this were to occur, the best hope would be that one could avoid such behavior by passing to a higher modification. One of our main results is that this is almost always possible.

Theorem A. *Let (X, x_0) be an irreducible germ of a normal complex surface at a point $x_0 \in X$, and let $f : (X, x_0) \rightarrow (X, x_0)$ be a dominant non-invertible holomorphic map. Assume we are not in the situation of a finite germ at a cusp singularity. Then for any modification $\pi : X_\pi \rightarrow (X, x_0)$, one can find a modification $\pi' : X_{\pi'} \rightarrow (X, x_0)$ dominating π with the following property: if E is any exceptional prime divisor of π' and $f_{\pi'} : X_{\pi'} \dashrightarrow X_{\pi'}$ is the (meromorphic) lift of f , then $f_{\pi'}^n(E)$ is an indeterminacy point of $f_{\pi'}$ for at most finitely many n . Moreover the space $X_{\pi'}$ may be taken to have at most cyclic quotient singularities.*

Recall that $f : (X, x_0) \rightarrow (X, x_0)$ is *finite* near x_0 when f does not contract any curve $C \subset X$ passing through x_0 to x_0 . In the smooth case when the surface germ is $(\mathbb{C}^2, 0)$, Theorem A was essentially proved in [35] for superattracting fixed point germs $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$. To be precise, it was shown that there exists *some* non-trivial model $\pi : X_\pi \rightarrow (\mathbb{C}^2, 0)$ where the orbits of exceptional prime divisors are well behaved, rather than arbitrarily high models, but the full theorem follows from a minor modification of the argument given there, see §7. In the semi-superattracting case, Theorem A follows similarly from the work in [57]. The results of these two articles will play a prominent role in this paper.

In both of these papers, the broad strategy to prove [Theorem A](#) is the same: instead of studying the dynamics of $f_\pi: X_\pi \dashrightarrow X_\pi$ on any model $\pi: X_\pi \rightarrow (X, x_0)$ of (X, x_0) individually, one analyzes a dynamical system $f_\bullet: \mathcal{V}_X \rightarrow \mathcal{V}_X$ on a space \mathcal{V}_X that encodes all bimeromorphic models $\pi: X_\pi \rightarrow (X, x_0)$ simultaneously. Results about the dynamics of f_\bullet can then often be translated into statements about the dynamics of f on bimeromorphically equivalent models of (X, x_0) . This \mathcal{V}_X is the space of (suitably normalized) centered, rank one semivaluations on the local ring \mathcal{O}_{X, x_0} of X at x_0 which are trivial on \mathbb{C} , see [§2](#) for definitions. If X is algebraic, \mathcal{V}_X can be viewed as a subset of the Berkovich analytification X^{an} of X , where the ground field \mathbb{C} is equipped with the trivial absolute value. This broad strategy was first employed by Favre-Jonsson [[29](#)] in the case when $X = \mathbb{C}^2$, and they have since successfully used it in a variety of settings [[28](#), [30](#), [25](#), [27](#), [10](#)].

In this article, we follow the same strategy, and our central result (from which [Theorem A](#) will follow) is a classification of the possible dynamics of $f_\bullet: \mathcal{V}_X \rightarrow \mathcal{V}_X$, the simplest possible summary of which is given below. A finer classification can (and will) be given if one has more information about the nature of the singularity (X, x_0) and the map f , but to avoid technicalities in the introduction the following less precise statement will suffice.

Theorem B. *Let (X, x_0) be an irreducible germ of a normal complex surface at a point $x_0 \in X$, and let $f: (X, x_0) \rightarrow (X, x_0)$ be a dominant non-invertible holomorphic map. Denote by $f_\bullet: \mathcal{V}_X \rightarrow \mathcal{V}_X$ the induced dynamical system on the space \mathcal{V}_X of normalized, centered, rank one semivaluations on \mathcal{O}_{X, x_0} . Then there is a subgraph $S \subset \mathcal{V}_X$, homeomorphic to either a point, a closed interval, or a circle, such that $f_\bullet(S) = S$ and for any quasimonomial valuation $\nu \in \mathcal{V}_X$ one has that $f_\bullet^n \nu \rightarrow S$ as $n \rightarrow \infty$.*

A special situation in the previous theorem is given when S is a circle, f is finite, and f_\bullet acts on S as a rotation of infinite order (irrational rotation). In this situation, the conclusion of [Theorem A](#) is simply false, see [§8.3](#).

As applications of [Theorems A](#) and [B](#) we will derive three additional interesting results, stated below. The first is related to the notion of *algebraic stability* of holomorphic dynamical systems, an important and actively studied topic in complex dynamics, see for instance [[17](#), [24](#), [4](#), [18](#), [20](#), [30](#), [42](#), [49](#)]. In our local setting, the classical notion of algebraic stability has meaning only when the germ $f: (X, x_0) \rightarrow (X, x_0)$ is *finite* near x_0 . In this case, given any modification $\pi: X_\pi \rightarrow (X, x_0)$, the lift $f_\pi: X_\pi \dashrightarrow X_\pi$ induces a \mathbb{Z} -linear pull-back operation $f^*: \mathcal{E}(\pi) \rightarrow \mathcal{E}(\pi)$ on the group $\mathcal{E}(\pi)$ of exceptional divisors of π , see [§4.2](#). However, due to the presence of indeterminacy points of f_π , it may happen that the pull-back operation is not functorial, that is, possibly $(f_\pi^n)^* \neq (f_\pi^*)^n$ for some or even all $n > 1$. On the other hand, the control on the orbits of exceptional prime divisors given by [Theorem A](#) allows us to obtain a sort of “asymptotic” functoriality.

Theorem C. *Let (X, x_0) be an irreducible germ of a normal complex surface at a point $x_0 \in X$, and let $f: (X, x_0) \rightarrow (X, x_0)$ be a finite non-invertible holomorphic map. Assume we are not in the case of f inducing an irrational rotation at a cusp singularity x_0 of X . Then for any modification $\pi: X_\pi \rightarrow (X, x_0)$ there exists a dominating modification $\pi': X_{\pi'} \rightarrow (X, x_0)$ and an integer $N = N(\pi') \geq 1$ such that $(f_{\pi'}^{N+n})^* = (f_{\pi'}^N)^*(f_{\pi'}^*)^n$ for all $n \geq 0$. Moreover, $X_{\pi'}$ may be taken to have at worst cyclic quotient singularities.*

The second application is related to the sequence of so-called *attraction rates* $c(f^n, \nu)$ of the iterates of f^n along a given valuation $\nu \in \mathcal{V}_X$. The attraction rate $c(f^n, \nu)$ is by definition

$$c(f^n, \nu) := \min_{\phi \in f^{n*} \mathfrak{m}} \nu(\phi),$$

where \mathfrak{m} is the maximal ideal of \mathcal{O}_{X, x_0} . The dynamical relevance of these attraction rates can be seen in the fact that understanding the values of $c(f^n, \nu)$ as $\nu \in \mathcal{V}_X$ varies allows one to recover the ideals $f^{n*} \mathfrak{m}$. Our last theorem describes the structure of the sequence $\{c(f^n, \nu)\}_{n \geq 1}$ for some $\nu \in \mathcal{V}_X$.

Theorem D. *Let (X, x_0) be an irreducible germ of a normal complex surface at a point $x_0 \in X$, and let $f : (X, x_0) \rightarrow (X, x_0)$ be a dominant non-invertible holomorphic function. Assume we are not in the case of a finite germ f at a cusp singularity (X, x_0) inducing an irrational rotation. Then for any quasimonomial valuation $\nu \in \mathcal{V}_X$, the sequence of attraction rates $c_n := c(f^n, \nu)$ eventually satisfies an integral linear recursion relation. More precisely, there exist integers a, b, N , and m with $N, m \geq 1$ such that $c_{n+2m} = ac_{n+m} + bc_n$ for all $n \geq N$.*

Again, the conclusions of [Theorems C](#) and [D](#) do not hold when f is a finite germ at a cusp singularity inducing an irrational rotation. We prove that this situation may happen for every cusp singularity, and in this case, no linear recursion relations are satisfied for the sequence of attraction rates. We deduce the non-existence of algebraically stable models in this case.

A coarser invariant than the sequence of attraction rates is given by the *first dynamical degree*, which measures the exponential growth rate of this sequence. It is defined by $c_\infty(f, \nu) = \lim_n (c(f^n, \nu))^{1/n}$ for any quasimonomial valuation ν . In fact it can be easily shown that this limit exists, and it does not depend on ν , so we simply denote it by $c_\infty(f)$. The arithmetical properties of the first dynamical degrees have been explored in several contexts, see e.g. [\[17, 29, 7\]](#); in the global setting, its logarithm gives an upper bound to the topological entropy, see [\[19\]](#). For superattracting germs at a smooth point $(\mathbb{C}^2, 0)$, Favre and Jonsson proved (see [\[29, Theorem A\]](#)) that the first dynamical degree is always a quadratic integer. This can also be deduced directly from the study of the recursion relation of the sequence of attraction rates, see [\[30, 35\]](#). We have seen that in the singular setting, such recursion relation is not always present. Nevertheless, the first dynamical degree still shares the same arithmetic properties.

Theorem E. *Let (X, x_0) be an irreducible germ of a normal complex surface at a point $x_0 \in X$, and let $f : (X, x_0) \rightarrow (X, x_0)$ be a dominant non-invertible holomorphic function. Then the first dynamical degree $c_\infty(f)$ is a quadratic integer.*

Each of the theorems stated above were shown for polynomial endomorphisms in [\[29, 30\]](#), and for superattracting germs $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ in [\[29, 35\]](#). The new content in the theorems is in the case when x_0 is a singular point of X . It is remarkable that the results from the smooth setting carry over so seamlessly to the singular setting. Moreover, the generality of these results is especially noteworthy given that very little is known about local dynamics near singularities. Because the techniques used in this article are valuative rather than complex analytic, at no point will we use in an essential way that we are working over the complex numbers, and in fact all of the theorems stated above hold equally well if \mathbb{C} is replaced by any field of characteristic 0. Some of the results carries over fields of positive characteristic as well (see [Remarks 5.16, 4.27](#) and [§9.10](#)). Nonetheless, we will continue to work exclusively over \mathbb{C} , as this is probably the most familiar setting to the audience.

The core of the paper is the proof of [Theorem B](#). The main technical tool is the construction of a suitable distance ρ on the set of normalized valuations \mathcal{V}_X associated to a normal surface singularity, that we call *angular distance*. We study the non-expanding properties of the action f_\bullet induced by non-invertible germs $f : (X, x_0) \rightarrow (Y, y_0)$ between normal surface singularities with respect to the angular distance (see [Theorem 4.24](#)). Understanding the behavior of the angular distance and related objects helps also to describe the geometry of surface singularities, see [\[33\]](#). When f is not finite, we show that the action of f_\bullet strictly decreases the angular distance. We deduce the existence and uniqueness of a fixed point for f_\bullet in this case. When f is finite, a theorem of Wahl [\[67\]](#) tells us that (X, x_0) is necessarily log canonical. We use the classification of log canonical surface singularities, and continue the analysis of finite germs developed in [\[25\]](#). In both cases, we need to control the behavior of the attraction rates of the iterates of f . We deduce some restrictions on the dynamics of f_\bullet for semi-superattracting germs (see [Proposition 5.14](#) and [Theorem 6.3](#)).

The structure of this article is as follows. In [§1](#) we recall a few facts on resolution of singularities and intersection theory. In [§2](#) we describe several properties of the valuation spaces associated to normal

surface singularities, and construct the angular distance. In §3 we recall the definition of log discrepancy of a valuation, study its properties on the valuation spaces, and recall the classification of log canonical surface singularities. In §4 we define the action f_\bullet induced by f on valuative spaces, and we study several properties. In particular we establish the contracting properties with respect to the angular distance, and state a more precise version of [Theorem B](#), namely [Theorem 4.41](#). Sections §5 and §6 are devoted to the proof of [Theorem B](#) in the non-finite and finite cases. In §7 we prove [Theorems A](#) and [C](#). In §8 we prove [Theorems D](#) and [E](#). Finally, §9 contains worked-out examples and remarks. We conclude with an appendix §A where we recall the arithmetical construction of cusp singularities, and the construction of finite endomorphisms on them given by [\[25\]](#).

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1. Normal surface singularities, resolutions, and intersection theory

In this section we recall some ideas and constructions from the theory of surface singularities and, in particular, the resolution of such singularities. These ideas will be essential to understanding the spaces of valuations we begin studying in the next section, the central objects of this article. Throughout this paper, we let (X, x_0) be an irreducible germ of a normal complex surface at a point $x_0 \in X$. We refer to (X, x_0) as a *normal surface singularity*. Let $R = R_X$ be the completed local ring $\hat{\mathcal{O}}_{X, x_0}$ and let $\mathfrak{m} = \mathfrak{m}_X$ be its maximal ideal.

Definition 1.1. A *modification* of (X, x_0) is a proper holomorphic map $\pi: X_\pi \rightarrow (X, x_0)$, where X_π is a normal complex surface and π is a biholomorphism over $X \setminus \{x_0\}$. A modification is a *resolution* of (X, x_0) if in addition X_π is regular. Such a resolution is *good* if the exceptional locus $\pi^{-1}(x_0)$ is a divisor whose support has simple normal crossings.

It is a fundamental fact that good resolutions of (X, x_0) always exist, and moreover for any \mathfrak{m} -primary ideal $\mathfrak{a} \subset R$, one can find good resolutions of (X, x_0) which resolve \mathfrak{a} in the following sense.

Definition 1.2. If $\mathfrak{a} \subset R$ is an \mathfrak{m} -primary ideal, that is, a proper ideal containing some power of \mathfrak{m} , then a *log resolution* of \mathfrak{a} is a good resolution π of (X, x_0) such that the ideal sheaf $\pi^*\mathfrak{a}$ is locally principal.

In the following, we will also need to solve the singularities of a curve inside (X, x_0)

Definition 1.3. Let $(C, x_0) \subset (X, x_0)$ be the germ of a (reduced) curve at x_0 sitting in a normal surface singularity (X, x_0) . Then a *embedded resolution* of (C, x_0) in (X, x_0) is a good resolution $\pi: X_\pi \rightarrow (X, x_0)$ so that $\pi^{-1}(C)$ has simple normal crossings.

A modification π' of (X, x_0) is said to *dominate* another modification π if there is a holomorphic map $\eta: X_{\pi'} \rightarrow X_\pi$ such that $\pi' = \pi \circ \eta$. Because this situation will appear frequently, it will be convenient to set once and for all the following notation.

Notation. If π and π' are two modifications of (X, x_0) , we write $\pi' \geq \pi$ to mean that π' dominates π , and in this case we will always denote by $\eta_{\pi\pi'}: X_{\pi'} \rightarrow X_\pi$ the holomorphic map $\eta_{\pi\pi'} = \pi^{-1} \circ \pi'$.

Modifications of (X, x_0) form a category, the morphisms being the maps $\eta_{\pi\pi'}$. Any two modifications of (X, x_0) are dominated by some good resolution, making good resolutions an inverse system within this category. A good resolution π_0 is said to be *minimal* if $\eta_{\pi\pi_0}$ is a biholomorphism whenever π_0 dominates another good resolution π . It is an important theorem that minimal good resolutions of (X, x_0) always exist, and that every good resolution of (X, x_0) dominates some minimal good resolution. More precisely, every good resolution of (X, x_0) is obtained from a minimal good resolution by a composition of point blowups [46, Chapter 5].

Remark 1.4. For normal surface singularities, there is a unique minimal good resolution, see [46, Theorem 5.12]. Notice that two exceptional primes in a minimal good resolution may intersect in more than one point. This may happen for example for cusp singularities like $X = \{x^2 + y^4 + z^6 = xyz\}$, where the exceptional divisor of the minimal good resolution consists of a cycle of two rational curves (which hence intersect transversely in two points p and q), see §9.7. A classical object associated to a good resolution $\pi: X_\pi \rightarrow (X, x_0)$ is the dual graph Γ , which is a simplicial graph that has as vertices the irreducible components of $\pi^{-1}(x_0)$ (also called exceptional primes), and an edge between two exceptional primes $E \neq F$ for each point in $E \cap F$. In particular, there could be several edges connecting two vertices E and F . By further blow-ups, we may also assume that any two (different) exceptional primes in a good resolution intersect in at most one point. In the cusp example above, this can be obtained by either blowing-up p or q . In particular, minimal good resolutions satisfying this stronger property are not unique in general. For such good resolutions, the edge between two vertices is uniquely determined by its endpoints, and notations can be eased. This stronger concept will be also useful to construct the dual graph as embedded in the vector space of exceptional \mathbb{R} -divisors, see §2.5.

One could equivalently work with good resolutions in this stronger sense throughout all the paper, since the uniqueness of the minimal good resolution will not play any role.

Remark 1.5. The modifications considered in this paper will nearly always be good resolutions. On the other hand, in order to prove [Theorem A](#) we will be forced to consider some modifications π for which X_π has some mild singularities, namely cyclic quotient singularities, see §3.1 for definitions. These spaces X_π still have very controlled geometry; for instance, they are \mathbb{Q} -factorial [2]. They can also be interpreted V-manifolds, first introduced by Satake [60], and then renamed *orbifolds* by Thurston [64].

1.1. The intersection theory of good resolutions

Let $\pi: X_\pi \rightarrow (X, x_0)$ be a good resolution. We now turn our attention to the intersection theory of X_π . Throughout this article, we will denote by $\mathcal{E}(\pi)$ the group of *exceptional divisors* of X_π , that is, the group of Weil divisors with support lying in the exceptional locus $\pi^{-1}(x_0)$. The prime divisors with support in $\pi^{-1}(x_0)$ are called the *exceptional primes* of π ; they form a \mathbb{Z} -basis of $\mathcal{E}(\pi)$. It will also be convenient to consider the vector spaces of exceptional \mathbb{R} - and \mathbb{Q} -divisors of π , which by definition are the elements of $\mathcal{E}(\pi)_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{E}(\pi)$ and $\mathcal{E}(\pi)_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{E}(\pi)$, respectively.

The intersection product on X_π induces a symmetric bilinear form $\mathcal{E}(\pi) \times \mathcal{E}(\pi) \rightarrow \mathbb{Z}$ that, crucially, is negative definite [36], thus giving rise to a canonical identification of $\mathcal{E}(\pi)_{\mathbb{R}}$ with its dual $\mathcal{E}(\pi)_{\mathbb{R}}^*$. If E_1, \dots, E_n are the exceptional primes of π , then we will always denote by $\check{E}_1, \dots, \check{E}_n \in \mathcal{E}(\pi)_{\mathbb{R}}$ the basis dual to E_1, \dots, E_n . Concretely, \check{E}_i is the unique \mathbb{R} -divisor with the property that $\check{E}_i \cdot E_j = \delta_{ij}$ for each j . This dual basis will appear prominently in later sections. One of its useful properties, an easy consequence of the projection formula, is that if π' is a good resolution dominating π and if F_i is the strict transform in $X_{\pi'}$ of E_i , then $\eta_{\pi\pi'}^* \check{E}_i = \check{F}_i$. Another important property is the following.

Proposition 1.6. *The divisors \check{E}_i are (strictly) anti-effective, that is, $\check{E}_i \cdot \check{E}_j < 0$ for each i and j .*

Remark 1.7. In fact, we can easily write \check{E}_i in coordinates with respect to the basis $\{E_j\}$, as

$$\check{E}_i = \sum_j (\check{E}_i \cdot \check{E}_j) E_j.$$

This formula can be easily proved by intersecting each side of the formula with \check{E}_j for all j . By duality, we also get

$$E_i = \sum_j (E_i \cdot E_j) \check{E}_j.$$

[Proposition 1.6](#) is a consequence of a rather useful linear algebra lemma.

Lemma 1.8. *Let V be a finite dimensional real inner product space and suppose that e_1, \dots, e_n is a basis of V with the property that $(e_i \cdot e_j) \leq 0$ for each $i \neq j$. Let $e_1^*, \dots, e_n^* \in V$ be the corresponding dual basis with respect to the inner product. Then the following statements hold.*

1. For all i and j , one has $(e_i^* \cdot e_j^*) \geq 0$.
2. Let Γ be the graph with vertices e_1, \dots, e_n and an edge connecting two vertices e_i and e_j if and only if $(e_i \cdot e_j) < 0$. Then $(e_i^* \cdot e_j^*) > 0$ if and only if the vertices e_i and e_j belong to the same connected component in Γ .

Proof. To start, we assume that Γ is connected. We wish to prove that $(e_i^* \cdot e_j^*) > 0$ for arbitrary i and j , but after re-indexing, we may assume without loss of generality that $j = 1$, and that the sequence of vertices e_1, \dots, e_i is a path in Γ from e_1 to e_i . If we perform the Gram-Schmidt orthogonalization process on the basis e_1, \dots, e_n , we obtain an orthogonal basis u_1, \dots, u_n defined recursively by $u_1 = e_1$ and

$$u_j = e_j - \sum_{k < j} \frac{(e_j \cdot u_k)}{|u_k|^2} u_k$$

for $j > 1$. Writing the u_j out in terms of the basis e_k then gives $u_j = \sum_{k \leq j} \lambda_{jk} e_k$ for some $\lambda_{jk} \in \mathbb{R}$. By a straightforward induction on j , which we leave to the reader, we in fact have (a) that $\lambda_{jk} \geq 0$ for all $k \leq j$, and (b) that if $j \leq i$, then $\lambda_{jk} > 0$ for all $k \leq j$. Note (a) is a consequence of the assumption that $(e_k \cdot e_l) \leq 0$ whenever $k \neq l$, and (b) needs the assumption that e_1, \dots, e_i is a path in Γ . Continuing the proof, if we now write the e_k^* in terms of the orthogonal basis u_j , we obtain

$$e_k^* = \sum_j \frac{(e_k^* \cdot u_j)}{|u_j|^2} u_j = \sum_{j \geq k} \frac{\lambda_{jk}}{|u_j|^2} u_j.$$

Using this and statements (a) and (b) above, one computes that

$$(e_i^* \cdot e_1^*) = \sum_{j \geq i} \frac{\lambda_{ji} \lambda_{j1}}{|u_j|^2} \geq \frac{\lambda_{ii} \lambda_{i1}}{|u_i|^2} > 0.$$

This completes the proof in the case when Γ is connected.

Now, let us consider general Γ . Up to re-indexing, assume that the basis e_1, \dots, e_n is ordered in blocks corresponding to the connected components of Γ . Then the matrix M whose ij -th entry is $(e_i \cdot e_j)$ is a block diagonal matrix, with each diagonal block corresponding to a connected component of Γ . The inverse M^{-1} is exactly the matrix whose ij -th entry is $(e_i^* \cdot e_j^*)$. Since M was block diagonal, M^{-1} will also be, and the diagonal blocks of M^{-1} will be the inverses of the diagonal blocks of M . By the connected case, each of the diagonal blocks of M^{-1} will have strictly positive entries, completing the proof. \square

Proof of Proposition 1.6. We apply Lemma 1.8 in the case when $V = \mathcal{E}(\pi)_{\mathbb{R}}$, the given basis e_1, \dots, e_n is the set of exceptional primes E_1, \dots, E_n , and the inner product is the negative of the intersection product. In this case, the graph $\Gamma = \Gamma_{\pi}$ is what is typically referred to as the *dual graph* of π ; it is the graph with vertices E_1, \dots, E_n and an edge connecting E_i and E_j if and only if the exceptional primes E_i and E_j intersect in X_{π} (see also §2.5). Because the dual graph of any good resolution π is connected, we conclude that $\check{E}_i \cdot \check{E}_j < 0$ for each i and j . \square

Important for us are notions of (relative) positivity for exceptional divisors, particularly the notions of relative ampleness and nefness, a good reference for which is [48, §1.7]. The relative ampleness of a divisor $D \in \mathcal{E}(\pi)$ is defined in terms of projective embeddings, but can be characterized numerically by having $D \cdot E_i > 0$ for each of the exceptional primes E_i , a characterization which extends to exceptional \mathbb{R} -divisors [32, Theorem B]. Similarly, an \mathbb{R} -divisor $D \in \mathcal{E}(\pi)_{\mathbb{R}}$ is *relatively nef* if $D \cdot E_i \geq 0$ for each i . Evidently, an exceptional \mathbb{R} -divisor D is relatively nef (resp. relatively ample) if and only if $D = \sum_i \lambda_i \check{E}_i$ with $\lambda_i \geq 0$ (resp. > 0). The sets $\text{Nef}(\pi)$ and $\text{Amp}(\pi)$ of relatively nef and ample \mathbb{R} -divisors are therefore cones; $\text{Nef}(\pi)$ is a strict convex polyhedral cone, $\text{Amp}(\pi)$ is its (nonempty) interior, and $\text{Nef}(\pi)$ is the closure $\overline{\text{Amp}(\pi)}$. It is then easy to construct bases of $\mathcal{E}(\pi)_{\mathbb{R}}$ consisting of integral relatively ample divisors, which after scaling can even be taken to be relatively very ample.

As an immediate consequence of Proposition 1.6, we obtain the following.

Corollary 1.9. *If D_1 and D_2 are nonzero relatively nef divisors, then $D_1 \cdot D_2 < 0$. In particular, nonzero relatively nef divisors are strictly anti-effective.*

We conclude this section with another extremely useful consequence of Lemma 1.8.

Proposition 1.10. *Let $\pi: X_{\pi} \rightarrow (X, x_0)$ be a good resolution, and let E, F and H be exceptional prime divisors of π . Then one has the inequality*

$$(\check{E} \cdot \check{F})(\check{E} \cdot \check{H}) \leq (\check{E} \cdot \check{E})(\check{F} \cdot \check{H}), \quad (1)$$

with equality if and only if $E = H$, $E = F$ or F and H do not belong to the same connected component of $\Gamma_{\pi} \setminus \{E\}$, where Γ_{π} is the dual graph of π .

Proof. If $E = F$, then (1) trivially holds as an equality, so we may assume that E and F are distinct. Let D_1, \dots, D_{n-2} be the exceptional primes of π that are not equal to E or F , and set

$$Q = \check{F} - \frac{\check{E} \cdot \check{F}}{\check{E} \cdot \check{E}} \check{E}.$$

Notice that (1) is equivalent to $Q \cdot \check{H} \leq 0$.

Consider the basis $D_1, \dots, D_{n-2}, F, -\check{E}$ of $\mathcal{E}(\pi)_{\mathbb{R}}$, which for simplicity we denote by u_1, \dots, u_n . This basis has the property that $u_i \cdot u_j \geq 0$ for all $i \neq j$ (while $u_i \cdot u_i < 0$ for all i). Moreover we have $Q \cdot u_i = 0$ for all $i \neq n-1$, and $Q \cdot u_{n-1} = 1$.

If we write $Q = a_1 u_1 + \dots + a_n u_n$, then

$$\begin{pmatrix} u_1 \cdot u_1 & \cdots & u_n \cdot u_1 \\ \vdots & \ddots & \vdots \\ u_1 \cdot u_n & \cdots & u_n \cdot u_n \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} Q \cdot u_1 \\ \vdots \\ Q \cdot u_n \end{pmatrix}. \quad (2)$$

By solving this linear system, we get $a_i = (u_i^* \cdot u_{n-1}^*)$. By Lemma 1.8 applied to the basis u_1, \dots, u_n , and the inner product given by the negative of the intersection product, the matrix $(u_i^* \cdot u_j^*)_{ij}$ has non-positive entries. We conclude that $a_i \leq 0$ for each i . Because each of the divisors u_i is effective, it follows

that $Q = \sum a_i u_i$ is anti-effective. That is to say, $Q \cdot \check{H} \leq 0$ for each exceptional prime H of π , and the inequality (1) is proved.

To determine when we have an equality, we need to be more careful. First notice that $a_n = 0$. In fact $u_j \cdot u_n = 0$ for all $j = 1, \dots, n-1$, while $u_n \cdot u_n = \check{E} \cdot \check{E} < 0$. We deduce that $a_n = 0$ from the last line of (2), and the fact that $Q \cdot u_n = 0$.

If we consider the first $n-1$ equations of the linear system (2) (for the variables a_i , $i = 1, \dots, n-1$) the associated matrix U_E is the intersection matrix of $\pi^{-1}(x_0) \setminus E$. By Lemma 1.8, $a_i = (u_i^* \cdot u_{n-1}^*) < 0$ for $i = 1, \dots, n-1$ if and only if u_i and F belong to the same connected component of $\pi^{-1}(x_0) \setminus E$. We conclude by noticing that $Q \cdot \check{H} = a_i$ if $H = u_i$. \square

1.2. Log resolutions and divisors

Suppose now that $\mathfrak{a} \subset R$ is an \mathfrak{m} -primary ideal, and that $\pi: X_\pi \rightarrow (X, x_0)$ is a log resolution of \mathfrak{a} . Then $\pi^* \mathfrak{a}$ is the ideal sheaf in \mathcal{O}_{X_π} of some exceptional divisor $D \in \mathcal{E}(\pi)$, that is, $\pi^* \mathfrak{a} = \mathcal{O}_{X_\pi}(-D)$. If, as before, we let E_1, \dots, E_n be the exceptional primes of π , the divisor D is given explicitly as $D = \sum_i b_i E_i$, where b_i is the minimum order of vanishing along E_i of all the functions $\phi \circ \pi$ as ϕ ranges over \mathfrak{a} , i.e.,

$$b_i = \min_{\phi \in \mathfrak{a}} \text{ord}_{E_i}(\phi \circ \pi). \quad (3)$$

Necessarily $b_i \geq 1$ for each i , since $\mathfrak{a} \subseteq \mathfrak{m}$.

The divisor $-D$ has positivity properties owing to the fact that, essentially by the definition of being a log resolution, the line bundle $L := \mathcal{O}_{X_\pi}(-D)$ is *relatively base point free*, meaning that L has sections which do not vanish at any prescribed point $x \in \pi^{-1}(x_0)$. In particular, the restrictions $L|_{E_i}$ are base point free for each i . With E_i being a smooth curve, this implies $\deg L|_{E_i} \geq 0$ for each i , or equivalently in the language of intersections $-D \cdot E_i \geq 0$ for each i . In other words, the divisor $-D$ is relatively nef. We will use the following notation for the divisor $-D$, in keeping with [41].

Definition 1.11. If π is a log resolution of an \mathfrak{m} -primary ideal \mathfrak{a} , then $Z_\pi(\mathfrak{a}) \in \mathcal{E}(\pi)$ will denote the divisor for which $\pi^* \mathfrak{a} = \mathcal{O}_{X_\pi}(Z_\pi(\mathfrak{a}))$.

Conversely, if π is a good resolution of (X, x_0) and $Z \in \mathcal{E}(\pi)$ is a divisor for which the line bundle $\mathcal{O}_{X_\pi}(Z)$ is relatively base point free, then π will be a log resolution of the ideal $\mathfrak{a} := \pi_* \mathcal{O}_{X_\pi}(Z)$, moreover with $Z = Z_\pi(\mathfrak{a})$. Note, the condition that $\mathcal{O}_{X_\pi}(Z)$ be relatively base point free is satisfied by definition when Z is relatively very ample, and thus every relatively very ample divisor $Z \in \mathcal{E}(\pi)$ is of the form $Z = Z_\pi(\mathfrak{a})$ for some \mathfrak{m} -primary ideal \mathfrak{a} . As we noted at the end of §1.1, one can find a basis of $\mathcal{E}(\pi)_\mathbb{R}$ consisting of relatively very ample divisors, thus proving the following.

Proposition 1.12. *For any good resolution π of (X, x_0) , there exist \mathfrak{m} -primary ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ for which π is a log resolution and for which the divisors $Z_\pi(\mathfrak{a}_i)$ form a basis of $\mathcal{E}(\pi)_\mathbb{R}$.*

Finally, let us point out that if π is a log resolution of \mathfrak{a} and π' is any good resolution dominating π , then π' is again a log resolution of \mathfrak{a} , and moreover $Z_{\pi'}(\mathfrak{a}) = \eta_{\pi\pi'}^* Z_\pi(\mathfrak{a})$.

2. Normal surface singularities and their valuation spaces

Let (X, x_0) be a normal surface singularity. In this section we will introduce and discuss the basic structure of an associated space \mathcal{V}_X of (semi-)valuations; it is by studying dynamics on this space that we will be able to deduce the main theorems of the paper. In the case when $X = \mathbb{C}^2$, the space $\mathcal{V}_{\mathbb{C}^2} = \mathcal{V}$

was introduced and described in detail by Favre-Jonsson, who called it the *valuative tree* [26, 29, 41]. Unfortunately in our singular setting the literature is less extensive. The spaces \mathcal{V}_X were briefly analyzed by Favre in [25], and they have appeared in a somewhat different vein in the works [22, 63, 16]. Our aim in this section is to give a fairly detailed treatment of the anatomy of \mathcal{V}_X , in the spirit of [41, §7]. We will also introduce in §2.7 a new tool we call the *angular metric*, which will prove to be quite useful in the context of dynamics. As in the previous section, we denote by R the completed local ring $\hat{\mathcal{O}}_{X,x_0}$ and by \mathfrak{m} its unique maximal ideal.

Definition 2.1. A *rank one semivaluation* on R is a function $\nu: R \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$ and $\nu(\phi + \psi) \geq \min\{\nu(\phi), \nu(\psi)\}$ for all $\phi, \psi \in R$. Such a semivaluation is said to be *centered* if in addition $\nu(\phi) \geq 0$ for all $\phi \in R$ and $\nu(\phi) > 0$ if and only if $\phi \in \mathfrak{m}$. The collection of all rank one centered semivaluations on R will be denoted $\hat{\mathcal{V}}_X$.

Equivalently, one may define rank one centered semivaluations on R not as functions on R itself, but on the set \mathcal{I} of ideals of R . In this formulation, the elements $\nu \in \hat{\mathcal{V}}_X$ are functions $\nu: \mathcal{I} \rightarrow \mathbb{R} \cup \{+\infty\}$ which satisfy $\nu(R) = 0$, $\nu(\mathfrak{m}) > 0$, $\nu(\mathfrak{a}) \geq \nu(\mathfrak{b})$ whenever $\mathfrak{a} \subseteq \mathfrak{b}$, $\nu(\mathfrak{a} + \mathfrak{b}) = \min\{\nu(\mathfrak{a}), \nu(\mathfrak{b})\}$, and $\nu(\mathfrak{a}\mathfrak{b}) = \nu(\mathfrak{a}) + \nu(\mathfrak{b})$. The equivalence between these two formulations is established by defining $\nu(\phi) = \nu(\phi R)$ for any $\phi \in R$ and exploiting the fact that ideals of R are finitely generated. In the following, we will go back and forth between these two perspectives without further comment.

Definition 2.2. A semivaluation $\nu \in \hat{\mathcal{V}}_X$ is *finite* if $\nu(\mathfrak{m}) < +\infty$ and *normalized* if $\nu(\mathfrak{m}) = 1$. We let $\hat{\mathcal{V}}_X^*$ and \mathcal{V}_X denote the subsets of $\hat{\mathcal{V}}_X$ consisting of all finite and normalized semivaluations, respectively. Note, there is exactly one semivaluation in $\hat{\mathcal{V}}_X$ which is not finite, which we denote by triv_{x_0} .

Remark 2.3. The interest in centered valuations is quite natural since we study the singularity (X, x_0) locally at x_0 . Nevertheless, the normalization with respect to the value on the maximal ideal, although very natural, is only one of the possible normalizations we could consider. Other normalizations appear in literature, and we could also work with different normalizations, for example with respect to other \mathfrak{m} -primary ideals, or with respect to irreducible curves. See §9.8 for further remarks.

The natural topology to put on $\hat{\mathcal{V}}_X$ is the so-called *weak topology*, which is the weakest topology such that for each $\phi \in R$ the evaluation map $\nu \in \hat{\mathcal{V}}_X \mapsto \nu(\phi) \in [0, +\infty]$ is continuous (or, equivalently, one may require evaluation on ideals $\mathfrak{a} \subseteq R$ to be continuous). It is not metrizable, but nonetheless possesses very useful properties: in the weak topology, \mathcal{V}_X is compact Hausdorff, sequentially compact, path connected, and locally contractible.

Note that any positive scalar multiple of a semivaluation $\nu \in \hat{\mathcal{V}}_X$ is again a semivaluation in $\hat{\mathcal{V}}_X$, and consequently any finite semivaluation ν can be scaled to obtain a normalized semivaluation $\nu(\mathfrak{m})^{-1}\nu \in \mathcal{V}_X$. In fact, scalar multiplication gives a homeomorphism $(0, +\infty) \times \mathcal{V}_X \rightarrow \hat{\mathcal{V}}_X^*$, so topologically $\hat{\mathcal{V}}_X^*$ is a cylinder over \mathcal{V}_X . Similarly, $\hat{\mathcal{V}}_X$ is a cone over \mathcal{V}_X with vertex triv_{x_0} .

2.1. Classification of finite semivaluations

In his foundational work on resolution of singularities in dimension two [68], Zariski classified centered semivaluations $\nu \in \hat{\mathcal{V}}_X^*$ into four types according to certain associated algebraic invariants of ν (the *rational rank* and *transcendence degree*), and then characterized these types geometrically. While we will not care about the algebraic invariants here, the geometric characterizations will be crucial. In the terminology of Favre-Jonsson, any $\nu \in \hat{\mathcal{V}}_X^*$ is either a divisorial valuation, an irrational valuation, an infinitely singular valuation, or a curve semivaluation.

A *divisorial* valuation $\nu \in \hat{\mathcal{V}}_X^*$ is a valuation which is proportional to the order of vanishing along some exceptional prime divisor in some modification of (X, x_0) . Explicitly, ν is divisorial if there exists some

modification $\pi: X_\pi \rightarrow (X, x_0)$, some exceptional prime divisor E of X_π , and some constant $\lambda > 0$ such that $\nu(\phi) = \lambda \text{ord}_E(\phi \circ \pi)$ for all $\phi \in R$. Note, the order of vanishing ord_E along E makes sense as we assume X_π is normal, and hence $\mathcal{O}_{X_\pi, E}$ is a DVR for each exceptional prime E of X_π . Observe also that if π' is a modification dominating π , and if F is the strict transform of E in $X_{\pi'}$, then $\nu(\phi) = \lambda \text{ord}_F(\phi \circ \pi')$ as well, so there is no harm in assuming for instance that π is a good resolution of (X, x_0) . Given an exceptional prime E , we denote by div_E the divisorial valuation defined by $\text{div}_E(\phi) = \pi_* \text{ord}_E(\phi) := \text{ord}_E(\phi \circ \pi)$, and by ν_E the associated normalized valuation $\nu_E = b_E^{-1} \text{div}_E$, where $b_E = \text{div}_E(\mathfrak{m})$ is called the *generic multiplicity* of E (see also §2.5). In the following, we say that a divisorial valuation $\lambda \nu_E$ is *realized* by π (or in X_π), if $\pi: X_\pi \rightarrow (X, x_0)$ is a modification and E is an exceptional prime of X_π .

As divisorial valuations are associated to exceptional primes of good resolutions of (X, x_0) , *irrational* valuations are associated to intersections of exceptional primes of good resolutions. Explicitly, suppose that π is a good resolution of (X, x_0) , and that E_1 and E_2 are two exceptional primes of π that intersect at a point $p \in X_\pi$. There then exist local holomorphic coordinates (z_1, z_2) of X_π around p for which E_i has defining equation $z_i = 0$. For any real $r, s \geq 0$ not both zero, one obtains a valuation $\nu_{r,s} \in \hat{\mathcal{V}}_X^*$ in the following manner. For any $\psi = \sum_{ij} a_{ij} z_1^i z_2^j \in \mathbb{C}[[z_1, z_2]]$, we consider the monomial valuation at p defined by $\mu_{r,s}(\psi) := \min\{ir + js : a_{ij} \neq 0\}$. Then $\nu_{r,s} = \pi_* \mu_{r,s}$.

If r and s are rationally independent, then $\nu_{r,s}$ is called an *irrational valuation*. On the other hand, if r and s are rationally dependent, then in fact $\nu_{r,s}$ is actually a divisorial valuation, though possibly not associated to any exceptional prime of π . In either case, we call $\nu_{r,s}$ the *monomial valuation* at p with *weights* r and s . Notice that in general $\nu_{r,s}$ is not normalized. In fact, if π is a log resolution of \mathfrak{m} , then $\nu_{r,s}(\mathfrak{m}) = b_{E_1} r + b_{E_2} s$.

Definition 2.4. Let $\pi: X_\pi \rightarrow (X, x_0)$ be a log resolution of \mathfrak{m} , and E, F two exceptional primes of X_π intersecting at a point p . We denote by $[\nu_E, \nu_F]_p$ (or simply $[\nu_E, \nu_F]$) the set of normalized monomial valuations $\nu_{r,s}$ at p . The *monomial parameterization* of $[\nu_E, \nu_F]_p$ is the map $w: [0, 1] \rightarrow [\nu_E, \nu_F]_p$, defined for all $t \in [0, 1]$ as $w(t) = \nu_{r(t), s(t)}$, where $r(t) = (1-t)b_E^{-1}$ and $s(t) = t b_F^{-1}$.

In the terminology of Favre-Jonsson, divisorial and irrational valuations make up the class of *quasimonomial* valuations, a term reflecting the geometry just described. They are also commonly referred to as *Abhyankar* valuations when thinking of them in terms of algebraic invariants, a name which alludes to the fact that the sum of the rational rank and transcendence degree is maximal for these valuations. As our perspective is geometric here, we will call them *quasimonomial* valuations.

A *curve semivaluation* is a semivaluation associated to an irreducible formal curve germ (C, x_0) in X passing through x_0 . It is well known that any such formal curve germ can be uniformized, that is, there exists a formal parameterization $h: (\mathbb{C}, 0) \rightarrow (C, x_0)$. Explicitly, a curve semivaluation associated to C is any semivaluation of the form $\nu(\phi) = \lambda \text{ord}_0(\phi|_C \circ h)$ for $\lambda > 0$. Observe that curve semivaluations are *not* valuations; in fact their kernel $\nu^{-1}(+\infty)$ is the prime ideal \mathfrak{p} corresponding to the curve germ (C, x_0) in X . We denote by int_C the curve semivaluation obtained setting $\lambda = 1$, and by ν_C the associated normalized valuation $\nu_C = m(C)^{-1} \text{int}_C$, where $m(C) = \text{int}_C(\mathfrak{m})$ is the *multiplicity* of C .

The only remaining $\nu \in \hat{\mathcal{V}}_X^*$ are the *infinitely singular* valuations. These have an interpretation in terms of Puiseux series which we will not need and thus omit. We point out, however, they are actual valuations instead of simply semivaluations. See [26] or [68, pp. 648-9] for details.

2.2. Dual divisors associated to valuations and b-divisors

Let $\nu \in \hat{\mathcal{V}}_X^*$, and let $\pi: X_\pi \rightarrow (X, x_0)$ be a modification of (X, x_0) . By the valuative criterion of properness, ν has a unique *center* in X_π , that is, a unique scheme-theoretic point $\xi \in X_\pi$ with the property that ν takes nonnegative values on the local ring $\mathcal{O}_{X_\pi, \xi}$ and strictly positive values exactly on its maximal ideal

m_ξ . We will write this $\xi = \text{cen}_\pi(\nu)$. Note, the map $\text{cen}_\pi: \mathcal{V}_X^* \rightarrow X_\pi$ is typically called the *reduction map* in the context of non-archimedean analytic geometry; it is anti-continuous in the sense that the inverse image of a Zariski open set is closed. If π' is a modification dominating π , then one easily sees that $\text{cen}_\pi = \eta_{\pi\pi'} \circ \text{cen}_{\pi'}$.

Suppose now that $\pi: X_\pi \rightarrow (X, x_0)$ is a good resolution of (X, x_0) . Then ν can be evaluated on the group $\mathcal{E}(\pi)$ of exceptional divisors by setting $\nu(D) := \nu(\phi)$, where $\phi \in \mathcal{O}_{X_\pi, \xi}$ is a defining equation of $D \in \mathcal{E}(\pi)$ around $\xi = \text{cen}_\pi(\nu)$. Clearly this evaluation $\nu: \mathcal{E}(\pi) \rightarrow \mathbb{R}$ is \mathbb{Z} -linear, and hence extends linearly to an evaluation map $\nu: \mathcal{E}(\pi)_\mathbb{R} \rightarrow \mathbb{R}$ on \mathbb{R} -divisors. Recalling that the intersection product on X_π gives a natural identification of $\mathcal{E}(\pi)_\mathbb{R}^*$ with $\mathcal{E}(\pi)_\mathbb{R}$, we conclude the following.

Proposition 2.5. *For each $\nu \in \mathcal{V}_X^*$ and each good resolution $\pi: X_\pi \rightarrow (X, x_0)$, there is a unique \mathbb{R} -divisor $Z_\pi(\nu) \in \mathcal{E}(\pi)_\mathbb{R}$ such that $\nu(D) = Z_\pi(\nu) \cdot D$ for each $D \in \mathcal{E}(\pi)_\mathbb{R}$.*

If the center $\xi = \text{cen}_\pi(\nu)$ lies within exactly one exceptional prime E of π , then $Z_\pi(\nu) = \lambda \check{E}$ for some $\lambda > 0$. Otherwise, ξ must be the intersection point of two exceptional primes, say E and F , in which case $Z_\pi(\nu) = r \check{E} + s \check{F}$ for some $r, s > 0$. Either way, we see that $Z_\pi(\nu)$ is relatively nef.

Given $\nu \in \mathcal{V}_X^*$ one can compute $Z_\pi(\nu)$ for any good resolution π of (X, x_0) , but conversely, knowing the divisors $Z_\pi(\nu)$ for every good resolution π allows one to recover the semivaluation ν . In fact, one can evaluate semivaluations on ideals using intersections in the following way.

Proposition 2.6. *Let $\nu \in \mathcal{V}_X^*$ and let $\mathfrak{a} \subset R$ be an \mathfrak{m} -primary ideal. Let $\pi: X_\pi \rightarrow (X, x_0)$ be a log resolution of \mathfrak{a} . Then $\nu(\mathfrak{a}) = -Z_\pi(\nu) \cdot Z_\pi(\mathfrak{a})$.*

Proof. Let ξ be the center of ν in X_π , so that ν defines a centered semivaluation on the local ring $\mathcal{O}_{X_\pi, \xi}$. Tautologically, the value of ν on $\mathfrak{a} \subset R$ agrees with the value of ν on the ideal $(\pi^*\mathfrak{a})_\xi \subset \mathcal{O}_{X_\pi, \xi}$. Because π is a log resolution of \mathfrak{a} , the ideal sheaf $\pi^*\mathfrak{a}$ is the ideal sheaf of the divisor $-Z_\pi(\mathfrak{a})$, and thus in particular $\nu((\pi^*\mathfrak{a})_\xi) = \nu(-Z_\pi(\mathfrak{a})) = -Z_\pi(\nu) \cdot Z_\pi(\mathfrak{a})$, completing the proof. \square

Corollary 2.7. *Let π and π' be good resolutions of (X, x_0) , with π' dominating π . Then $(\eta_{\pi\pi'})_* Z_{\pi'}(\nu) = Z_\pi(\nu)$.*

Proof. It suffices to check that $(\eta_{\pi\pi'})_* Z_{\pi'}(\nu) \cdot Z = Z_\pi(\nu) \cdot Z$ for all divisors Z in a given basis Z_1, \dots, Z_n of $\mathcal{E}(\pi)_\mathbb{R}$. Using [Proposition 1.12](#), we can take the Z_i to be of the form $Z_i = Z_\pi(\mathfrak{a}_i)$ for some \mathfrak{m} -primary ideals \mathfrak{a}_i of R . Then applying the projection formula and [Proposition 2.6](#) twice yields

$$(\eta_{\pi\pi'})_* Z_{\pi'}(\nu) \cdot Z_\pi(\mathfrak{a}_i) = Z_{\pi'}(\nu) \cdot \eta_{\pi\pi'}^* Z_\pi(\mathfrak{a}_i) = Z_{\pi'}(\nu) \cdot Z_{\pi'}(\mathfrak{a}_i) = -\nu(\mathfrak{a}_i) = Z_\pi(\nu) \cdot Z_\pi(\mathfrak{a}_i),$$

as desired. \square

Note, in general it is not true that $Z_{\pi'}(\nu) = \eta_{\pi\pi'}^* Z_\pi(\nu)$. There is, on the other hand, one case in which this equality does hold, namely when ν is a divisorial valuation associated to an exceptional prime E of π . In this case, $\nu = \lambda \text{div}_E$ for some $\lambda > 0$, and $Z_\pi(\nu) = \lambda \check{E}$. If F is the strict transform of E in $X_{\pi'}$, then similarly $\nu = \lambda \text{div}_F$ and $Z_{\pi'}(\nu) = \lambda \check{F}$. The equality $Z_{\pi'}(\nu) = \eta_{\pi\pi'}^* Z_\pi(\nu)$ thus follows from the fact that $\eta_{\pi\pi'}^* \check{E} = \check{F}$, which is in turn a consequence of the projection formula.

We may define $Z_\pi = Z_\pi(\nu)$ or $Z_\pi = Z_\pi(\mathfrak{a})$ for all modifications π , by taking any high enough good resolution π' dominating π where $Z_{\pi'}$ is defined, and setting $Z_\pi = (\eta_{\pi\pi'})_* Z_{\pi'}$. As a consequence of [Corollary 2.7](#), Z_π is well defined, as it does not depend on the choice of π' . The family of divisors $Z = (Z_\pi)_\pi$ is sometimes called a *b-divisor* in the sense of Shokurov [[61](#)] (see also [[9](#), [25](#)]).

Definition 2.8. Let (X, x_0) be a normal surface singularity. A (Weil) (exceptional) *b-divisor* Z on (X, x_0) is a collection $Z = (Z_\pi)_\pi$ of Weil \mathbb{R} -divisors $Z_\pi \in \mathcal{E}(\pi)_\mathbb{R}$ for any modification $\pi: X_\pi \rightarrow (X, x_0)$, satisfying

the relation $(\eta_{\pi\pi'})_*Z_{\pi'} = Z_\pi$ for all modifications π, π' with π' dominating π . The divisor Z_π is called the *incarnation* of Z in the model π .

If moreover there exists a modification π so that $Z_{\pi'} = \eta_{\pi\pi'}^*Z_\pi$ for all modifications π' dominating π , we say that Z is a *Cartier b-divisor*, *determined* by the model π .

We denote by $\mathcal{E}(X)$ the set of b-divisors, and by $\mathcal{E}_C(X)$ its subset of *Cartier* b-divisors.

Finally, we say that a b-divisor $Z = (Z_\pi)_\pi$ is *nef* if Z_π is nef for all modifications π .

Notice that, to make sense of the definition of Cartier divisors, we have to be able to pull back divisors. Since (X, x_0) is a surface, we can do it numerically in the sense of Mumford. Alternatively, we may notice that if π is a modification which is a determination of a Cartier b-divisor Z , then any other modification π' dominating π is a determination of Z . We may hence assume that π is a resolution of (X, x_0) in the definition of Cartier b-divisors. In the following (see §4.3), we will consider a more general class of b-divisors. The b-divisors defined here will be the subset of *exceptional* b-divisors, which have the property of having every incarnation exceptional.

Remark 2.9. We describe here some properties of b-divisors, see [9] for proofs. The set $\mathcal{E}_C(X)$ of Cartier b-divisor has a natural structure of infinite dimensional \mathbb{R} -vector space. Denote by Γ_X^* the set of exceptional primes over X , which is in bijection with the set of divisorial valuations $\mathcal{V}_X^{\text{div}}$, through the identification $E \mapsto \nu_E$. A basis for $\mathcal{E}_C(X)$ is naturally given by dual divisors \check{E} , with E varying in Γ_X^* . Analogously, one can consider the basis given by $Z(\nu_E)$, where ν_E varies among normalized divisorial valuations in $\mathcal{V}_X^{\text{div}}$. Another natural basis for $\mathcal{E}_C(X)$ can be constructed as follows. Since X is a surface, for any exceptional prime $E \in \Gamma_X^*$, there is a minimal resolution $\pi: X_\pi \rightarrow (X, x_0)$ realizing ν_E . We set $Z(E)$ to be the b-divisor determined by E in the model π . The set $\{Z(E) \mid E \in \Gamma_X^*\}$ forms a basis of $\mathcal{E}_C(X)$. This latter basis has the property of being orthogonal with respect to the intersection form *when* (X, x_0) is *smooth*. This no longer holds for singularities, since $Z(E) \cdot Z(F) \neq 0$ if E and F are two exceptional primes realized and intersecting in the minimal resolution of (X, x_0) . Nevertheless, it allows to describe the space of Weil (exceptional) b-divisors $\mathcal{E}(X)$, which is isomorphic to the vector space of functions $\Gamma_X^* \rightarrow \mathbb{R}$. In fact, given a b-divisor Z , we may associate to it the function sending E to $\check{E} \cdot Z := \text{ord}_E(Z_\pi)$, where Z_π is the incarnation of Z in any model π realizing ν_E .

Finally, we may endow $\mathcal{E}(X)$ with a topology, by stating that a sequence of b-divisors Z_n converges to Z if for any $E \in \Gamma_X^*$ we have $\check{E} \cdot Z_n \rightarrow \check{E} \cdot Z$.

With respect to this topology, and the weak topology on the valuative space, the map $\hat{\mathcal{V}}_X^* \rightarrow \mathcal{E}(X)$ given by $\nu \mapsto Z(\nu)$ is continuous (see [Proposition 2.26](#)).

Remark 2.10. [Proposition 2.5](#) can be used to define the b-divisor associated to a non-necessarily \mathfrak{m}_X -primary ideal \mathfrak{a} . Let $\pi: X_\pi \rightarrow (X, x_0)$ be any good resolution. We define $Z_\pi(\mathfrak{a})$ as the unique divisor supported on $\pi^{-1}(0)$ so that $\text{ord}_E(\pi^*\mathfrak{a}) = -\check{E} \cdot Z_\pi(\mathfrak{a})$ for all exceptional primes E of π . The family $Z(\mathfrak{a}) = (Z_\pi(\mathfrak{a}))_\pi$ defines a nef b-divisor which is Cartier if and only if \mathfrak{a} is \mathfrak{m}_X -primary.

2.3. Intersection theory and skewness

Having the family of divisors $Z_\pi(\nu)$ associated to semivaluations ν allows us to study semivaluations using the geometry of the vector spaces $\mathcal{E}(\pi)_\mathbb{R}$ in which the $Z_\pi(\nu)$ reside. For instance, given two semivaluations ν and μ it is natural to consider the quantities

$$\alpha_\pi(\nu) := -Z_\pi(\nu) \cdot Z_\pi(\nu) \quad \text{and} \quad \beta_\pi(\nu \mid \mu) := \frac{Z_\pi(\nu) \cdot Z_\pi(\nu)}{Z_\pi(\nu) \cdot Z_\pi(\mu)}. \quad (4)$$

These are both positive. The former is of course the norm squared of $Z_\pi(\nu)$ with respect to the inner product given by the negative of the intersection product, and thus gives some measure of the size of

ν . The latter measures the component of $Z_\pi(\mu)$ in the direction of $Z_\pi(\nu)$, giving a means of comparing ν and μ .

Proposition 2.11. *Let $\nu, \mu \in \hat{\mathcal{V}}_X^*$. Then for high enough good resolutions π of (X, x_0) , the quantities $\alpha_\pi(\nu)$ and $\beta_\pi(\nu | \mu)$ increase monotonically to some values $\alpha(\nu)$ and $\beta(\nu | \mu)$ belonging to $(0, +\infty]$.*

Proof. First we consider the quantities $\beta_\pi(\nu | \mu)$. If $\mu = \lambda \nu$ for some $\lambda > 0$, then $\beta_\pi(\nu | \mu) = 1/\lambda$ for all good resolutions π , so the convergence is clear. Assume therefore that μ is not a scalar multiple of ν . Then necessarily for some good resolution π_0 of (X, x_0) the centers $\xi = \text{cen}_{\pi_0}(\nu)$ and $\zeta = \text{cen}_{\pi_0}(\mu)$ will be distinct. Suppose that π is a good resolution dominating π_0 . Then $\eta_{\pi_0\pi}$ factors as $\eta_{\pi_0\pi} = \eta_2 \circ \eta_1$, where $\eta_1: X_{\pi_1} \rightarrow X_{\pi_0}$ is a sequence of point blowups over ξ and $\eta_2: X_\pi \rightarrow X_{\pi_1}$ is a sequence of point blowups over ζ . Since η_1 is an isomorphism over a neighborhood of ζ , we have that $Z_{\pi_1}(\mu) = \eta_1^* Z_{\pi_0}(\mu)$ and thus by the projection formula

$$Z_{\pi_1}(\nu) \cdot Z_{\pi_1}(\mu) = Z_{\pi_1}(\nu) \cdot \eta_1^* Z_{\pi_0}(\mu) = \eta_{1*} Z_{\pi_1}(\nu) \cdot Z_{\pi_0}(\mu) = Z_{\pi_0}(\nu) \cdot Z_{\pi_0}(\mu).$$

The exact same argument used for η_2 allows us to conclude that

$$Z_\pi(\nu) \cdot Z_\pi(\mu) = Z_{\pi_1}(\nu) \cdot Z_{\pi_1}(\mu) = Z_{\pi_0}(\nu) \cdot Z_{\pi_0}(\mu).$$

We have therefore shown that the quantity $Z_\pi(\nu) \cdot Z_\pi(\mu)$ is constant in π , so long as π dominates π_0 . As a consequence, the denominator in the definition (4) of $\beta_\pi(\nu | \mu)$ is constant for high enough π , and the proposition will be proved if we can show that $\alpha_\pi(\nu)$ increases monotonically with π . But this follows from [Corollary 2.7](#) and the next lemma. \square

Lemma 2.12. *Let π and π' be good resolutions of (X, x_0) , with π' dominating π . For any $D \in \mathcal{E}(\pi')_{\mathbb{R}}$, one has $D \cdot D \leq (\eta_{\pi\pi'})_* D \cdot (\eta_{\pi\pi'})_* D$.*

Proof. To ease notation, let $\eta = \eta_{\pi\pi'}$. The point now is that by the projection formula $\eta^* \eta_* D$ is the orthogonal projection of D onto the subspace $\eta^* \mathcal{E}(\pi)_{\mathbb{R}} \subset \mathcal{E}(\pi')_{\mathbb{R}}$. Thus $D = \eta^* \eta_* D + R$, where R is orthogonal to $\eta^* \mathcal{E}(\pi)_{\mathbb{R}}$, so

$$D \cdot D = \eta^* \eta_* D \cdot \eta^* \eta_* D + R \cdot R = \eta_* D \cdot \eta_* D + R \cdot R \leq \eta_* D \cdot \eta_* D,$$

with the second equality by the projection formula and the inequality because the intersection product is negative definite. \square

[Proposition 2.11](#) can also be interpreted in terms of b-divisors. In fact, given two nef b-divisors $Z, W \in \mathcal{E}(X)$, we may define their intersection as $Z \cdot W = \lim_{\pi} Z_\pi \cdot W_\pi \in [-\infty, 0)$. The nef hypothesis assures that the value $Z_\pi \cdot W_\pi$ is non-increasing with respect to π , and allows the limit to exist. Notice that by the projection formula, $Z \cdot W$ is always finite as far as at least one between Z and W is Cartier. In this case, $Z \cdot W = Z_\pi \cdot W_\pi$ for any determination π of the Cartier divisor. More generally, from the proof of [Proposition 2.11](#), we deduce the following property.

Proposition 2.13. *Let (X, x_0) be a normal surface singularity, and $\nu, \mu \in \hat{\mathcal{V}}^*$ be two semivaluations. If ν and μ have different centers in X_π for a good resolution $\pi: X_\pi \rightarrow (X, x_0)$, then*

$$Z(\nu) \cdot Z(\mu) = Z_\pi(\nu) \cdot Z_\pi(\mu).$$

Analogously, [Proposition 1.10](#) can be stated in terms of b-divisors. To ease statements, we first introduce a definition.

Definition 2.14. Let (X, x_0) be a normal surface singularity, and let $\nu, \mu_1, \mu_2 \in \mathcal{V}_X$ be three semivaluations. We say that ν *disconnects* μ_1 and μ_2 if either $\nu = \mu_1$, $\nu = \mu_2$, or μ_1 and μ_2 belong to different connected components of $\mathcal{V}_X \setminus \{\nu\}$.

Proposition 2.15. Let (X, x_0) be a normal surface singularity, and $\nu, \mu_1, \mu_2 \in \hat{\mathcal{V}}_X^*$ be three semivaluations. Set $Z = Z(\nu)$ and $W_j = Z(\mu_j)$ for $j = 1, 2$. Then we have

$$(Z \cdot W_1)(Z \cdot W_2) \leq (Z \cdot Z)(W_1 \cdot W_2). \quad (5)$$

Moreover (taking ν, μ_1, μ_2 normalized) the equality holds if and only if ν disconnects μ_1 and μ_2 .

Proof. First notice that the statement is homogeneous on ν, μ_1 and μ_2 . If the three valuations are divisorial, the statement is just a rephrased version of [Proposition 1.10](#). The general case is obtained taking the limit over the modifications π , and using the structure of \mathcal{V}_X as a real graph. \square

Notice that (5) still holds if we replace W_1 and W_2 with any nef b-divisors. In fact any nef b-divisor is a non-negative linear combination of divisors of the form \check{E} for some exceptional primes E .

We now come back to the quantity $\alpha(\nu)$ and $\beta(\nu \mid \mu)$. The former is called the *skewness* of ν in [\[26, 29, 27, 30, 41\]](#), though it should be noted that in some of these works the definition of skewness differs from ours by a sign. We will call $\beta(\nu \mid \mu)$ the *relative skewness* of ν relative to μ . It is worth pointing out that the proof of [Proposition 2.11](#) gives a way of relating skewness and relative skewness.

Corollary 2.16. Let $\nu, \mu \in \hat{\mathcal{V}}_X^*$. If $\pi: X_\pi \rightarrow (X, x_0)$ is a good resolution of (X, x_0) for which ν and μ have distinct centers in X_π , then

$$\alpha(\nu) = \beta(\nu \mid \mu)[-Z_\pi(\nu) \cdot Z_\pi(\mu)]. \quad (6)$$

In particular, if ν and μ are not proportional (so that such a good resolution π exists) then $\alpha(\nu)$ is finite if and only if $\beta(\nu \mid \mu)$ is finite.

Notice that for non-divisorial valuations $\nu \in \hat{\mathcal{V}}_X^*$, the skewness $\alpha(\nu)$ may be infinite. It turns out that the skewness is always finite for quasimonomial valuations. In fact, one can give a very explicit formula for $\alpha(\nu)$ when ν is quasimonomial using the next proposition.

Proposition 2.17. Let $\pi: X_\pi \rightarrow (X, x_0)$ be a good resolution of (X, x_0) , and suppose that E_1 and E_2 are two exceptional primes of π which intersect in a point $p \in X_\pi$. Let ν_1 and ν_2 be the monomial valuations at p with weights r_1, s_1 and r_2, s_2 , respectively, as defined in [§2.1](#). Then

$$\lim_{\pi' \geq \pi} Z_{\pi'}(\nu_1) \cdot Z_{\pi'}(\nu_2) = Z_\pi(\nu_1) \cdot Z_\pi(\nu_2) - \min\{r_1 s_2, r_2 s_1\},$$

the limit taken over all good resolutions $\pi' \geq \pi$.

Proof. After reversing the roles of E_1 and E_2 and/or the roles of ν_1 and ν_2 , we may assume we are in one of the following two cases:

1. $r_1 \geq s_1$ and $s_2 > r_2$, or
2. $r_1 \geq s_1$ and $r_2 \geq s_2$.

Let π_1 be the good resolution of (X, x_0) obtained by blowing up $p \in X_\pi$, and let F be the new exceptional prime thus obtained. Abusing notation, let E_1 and E_2 also denote the strict transforms of E_1 and E_2 in X_{π_1} . Let p_i be the point of intersection of F with E_i for $i = 1, 2$.

Assume first we are in case 1. Then one easily checks that ν_1 is the monomial valuation at p_1 giving weight $r_1 - s_1$ to E_1 and s_1 to F , whereas ν_2 is the monomial valuation at p_2 giving weight $s_2 - r_2$ to E_2 and r_2 to F . In particular, ν_1 and ν_2 have different centers in π_1 , and by [Proposition 2.13](#) we get

$$\lim_{\pi' \geq \pi} Z_{\pi'}(\nu_1) \cdot Z_{\pi'}(\nu_2) = Z_{\pi_1}(\nu_1) \cdot Z_{\pi_1}(\nu_2) = [(r_1 - s_1)\check{E}_1 + s_1\check{F}] \cdot [(s_2 - r_2)\check{E}_2 + r_2\check{F}].$$

Using the (easy) relation $\check{F} = \check{E}_1 + \check{E}_2 - F$, the right hand side of this equation becomes

$$\begin{aligned} [r_1\check{E}_1 + s_1\check{E}_2 - s_1F] \cdot [r_2\check{E}_1 + s_2\check{E}_2 - r_2F] &= [r_1\check{E}_1 + s_1\check{E}_2] \cdot [r_2\check{E}_1 + s_2\check{E}_2] - s_1r_2 \\ &= Z_{\pi}(\nu_1) \cdot Z_{\pi}(\nu_2) - s_1r_2, \end{aligned}$$

the last equality by the projection formula. This completes the proof in case 1.

Assume next we are in case 2. Now ν_1 and ν_2 are both monomial valuations at p_1 giving weights $r_{11} := r_1 - s_1$ and $r_{21} := r_2 - s_2$ to E_1 , respectively, and weights $s_{11} := s_1$ and $s_{21} := s_2$ to F , respectively. A similar computation to the one made in case 1 shows that

$$Z_{\pi_1}(\nu_1) \cdot Z_{\pi_1}(\nu_2) = Z_{\pi}(\nu_1) \cdot Z_{\pi}(\nu_2) - s_1s_2,$$

from which it follows that

$$Z_{\pi_1}(\nu_1) \cdot Z_{\pi_1}(\nu_2) - \min\{r_{11}s_{21}, r_{21}s_{11}\} = Z_{\pi}(\nu_1) \cdot Z_{\pi}(\nu_2) - \min\{r_1s_2, r_2s_1\}.$$

It therefore suffices to prove the proposition for π_1 instead of π .

By iterating the above argument, one obtains a (possibly finite) sequence of good resolutions $\pi_n \geq \pi$ such that for each n the valuations ν_1 and ν_2 are monomial valuations at some point $p_n \in X_{\pi_n}$ with weights r_{1n}, s_{1n} and r_{2n}, s_{2n} , respectively, and such that π_{n+1} is obtained from π_n by blowing up p_n . Moreover, we have inductively that

$$Z_{\pi_n}(\nu_1) \cdot Z_{\pi_n}(\nu_2) - \min\{r_{1n}s_{2n}, r_{2n}s_{1n}\} = Z_{\pi}(\nu_1) \cdot Z_{\pi}(\nu_2) - \min\{r_1s_2, r_2s_1\} \quad (7)$$

for each n . The sequence π_n terminates at some π_N if ν_1 and ν_2 have different centers after blowing up p_N ; in this case the proposition is proved by case 1 above. If the sequence π_n does not terminate, then ν_1 and ν_2 have the same center in every good resolution of (X, x_0) , and hence are proportional. We may therefore assume with no loss in generality that $\nu_1 = \nu_2$, and thus that $r_{1n} = r_{2n}$ and $s_{1n} = s_{2n}$ for every n . The identity (7) then becomes

$$Z_{\pi_n}(\nu_1) \cdot Z_{\pi_n}(\nu_2) - r_{1n}s_{1n} = Z_{\pi}(\nu_1) \cdot Z_{\pi}(\nu_2) - r_1s_1$$

for all $n \geq 1$. Since $r_{1n}s_{1n} \rightarrow 0$ as $n \rightarrow \infty$, we obtain in the limit that

$$\lim_{\pi' \geq \pi} Z_{\pi'}(\nu_1) \cdot Z_{\pi'}(\nu_2) = \lim_{n \rightarrow \infty} Z_{\pi_n}(\nu_1) \cdot Z_{\pi_n}(\nu_2) = Z_{\pi}(\nu_1) \cdot Z_{\pi}(\nu_2) - r_1s_1,$$

completing the proof. □

Corollary 2.18. *With the same setup as [Proposition 2.17](#), we have the identities*

$$\alpha(\nu_i) = -Z_{\pi}(\nu_i) \cdot Z_{\pi}(\nu_i) + r_i s_i \quad (8)$$

and

$$\beta(\nu_1 | \nu_2) = \frac{-Z_{\pi}(\nu_1) \cdot Z_{\pi}(\nu_1) + r_1 s_1}{-Z_{\pi}(\nu_1) \cdot Z_{\pi}(\nu_2) + \min\{r_1 s_2, s_1 r_2\}}. \quad (9)$$

We denote by \mathcal{V}_X^α the set of normalized valuations with finite skewness. As a consequence of [Corollary 2.18](#), \mathcal{V}_X^α contains the set of quasimonomial valuations $\mathcal{V}_X^{\text{qm}}$.

We conclude with an important alternate characterization of the relative skewness $\beta(\nu | \mu)$, as what is sometimes called the *relative Izumi constant* of ν with respect to μ .

Proposition 2.19. *Let $\nu, \mu \in \hat{\mathcal{V}}_X^*$. Then $\beta(\nu | \mu)$ can also be computed as*

$$\beta(\nu | \mu) = \sup_{\mathfrak{a}} \frac{\nu(\mathfrak{a})}{\mu(\mathfrak{a})},$$

the supremum being taken over all \mathfrak{m} -primary ideals \mathfrak{a} of R .

Proof. Suppose that \mathfrak{a} is an \mathfrak{m} -primary ideal, and let π be a log resolution of \mathfrak{a} . By [Proposition 2.6](#),

$$\frac{\nu(\mathfrak{a})}{\mu(\mathfrak{a})} = \frac{Z_\pi(\nu) \cdot Z_\pi(\mathfrak{a})}{Z_\pi(\mu) \cdot Z_\pi(\mathfrak{a})}.$$

Consider the hyperplane $H \subset \mathcal{E}(\pi)_{\mathbb{R}}$ consisting of all divisors D such that $Z_\pi(\mu) \cdot D = Z_\pi(\mu) \cdot Z_\pi(\mathfrak{a})$. It follows from [Corollary 1.9](#), that $H \cap \text{Nef}(\pi)$ is a cross-section of the nef cone in $\mathcal{E}(\pi)_{\mathbb{R}}$. The map

$$D \in H \cap \text{Nef}(\pi) \mapsto \frac{Z_\pi(\nu) \cdot D}{Z_\pi(\mu) \cdot D} = \frac{Z_\pi(\nu) \cdot D}{Z_\pi(\mu) \cdot Z_\pi(\mathfrak{a})}$$

is maximized by Cauchy-Schwarz precisely when D is proportional to $Z_\pi(\nu)$, and in this case

$$\frac{Z_\pi(\nu) \cdot D}{Z_\pi(\mu) \cdot D} = \frac{Z_\pi(\nu) \cdot Z_\pi(\nu)}{Z_\pi(\mu) \cdot Z_\pi(\nu)} = \beta_\pi(\nu | \mu).$$

Thus proves that $\nu(\mathfrak{a})/\mu(\mathfrak{a}) \leq \beta_\pi(\nu | \mu)$, and we can conclude that $\sup_{\mathfrak{a}} \nu(\mathfrak{a})/\mu(\mathfrak{a}) \leq \beta(\nu | \mu)$. To show the opposite inequality, fix a good resolution π of (X, x_0) . Since relatively ample \mathbb{Q} -divisors are dense in the nef cone of π , there exist relatively ample \mathbb{Q} -divisors D_n with $D_n \rightarrow Z_\pi(\nu)$ as $n \rightarrow \infty$. For each n , let $k_n \in \mathbb{N}$ be a large enough integer that $k_n D_n$ is an integral relatively very ample divisor, and hence $k_n D_n = Z_\pi(\mathfrak{a}_n)$ for some \mathfrak{m} -primary ideal \mathfrak{a}_n . Then

$$\frac{\nu(\mathfrak{a}_n)}{\mu(\mathfrak{a}_n)} = \frac{Z_\pi(\nu) \cdot Z_\pi(\mathfrak{a}_n)}{Z_\pi(\mu) \cdot Z_\pi(\mathfrak{a}_n)} = \frac{Z_\pi(\nu) \cdot D_n}{Z_\pi(\mu) \cdot D_n} \rightarrow \frac{Z_\pi(\nu) \cdot Z_\pi(\nu)}{Z_\pi(\mu) \cdot Z_\pi(\nu)} = \beta_\pi(\nu | \mu).$$

Since $\beta(\nu | \mu) = \sup_{\mathfrak{a}} \beta_\pi(\nu | \mu)$, it follows immediately that $\beta(\nu | \mu) \leq \sup_{\mathfrak{a}} \nu(\mathfrak{a})/\mu(\mathfrak{a})$. \square

Remark 2.20. A similar argument can be deployed to prove an analogous statement for the skewness α . In fact, the intersection $-Z_\pi(\mathfrak{m}) \cdot Z_\pi(\mathfrak{a})$ does not depend on the chosen log resolution π of \mathfrak{m} . One can define $m(\mathfrak{a})$ as such value. Then one can prove

$$\frac{\alpha(\nu)}{\nu(\mathfrak{m})} = \sup_{\mathfrak{a}} \frac{\nu(\mathfrak{a})}{m(\mathfrak{a})}.$$

It suffices to replace $Z_\pi(\mu)$ by $Z_\pi(\mathfrak{m})$ in the proof of [Proposition 2.19](#), and recall that $\nu(\mathfrak{m}) = -Z_\pi(\mathfrak{m}) \cdot Z_\pi(\nu)$ for any log resolution π of \mathfrak{m} . One can also replace the supremum over \mathfrak{m} -primary ideals \mathfrak{a} with the supremum over elements ϕ in \mathfrak{m} , and accordingly $\nu(\mathfrak{a})$ with $\nu(\phi)$. In fact, by [Proposition 1.12](#), for any good desingularization π one can find a \mathfrak{m} -primary ideal \mathfrak{a}_π so that $\pi^* \phi = \mathcal{O}_{X_\pi}(Z_\pi(\mathfrak{a}_\pi))$. Moreover, if π' dominates π , then $\eta_{\pi\pi'}^* Z_\pi(\mathfrak{a}_\pi) - Z_{\pi'}(\mathfrak{a}_{\pi'})$ is effective (see also [Proposition 2.32](#)). Alternatively, one can notice that $\nu(\phi)$ can be computed as the limit (the supremum) over good resolutions π of $-Z_\pi(\nu) \cdot Z_\pi(\mathfrak{a}_\pi)$, where $\mathfrak{a}_\pi = (\phi) + \mathfrak{m}^n$ for n big enough (depending on π).

Finally, if ν is a curve semivaluation and μ is another semivaluation, then $\beta(\nu | \mu) = +\infty$; indeed, if $\mathfrak{p} \subset R$ is the kernel of ν , one need only note that for the primary ideals $\mathfrak{a}_n = \mathfrak{p} + \mathfrak{m}^n$ one has $\nu(\mathfrak{a}_n)/\mu(\mathfrak{a}_n) \rightarrow +\infty$ as $n \rightarrow \infty$. Since $\beta(\nu | \mu)$ is finite if and only if $\alpha(\nu)$ is finite, we have proved that curve semivaluations always have infinite skewness. On the other hand, infinitely singular valuations can have either finite or infinite skewness. In particular, having finite skewness is not equivalent to being quasimonomial.

2.4. Weak topology and tangent vectors

We now focus our attention to the space of normalized semivaluations \mathcal{V}_X . The utility of this space lies in its topological properties (it is compact Hausdorff) as well as the fact that its structure reflects the combinatorics of good resolutions of (X, x_0) . We start by describing in more details the weak topology on \mathcal{V}_X introduced above.

Definition 2.21. Let (X, x_0) be a normal surface singularity, $\pi: X_\pi \rightarrow (X, x_0)$ be a proper birational map, and $p \in \pi^{-1}(x_0)$ a point in the exceptional divisor of π . We set

$$U_\pi(p) = \{\nu \in \mathcal{V}_X \mid \text{cen}_\pi(\nu) = p\}.$$

Notice that the point p does not need to be a smooth point of X_π . The family of sets $U_\pi(p)$ where π varies among good resolutions and p among points in $\pi^{-1}(x_0)$ form a prebasis for the weak topology on \mathcal{V}_X . A (non-empty) connected weak open subset of \mathcal{V}_X is the connected component of the complement of a finite set in \mathcal{V}_X . We will need a notation for such connected weak open sets. In what follows, a *connected subgraph* of \mathcal{V}_X is a weakly closed connected subset of \mathcal{V}_X whose set of endpoints ∂S is finite. An *endpoint* of S is a point $s \in S$ that does not disconnect any connected subset U of S , i.e., for any connected subset $U \subseteq S$, $U \setminus \{s\}$ remains connected.

Definition 2.22. Let (X, x_0) be a normal surface singularity, and let $S \subset \mathcal{V}_X$ be a connected subgraph. We denote by $U(S)$ the connected weak open subset of \mathcal{V}_X given by the connected component of $\mathcal{V}_X \setminus \partial S$ containing $S \setminus \partial S$.

We will use extensively this notation when I is an interval with divisorial endpoints. In this case, $U(I) = U_\pi(p)$, where $\pi: X_\pi \rightarrow (X, x_0)$ is a modification obtained from any good resolution $\pi': X_{\pi'} \rightarrow (X, x_0)$ that realizes the endpoints of I , and contracting the chain of exceptional primes E of $X_{\pi'}$ so that ν_E belongs to the interior part of I . In this case p is the image through $\eta_{\pi\pi'}$ of the contracted chain of exceptional primes.

This description of the weak topology is related to the construction of tangent vectors attached to (divisorial) valuations, see [29, Section 1.6] for the analogous construction in the smooth case. Let $\nu \in \mathcal{V}_X$ be any normalized semivaluation. Let U be any connected weak open neighborhood of ν , and let $T_\nu(U)$ be the set of connected components of $U \setminus \{\nu\}$. Given another connected open neighborhood $V \subset U$, there is a natural map $j_{V,U}: T_\nu(V) \rightarrow T_\nu(U)$ which sends a connected component W of $V \setminus \{\nu\}$ to the unique connected component W' of $U \setminus \{\nu\}$ which contains W . The map $j_{V,U}$ is always surjective but not necessarily injective, the obstruction given by the presence of circles in \mathcal{V}_X . The maps $j_{V,U}$ while U, V vary among nested connected weakly open neighborhoods of ν form an inverse system, which allows the following definition.

Definition 2.23. Let (X, x_0) be a normal surface singularity, and let \mathcal{V}_X be its space of normalized semivaluations. The *tangent space* $T_\nu \mathcal{V}_X$ of \mathcal{V}_X at ν is given by the projective limit

$$T_\nu \mathcal{V}_X = \varprojlim_U T_\nu(U).$$

Its elements are called *tangent vectors*.

Remark 2.24. The definition of tangent vector given above works in more general settings. For valuation spaces of normal surface singularities, the tangent space can be constructed also without taking projective limits. This follows from the fact that the maps $j_{V,U}$ stabilize in the projective limit, meaning that there exists U_0 connected weakly open neighborhood of ν for which $j_{V,U}$ are bijections for every $V \subseteq U \subseteq U_0$. Such open U_0 can be constructed by taking any good resolution $\pi: X_\pi \rightarrow (X, x_0)$, and by letting U_0 be the connected component of $(\mathcal{V}_X \setminus \mathcal{S}_\pi^*) \cup \{\nu\}$ containing ν . Here \mathcal{S}_π^* denotes the set of divisorial valuations ν_E associated to exceptional primes E of X_π .

One can distinguish the type of valuation ν in \mathcal{V}_X (non quasimonomial, irrational and divisorial respectively) according to the number of elements in $T_\nu \mathcal{V}_X$ (one, two, or infinitely many respectively). In particular, for a divisorial valuation $\nu = \nu_E$, there is a natural 1-to-1 correspondence between points in E and tangent vectors in $T_\nu \mathcal{V}_X$. Explicitly, the point p corresponds to the tangent vector determined by the set of valuations $U_\pi(p)$ whose center in X_π is p .

2.5. Dual graphs and the structure of \mathcal{V}_X

We investigate here the relations between the dual graphs associated to good resolutions and the structure of valuation spaces. Let $\pi: X_\pi \rightarrow (X, x_0)$ be any log resolution of \mathfrak{m} . From (3), we know $Z_\pi(\mathfrak{m}) = -\sum b_E E$, where the sum is taken over all exceptional primes E of π and where $b_E = \text{div}_E(\mathfrak{m}) \geq 1$ is the minimal order of vanishing of any function in \mathfrak{m} along E . This positive integer is called the *generic multiplicity* of E . Recall that the dual graph Γ_π of π is defined abstractly as the graph whose vertices are the exceptional primes of π and with an edge connecting two distinct vertices E and F if and only if E and F intersect in X_π . For us, it will be more convenient to realize the dual graph explicitly as a subset $\Gamma_\pi \subset \mathcal{E}(\pi)_\mathbb{R}$ in the following manner. In this section, we will assume the good resolutions $\pi: X_\pi \rightarrow (X, x_0)$ to satisfy the stronger property of having any two exceptional primes intersecting in at most one point (see Remark 1.4).

Definition 2.25. The *dual graph* Γ_π of π is the subset of $\mathcal{E}(\pi)_\mathbb{R}$ consisting of the divisors $b_E^{-1}\check{E}$ for each of the exceptional primes E , as well as the straight line segment connecting $b_E^{-1}\check{E}$ and $b_F^{-1}\check{F}$ if E and F intersect in X_π .

Proposition 2.26. *The map $\text{ev}_\pi: \mathcal{V}_X \rightarrow \mathcal{E}(\pi)_\mathbb{R}$ which takes $\nu \in \mathcal{V}_X$ to its associated divisor $Z_\pi(\nu)$ is continuous and has Γ_π as its image. Moreover, there is a continuous embedding $\text{emb}_\pi: \Gamma_\pi \rightarrow \mathcal{V}_X$ such that $\text{ev}_\pi \circ \text{emb}_\pi$ is the identity on Γ_π .*

The notation ev_π and emb_π is chosen to agree with that of [41], where ev_π is called *evaluation*.

Proof. To see that ev_π is continuous it suffices to check that $Z_\pi(\nu) \cdot Z$ varies continuously with ν for all divisors Z in a basis $Z_1, \dots, Z_n \in \mathcal{E}(\pi)_\mathbb{R}$. By Proposition 1.12, we may choose such a basis consisting of divisors of the form $Z_i = Z_\pi(\mathfrak{a}_i)$ for \mathfrak{m} -primary ideals $\mathfrak{a}_i \subset R$. Applying Proposition 2.6, we have $Z_\pi(\nu) \cdot Z_i = -\nu(\mathfrak{a}_i)$, which varies continuously in ν by the definition of the weak topology. Therefore ev_π is continuous.

Let $\nu \in \mathcal{V}_X$ and let $\xi = \text{cen}_\pi(\nu)$. Assume first that ξ lies within a unique exceptional prime E of π . In this case we have seen $Z_\pi(\nu) = \lambda \check{E}$ for some $\lambda > 0$. However, because ν is normalized, we must have $1 = \nu(\mathfrak{m}) = -Z_\pi(\nu) \cdot Z_\pi(\mathfrak{m}) = \lambda b_E$, proving that $Z_\pi(\nu) = b_E^{-1}\check{E} \in \Gamma_\pi$. Assume next that ξ is the intersection point of two exceptional primes E and F of π . Then $Z_\pi(\nu) = r\check{E} + s\check{F}$ for some $r, s > 0$ satisfying the normalization condition $1 = \nu(\mathfrak{m}) = -Z_\pi(\nu) \cdot Z_\pi(\mathfrak{m}) = r b_E + s b_F$. This says exactly that $Z_\pi(\nu)$ lies on the straight line segment between $b_E^{-1}\check{E}$ and $b_F^{-1}\check{F}$, and thus lies within Γ_π . To complete the proof, we need only construct the embedding emb_π . This is done as follows. First, the vertex points $b_E^{-1}\check{E}$ in Γ_π are mapped by emb_π to the normalized divisorial valuation $b_E^{-1}\text{div}_E$. Then, the edge points

$r\check{E} + s\check{F}$ are mapped by emb_π to the monomial valuation $\nu_{r,s}$ at the point $E \cap F$ that was defined in §2.1. It is trivial to check that emb_π has the desired properties. \square

Observe that if π' is a good resolution dominating π , then $\text{ev}_\pi = (\eta_{\pi\pi'})_* \circ \text{ev}_{\pi'}$ by Corollary 2.7. In particular, $(\eta_{\pi\pi'})_*$ maps $\Gamma_{\pi'}$ onto Γ_π . In this way, the dual graphs Γ_π and the maps $(\eta_{\pi\pi'})_*$ form an inverse system, allowing us to speak of the inverse limit $\varprojlim \Gamma_\pi$. Notice that by construction $\varprojlim \Gamma_\pi$ naturally embeds in the set of (exceptional) \mathbb{R} -b-divisors $\mathcal{E}(X)$. Moreover, by the universal property of the inverse limit, the maps ev_π induce a continuous map $\text{ev}: \mathcal{V}_X \rightarrow \varprojlim \Gamma_\pi$. The structure of \mathcal{V}_X is then given by the following theorem.

Theorem 2.27. *The map $\text{ev}: \mathcal{V}_X \rightarrow \varprojlim \Gamma_\pi$ is a homeomorphism.*

The proof of this theorem can be found in [41, Theorem 7.9] for $X = \mathbb{C}^2$, but in fact the argument works equally well in our singular setting. As a consequence of the theorem, the map $r_\pi := \text{emb}_\pi \circ \text{ev}_\pi$ is a retraction of \mathcal{V}_X onto the *embedded dual graph* $\mathcal{S}_\pi := \text{emb}_\pi(\Gamma_\pi) \subset \mathcal{V}_X$. More difficult to see is that in fact \mathcal{V}_X deformation retracts onto \mathcal{S}_π (see [5, 63]) and thus in particular has the homotopy type of the finite graph $\mathcal{S}_\pi \cong \Gamma_\pi$.

Notice that the map ev coincides with the restriction of the map $\hat{e}\nu: \hat{\mathcal{V}}_X \rightarrow \mathcal{E}(X)$ considered at the end of §2.2 to the set of normalized semivaluations \mathcal{V}_X .

Remark 2.28. If π_0 is a minimal good resolution of (X, x_0) dominated by π , then, using the fact that π is obtained from π_0 by a composition of point blowups, one sees that Γ_π deformation retracts onto a subset homeomorphic to the dual graph of π_0 , and thus in fact \mathcal{V}_X has the homotopy type of the dual graph of any minimal good resolution. Note, any two minimal good resolutions of (X, x_0) have homeomorphic dual graphs, an easy consequence of [46, Theorem 5.12]. Moreover, minimal good resolutions are obtained from the unique good resolution in the sense of Laufer by a sequence of blowups of *satellite points* (see Remark 1.4). Hence the image through emb_π of the dual graph Γ_{π_0} of any minimal good resolution π_0 does not depend on π . We denote such image as \mathcal{S}_X , and refer to it as the *skeleton* of \mathcal{V}_X .

For any log resolution π of \mathfrak{m} , we make Γ_π into a *metric* graph by specifying the lengths of its edges: if E and F are distinct intersecting exceptional primes of π , the length of the edge from $b_E^{-1}\check{E}$ to $b_F^{-1}\check{F}$ is set to be $1/(b_E b_F)$. The metric has the very useful property that if π' is a good resolution dominating π , then the continuous map $\text{ev}_{\pi'} \circ \text{emb}_\pi$ is an isometric embedding $i_{\pi'\pi}: \Gamma_\pi \rightarrow \Gamma_{\pi'}$. One can prove this easily when π' is obtained from π by a point blowup; the general case then follows by induction. Also by induction one sees that the inclusions $i_{\pi'\pi}$ are compatible in that $\text{emb}_\pi = \text{emb}_{\pi'} \circ i_{\pi'\pi}$. It therefore makes sense to speak of a direct limit $\varinjlim \Gamma_\pi$ and an induced continuous map $\text{emb}: \varinjlim \Gamma_\pi \rightarrow \mathcal{V}_X$. Note that by definition a valuation $\nu \in \mathcal{V}_X$ is quasimonomial if and only if it lies in the image \mathcal{S}_π of emb_π for some π , so the image of $\text{emb}: \varinjlim \Gamma_\pi \rightarrow \mathcal{V}_X$ is precisely the set $\mathcal{V}_X^{\text{qm}}$ of quasimonomial valuations. It follows from this and Theorem 2.27 that quasimonomial valuations are dense in \mathcal{V}_X . In fact, one can show that all four types of semivaluations are dense in \mathcal{V}_X .

Theorem 2.29. *The map $\text{emb}: \varinjlim \Gamma_\pi \rightarrow \mathcal{V}_X^{\text{qm}}$ is a continuous bijection, but is not a homeomorphism.*

Of course, the map emb cannot be a homeomorphism, since $\varinjlim \Gamma_\pi$ is by construction a metric space, whereas the weak topology is not metrizable. However we may push forward the metric through emb and obtain a new, strictly stronger topology on the space of quasimonomial valuations.

Definition 2.30. Throughout this article, we denote by d the metric on $\mathcal{V}_X^{\text{qm}}$ constructed in this manner. Its induced topology will be called the *strong topology*.

Proposition 2.31. *The skewness $\alpha: \mathcal{V}_X \rightarrow (0, +\infty]$ and relative skewness $\beta: \mathcal{V}_X \times \mathcal{V}_X \rightarrow (0, +\infty]$ functions are lower semicontinuous. When α is restricted to $\mathcal{V}_X^{\text{qsm}}$ and β is restricted to $\mathcal{V}_X^{\text{qsm}} \times \mathcal{V}_X^{\text{qsm}}$, they are both continuous in the strong topology.*

Proof. For each log resolution π of \mathfrak{m} , the evaluation map ev_π is continuous, and hence so are the maps α_π and β_π defined in (4). Since α and β are the supremum of α_π and β_π , respectively, they are lower semicontinuous. The continuity in the strong topology on quasimonomial valuations is immediate from (8) for α and (9) for β . \square

2.6. Partial order, trees, and parameterizations

The space \mathcal{V}_X carries a natural partial ordering, namely where $\nu \geq \mu$ if and only if $\nu(\mathfrak{a}) \geq \mu(\mathfrak{a})$ for all \mathfrak{m} -primary ideals \mathfrak{a} of R . One can give an equivalent geometric interpretation of this partial order in the following manner.

Proposition 2.32. *For semivaluations $\nu, \mu \in \mathcal{V}_X$, one has $\nu \geq \mu$ if and only if the divisor $Z_\pi(\mu) - Z_\pi(\nu)$ is effective for each good resolution π of (X, x_0) .*

Proof. Suppose first that $Z_\pi(\mu) - Z_\pi(\nu)$ is effective for each π . Let \mathfrak{a} be any \mathfrak{m} -primary ideal of R , and let π be a log resolution of \mathfrak{a} . Then by Proposition 2.6 we see that $\nu(\mathfrak{a}) - \mu(\mathfrak{a}) = [Z_\pi(\mu) - Z_\pi(\nu)] \cdot Z_\pi(\mathfrak{a})$. Since $Z_\pi(\mathfrak{a})$ is relatively nef and $Z_\pi(\mu) - Z_\pi(\nu)$ is effective, this quantity is ≥ 0 , as desired.

Conversely, assume that $\nu \geq \mu$, and let π be any good resolution of (X, x_0) . Let E be an exceptional prime of π . Since relatively ample \mathbb{Q} -divisors are dense in $\text{Nef}(\pi)$, we can find a sequence of relatively ample divisors $D_n \in \mathcal{E}(\pi)_{\mathbb{Q}}$ such that $D_n \rightarrow \check{E}$ as $n \rightarrow \infty$. Let $k_n \geq 1$ be an integer for which $k_n D_n$ is an integral relatively very ample divisor, and thus such that $k_n D_n = Z_\pi(\mathfrak{a}_n)$ for some \mathfrak{m} -primary ideal \mathfrak{a}_n . We then have by Proposition 2.6 that

$$1 \geq \frac{\mu(\mathfrak{a}_n)}{\nu(\mathfrak{a}_n)} = \frac{Z_\pi(\mu) \cdot Z_\pi(\mathfrak{a}_n)}{Z_\pi(\nu) \cdot Z_\pi(\mathfrak{a}_n)} = \frac{Z_\pi(\mu) \cdot D_n}{Z_\pi(\nu) \cdot D_n} \rightarrow \frac{Z_\pi(\mu) \cdot \check{E}}{Z_\pi(\nu) \cdot \check{E}},$$

proving that $[Z_\pi(\mu) - Z_\pi(\nu)] \cdot \check{E} \geq 0$. As E was an arbitrary exceptional prime of π , we conclude that $Z_\pi(\mu) - Z_\pi(\nu)$ is effective. \square

Given this geometric characterization of the partial order, it makes sense to expect a tight relationship between the partial ordering and the behavior of the skewness and relative skewness functions α and β , which were defined in terms of the geometry of the dual divisors $Z_\pi(\nu)$. We investigate this interaction now.

Lemma 2.33. *Let $\nu, \mu \in \mathcal{V}_X$ be semivaluations. Then:*

1. *The relative skewness β satisfies $\beta(\mu | \nu) \geq 1$ with equality if and only if $\nu \geq \mu$.*
2. *If $\nu > \mu$, then $\alpha(\mu) = -Z_\pi(\nu) \cdot Z_\pi(\mu)$ for any good resolution $\pi: X_\pi \rightarrow (X, x_0)$ such that μ and ν have distinct centers in X_π .*
3. *The skewness function is monotonic in that $\alpha(\nu) > \alpha(\mu)$ if $\nu > \mu$.*

Proof. The first statement follows immediately from Proposition 2.19. Note, the inequality $\beta(\mu | \nu) \geq 1$ is using the assumption that μ and ν are normalized, so that $\mu(\mathfrak{m})/\nu(\mathfrak{m}) = 1$. The second statement is then a consequence of the first and Corollary 2.16. For the third, take a good resolution $\pi: X_\pi \rightarrow (X, x_0)$ such that ν and μ have different centers in X_π and apply Corollary 2.16 again to derive

$$\alpha(\nu) = \beta(\nu | \mu)[-Z_\pi(\nu) \cdot Z_\pi(\mu)] = \beta(\nu | \mu)\alpha(\mu) > \alpha(\mu),$$

completing the proof. \square

We focus our attention to skewness and relative skewness of normalized valuations which are monomial with respect to some fixed infinitely near point p . The general situation can be described as follows.

Proposition 2.34. *Let $\pi: X_\pi \rightarrow (X, x_0)$ be a log resolution of \mathfrak{m} , and suppose that E and F are two exceptional primes of π which intersect at a point $p \in X_\pi$. Let $w: [0, 1] \rightarrow [\nu_E, \nu_F]_p$ be the monomial parameterization at p , and let $\mu \in \mathcal{V}_X$ be any valuation of finite skewness.*

1. *The function $[0, 1] \ni t \mapsto \alpha(w(t))$ is a polynomial map of degree one or two.*
2. *The function $[0, 1] \ni t \mapsto \beta(w(t) \mid \mu)$ is a piecewise rational map of degree ≤ 2 .*
3. *The function $[0, 1] \ni t \mapsto \beta(\mu \mid w(t))$ is a piecewise rational map of degree ≤ 1 .*

Moreover, stationary points for these maps correspond to rational parameters.

Proof. Throughout the proof, we will let Δ denote the constant

$$\Delta := \frac{\check{E}^2}{b_E^2} - \frac{\check{E} \cdot \check{F}}{b_E b_F}.$$

Observe that $\Delta \leq 0$ with equality if and only if $\nu_E \leq \nu_F$; indeed, this is simply another way of writing the fact that $\beta(\nu_E \mid \nu_F) \geq 1$, with equality if and only if $\nu_E \leq \nu_F$. We now rewrite (8) and (9) for valuations in $[\nu_E, \nu_F]_p$ in terms of Δ :

$$\alpha(w(t)) = \left[\alpha(\nu_F) - \alpha(\nu_E) - 2\Delta - \frac{1}{b_E b_F} \right] t^2 + \left[2\Delta + \frac{1}{b_E b_F} \right] t + \alpha(\nu_E) \quad (10)$$

and

$$\beta(w(t) \mid w(s)) = \frac{\left[\alpha(\nu_F) - \alpha(\nu_E) - 2\Delta - \frac{1}{b_E b_F} \right] t^2 + \left[2\Delta + \frac{1}{b_E b_F} \right] t + \alpha(\nu_E)}{\left[\alpha(\nu_F) - \alpha(\nu_E) - 2\Delta - \frac{1}{b_E b_F} \right] s t + \left[\Delta s + \Delta t + \frac{\min\{t, s\}}{b_E b_F} \right] + \alpha(\nu_E)}. \quad (11)$$

The first assertion is a direct consequence of (10), while the last two assertions directly follow from (11) if $\mu = w(s)$ for some $s \in [0, 1]$.

If it is not the case, then up to taking another good resolution π' dominating π , we may assume that μ and $w(t)$ have distinct centers for any $t \in [0, 1]$. Then the result follows from [Corollary 2.16](#). \square

Once the general behavior of skewness and relative skewness is understood, we now investigate what happens when the monomial parameterization is monotonic. In this case, the situation is quite simpler, and more similar to what happens in the smooth setting.

Proposition 2.35. *Let $\pi: X_\pi \rightarrow (X, x_0)$ be a log resolution of \mathfrak{m} , and suppose that E and F are exceptional primes of π which intersect in a point p . Let $w: [0, 1] \rightarrow [\nu_E, \nu_F]_p$ be the monomial parameterization at p . The following are equivalent.*

1. *The map $w(t)$ is monotonic increasing with respect to the partial order on \mathcal{V}_X .*
2. *There exist $t_1, t_2 \in [0, 1]$ with $0 < t_1 < t_2$ such that $w(t_1) < w(t_2)$.*
3. *The function $\alpha(w(t))$ is affine linear and increasing with t .*
4. *The distance $d(\nu_E, \nu_F) := 1/b_E b_F$ is equal to $\alpha(\nu_F) - \alpha(\nu_E)$.*

Proof. We use here the same notations as in the proof of [Proposition 2.34](#). The implication (1 \Rightarrow 2) is trivial.

(2 \Rightarrow 3) Let $f : [t_1, 1] \rightarrow \mathbb{R}$ be the function $f(s) = \beta(w(t_1) \mid w(s))$. By (11), this a rational function of degree at most one. On the other hand, f takes the value 1 twice: $f(t_1) = 1$ by definition and $f(t_2) = 1$ since we have assumed $w(t_1) < w(t_2)$. Thus $f \equiv 1$, proving that $w(t_1) < w(s)$ for all $s > t_1$. Now let $g : [0, 1] \rightarrow \mathbb{R}$ be the function $g(t) = \beta(w(t) \mid \nu_F)$. By what we have just shown it takes the value 1 at two distinct t , namely at $t = t_1$ and $t = 1$. On the other hand, since t_1 lies in the interior of $[0, 1]$ and $g(t) \geq 1$ everywhere, it must be that $g'(t_1) = 0$. We have therefore found (with multiplicity) at least three solutions to the equation $g(t) = 1$. By (11), however, g is a rational function of degree at most 2, so this is possible only if $g \equiv 1$. This proves that $\nu_F \geq w(t)$ for all t . By [Lemma 2.33\(2\)](#),

$$\alpha(w(t)) = -Z_\pi(w(t)) \cdot Z_\pi(\nu_F) = -(1-t) \frac{\check{E} \cdot \check{F}}{b_E b_F} - t \frac{\check{F}^2}{b_F^2},$$

an affine linear function. Since α is monotonic by [Lemma 2.33\(3\)](#), necessarily $\alpha(w(t))$ is increasing.

(3 \Rightarrow 4) The assumption that $\alpha(w(t))$ is affine linear tells us that the quadratic term in (10) vanishes. Given this, it is clear we only need to prove that $\Delta = 0$. Again, we consider $g(t) := \beta(w(t) \mid \nu_F)$. From (11), we have $g(t) = \alpha(w(t))/h(t)$, where $h(t)$ is the nonnegative affine linear function

$$h(t) = \left[\Delta + \frac{1}{b_E b_F} \right] t + \alpha(\nu_E) + \Delta.$$

The second derivative of g is easily computed to be

$$g''(t) = \frac{-2\Delta\alpha(\nu_F) \left[\Delta + \frac{1}{b_E b_F} \right]}{h(t)^3} = \frac{-2\Delta\alpha(\nu_F) [\alpha(\nu_F) - \alpha(\nu_E) - \Delta]}{h(t)^3},$$

the equalities coming from the vanishing of the quadratic term of (10). Given the fact that $\Delta \leq 0$ and, by our assumption that α is increasing, $\alpha(\nu_F) > \alpha(\nu_E)$, we conclude that $g''(t) \geq 0$, with equality only if $\Delta = 0$. It therefore suffices to show that g cannot be a strictly convex function. Let π' be the good resolution of (X, x_0) obtained by blowing up the point p , and let G be the new exceptional prime thus obtained. Clearly for any \mathfrak{m} -primary ideal \mathfrak{a} of R , we have $\text{div}_G(\mathfrak{a}) \geq \text{div}_E(\mathfrak{a}) + \text{div}_F(\mathfrak{a})$. The normalized divisorial valuation $\nu_G = (b_E + b_F)^{-1} \text{div}_G$ associated to G is exactly $w(t_0)$ with $t_0 = b_F/(b_E + b_F)$, so we can rewrite this inequality as

$$w(t_0)(\mathfrak{a}) \geq (1-t_0)\nu_E(\mathfrak{a}) + t_0\nu_F(\mathfrak{a}).$$

Applying [Proposition 2.19](#), this proves that $g(t_0) \geq (1-t_0)g(0) + t_0g(1)$. Thus g is not strictly convex, completing the proof.

(4 \Rightarrow 1) We wish to prove that $\beta(w(t) \mid w(s)) = 1$ whenever $s \geq t$. Since statement 4 is assumed to hold, the function $\beta(w(t) \mid w(s))$ takes the simple form

$$\beta(w(t) \mid w(s)) = \frac{-2\Delta t^2 + 2\Delta t + \frac{t}{b_E b_F} + \alpha(\nu_E)}{-2\Delta s t + \Delta s + \Delta t + \frac{t}{b_E b_F} + \alpha(\nu_E)} \quad (12)$$

whenever $s \geq t$. The fact that this quantity must be ≥ 1 implies that necessarily

$$-2\Delta t^2 + 2\Delta t \geq -2\Delta s t + \Delta s + \Delta t,$$

or equivalently that $\Delta(2t-1)(s-t) \geq 0$ whenever $1 \geq s \geq t \geq 0$. Since $\Delta \leq 0$, this is only possible if $\Delta = 0$. But then from (12) we get $\beta(w(t) \mid w(s)) = 1$ whenever $s \geq t$, completing the proof. \square

Remark 2.36. One cannot remove the assumption in statement 2 of [Proposition 2.35](#) that $t_1 > 0$. One example of this is the cusp singularity $(X, 0)$ with $X = \{x^2 + y^3 + z^9 - xyz = 0\} \subset \mathbb{C}^3$, see [§9.6](#). It has a unique minimal good resolution π , with three rational exceptional primes E_1, E_2 , and E_3 that pairwise intersect in X_π . Their self-intersections are $E_1^2 = E_2^2 = -2$, and $E_3^2 = -3$, and the generic multiplicities of all three equal 1. The resolution π is not a log resolution of the maximal ideal \mathfrak{m}_X , but there is a unique base point, which is a *free* point in E_3 (see below). It follows that $\check{E}_3^2 = \check{E}_3 \cdot \check{F}_1 = -1$, and thus $\beta(\nu_{E_3} | \nu_{E_1}) = 1$, saying precisely that $\nu_{E_3} \leq \nu_{E_1}$. On the other hand, $\alpha(\nu_{E_1}) - \alpha(\nu_{E_3}) = 2/3 \neq 1 = d(\nu_{E_3}, \nu_{E_1})$, so the equivalent conditions of [Proposition 2.35](#) are not satisfied.

Recall that for each log resolution π of \mathfrak{m} , the map emb_π gives a homeomorphism of the dual graph $\Gamma_\pi \subset \mathcal{E}(\pi)_\mathbb{R}$ onto a subset $\mathcal{S}_\pi \subset \mathcal{V}_X$. Moreover, if $b_E^{-1}\check{E}$ and $b_F^{-1}\check{F}$ are adjacent vertices in Γ_π , then the map $w: [0, 1] \rightarrow \mathcal{V}_X$ given by $w(t) = \text{emb}_\pi((1-t)b_E^{-1}\check{E} + t b_F^{-1}\check{F})$ is precisely monomial parameterization at p . Thus w gives a continuous parameterization of the edge in \mathcal{S}_π between the adjacent vertices ν_E and ν_F .

Suppose now that π is a log resolution of \mathfrak{m} , and π' is the good resolution of (X, x_0) obtained from π by blowing up a single point $p \in X_\pi$. If p is a *satellite point* of π (that is, if p is the intersection of two distinct exceptional primes of π), then $\mathcal{S}_{\pi'} = \mathcal{S}_\pi$. If p is a *free point* of π (that is, if p lies on a unique exceptional prime E of π), then $\mathcal{S}_\pi \subsetneq \mathcal{S}_{\pi'}$; more precisely, $\mathcal{S}_{\pi'}$ is obtained from \mathcal{S}_π by adjoining the edge from $\nu_E \in \mathcal{S}_\pi$ to a new vertex ν_F , with F the exceptional divisor of $\eta_{\pi\pi'}$. We claim that the parameterization w of this new edge satisfies the equivalent conditions of [Proposition 2.35](#). Indeed, it is easy to see that $b_F = b_E$ and that in $\mathcal{E}(\pi')_\mathbb{R}$ one has $\check{F} = \check{E} - F$. It follows that

$$\alpha(\nu_F) = -\frac{\check{F}^2}{b_F^2} = -\frac{(\check{E} - F)^2}{b_E^2} = -\frac{\check{E}^2}{b_E^2} - \frac{1}{b_E^2} = \alpha(\nu_E) + \frac{1}{b_E^2},$$

which is exactly condition 4 of [Proposition 2.35](#). Since any good resolution π' of (X, x_0) dominating π is obtained from π by a composition of point blowups, the above analysis applied inductively gives that $\mathcal{S}_{\pi'}$ is obtained from \mathcal{S}_π by adjoining a collection of ordered trees. More precisely, the following holds.

Proposition 2.37. *Let π_0 be a fixed log resolution of \mathfrak{m} , and suppose $\hat{\nu} \in \mathcal{S}_{\pi_0}$ is a chosen point. Let π be any good resolution of (X, x_0) dominating π_0 , and let $\mathcal{S}_{\pi, \hat{\nu}} \subseteq \mathcal{S}_\pi$ denote the set of valuations ν whose retraction $r_{\pi_0}(\nu)$ onto \mathcal{S}_{π_0} is the point $\hat{\nu}$. Then $\mathcal{S}_{\pi, \hat{\nu}}$ is a rooted tree with respect to the partial order \geq , having root $\hat{\nu}$. Moreover, the skewness function α is a parameterization of $\mathcal{S}_{\pi, \hat{\nu}}$ inducing the metric d .*

Here the terms *rooted tree* and *parameterization* are in the sense of [\[41, §2\]](#). Explicitly, a rooted tree is a partially ordered set (\mathcal{T}, \leq) in which

1. there exists a unique minimal element \hat{x} , called the *root*,
2. any two elements $x, y \in \mathcal{T}$ have an infimum $x \wedge y \in \mathcal{T}$,
3. the set $\{x \in \mathcal{T} : x \leq y\}$ is order isomorphic to $[0, 1]$ when $y \neq \hat{x}$, and
4. any nonempty totally ordered subset of \mathcal{T} has a least upper bound in \mathcal{T} .

A *parameterization* of a rooted tree (\mathcal{T}, \leq) is a function $\sigma: \mathcal{T} \rightarrow \mathbb{R} \cup \{+\infty\}$ which is monotonically increasing and which maps intervals $[x, y] := \{z \in \mathcal{T} : x \leq z \leq y\}$ order isomorphically onto their images $[\sigma(x), \sigma(y)]$. Any such parameterization gives rise to a metric d_σ on the set of all $x \in \mathcal{T}$ with $\sigma(x) < +\infty$, namely the metric defined by $d_\sigma(x, y) = \sigma(x) + \sigma(y) - 2\sigma(x \wedge y)$.

With the same setup as [Proposition 2.37](#), let $\mathcal{V}_{X, \hat{\nu}}$ denote the collection of all semivaluations $\nu \in \mathcal{V}_X$ whose retraction $r_{\pi_0}(\nu)$ onto \mathcal{S}_{π_0} is the point $\hat{\nu}$. More or less immediately from [Theorem 2.27](#) we see that $\mathcal{V}_{X, \hat{\nu}} \cong \varprojlim \mathcal{S}_{\pi, \hat{\nu}}$. It is not hard to see that the rooted tree structure of the the $\mathcal{S}_{\pi, \hat{\nu}}$ induces in the limit a rooted tree structure on $\mathcal{V}_{X, \hat{\nu}}$.

Proposition 2.38. *The set $\mathcal{V}_{X, \hat{\nu}}$ is a rooted tree with respect to the partial order \geq , having root $\hat{\nu}$. The skewness function α is a parameterization of $\mathcal{V}_{X, \hat{\nu}}$ whose induced metric d_α agrees with d on the set $\mathcal{V}_{X, \hat{\nu}}^{\text{qm}}$ of quasimonomial valuations.*

Proposition 2.38 generalizes the fact that \mathcal{V}_X is a rooted tree in the smooth setting when $X = \mathbb{C}^2$, see [41, §7.6] or [26, §3.2]. Finally, we note that Proposition 2.38 gives a natural way of extending the metric d to the set of all finite skewness valuations of \mathcal{V}_X . Explicitly, if ν and μ are finite skewness valuations that lie in $\mathcal{V}_{X, \hat{\nu}}$ for some $\hat{\nu} \in \mathcal{S}_{\pi_0}$, then $d(\nu, \mu) := d_\alpha(\nu, \mu)$, whereas if ν and μ lie in $\mathcal{V}_{X, \hat{\nu}}$ and $\mathcal{V}_{X, \hat{\mu}}$ for distinct $\hat{\nu}$ and $\hat{\mu}$, then $d(\nu, \mu) := d_\alpha(\nu, \hat{\nu}) + d(\hat{\nu}, \hat{\mu}) + d_\alpha(\hat{\mu}, \mu)$. Henceforth, we will extend d in this way, and call the resulting topology the *strong topology* of finite skewness valuations.

2.7. The angular metric

The metric d constructed above has proved useful a number of times in previous works for studying the geometry of valuation spaces and developing potential theory on them, see [26, 29, 27, 28, 30, 11, 3]. For us, however, another metric, which we will call the *angular metric*, shall prove to be of greater use in our dynamical setting.

Definition 2.39. The *angular distance* $\rho(\nu, \mu)$ between two semivaluations $\nu, \mu \in \mathcal{V}_X$ is given by

$$\rho(\nu, \mu) := \log[\beta(\nu | \mu)\beta(\mu | \nu)]. \quad (13)$$

This quantity may of course be $+\infty$.

The term angular distance alludes to the fact that $\beta(\nu | \mu)\beta(\mu | \nu)$ is the limit over good resolutions π of $\beta_\pi(\nu | \mu)\beta_\pi(\mu | \nu)$, which in turn is a measure of the angle between the divisors $Z_\pi(\nu)$ and $Z_\pi(\mu)$ as vectors in $\mathcal{E}(\pi)_{\mathbb{R}}$, see (4).

Proposition 2.40. *The angular distance ρ gives a metric on the set of valuations of finite skewness in \mathcal{V}_X .*

Proof. If $\nu, \mu \in \mathcal{V}_X$ are both valuations of finite skewness, then $\rho(\nu, \mu)$ is finite by Corollary 2.16. Clearly ρ is symmetric. It is positive definite by Lemma 2.33(1), so we need only to prove the triangle inequality. If $\gamma \in \mathcal{V}_X$ is another valuation of finite skewness, then by Proposition 2.19

$$\begin{aligned} \beta(\nu | \mu)\beta(\mu | \nu) &= \sup_a \left(\frac{\nu(a)\gamma(a)}{\mu(a)\gamma(a)} \right) \times \sup_a \left(\frac{\mu(a)\gamma(a)}{\nu(a)\gamma(a)} \right) \\ &\leq \sup_a \left(\frac{\nu(a)}{\gamma(a)} \right) \times \sup_a \left(\frac{\gamma(a)}{\mu(a)} \right) \times \sup_a \left(\frac{\mu(a)}{\gamma(a)} \right) \times \sup_a \left(\frac{\gamma(a)}{\nu(a)} \right) \\ &= \beta(\nu | \gamma)\beta(\gamma | \mu)\beta(\mu | \gamma)\beta(\gamma | \nu). \end{aligned}$$

Upon taking logarithms, this gives $\rho(\nu, \mu) \leq \rho(\nu, \gamma) + \rho(\gamma, \mu)$, as desired. \square

Remark 2.41. In a recent work [33], the authors introduce a distance, called *determinant distance*, on the set \mathcal{S}_π^* of divisorial valuations realized in a given good resolution $\pi: X_\pi \rightarrow (X, x_0)$ of an *arborescent singularity*. An arborescent singularity is a normal surface singularity whose dual graph of any good resolution is a tree.

Up to a rescaling factor, their determinant distance coincide with the restriction of the angular distance on S_π^* . With this point of view, the angular distance can be seen as an inverse limit of the determinant distances along all good resolutions π of (X, x_0) .

We spend the remainder of the section comparing the two metrics d and ρ . We saw in §2.6 that the metric d is the induced by the skewness parameterization on subsets of the form $\mathcal{V}_{X, \hat{\nu}}$. As we will now see, the angular metric is also induced by a parameterization on these sets, namely the parameterization by $\log \alpha$.

Proposition 2.42. *Let π_0 be a fixed log resolution of \mathfrak{m} , and suppose $\hat{\nu} \in \mathcal{S}_{\pi_0}$ is a chosen point. Then $\log \alpha$ is a parameterization on the rooted tree $\mathcal{V}_{X, \hat{\nu}}$, that induces the angular metric on the set of finite skewness valuations.*

Proof. That $\log \alpha$ is a parameterization of $\mathcal{V}_{X, \hat{\nu}}$ is immediate from the fact that α is a parameterization of $\mathcal{V}_{X, \hat{\nu}}$. To see it induces the angular metric, we need only show that if $\nu, \mu \in \mathcal{V}_{X, \hat{\nu}}$ are such that $\mu < \nu$, then $\rho(\mu, \nu) = \log \alpha(\nu) - \log \alpha(\mu) = \log[\alpha(\nu)/\alpha(\mu)]$. Indeed, if π is a good resolution dominating π_0 in which the centers of ν and μ are distinct, we have by Lemma 2.33 that

$$\rho(\nu, \mu) = \log[\beta(\mu | \nu)\beta(\nu | \mu)] = \log \beta(\nu | \mu) = \log \frac{\alpha(\nu)}{-Z_\pi(\nu) \cdot Z_\pi(\mu)} = \log \frac{\alpha(\nu)}{\alpha(\mu)},$$

completing the proof. \square

It follows essentially immediately that when restricted to sets of the form $\mathcal{V}_{X, \hat{\nu}}$ the metrics d and ρ are equivalent in that they both induce the strong topology on finite skewness valuations. To compare the metrics on all of \mathcal{V}_X , we need the following lemma.

Lemma 2.43. *Suppose that $\mu, \nu \in \mathcal{V}_X$ are finite skewness valuations that lie in $\mathcal{V}_{X, \hat{\mu}}$ and $\mathcal{V}_{X, \hat{\nu}}$, respectively, where $\hat{\mu} \neq \hat{\nu}$. Then $\rho(\nu, \mu) = \rho(\nu, \hat{\nu}) + \rho(\hat{\nu}, \hat{\mu}) + \rho(\hat{\mu}, \mu)$.*

Proof. After possibly replacing π_0 with a good resolution obtained from π_0 by a sequence of satellite point blowups (which do not change \mathcal{S}_{π_0}), we may assume without loss of generality that the centers of $\hat{\nu}$ and $\hat{\mu}$ are distinct in X_{π_0} . By definition, $Z_{\pi_0}(\nu) = Z_{\pi_0}(\hat{\nu})$ and similarly $Z_{\pi_0}(\mu) = Z_{\pi_0}(\hat{\mu})$, so applying Lemma 2.33 again, we see

$$\beta(\nu | \mu)\beta(\mu | \nu) = \frac{\alpha(\nu)\alpha(\mu)}{(Z_{\pi_0}(\hat{\nu}) \cdot Z_{\pi_0}(\hat{\mu}))^2} = \frac{\alpha(\nu)\alpha(\mu)}{\alpha(\hat{\nu})\alpha(\hat{\mu})} \cdot \frac{\alpha(\hat{\nu})\alpha(\hat{\mu})}{(Z_{\pi_0}(\hat{\nu}) \cdot Z_{\pi_0}(\hat{\mu}))^2} = \frac{\alpha(\nu)\alpha(\mu)}{\alpha(\hat{\nu})\alpha(\hat{\mu})} \beta(\hat{\nu} | \hat{\mu})\beta(\hat{\mu} | \hat{\nu}).$$

We saw in Proposition 2.42 that $\rho(\nu, \hat{\nu}) = \log[\alpha(\nu)/\alpha(\hat{\nu})]$, and similarly for μ , so upon taking logarithms we get exactly the desired equality. \square

We saw in §2.6 that the analogous formula holds for d . Given that d and ρ are known to be equivalent on $\mathcal{V}_{X, \hat{\nu}}$ and $\mathcal{V}_{X, \hat{\mu}}$, it will therefore follow that d and ρ are equivalent on all of \mathcal{V}_X if we can show that d and ρ are equivalent on \mathcal{S}_{π_0} . The metric d induces the Euclidean topology on $\mathcal{S}_{\pi_0} \cong \Gamma_{\pi_0}$ by definition. The fact that ρ also induces the Euclidean topology on \mathcal{S}_{π_0} follows immediately from (9). We conclude that d and ρ both induce the strong topology on the set of finite skewness valuations of \mathcal{V}_X .

3. Log discrepancy, essential skeleta, and special singularities

In the proofs of our main theorems, the most important singularities one needs to consider turn out to be the *log canonical* singularities. In this section we review the classification of such singularities and discuss the related notion of log discrepancy for valuations. As usual, (X, x_0) denotes a normal surface singularity and (R, \mathfrak{m}) denotes the completed local ring $\hat{\mathcal{O}}_{X, x_0}$. We also fix a nontrivial holomorphic 2-form ω on (X, x_0) ; the vanishing of ω defines a Weil divisor on X we denote by $\text{Div}(\omega)$.

Given a good resolution $\pi: X_\pi \rightarrow (X, x_0)$ of (X, x_0) , the *relative canonical divisor* of π is the divisor $K_\pi \in \mathcal{E}(\pi)_\mathbb{Q}$ defined by the equality $\text{Div}(\pi^*\omega) = \pi^*\text{Div}(\omega) + K_\pi$, where here $\pi^*\text{Div}(\omega)$ refers to the Mumford pull-back of $\text{Div}(\omega)$, see [50, p. 195]. The relative canonical divisor K_π does not depend on the choice of ω . Indeed, the Mumford pull-back $\pi^*\text{Div}(\omega)$ is numerically trivial on $\mathcal{E}(\pi)$ by definition, so the adjunction formula for X_π gives

$$K_\pi \cdot E = 2g(E) - 2 - E^2 \quad (14)$$

for every exceptional prime E of π independently of ω , where here $g(E)$ denotes the genus of E . Since the intersection form on $\mathcal{E}(\pi)_\mathbb{R}$ is definite, this uniquely determines K_π independently of ω .

If E is an exceptional prime of a good resolution π , the *log discrepancy* of E is $A(E) := 1 + \text{ord}_E(K_\pi)$. More generally, one can define a log discrepancy function $A: \mathcal{V}_X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ on valuations which is uniquely characterized by the following properties:

1. If E_1 and E_2 are exceptional primes of a good resolution π intersecting in a point p , and if $\nu_{r,s}$ is the monomial valuation at p with weights r and s , then $A(\nu_{r,s}) = rA(E) + sA(F)$.
2. A is lower semicontinuous on \mathcal{V}_X^* .

Taking $r = 1$ and $s = 0$ in the first statement, we derive that $A(\text{div}_E) = A(E)$ for any exceptional prime E of a good resolution. Note also that A is homogeneous in the sense that $A(\lambda\nu) = \lambda A(\nu)$, and thus if ν_E is the normalized divisorial valuation $\nu_E = b_E^{-1}\text{div}_E$ we have $A(\nu_E) = b_E^{-1}A(E)$. For the details of the construction of A as well as some other properties, see [25]. Here we will call $A(\nu)$ the *log discrepancy* of ν , in agreement with [8, 41], but it should be noted that $A(\nu)$ has also been called the *thinness* of ν [26, 29, 25, 10] and the *weight* of ν [53, 54, 3].

Suppose that π is a log resolution of \mathfrak{m} , that E is an exceptional prime of π , and that π' is the good resolution obtained from π by blowing up a free point $p \in E$. If F is the exceptional locus of $\eta_{\pi\pi'}$, then it is easy to verify that $A(F) = A(E) + 1$ and $b_F = b_E$. Therefore, if ν_E and ν_F are the normalized divisorial valuations associated to E and F , we derive that $A(\nu_F) = A(\nu_E) + b_E^{-1}$. We showed in §2.6 that $\alpha(\nu_F) = \alpha(\nu_E) + b_E^{-2}$, so we see $A(\nu_F) - A(\nu_E) \geq \alpha(\nu_F) - \alpha(\nu_E)$. Applying this analysis inductively on point blowups, we obtain the following analogue of Proposition 2.38 for log discrepancy.

Proposition 3.1. *Let π_0 be a log resolution of \mathfrak{m} and let $\hat{\nu}$ be a chosen point of \mathcal{S}_{π_0} . Then the function A gives a parameterization of $\mathcal{V}_{X,\hat{\nu}}$. Moreover, if d_A denotes the metric induced by this parameterization, then $d_A(\nu, \mu) \geq d(\nu, \mu)$ for all $\nu, \mu \in \mathcal{V}_{X,\hat{\nu}}$. In particular, if $A(\nu) < +\infty$, then $\alpha(\nu) < +\infty$ as well.*

Note, it is possible for $\alpha(\nu) < +\infty$ while $A(\nu) = +\infty$, so the metrics d and d_A are not equivalent at the ends of the tree $\mathcal{V}_{X,\hat{\nu}}$. On the other hand, for any good resolution $\pi \geq \pi_0$, the parameterizations α and A both induce the Euclidean topology on the finite tree $\mathcal{S}_{\pi,\hat{\nu}} := \mathcal{S}_\pi \cap \mathcal{V}_{X,\hat{\nu}}$, and thus α and A both induce the strong topology on quasimonomial valuations $\mathcal{V}_{X,\hat{\nu}}^{\text{qm}} \cong \varinjlim \mathcal{S}_{\pi,\hat{\nu}}$.

3.1. Log canonical and log terminal singularities

Because A is lower semicontinuous and \mathcal{V}_X is compact, the log discrepancy function A necessarily takes a minimum value on \mathcal{V}_X , which is called the *log canonical threshold* of \mathfrak{m} . If this minimum is nonnegative, then (X, x_0) is said to be *log canonical* (lc); if it is strictly positive, then (X, x_0) is *log terminal* (lt). Log canonical and log terminal singularities feature prominently in the minimal model program, and thus have been studied intensively. They are relatively mild singularities, and in fact in our two-dimensional setting they are completely classified. The details of this classification, due originally to Kawamata [43, §10], are discussed in depth in [50, Ch. 4], see also [45, §3] or [44, Ch. 3] for an alternative approach due to Alexeev. We provide now a rough summary of this classification sufficient for our purposes.

Definition 3.2. We say (X, x_0) is a *finite quotient* of another irreducible normal surface germ (Y, y_0) if there is a finite group G acting on (Y, y_0) by analytic automorphisms, the action being free on $Y \setminus \{y_0\}$, such that $(X, x_0) \cong (Y, y_0)/G$.

If (X, x_0) is a finite quotient of a normal surface germ (Y, y_0) , then (X, x_0) is lc (resp. lt) if and only if (Y, y_0) is lc (resp. lt). This can be seen, for instance, by applying the Jacobian formula (see §4.5) to the quotient map $(Y, y_0) \rightarrow (X, x_0)$.

Theorem 3.3. *Every log canonical normal surface germ (X, x_0) is a finite quotient $(Y, y_0)/G$, where (Y, y_0) is either nonsingular, a cusp singularity, or a simple elliptic singularity. Moreover, (X, x_0) is log terminal if and only if (Y, y_0) is nonsingular.*

Cusp and simple elliptic singularities are defined in terms of the geometry of the exceptional loci of their minimal good resolutions π_0 . We say (X, x_0) is a *cusp singularity* if the exceptional primes of π_0 are $n \geq 2$ rational curves E_1, \dots, E_n which intersect in a *cycle*, that is, for which up to re-indexing one has $E_i \cdot E_j = 1$ if and only if $i - j \equiv \pm 1 \pmod{n}$. We say (X, x_0) is a *simple elliptic singularity* if the exceptional locus of π_0 consists of a single genus one curve E . In particular, the dual graph Γ_{π_0} of π_0 is homeomorphic to a circle for cusp singularities and homeomorphic to a point for simple elliptic singularities.

Theorem 3.3 gives that the class of lt surface singularities (X, x_0) coincides with the class of *quotient surface singularities*, that is, finite quotients $(\mathbb{C}^2, 0)/G$. The group G can always be taken to be a subgroup of $GL_2(\mathbb{C})$ acting linearly on $(\mathbb{C}^2, 0)$, so one can classify quotient singularities by understanding the finite subgroups of $GL_2(\mathbb{C})$, see [12]. Of particular interest to us are the *cyclic quotient singularities*, where G is a cyclic group; these are sometimes called *Hirzebruch-Jung quotient singularities*. It can be shown that (X, x_0) is a cyclic quotient singularity if and only if the exceptional primes of the (unique) minimal good resolution π_0 of (X, x_0) are $n \geq 1$ rational curves E_1, \dots, E_n intersecting in a *chain*, that is, for which up to re-indexing one has $E_i \cdot E_j = 1$ if and only if $|i - j| = 1$. In particular, the dual graph Γ_{π_0} for a cyclic quotient singularity is homeomorphic to either a point or an interval. If (X, x_0) is a non-cyclic quotient singularity, then its (unique) minimal good resolution π_0 again only has rational exceptional primes, but now Γ_{π_0} is homeomorphic to a tree with one degree-three *fork*.

The remaining lc singularities (X, x_0) are quotients $(Y, y_0)/G$ of cusps or simple elliptic singularities. Suppose that (X, x_0) is a quotient of a cusp. It may be that (X, x_0) is again a cusp, but if it is not, then necessarily $G \cong \mathbb{Z}/2\mathbb{Z}$, the (unique) minimal good resolution π_0 of (X, x_0) has only rational exceptional primes, and Γ_{π_0} is homeomorphic to a tree with two forks, both of which have degree three. These singularities are usually referred to as *quotient-cusp* singularities.

If (X, x_0) is a non-trivial quotient of a simple elliptic singularity, then necessarily $G \cong \mathbb{Z}/n\mathbb{Z}$ with $n \in \{2, 3, 4, 6\}$. In all of these cases, the (unique) minimal good resolution π_0 of (X, x_0) has only rational exceptional primes and Γ_{π_0} is homeomorphic to a tree with one fork. If $n = 2$, this fork has degree four, while if $n \in \{3, 4, 6\}$ this fork has degree three.

We have now determined the possible homeomorphism types of all the dual graphs Γ_{π_0} of minimal good resolutions π_0 of lc singularities (X, x_0) . If π is a log resolution of \mathfrak{m} dominating π_0 , then emb_π maps Γ_{π_0} homeomorphically onto the skeleton $\mathcal{S}_X = \mathcal{S}_{\pi_0} \subset \mathcal{V}_X$. The log discrepancy A will be strictly positive off of \mathcal{S}_X , but if (X, x_0) is not lt then A must vanish somewhere on \mathcal{S}_X . In [Figure 1](#) we have drawn all possible homeomorphism types of \mathcal{S}_X for lc singularities (X, x_0) , and marked exactly where the log discrepancy function vanishes. Notice that \mathcal{S}_X is not contractible only when (X, x_0) is a cusp singularity.

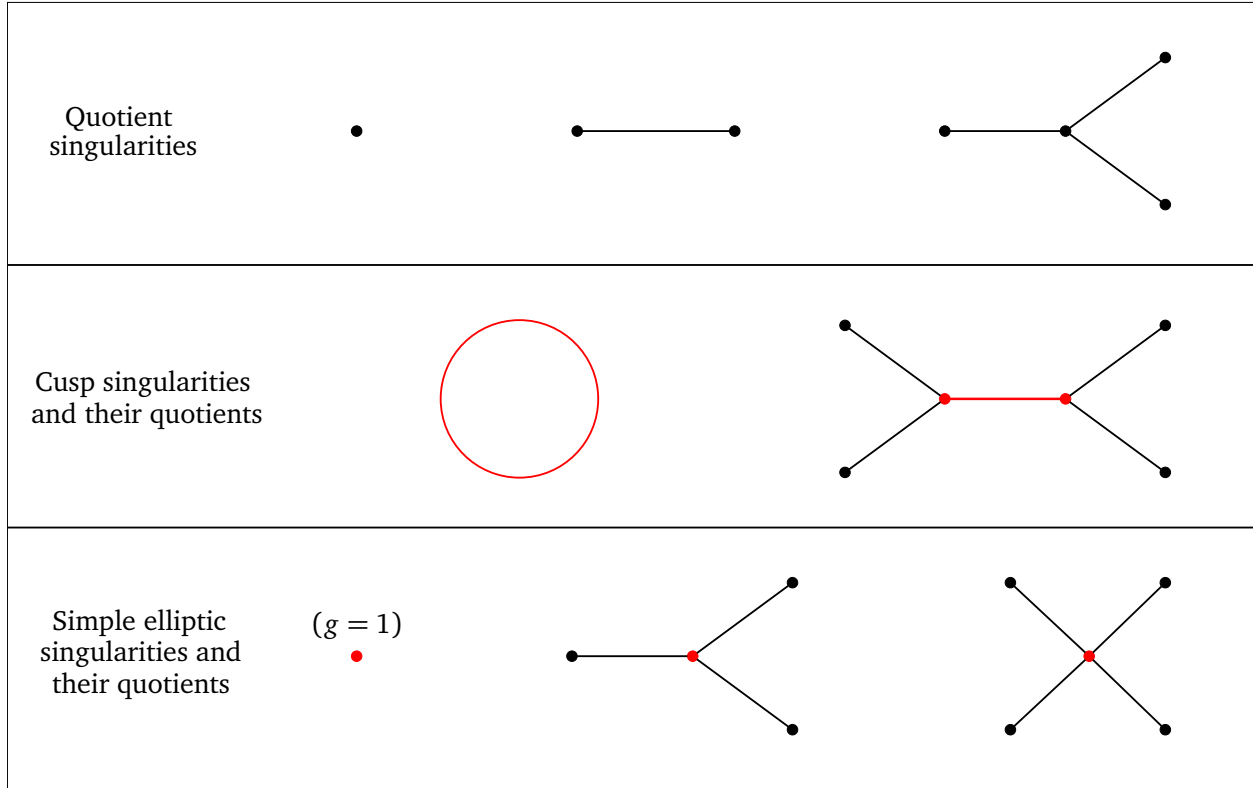


Figure 1: The possible skeleta $\mathcal{S}_X \subset \mathcal{V}_X$ of lc singularities (X, x_0) . Here dots are drawn only at forks and endpoints of \mathcal{S}_X , and a point \mathcal{S}_X is red if and only if $A(\nu) = 0$.

3.2. The essential skeleton

Now suppose that (X, x_0) is any irreducible normal surface germ, and let π_0 be a minimal good resolution of (X, x_0) . Within its dual graph Γ_{π_0} , let $\Gamma_X \subseteq \Gamma_{\pi_0}$ denote the smallest connected subgraph that contains all cycles, all forks, and all vertices corresponding to exceptional primes E of genus $g(E) > 0$. If π is any log resolution of \mathfrak{m} which dominates π_0 , let $\mathcal{S}_{\hat{X}} \subset \mathcal{V}_X$ denote the homeomorphic image of Γ_X under the embedding emb_{π} ; this set is independent of both the choice of π_0 and the choice of π .

Definition 3.4. The subgraph $\mathcal{S}_{\hat{X}} \subset \mathcal{V}_X$ is called the *essential skeleton* of (X, x_0) .

An illustration of the essential skeleton is given in [Figure 2](#). We make a couple of easy observations. First, the essential skeleton $\mathcal{S}_{\hat{X}}$ may be empty, but this happens if and only if (X, x_0) is nonsingular or a cyclic quotient singularity. Second, the connected components of $\mathcal{S}_X \setminus \mathcal{S}_{\hat{X}}$ are always intervals, and the vertices appearing in any such interval correspond to chains of rational curves in X_{π_0} . In fact, one should think of $\mathcal{S}_{\hat{X}}$ as the (embedded) dual graph of the space \hat{X} obtained from X_{π_0} by contracting the chains of rational curves corresponding to the components of $\mathcal{S}_X \setminus \mathcal{S}_{\hat{X}}$. This space \hat{X} is a modification of (X, x_0) which has only cyclic quotient singularities (one for each of the contracted chains), and, assuming $\mathcal{S}_{\hat{X}}$ is nonempty, each of these singular points will lie on exactly one exceptional prime of \hat{X} .

Remark 3.5. It would be more natural to modify the definition of the essential skeleton $\mathcal{S}_{\hat{X}}$ and the space \hat{X} in the following manner. We let $\pi: \hat{X} \rightarrow (X, x_0)$ be the minimal modification of (X, x_0) for which the pair (\hat{X}, B) is divisorial log terminal (dlt), where here B is the reduced divisor supported on the exceptional locus of π . We then set $\mathcal{S}_{\hat{X}}$ to be the dual graph of π embedded into \mathcal{V}_X in the usual

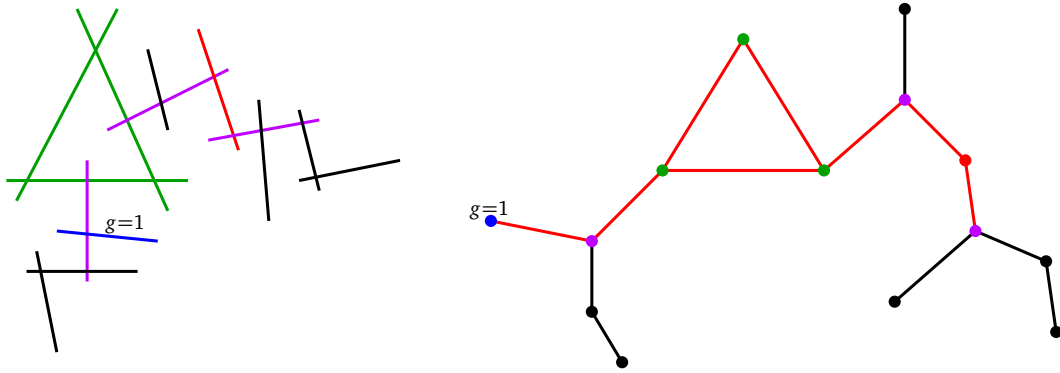


Figure 2: On the left is the exceptional locus of some minimal good resolution π_0 . On the right is the corresponding dual graph Γ_{π_0} . The edges marked in red are the edges corresponding to the essential skeleton $\mathcal{S}_{\hat{X}}$.

manner. This agrees with the previous definition of \hat{X} and $\mathcal{S}_{\hat{X}}$ in all but one case, namely when (X, x_0) is a non-cyclic quotient singularity; in this case, by the previous definition $\mathcal{S}_{\hat{X}}$ would consist of a single point but with this new definition $\hat{X} = X$ and $\mathcal{S}_{\hat{X}}$ is empty. While this new definition is more natural, we will stick to the previous definition simply to avoid a discussion of log discrepancies for pairs and dlt models, of which we will have no need.

The main result of the section is the following.

Theorem 3.6. *If (X, x_0) is not log terminal, then $A(v) \leq 0$ for all $v \in \mathcal{S}_{\hat{X}}$.*

This theorem was proved by Veys in [65]. Note, if we used the definition of $\mathcal{S}_{\hat{X}}$ in Remark 3.5, then the theorem holds without the assumption that (X, x_0) is not lt, because in this case $\mathcal{S}_{\hat{X}}$ is empty for lt singularities. We give a brief outline of the proof below. The key tool is the adjunction formula for the space \hat{X} . Since \hat{X} has singular points, the usual adjunction formula is invalid, and one has to modify it by adding certain correction terms for each of the singular points p_1, \dots, p_r of \hat{X} . Recall that each p_i is a cyclic quotient singularity, say $(\hat{X}, p_i) \cong (\mathbb{C}^2, 0)/G_i$ where G_i is a cyclic group. The adjunction formula for \hat{X} is the following. For each exceptional prime E of \hat{X} ,

$$K_{\hat{X}} \cdot E = 2g(E) - 2 - E^2 + \sum_{p_i \in E} \left(1 - \frac{1}{|G_i|}\right) \deg_E[p_i], \quad (15)$$

see [44, Thm. 3.36] for a proof.

Proof sketch for Theorem 3.6. It suffices to check that $A(E) \leq 0$ for every exceptional prime E of \hat{X} . Let E_1, \dots, E_n be the exceptional primes of \hat{X} , and let a be the column vector $a = (A(E_1), \dots, A(E_n))^t$. If M is the intersection matrix $M = (E_i \cdot E_j)_{ij}$, then the adjunction formula (15) says exactly that $Ma = b$, where $b = (b_1, \dots, b_n)^t$ is the column vector with entries

$$b_j = 2g(E_j) - 2 + d_j + \sum_{p_i \in E_j} \left(1 - \frac{1}{|G_i|}\right) \deg_{E_j}[p_i],$$

with d_j being the degree of the vertex in $\mathcal{S}_{\hat{X}}$ corresponding to E_j . Note, each term in the expression for b_j is nonnegative except for the term -2 ; thus if either $g(E_j) \geq 1$ or $d_j \geq 2$, then necessarily $b_j \geq 0$.

Assume then that $g(E_j) = 0$ and $d_j \in \{0, 1\}$. The case when $d_j = 0$ is special, since this says that E_j is the only exceptional prime of \hat{X} ; in this case the theorem is proved if $A(E_j) \leq 0$, and if $A(E_j) > 0$ then in fact $A(E) > 0$ for each of the contracted primes E of π_0 , so that (X, x_0) is log terminal. We may assume therefore that $d_j = 1$. Then by the definition of $\mathcal{S}_{\hat{X}}$, there must be at least two singular points p_i lying within E_j . Since $2g(E_j) - 2 + d_j = -1$ and for each of the (at least two) singular points $p_i \in E_j$ the term $(1 - |G_i|^{-1}) \deg_{E_j}[p_i]$ is at least $1/2$, we conclude that again $b_j \geq 0$. To sum up, this proves that the column vector b has nonnegative entries in all cases except for if (X, x_0) is log terminal, when it necessarily consists of a single negative entry. Note, the inverse M^{-1} has strictly negative entries by [Lemma 1.8](#), so unless (X, x_0) is lt, $a = M^{-1}b$ must have nonpositive entries, completing the proof. \square

It is worth pointing out that the converse is false: it is not generally true that if $A(\nu) \leq 0$ then $\nu \in \mathcal{S}_{\hat{X}}$. For example, let (X, x_0) be a cone singularity over a hyperbolic Riemann surface E . This means that (X, x_0) is obtained from the total space $X(L)$ of a line bundle L of degree ≤ -1 over E by contracting the zero section of L . The minimal resolution of (X, x_0) is then $X(L)$, and the essential skeleton $\mathcal{S}_{\hat{X}}$ consists only of the divisorial point ν_E corresponding to the zero section of L . As the log discrepancy $A(\nu_E)$ is strictly negative and A is continuous on quasimonomial valuations, there is a whole neighborhood of ν_E in the strong topology where A is strictly negative.

4. Dynamics on valuation spaces

4.1. Induced maps on valuation spaces

Suppose now that (X, x_0) and (Y, y_0) are two normal surface singularities and $f : (X, x_0) \rightarrow (Y, y_0)$ is a dominant holomorphic map between them. In this section, we will define an associated continuous map $f_\bullet : \mathcal{V}_X \rightarrow \mathcal{V}_Y$, and discuss how this map f_\bullet interacts with the various constructions from [§2](#) and [§3](#). Note, in the dynamical setting when $(Y, y_0) = (X, x_0)$, we obtain a dynamical system $f_\bullet : \mathcal{V}_X \rightarrow \mathcal{V}_X$, which is the focus of our [Theorem B](#). In this section we will state a precise version of this theorem, while the next two sections are devoted to its proof. Let (R_X, \mathfrak{m}_X) and (R_Y, \mathfrak{m}_Y) denote the completed local rings $\hat{\mathcal{O}}_{X, x_0}$ and $\hat{\mathcal{O}}_{Y, y_0}$, respectively, and let $f^* : R_Y \rightarrow R_X$ be the induced local ring homomorphism.

Definition 4.1. The holomorphic map f is said to be *finite* if it is finite-to-one in a neighborhood of x_0 .

There are many other equivalent characterizations of finiteness. Algebraically, f is finite if and only if $f^*\mathfrak{m}_Y$ is an \mathfrak{m}_X -primary ideal. Geometrically, f is finite if and only if it has no *contracted curves*, that is, no formal curve germs $(C, x_0) \subset (X, x_0)$ for which $f(C) = \{y_0\}$. Note, any contracted curve (C, x_0) would necessarily have to be a holomorphic curve instead of simply a formal curve, since it would be critical for f , see [Remark 4.34](#). Therefore f is finite if and only if it contracts no holomorphic curve C through $\{x_0\}$ to a point. Observe that there are at most finitely many contracted curve semivaluations, since $f^*\mathfrak{m}_Y$ is finitely generated. In the following, we denote by Cc_f the set of irreducible curves C with $f(C) = y_0$, and by \mathcal{C}_f the associated set of contracted curve valuations ν_C .

Associated to f is a map $f_* : \hat{\mathcal{V}}_X \rightarrow \hat{\mathcal{V}}_Y$ of valuation spaces, defined by $f_*\nu := \nu \circ f^*$. This map f_* is continuous with respect to the weak topologies on $\hat{\mathcal{V}}_X$ and $\hat{\mathcal{V}}_Y$ and commutes with scalar multiplication of semivaluations. If $\nu \in \hat{\mathcal{V}}_X^*$ is a finite semivaluation then $f_*\nu$ will again be finite, except in exactly one situation: if ν is a curve semivaluation associated to a curve germ $(C, x_0) \subset (X, x_0)$ which is contracted by f , then $f_*\nu$ will not be finite. In particular, f is finite if and only if f_* maps finite semivaluations to finite semivaluations.

Definition 4.2. Let $\nu \in \mathcal{V}_X$ be a semivaluation. The *attraction rate* $c(f, \nu)$ of f along ν is the quantity $c(f, \nu) := (f_*\nu)(\mathfrak{m}_Y) = \nu(f^*\mathfrak{m}_Y)$.

By definition $c(f, \nu) < +\infty$ if and only if $f_*\nu$ is finite, that is, if and only if ν is not a contracted curve semivaluation. Assuming $c(f, \nu) < +\infty$, the semivaluation $f_\bullet \nu := c(f, \nu)^{-1} f_* \nu \in \mathcal{V}_Y$ is normalized. In this way we obtain a map $f_\bullet : \mathcal{V}_X \dashrightarrow \mathcal{V}_Y$, defined and continuous away from the finite set of contracted curve semivaluations. As the next proposition shows, f_\bullet can be extended naturally to all of \mathcal{V}_X .

Proposition 4.3. *There is a unique continuous extension $f_\bullet : \mathcal{V}_X \rightarrow \mathcal{V}_Y$ of f_\bullet to all of \mathcal{V}_X .*

Proof. Suppose that $(C, x_0) \subset (X, x_0)$ is an irreducible curve germ which is contracted by f , and let ν_C be the associated normalized curve semivaluation. Let $\mathfrak{p} \subset R_X$ be the prime ideal of C . Since (X, x_0) is normal, the local ring $\mathcal{O}_{X,C}$ is a DVR, and thus we get a valuation ord_C on R_X associated to C , measuring the order of vanishing of functions along C . While ord_C is not centered at x_0 and thus does not belong to \mathcal{V}_X , its image $f_*\text{ord}_C := \text{ord}_C \circ f^*$ is centered at y_0 because C is contracted by f . The rational rank and transcendence degree of $f_*\text{ord}_C$ are both 1, so $f_*\text{ord}_C$ is a divisorial valuation λdiv_E associated to some exceptional prime E of some good resolution of (Y, y_0) . We define $f_\bullet \nu_C$ to be the normalized divisorial valuation $\nu_E = b_E^{-1} \text{div}_E$. To check that f_\bullet defined in this way is continuous at ν_C , take any sequence $\{\nu_i\}$ of semivaluations $\nu_i \in \mathcal{V}_X \setminus \{\nu_C\}$ converging to ν_C . Then, since $\nu_i(\mathfrak{p}) \rightarrow \nu_C(\mathfrak{p}) = +\infty$, we see that the sequence $\nu_i/\nu_i(\mathfrak{p})$ converges to ord_C . Because f_* is continuous, it follows that

$$\frac{c(f, \nu_i)}{\nu_i(\mathfrak{p})} f_\bullet \nu_i = f_* \left(\frac{\nu_i}{\nu_i(\mathfrak{p})} \right) \rightarrow f_* \text{ord}_C = \lambda b_E \nu_E.$$

Since evaluating semivaluations on ideals is weakly continuous, evaluating the above limit on \mathfrak{m}_Y yields that $c(f, \nu_i)/\nu_i(\mathfrak{p}) \rightarrow \lambda b_E > 0$. Because scalar multiplication is continuous, we conclude that

$$f_\bullet \nu_i = \frac{\nu_i(\mathfrak{p})}{c(f, \nu_i)} \times \frac{c(f, \nu_i)}{\nu_i(\mathfrak{p})} f_\bullet \nu_i \rightarrow \frac{1}{\lambda b_E} \times \lambda b_E \nu_E = \nu_E,$$

completing the proof. □

In the next proposition, we collect some easy facts about the map f_\bullet and the attraction rates $c(f, \nu)$. Each of them is proved in [25] or in [29] in the case when $(X, x_0) = (Y, y_0) = (\mathbb{C}^2, 0)$, but these proofs are very general and apply equally in our setting; the first three are essentially translations of standard results in valuation theory about how valuations on a field K extend to valuations on a finite extension L of K , and the last two are just a matter of unraveling definitions.

Proposition 4.4. *The following hold for any dominant holomorphic germ $f : (X, x_0) \rightarrow (Y, y_0)$.*

1. *If $\nu \in \mathcal{V}_X$ is not a contracted curve semivaluation, then $f_\bullet \nu$ is of the same type (divisorial, irrational, infinitely singular, curve) as ν . If ν is a contracted curve semivaluation, then $f_\bullet \nu$ is divisorial.*
2. *Every $\nu \in \mathcal{V}_Y$ has at most N preimages under f_\bullet , where $N = [\text{Frac}(R_X) : f^* \text{Frac}(R_Y)]$ is the degree of the field extension $\text{Frac}(R_X)/f^* \text{Frac}(R_Y)$.*
3. *If f is finite, then f_\bullet is surjective.*
4. *The inequality $c(f, \nu) \geq 1$ holds for all $\nu \in \mathcal{V}_X$.*
5. *Attraction rates are multiplicative in the sense that if $g : (Y, y_0) \rightarrow (Z, z_0)$ is another dominant holomorphic germ between normal surface singularities, then $c(g \circ f, \nu) = c(g, f_\bullet \nu) c(f, \nu)$.*

As a special case of the first statement, a divisorial valuation $\nu_E \in \mathcal{V}_X$ necessarily maps to a divisorial valuation $\nu_{E'} \in \mathcal{V}_Y$, while a curve semivaluation $\nu_C \in \mathcal{V}_X$ maps to a curve semivaluation $\nu_{C'} \in \mathcal{V}_Y$, unless $f(C) = y_0$, and in this case it is mapped to a divisorial valuation $\nu_G \in \mathcal{V}_Y$.

In the following, we will often consider modifications (or good resolutions) realizing a finite set of divisorial valuations, and/or giving embedded resolutions of some curves, or with other features regarding the dynamics. We introduce here some notation to avoid long statements afterwards.

Definition 4.5. Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ, and let $V \subset \mathcal{V}_X$ be a finite set of divisorial or curve (semi-)valuations. We say that two modifications $\pi : X_\pi \rightarrow (X, x_0)$ and $\varpi : Y_\varpi \rightarrow (Y, y_0)$ give a *resolution of f with respect to V* if the following conditions are satisfied :

- For any divisorial valuation $\nu_E \in V$, ν_E is realized by π , and $f_\bullet \nu_E$ is realized by ϖ .
- For any curve valuation $\nu_C \in V$ non contracted by f , π is an embedded resolution of $(C, x_0) \subset (X, x_0)$ and ϖ is an embedded resolution of $(f(C), y_0) \subset (Y, y_0)$.
- For any contracted curve valuation $\nu_C \in V$, π is an embedded resolution of $(C, x_0) \subset (X, x_0)$ and $f_\bullet \nu_C$ is realized by ϖ .
- The lift $\tilde{f} = \varpi^{-1} \circ f \circ \pi : X_\pi \rightarrow Y_\varpi$ is regular.

Notice that such modifications always exists, and can be taken dominating any two given modifications. In fact, since V is finite, we can easily find good resolutions π and ϖ satisfying the first three conditions. By taking an opportune good resolution π' dominating π , we can assure that the lift \tilde{f} is also regular.

We now come back to the geometric interpretation of the action of f_\bullet on divisorial and curve valuations. Consider two good resolutions $\pi : X_\pi \rightarrow (X, x_0)$ and $\varpi : Y_\varpi \rightarrow (Y, y_0)$ giving a resolution of f with respect to the empty set (meaning that we just ask the lift $\tilde{f} = \varpi^{-1} \circ f \circ \pi$ to be regular). Let D be a prime divisor in X_π (it could be an exceptional prime E , or the strict transform C_π of an irreducible curve (C, x_0)), and D' be a prime divisor in Y_ϖ . We denote by $k_D \geq 0$ the coefficient of D in the divisor $\tilde{f}^* D'$. Notice that $k_D \geq 1$ if and only if $\tilde{f}(D) \subset D'$. The integer k_D can also be characterized in the language of valuation theory as the ramification index $e(\text{div}_D / f_* \text{div}_D)$, i.e., the index of the value group of $f_* \text{div}_D$ within the value group of div_D . Note that k_D does not depend on the choice of the good resolutions π and ϖ , since it is a quantity associated to the valuation associated to D . If we have to keep track of the prime divisor D' , we will write $k_{D \rightarrow D'}$ or $k_{D \rightarrow D'}^{\tilde{f}}$ instead of k_D .

Analogously, if $f(D) = D'$, we denote by $e_D = e_{D \rightarrow D'} = e_{D \rightarrow D'}^{\tilde{f}}$ the topological degree of the map $\tilde{f}|_D : D \rightarrow D'$.

Remark 4.6. Notice that the values $k_{D \rightarrow D'}$ and $e_{D \rightarrow D'}$ do not depend on the good resolutions π and ϖ chosen (as far as they realize the valuations ν_E and $\nu_{E'}$). In fact they are quantities associated to the divisorial or curve semivaluations associated more than to the primes D and D' themselves.

Notice that the values k_D and e_D do not depend on the choice of π and ϖ satisfying the above conditions.

We can now state the geometrical interpretation of the action of f_\bullet and of the attraction rate $c(f, \nu)$. Some of these results have been stated and proved in [29] in the smooth setting, but the proofs easily generalize to the singular setting. We start with divisorial valuations.

Proposition 4.7 ([29, Prop. 2.5]). *Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ, $\nu_E \in \mathcal{V}_X$ a divisorial valuation, and $\nu_{E'} = f_\bullet \nu_E$. Then for any modifications $\pi : X_\pi \rightarrow (X, x_0)$ and $\varpi : Y_\varpi \rightarrow (Y, y_0)$ realizing ν_E and $\nu_{E'}$ respectively, the lift $\tilde{f} = \varpi^{-1} \circ f \circ \pi$ satisfies $\tilde{f}|_E(E) = E'$. Moreover*

$$c(f, \nu_E) = \frac{b_{E'}}{b_E} k_{E \rightarrow E'}, \quad (16)$$

This geometrical interpretation of the image of divisorial valuations allows to describe tangent maps on valuation spaces. In fact, in the situation of [Proposition 4.7](#), we may define the map $df_\bullet : T_{\nu_E} \mathcal{V}_X \rightarrow T_{\nu_{E'}} \mathcal{V}_Y$ by sending the tangent vector $\vec{v}_p \in T_{\nu_E} \mathcal{V}_X$ corresponding to a point $p \in E$, to the tangent vector

$df_{\bullet} \vec{v}_p \in T_{v_{E'}} \mathcal{V}_Y$ corresponding to the point $q = \tilde{f}|_E(p)$. Notice that if π and ϖ give a resolution of f with respect to $\{v_E\}$, again by [Proposition 4.7](#) we have $f_{\bullet}(U_{\pi}(p)) \subseteq U_{\pi'}(q)$. This corresponds to the description of tangent maps on real trees described in [\[29\]](#).

We now focus on the geometric interpretation of the image of a curve semivaluation v_C . We first assume that C is not contracted by f .

Proposition 4.8 ([\[29, Prop. 2.6\]](#)). *Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ, $v_C \in \mathcal{V}_X$ a non-contracted curve valuation, and $v_{C'} = f_{\bullet} v_C$. Then $f(C) = C'$, and moreover*

$$c(f, v_C) = \frac{m(C')}{m(C)} e_{C \rightarrow C'}. \quad (17)$$

For contracted curve valuations, we need some notations before proceeding.

Definition 4.9. Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ, and $v_C \in \mathcal{V}_X$ be a normalized curve semivaluation. We set

$$c_{\alpha}(f, v_C) = \lim_{v \rightarrow v_C} \frac{c(f, v)}{\alpha(v)}, \quad c_A(f, v_C) = \lim_{v \rightarrow v_C} \frac{c(f, v)}{A(v)}.$$

Notice that for any curve semivaluation v_C , there exists a subtree $\mathcal{V}_{X, \hat{v}}$ which contains v_C . In particular the interval $I = [\hat{v}, v_C]$ is totally ordered, and the limit can be taken over I . Moreover, the skewness α and the log discrepancy A give a parameterization of such interval I . So $c_{\alpha}(f, v_C)$ and $c_A(f, v_C)$ can be seen as the derivatives of $c(f, v)$ at v_C with respect to such parameterizations.

Finally, recall that for curve semivaluations, $\alpha(v_C) = A(v_C) = +\infty$. So $c_{\alpha}(f, v_C) = c_A(f, v_C) = 0$ unless v_C is a contracted curve valuation. In the latter case, $c(f, v_C)$ is affine with respect to both the α and A parameterizations. Then the derivative of $c(f, v)$ is just the coefficient of the linear part.

Proposition 4.10. *Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a non-finite germ between normal surface singularities. Let $v_C \in \mathcal{C}_f$ be a contracted curve valuation, and set $v_G = f_{\bullet} v_C \in \mathcal{V}_Y$. Then*

$$c_{\alpha}(f, v_C) = m(C) b_G k_{C \rightarrow G}, \quad c_A(f, v_C) = b_G k_{C \rightarrow G},$$

Proof. Let $\pi: X_{\pi} \rightarrow (X, x_0)$ and $\varpi: Y_{\varpi} \rightarrow (Y, y_0)$ give a resolution of f with respect to $\{v_C\}$. Up to taking higher good resolutions, we may also assume that π is a log resolution of m_X . The strict transform C_{π} of C intersects $\pi^{-1}(x_0)$ in a unique exceptional prime H , transversely at a point p . Notice that $b_H = m(C) =: m$. Take local coordinates (x, y) at p so that $H = \{x = 0\}$ and $C = \{y = 0\}$. For any $t \geq 0$, let μ_t be the monomial valuation at p of weights $\frac{1}{m}$ and t with respect to the coordinates (x, y) , and set $v_t = \pi_{*} \mu_t \in \mathcal{V}_X$. Notice that v_t is normalized thanks to the choice of the weight on x . By direct computation, we get that $A(v_t) = A(v_0) + t$, while $\alpha(v_t) = \alpha(v_0) + \frac{t}{m}$. In particular $c_{\alpha}(f, v_C) = m c_A(f, v_C)$.

By [Proposition 4.3](#), $\tilde{f}_{*} \text{ord}_{C_{\pi}} = \text{kord}_G$, where $k = k_{C_{\pi} \rightarrow G}$. By [Proposition 4.7](#) we get $\tilde{f}(C_{\pi}) = G$. Set $q = \tilde{f}(p)$, and pick local coordinates (z, w) at q , so that $G = \{w = 0\}$.

We consider the sets of valuations $\mathcal{V}_p^{x^m}$ at p normalized with respect to the value at x^m , and analogously \mathcal{V}_q^w . Notice that with our choices, $\mu_t \in \mathcal{V}_p^{x^m}$ is normalized, and we have

$$c(f, v_t) = \mu_t(\tilde{f}^* w) \cdot \mu'_t(\varpi^* m_Y),$$

where μ'_t is the valuation in \mathcal{V}_q^w proportional to $\tilde{f}_{*} \mu_t$. Moreover, since $f \circ \pi = \varpi \circ \tilde{f}$, we infer that $\varpi_{*} \mu'_t$ is proportional to $f_{\bullet} v_t$. In particular, for t big enough, $\mu'_t(\varpi^* m_Y) = \text{ord}_G(\varpi^* m_Y) = b_G$. We conclude by noticing that $\mu_t(\tilde{f}^* w) = kt$ for t big enough. \square

Remark 4.11. In the proof of [Proposition 4.10](#), we may also consider the derivative of $c(f, \nu)$ with respect to the parameterization given by the skewness α_p computed at p of the corresponding normalized valuation $m\nu_t$, instead of the skewness α computed at x_0 . In this case we have $\alpha_p(m\nu_t) = mt$ (for $t \geq 1/m$). In particular, the derivative $c^\alpha(f, \nu_C)$ of $c(f, \nu)$ with respect to α_p satisfies

$$c^\alpha(f, \nu_C) = \frac{b_G}{m(C)} k_{C \rightarrow G}.$$

We end this section by recalling a criterion that allows to recover all the indeterminacy points the lift \tilde{f} of a dominant germ $f : (X, x_0) \rightarrow (Y, y_0)$ with respect to given modifications $\pi : X_\pi \rightarrow (X, x_0)$ and $\varpi : Y_\varpi \rightarrow (Y, y_0)$. Recall that in general $\tilde{f} = \varpi^{-1} \circ f \circ \pi : X_\pi \dashrightarrow Y_\varpi$ is a meromorphic map. If $p \in \pi^{-1}(x_0)$ is a closed point, we denote by $U_X(p) \subset \mathcal{V}_X$ the subset of semivaluations ν whose center in X_π is p . This is a weak open set for all closed points p . Similarly, if $q \in \varpi^{-1}(y_0)$ is a closed point, then $U_Y(q)$ will denote those semivaluations $\nu \in \mathcal{V}_Y$ with center q in Y_ϖ . Certainly if \tilde{f} is holomorphic at a closed point $p \in \pi^{-1}(x_0)$, and if $q = \tilde{f}(p)$, then $f_\bullet(U_X(p)) \subseteq U_Y(q)$; this is simply a matter of unraveling definitions. More importantly for us, the converse is also true, see [[29](#), Prop. 3.2] for a proof.

Proposition 4.12. *The lift \tilde{f} is holomorphic at a closed point $p \in \pi^{-1}(x_0)$ and has $\tilde{f}(p) = q$ if and only if $f_\bullet(U_X(p)) \subseteq U_Y(q)$.*

4.2. Action on dual divisors

Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant map between two normal surface singularities. Pick good resolutions $\pi : X_\pi \rightarrow (X, x_0)$ and $\varpi : Y_\varpi \rightarrow (Y, y_0)$ such that the lift $\tilde{f} := \varpi^{-1} \circ f \circ \pi$ is holomorphic. If f is finite, then \tilde{f} induces \mathbb{Z} -linear push-forward and pull-back operations $\tilde{f}_* : \mathcal{E}(\pi) \rightarrow \mathcal{E}(\varpi)$ and $\tilde{f}^* : \mathcal{E}(\varpi) \rightarrow \mathcal{E}(\pi)$ in the usual way; note, the finiteness of f is needed in order for \tilde{f}^* to map into $\mathcal{E}(\pi)$. These operations satisfy the projection formula

$$\tilde{f}_* D_1 \cdot D_2 = D_1 \cdot \tilde{f}^* D_2 \tag{18}$$

for all divisors $D_1 \in \mathcal{E}(\pi)$ and $D_2 \in \mathcal{E}(\varpi)$. In this case, one can compute the push-forward and the pull-back of dual divisors, and more generally, divisors associated to valuations (see [[25](#), Lemma 1.10]).

In fact, if E is an exceptional prime of π which is not contracted by \tilde{f} , then $E' = \tilde{f}(E)$ is an exceptional prime of ϖ and a straightforward computation shows $\tilde{f}_* \check{E} = k_{E \rightarrow E'} \check{E}'$.

Similarly, if E' is an exceptional prime of ϖ and E_1, \dots, E_n are all of the exceptional primes of π for which $\tilde{f}(E_i) = E'$, then $\tilde{f}^* \check{E}' = \sum_i e_{E_i \rightarrow E'} \check{E}_i$.

Finally, let us note that even if the lift $\tilde{f} = \varpi^{-1} \circ f \circ \pi$ is not holomorphic, we still get well-defined \mathbb{Z} -linear push-forward and pull-back operations $f_* : \mathcal{E}(\pi) \rightarrow \mathcal{E}(\varpi)$ and $f^* : \mathcal{E}(\varpi) \rightarrow \mathcal{E}(\pi)$, simply by passing to a good resolution $\pi' \geq \pi$ for which the lift $g := \varpi^{-1} \circ f \circ \pi'$ is holomorphic and defining $f_* := g_* \circ \eta_{\pi\pi'}^*$ and $f^* = (\eta_{\pi\pi'})_* \circ g^*$. These operations f_* and f^* again satisfy the projection formula.

We want to generalize these formulas to non-finite maps. In this case, the pull-back \tilde{f}^* acting on $\mathcal{E}(\pi)$ does not give an element of $\mathcal{E}(\pi)$, but in general a divisor supported in $\pi^{-1}(f^{-1}(y_0))$. To deal with this situation, we need a few more notations. For any curve $(C, x_0) \subset (X, x_0)$, and any modification $\pi : X_\pi \rightarrow (X, x_0)$, we denote by C_π the strict transform of C by π .

Let now $\pi : X_\pi \rightarrow (X, x_0)$ be a good resolution. We denote by $\mathcal{D}(\pi)$ the set of Weil divisors of X_π , i.e., a formal sum of the form

$$D = \sum_{E \in \Gamma_\pi^*} a_E E + \sum_C a_C C_\pi,$$

where E varies in the set Γ_π^* of all exceptional primes of π , C among all irreducible curves $(C, x_0) \subset (X, x_0)$, and $a_E, a_C \in \mathbb{Z}$, all zero but for a finite number of indices. We may define analogously the set

of \mathbb{Q} -divisors $\mathcal{D}(\pi)_{\mathbb{Q}}$ and of \mathbb{R} -divisors $\mathcal{D}(\pi)_{\mathbb{R}}$. Notice that the intersection product of two elements in $\mathcal{D}(\pi)$ is not well defined in general, since the self-intersection of non-exceptional divisors is not. For any divisor $D \in \mathcal{D}(\pi)_{\mathbb{R}}$ as above, we denote by $\mathcal{E}_{\pi}(D) = \sum_E a_E E$ its exceptional part, and by $\mathcal{R}_{\pi}(D) = D - \mathcal{E}_{\pi}(D)$ the non-exceptional part.

To any irreducible curve $(C, x_0) \subset (X, x_0)$, we associate a divisor $\check{C} \in \mathcal{D}(\pi)_{\mathbb{Q}}$ as the only divisor so that $\check{C} + C_{\pi} \in \mathcal{E}(\pi)_{\mathbb{Q}}$ and $\check{C} \cdot E = 0$ for all $E \in \Gamma_{\pi}^*$. We denote $\check{C} = \hat{Z}_{\pi}(\text{int}_C)$ if we need to remember the good resolution π we are working with.

Notice that if $\pi: X_{\pi} \rightarrow (X, x_0)$ is an embedded resolution of $(C, x_0) \subset (X, x_0)$, then C_{π} intersects a unique exceptional prime H transversely, and $C + \check{C} = \check{H} = Z_{\pi}(\text{int}_C)$.

As before, a (not necessarily finite) map $f: (X, x_0) \rightarrow (Y, y_0)$ induces \mathbb{Z} -linear push-forward and pull-back operators $\tilde{f}_*: \mathcal{D}(\pi) \rightarrow \mathcal{D}(\varpi)$ and $\tilde{f}^*: \mathcal{D}(\varpi) \rightarrow \mathcal{D}(\pi)$, and analogously for \mathbb{Q} - and \mathbb{R} -divisors. Push-forward and pull-back operators satisfy the projection formula $\tilde{f}_* D_1 \cdot D_2 = D_1 \cdot \tilde{f}^* D_2$ for all divisors $D_1 \in \mathcal{E}(\pi)$ and $D_2 \in \mathcal{E}(\varpi)$. Note that the right-hand side of the projection formula (18) is well defined only as far as the supports of D_1 and $\tilde{f}^* D_2$ have no common non-exceptional components. When this happens, the projection formula still holds.

In this more general setting, we obtain the following formulae for pushforward and pullback of dual divisors.

Proposition 4.13. *Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ, and $\nu_E \in \mathcal{V}_X$ be a divisorial valuation. Let $\pi: X_{\pi} \rightarrow (X, x_0)$ and $\varpi: Y_{\varpi} \rightarrow (Y, y_0)$ give a resolution of f with respect to $\mathcal{C}_f \cup \{\nu_E\}$. Then we have*

$$\tilde{f}_* \check{E} = k_{E \rightarrow E'} \check{E}' - \sum_{C \in \text{Cc}_f} (-C_{\pi} \cdot \check{E}) k_{C \rightarrow G_C} \check{G}_C, \quad (19)$$

where $\tilde{f} = \varpi^{-1} \circ f \circ \pi$ is the lift of f , and E' and G_C satisfy $\nu_{E'} = f_{\bullet} \nu_E$ and $f_{\bullet} \nu_C = \nu_{G_C}$ for all $C \in \text{Cc}_f$.

Proof. It suffices to prove that the intersections of both sides of (19) with any exceptional prime D' in Y_{ϖ} coincide. Notice that for any $D' \in \Gamma_{\varpi}^*$, we have

$$\tilde{f}^* D' = \sum_{D \in \Gamma_{\pi}^*} k_{D \rightarrow D'} D + \sum_{C \in \text{Cc}_f} \mathbb{1}_{D'}^{G_C} k_{C \rightarrow G_C} C_{\pi},$$

where $\mathbb{1}$ denotes the Kronecker's delta function. The result easily follows. \square

Proposition 4.14. *Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ, and $\nu_{E'} \in \mathcal{V}_Y$ be a divisorial valuation. Let $\pi: X_{\pi} \rightarrow (X, x_0)$ and $\varpi: Y_{\varpi} \rightarrow (Y, y_0)$ give a resolution of f with respect to $\mathcal{C}_f \cup f_{\bullet}^{-1}(\{\nu_{E'}\})$. Then we have*

$$\tilde{f}^* \check{E}' = \sum_{i=1}^r e_{E_i \rightarrow E'} \check{E}'_i + \sum_{C \in \text{Cc}_f} (-\check{G}_C \cdot \check{E}') k_{C \rightarrow G_C} \check{C}, \quad (20)$$

where $\tilde{f} = \varpi^{-1} \circ f \circ \pi$ is the lift of f , $\nu_{E_1}, \dots, \nu_{E_r}$ are the divisorial valuations in $f_{\bullet}^{-1}(\{\nu_{E'}\})$, and G_C satisfies $f_{\bullet} \nu_C = \nu_{G_C}$ for all $C \in \text{Cc}_f$.

Proof. First, let us compute the non-exceptional part of $\tilde{f}^* \check{E}'$. For any $C \in \text{Cc}_f$, the C_{π} -coefficient of $\tilde{f}^* \check{E}'$ is given by $k_{C \rightarrow G_C}$ times the G_C -coefficient of \check{E}' . This last coefficient is given by $\check{G}_C \cdot \check{E}'$ (see Remark 1.7). Recalling that the C_{π} -coefficient of \check{C} is -1 , it follows that

$$L := \tilde{f}^* \check{E}' + \sum_{C \in \text{Cc}_f} (\check{G}_C \cdot \check{E}') k_{C \rightarrow G_C} \check{C} \in \mathcal{E}(\pi).$$

Now we may compute the intersection of L with any exceptional prime D of X_π . Since $\check{C} \cdot D = 0$ for all $D \in \mathcal{E}(X)$, using the projection formula we get

$$D \cdot L = D \cdot \tilde{f}^* \check{E}' = \tilde{f}_*(D) \cdot \check{E}' = e_{D \rightarrow \tilde{f}(D)} \tilde{f}(D) \cdot \check{E}' = \mathbb{1}_D^{E_i} e_{E_i \rightarrow E'},$$

and we are done. \square

Notice that in the expressions (19) and (20), the coefficients $-C_\pi \cdot \check{E}$ and $-\check{G}_C \cdot \check{E}'$ are always positive.

4.3. Action on b-divisors

This section is devoted to generalize equations (19) and (20) to the setting of b-divisors.

Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ between normal surface singularities. As for divisors, we will have to consider a more general class of b-divisors, whose support is not over the singular point.

Definition 4.15. Let (X, x_0) be a normal surface singularity. A (not necessarily exceptional) b-divisor is a collection $Z = (Z_\pi)_\pi$ of divisors $Z_\pi \in \mathcal{D}(\pi)_\mathbb{R}$ for all modification π , satisfying $Z_\pi = (\eta_{\pi\pi'})_* Z_{\pi'}$ whenever π' is a modification dominating π .

We say that Z is *Cartier* if there exists a modification π so that $Z'_\pi = \eta_{\pi\pi'}^* Z_\pi$ for all modifications π' dominating π .

We denote by $\mathcal{D}(X)$ the set of all b-divisors over X and by $\mathcal{D}_C(X)$ the subset of Cartier b-divisors.

A b-divisor Z is *nef* if $Z_\pi \cdot D \geq 0$ for any effective divisor $D \in \mathcal{E}(\pi)_\mathbb{R}$. It is *exceptional* if $Z_\pi \in \mathcal{E}(\pi)_\mathbb{R}$ for any modification π .

Remark 4.16. Given a b-divisor $Z \in \mathcal{D}(X)$, we may define its *exceptional part* $\mathcal{E}(Z) = (\mathcal{E}_\pi(Z_\pi))_\pi \in \mathcal{E}(X)$. Notice that $\mathcal{E}(Z)$ could be non-Cartier even though Z is. A nice example of this phenomenon is given by b-divisors associated to curve semivaluations. In fact, to a curve semivaluation int_C , we may associate a Cartier b-divisor $\hat{Z}(\text{int}_C)$, determined by any embedded resolution of $(C, x_0) \subset (X, x_0)$. In this case, $\mathcal{E}(\hat{Z}(\text{int}_C)) = Z(\text{int}_C)$, which is not Cartier.

In general, since modifications are isomorphisms outside the exceptional divisor, for any b-divisor $X \in \mathcal{D}(X)$ we have

$$Z - \mathcal{E}(Z) = \sum_C -a_C Z(C),$$

where C varies among irreducible curves in (X, x_0) , a_C vanishes for all but a finite number of C , and $Z(C) = (C_\pi)_\pi$ denotes the Weil b-divisor whose incarnation in π is the strict transform C_π of C . Notice that $Z(C)$ is nef. We set $\text{Supp}(Z) = \{x_0\} \cup \bigcup_{a_C \neq 0} \text{Supp}(C)$ the *support* of Z . It follows that $\mathcal{D}(X) =$

$\mathcal{R}(X) \oplus \mathcal{E}(X)$, and $\mathcal{D}_C(X) = \mathcal{R}(X) \oplus \mathcal{E}_C(X)$, where $\mathcal{R}(X) = \bigoplus_C \check{C}$, with C varying among irreducible curves in (X, x_0) , is an infinite dimensional \mathbb{R} -vector space. In particular, any Weil b-divisor $Z \in \mathcal{D}(X)$ can be uniquely written as

$$Z = \sum_C a_C \check{C} + W, \quad W \in \mathcal{E}(X).$$

Notice that $W \neq \mathcal{E}(Z)$. We will use the notation $W = \text{pr}_\mathcal{E}(Z)$, and call it the *projection* of Z on $\mathcal{E}(X)$. Analogously, we set $\text{pr}_\mathcal{R}(Z) = Z - \text{pr}_\mathcal{E}(Z)$.

Remark 4.17. The construction of the non-exceptional b-divisor associated to a curve semivaluation can be generalized to the case of non \mathfrak{m}_X -primary ideals \mathfrak{a} . In fact, given such an ideal \mathfrak{a} , we may consider any log resolution $\pi : X_\pi \rightarrow (X, x_0)$ of \mathfrak{a} . In this case we have $\pi^* \mathfrak{a} = \mathcal{O}_{X_\pi}(-D_\pi)$ for some divisor $D_\pi \in \mathcal{D}(\pi)$. The projection of D_π to (X, x_0) gives a germ of curve C , which induces a decomposition of D_π in its exceptional and non-exceptional parts $D_\pi = E_\pi + C_\pi$, where C_π stands for the strict transform

of C . We set $\hat{Z}_\pi(\mathfrak{a}) := -D_\pi$, and we denote by $\hat{Z}(\mathfrak{a})$ the associated b-divisor. As for curve semivaluations, in this case we have $\mathcal{E}(\hat{Z}(\mathfrak{a})) = Z(\mathfrak{a})$, where $Z(\mathfrak{a})$ is the b-divisor defined in [Remark 2.10](#)

We are ready to define pushforward and pullback of b-divisors (see [\[9\]](#)).

We first define the pushforward $f_* : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ as follows. Let $Z \in \mathcal{D}(X)$ be a b-divisor, and fix a modification $\varpi : Y_\varpi \rightarrow (Y, y_0)$. Pick a modification $\pi : X_\pi \rightarrow (X, x_0)$ so that the lift $\tilde{f} = \varpi^{-1} \circ f \circ \pi : X_\pi \rightarrow Y_\varpi$ of f is holomorphic, and set $(f_*Z)_\varpi := \tilde{f}_*Z_\pi$. This defines a continuous map $f_* : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$. Moreover, if $Z \in \mathcal{D}_C(X)$ is Cartier, then $f_*(Z)$ is Cartier, determined by any $\varpi : Y_\varpi \rightarrow (Y, y_0)$ so that $\pi : X_\pi \rightarrow (X, x_0)$ is a determination of Z and $\tilde{f} = \varpi^{-1} \circ f \circ \pi : X_\pi \rightarrow Y_\varpi$ does not contract any exceptional prime of X_π or (the strict transform of) any curve in the support of Z (see [\[9, Corollary 2.6\]](#)).

We now define the pullback $f^* : \mathcal{D}(Y) \rightarrow \mathcal{D}(X)$. Let $Z' \in \mathcal{D}(Y)$ be a b-divisor, and fix a modification $\pi : X_\pi \rightarrow (X, x_0)$. Pick a modification $\varpi : Y_\varpi \rightarrow (Y, y_0)$ so that the lift $\tilde{f} = \varpi^{-1} \circ f \circ \pi : X_\pi \rightarrow Y_\varpi$ of f does not contract any (exceptional or not) irreducible component of $(f \circ \pi)^{-1}(\text{Supp}(Z'))$. Then we set $(f^*Z')_\pi := \tilde{f}^*Z'_\varpi$. This defines a continuous map $f^* : \mathcal{D}(Y) \rightarrow \mathcal{D}(X)$ (see [\[9, Corollary 2.5\]](#)), which preserves Cartier b-divisors. In fact if Z' is determined by ϖ , then Z is determined by a modification π so that \tilde{f} is regular.

Theorem 4.18. *Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ, and $\nu \in \mathcal{V}_X$ and $\nu' \in \mathcal{V}_Y$ be normalized valuations. Then we have*

$$f_*Z(\nu) = c(f, \nu)Z(f \bullet \nu) - \sum_{\nu_C \in \mathcal{C}_f} (-Z(\nu_C) \cdot Z(\nu))c_\alpha(f, \nu_C)Z(f \bullet \nu_C), \quad (21)$$

$$f^*Z(\nu') = \sum_{f \bullet \nu = \nu'} \frac{m(f, \nu)}{c(f, \nu)}Z(\nu) + \sum_{\nu_C \in \mathcal{C}_f} (-Z(f \bullet \nu_C) \cdot Z(\nu'))c_\alpha(f, \nu_C)\hat{Z}(\nu_C), \quad (22)$$

where $m(f, \nu) \in \mathbb{N}^*$, and $\sum_{f \bullet \nu = \nu'} m(f, \nu)$ is bounded by the local topological degree of f .

Proof. By continuity we may assume that $\nu = \nu_E$ and $\nu' = \nu_{E'}$ are divisorial. To obtain [\(21\)](#), we divide both sides of [\(19\)](#) by b_E , and express the equation with respect to b-divisors associated to normalized valuations. We conclude by applying [Proposition 4.7](#), [Proposition 4.10](#), and the fact that $C_\pi \cdot Z_\pi(\nu_E) = Z_\pi(\text{int}_C) \cdot Z_\pi(\nu_E) = m(C)Z_\pi(\nu_C) \cdot Z_\pi(\nu_E)$, since $Z_\pi(\nu_E)$ is an exceptional divisor.

We proceed analogously for the pullback, dividing both sides of [\(20\)](#) by $b_{E'}$. First, we analyze the projection of $f^*Z(\nu_{E'})$ to $\mathcal{D}(X)$. We get in this way the second sum of [\(22\)](#). Notice that it varies continuously on ν' . It remains to study the projection of $f^*Z(\nu_{E'})$ to $\mathcal{E}(X)$, which is given by

$$\sum_{i=1}^r e_{E_i} \frac{b_{E_i}}{b_{E'}} Z(\nu_{E_i}) = \sum_{i=1}^r \frac{e_{E_i} k_{E_i}}{c(f, \nu_{E_i})} Z(\nu_{E_i})$$

by [\(16\)](#), where $e_{E_i} = e_{E_i \rightarrow E'}$ and $k_{E_i} = k_{E_i \rightarrow E'}$. We set $m(f, \nu_E) = e_E k_E \in \mathbb{N}^*$, and get the same expression as in [\(22\)](#), with possibly one exception. In fact, if $\nu_{E'} = f \bullet \nu_C$ then the two expressions differ for a term $\frac{m(f, \nu_C)}{c(f, \nu_C)}Z(\nu_C)$. Assume we can prove that $m(f, \nu)$ is uniformly bounded. Then since C is a contracted curve, $c(f, \nu_C) = +\infty$ and $\frac{m(f, \nu_C)}{c(f, \nu_C)}$ vanishes, giving [\(22\)](#).

Set now $M(\nu') := \sum_{f \bullet \nu = \nu'} m(f, \nu)$. Since the pull back of b-divisors is continuous, the map

$$\nu \mapsto \sum_{f \bullet \nu = \nu'} \frac{m(f, \nu)}{c(f, \nu)}Z(\nu) \quad (23)$$

is continuous. By intersecting with $Z(\mathfrak{m}_X)$, we deduce that $\nu' \mapsto M(\nu')$ is continuous as far as $c(f, \nu) < +\infty$, i.e., on $\mathcal{V}_Y \setminus f_\bullet(\mathcal{C}_f)$. We want to prove that $M(\nu') \leq e(f)$ the topological degree of f , concluding the proof.

Pick good resolutions $\pi: X_\pi \rightarrow (X, x_0)$ and $\varpi: Y_\varpi \rightarrow (Y, y_0)$ that give a resolution of f with respect to $f_\bullet^{-1}(\nu')$. We want to compute the number of preimages for a generic point $q \in Y_\varpi$ close to E' . The map $\tilde{f}|_E: E \rightarrow E'$ has topological degree $e_E = e_{E \rightarrow E'}$, hence any point q in E' has e_E preimages in E (counted with multiplicities). For any such preimage p , pick local coordinates (x, y) at p and (z, w) at q so that $E = \{x = 0\}$ and $E' = \{z = 0\}$. Up to taking higher models, we may assume that \tilde{f} is monomial. In these coordinates it is of the form

$$\tilde{f}(x, y) = (x^k y^b \varepsilon_1(x, y), y^d \varepsilon_2(x, y)),$$

for suitable ε_i non-vanishing at 0, $b \in \mathbb{N}$ and $d \in \mathbb{N}^*$. Notice that d is the multiplicity of p as a solution of $\tilde{f}|_E(p) = q$. It follows that the number of preimages near p of a generic point near q is kd . If we sum over all preimages p of q , we get ke preimages of a (generic) point near q . Summing up for all preimages ν_E of $\nu_{E'}$, we obtain the value $M(\nu')$, which is hence bounded by the topological degree. \square

Remark 4.19. When $f: (X, x_0) \rightarrow (Y, y_0)$ is finite, $\nu' \mapsto M(\nu')$ is a constant map in the whole space \mathcal{V}_Y , which coincides with the topological degree of f . From the proof of [Theorem 4.18](#), we also deduce that for (possibly) non-finite maps, the topological degree can still be computed as $e(f) = \max\{M(\nu') \mid \nu' \in \mathcal{V}_Y\}$. One can also study how $m(f, \nu)$ varies when ν varies. Its properties are related to the behavior of tangent maps df_\bullet at ν (see [Remarks 9.1](#) and [9.2](#)).

Definition 4.20. Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ, and $\nu \in \mathcal{V}_X$ be a normalized valuation. The value $m(f, \nu)$ of [Theorem 4.18](#) is called *multiplicity of f at ν* .

Remark 4.21. Analogous formulae can be obtained for pushforwards and pullbacks of any b-divisor. In particular, in the setting of [Theorem 4.18](#), for any exceptional nef b-divisor $Z \in \mathcal{E}(Y)$ we get

$$\mathcal{R}(f^*Z) = \sum_{\nu_C \in \mathcal{C}_f} (-Z(f_\bullet \nu_C) \cdot Z) c_a(f, \nu_C) \frac{-Z(C)}{m(C)}. \quad (24)$$

4.4. Angular distance is non-increasing

We come now to a remarkable property of the maps $f_\bullet: \mathcal{V}_X \rightarrow \mathcal{V}_Y$, a property that makes a classification result like [Theorem B](#) possible: f_\bullet is distance non-increasing with respect to the angular metrics on \mathcal{V}_X and \mathcal{V}_Y .

Theorem 4.22. For each $\nu_1, \nu_2 \in \mathcal{V}_X$ of finite skewness one has $\rho(f_\bullet \nu_1, f_\bullet \nu_2) \leq \rho(\nu_1, \nu_2)$.

Proof. Using [Proposition 2.19](#), we have

$$\rho(f_\bullet \nu_1, f_\bullet \nu_2) = \log \left(\sup_a \frac{f_\bullet \nu_1(a)}{f_\bullet \nu_2(a)} \times \sup_a \frac{f_\bullet \nu_2(a)}{f_\bullet \nu_1(a)} \right) = \log \left(\sup_a \frac{\nu_1(f^*a)}{\nu_2(f^*a)} \times \sup_a \frac{\nu_2(f^*a)}{\nu_1(f^*a)} \right),$$

the suprema taken over all \mathfrak{m}_Y -primary ideals $\mathfrak{a} \subset R_Y$. The ideals $f^*\mathfrak{a}$ will not be \mathfrak{m}_X -primary if f is not finite, but in any case for large enough $n \in \mathbb{N}$ the ideals $\mathfrak{a}_n := f^*\mathfrak{a} + \mathfrak{m}_X^n$ are \mathfrak{m}_X -primary and satisfy $\nu_i(\mathfrak{a}_n) = \nu_i(\mathfrak{a})$ for $i = 1, 2$, and thus we can conclude that

$$\rho(f_\bullet \nu_1, f_\bullet \nu_2) \leq \log \left(\sup_b \frac{\nu_1(b)}{\nu_2(b)} \times \sup_b \frac{\nu_2(b)}{\nu_1(b)} \right) = \rho(\nu_1, \nu_2),$$

the suprema taken over all \mathfrak{m}_X -primary ideals $\mathfrak{b} \subset R_X$. This completes the proof. \square

We emphasize that this holds for *every* dominant holomorphic map $f : (X, x_0) \rightarrow (Y, y_0)$. In particular, if we are in the dynamical setting where $(Y, y_0) = (X, x_0)$, then $\rho(f_{\bullet}^n \nu_1, f_{\bullet}^n \nu_2) \leq \rho(\nu_1, \nu_2)$ for all $n \geq 1$. This says that the family of maps $\{f_{\bullet}^n\}_{n \geq 1}$ is ρ -equicontinuous in a very strong sense.

Remark 4.23. [Theorem 4.22](#) is analogous to the well-known result in the theory of Riemann surfaces that if $f : \Sigma_1 \rightarrow \Sigma_2$ is a holomorphic map between hyperbolic Riemann surfaces, then f does not increase distances with respect to the hyperbolic metrics on Σ_1 and Σ_2 . Using this fact and Montel's theorem, Fatou gave a classification of all the dynamics possible on a hyperbolic Riemann surface, see [51, 14]. With this in mind, our [Theorem B](#) can be viewed as an analogue of Fatou's classification theorem.

In the following, we need a stronger characterization of when the map f_{\bullet} is strictly decreasing the angular distance.

Given two distinct valuations $\nu, \mu \in \mathcal{V}_X$, we denote by $U_{\nu}(\mu)$ the connected component of $\mathcal{V}_X \setminus \{\nu\}$ containing μ .

Theorem 4.24. *Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant non-invertible germ between two normal surface singularities. For any $\nu \neq \mu \in \mathcal{V}_X$ of finite skewness, we have $\rho(f_{\bullet} \nu, f_{\bullet} \mu) \leq \rho(\nu, \mu)$. The equality holds if and only if f is finite, and moreover*

- ν disconnects μ and any preimage of $f_{\bullet} \nu$, and symmetrically
- μ disconnects ν and any preimage of $f_{\bullet} \mu$.

Proof. We want to prove the same estimate as in [Theorem 4.22](#) in terms of b-divisors. This will allow to study the equality cases by [Proposition 2.15](#). In particular, we will show that $\beta(f_{\bullet} \nu | f_{\bullet} \mu) \leq \frac{c(f, \mu)}{c(f, \nu)} \beta(\nu | \mu)$, which gives the desired inequality by symmetry. This relations is equivalent to proving

$$(Z(f_{\bullet} \nu) \cdot c(f, \nu) Z(f_{\bullet} \nu))(Z(\nu) \cdot Z(\mu)) \leq (Z(f_{\bullet} \nu) \cdot c(f, \mu) Z(f_{\bullet} \mu))(Z(\nu) \cdot Z(\nu)). \quad (25)$$

By (21), we get

$$\begin{aligned} c(f, \nu) Z(f_{\bullet} \nu) &= f_* Z(\nu) + \sum_{\nu_C \in \mathcal{C}_f} (-Z(\nu_C) \cdot Z(\nu)) c_{\alpha}(f, \nu_C) Z(f_{\bullet} \nu_C), \\ c(f, \mu) Z(f_{\bullet} \mu) &= f_* Z(\mu) + \sum_{\nu_C \in \mathcal{C}_f} (-Z(\nu_C) \cdot Z(\mu)) c_{\alpha}(f, \nu_C) Z(f_{\bullet} \nu_C). \end{aligned}$$

Notice that all coefficients of $Z(f_{\bullet} \nu_C)$ in the two expressions are positive. By plugging these expressions in (25), we get to the equivalent system of inequalities:

$$(Z(f_{\bullet} \nu) \cdot f_* Z(\nu))(Z(\mu) \cdot Z(\nu)) \leq (Z(f_{\bullet} \nu) \cdot f_* Z(\mu))(Z(\nu) \cdot Z(\nu)), \quad (26)$$

$$(Z(\nu_C) \cdot Z(\nu))(Z(\mu) \cdot Z(\nu)) \leq (Z(\nu_C) \cdot Z(\mu))(Z(\nu) \cdot Z(\nu)), \quad (27)$$

where the equality holds for (25) if and only if the equality holds for all such equations. Recall that by (22), we have

$$f_* Z(f_{\bullet} \nu) = \sum_{i=1}^r a_i Z(\nu_i) + \sum_{\nu_C \in \mathcal{C}_f} a_C Z(f_{\bullet} \nu_C)$$

for suitable $a_i, a_C > 0$, where $\{\nu_1, \dots, \nu_r\}$ are the preimages of $f_{\bullet} \nu$. By using the projection formula and plugging in this pullback formula, (26) is equivalent to the system of inequalities

$$(Z(\nu_i) \cdot Z(\nu))(Z(\mu) \cdot Z(\nu)) \leq (Z(\nu_i) \cdot Z(\mu))(Z(\nu) \cdot Z(\nu)), \quad (28)$$

$$(Z(f_{\bullet} \nu_C) \cdot Z(\nu))(Z(\mu) \cdot Z(\nu)) \leq (Z(f_{\bullet} \nu_C) \cdot Z(\mu))(Z(\nu) \cdot Z(\nu)). \quad (29)$$

All these inequalities (27), (28) and (29) hold by [Proposition 2.15](#), and we deduce by symmetry that $\rho(f_\bullet \nu, f_\bullet \mu) \leq \rho(\nu, \mu)$.

Notice that if f is finite, the set \mathcal{C}_f is empty, and the inequalities (27) and (29) do not occur. By [Proposition 2.15](#), the equality holds in (28) if and only if ν disconnects μ and ν_i . By symmetry, we get the statement for finite maps.

Assume now that f is not finite. Again by [Proposition 2.15](#), the equality in (27) holds if and only if ν disconnects ν_C and μ . We apply the same argument with the role of ν and μ interchanged, and for the angular distance to be preserved, we need that ν disconnects ν_C and μ , and μ disconnects ν_C and ν . It is easy to check that this cannot happen, and the angular distance strictly decreases for non-finite maps. \square

We state here more explicitly what we proved for non-finite germs.

Corollary 4.25. *Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant non-finite germ. Then $\rho(f_\bullet \nu, f_\bullet \mu) < \rho(\nu, \mu)$ for all valuations $\nu \neq \mu \in \mathcal{V}_X^\alpha$ of finite skewness.*

4.5. The Jacobian formula

Another important tool in understanding the maps f_\bullet on valuation spaces is the *Jacobian formula*, which says precisely how the log discrepancy function A behaves with respect to this action. This proof has been suggested by Charles Favre; a version for finite maps can be found in [25, Prop. 1.9].

Proposition 4.26 (The Jacobian Formula). *Let (X, x_0) and (Y, y_0) be two normal surface singularities, and let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant holomorphic map. Then there exists a Weil divisor R_f on X so that*

$$A(f_* \nu) = A(\nu) + \nu(R_f) \quad (30)$$

for any $\nu \in \hat{\mathcal{V}}_X^*$.

We call R_f the *Jacobian divisor* of f .

Proof. By density, we can reduce ourselves to the case of divisorial valuations. Moreover, since (30) is linear homogeneous, we may assume that $\nu = \text{div}_E$ for some exceptional prime E . Let $\pi : X_\pi \rightarrow (X, x_0)$ and $\varpi : Y_\varpi \rightarrow (Y, y_0)$ give a resolution of f with respect to $\{\nu_E\}$. By homogeneity of the log discrepancy, equation (30) in this setting becomes

$$k_E(1 + \text{ord}_{E'}(K_\varpi)) = 1 + \text{ord}_E(K_\pi) + \text{ord}_E(\pi^* R_f), \quad (31)$$

where K_π and K_ϖ denote the relative canonical divisors of π and ϖ respectively.

Pick $\omega_Y = \omega$ a non-trivial 2-form on (Y, y_0) , and set $\omega_X = f^* \omega$. We may define canonical divisors as $K_X = \text{Div}(\omega_X)$, $K_{X_\pi} = \text{Div}(\pi^* \omega_X)$, and analogously for K_Y , K_{Y_ϖ} , so that, by definition,

$$K_\pi = K_{X_\pi} - \pi^* K_X \quad \text{and} \quad K_\varpi = K_{Y_\varpi} - \varpi^* K_Y.$$

Since $\tilde{f} : X_\pi \rightarrow Y_\varpi$ is holomorphic, there exists an effective divisor $J_{\tilde{f}}$ of X_π so that

$$K_{X_\pi} - \tilde{f}^* K_{Y_\varpi} = J_{\tilde{f}}.$$

We plug in all expressions together, and we get

$$\begin{aligned}
A(\nu) &= 1 + \text{ord}_E(K_\pi) = 1 + \text{ord}_E(K_{X_\pi}) - \text{ord}_E(\pi^*K_X) \\
&= 1 + \text{ord}_E(\tilde{f}^*K_{Y_\varpi}) + \text{ord}_E(J_{\tilde{f}}) - \text{ord}_E(\pi^*K_X) \\
&= 1 + k_E \text{ord}_{E'}(K_{Y_\varpi}) + k_E - 1 - \text{ord}_E(\pi^*K_X) \\
&= k_E [1 + \text{ord}_{E'}(K_{Y_\varpi}) + \text{ord}_{E'}(\varpi^*K_Y)] - \text{ord}_E(\pi^*K_X) \\
&= k_E A(\text{div}_{E'}) + k_E \text{ord}_{E'}(\varpi^*K_Y) - \text{ord}_E(\pi^*K_X) \\
&= A(f_*\nu) + \text{ord}_E(\tilde{f}^*\varpi^*K_Y) - \text{ord}_E(\pi^*K_X).
\end{aligned}$$

Set $R_{\tilde{f}} = \pi^*K_X - \tilde{f}^*\varpi^*K_Y$, so that the previous equations translate to

$$A(f_*\nu) = A(\nu) + \text{ord}_E(R_{\tilde{f}}).$$

Notice that

$$R_{\tilde{f}} = \pi^*K_X - \tilde{f}^*\varpi^*K_Y = \pi^*\text{Div}(f^*\omega) - \tilde{f}^*\varpi^*\text{Div}(\omega) = \pi^*[\text{Div}(f^*\omega) - f^*\text{Div}(\omega)] = \pi^*R_f,$$

where we set $R_f = \text{Div}(f^*\omega) - f^*\text{Div}(\omega)$. Notice that R_f does not depend on the choice of π and ϖ , since if $\pi' \geq \pi$ and $\varpi' \geq \varpi$ are two higher good resolutions so that $\tilde{f}' = (\varpi')^{-1} \circ f \circ \pi'$ is regular, then $R_{\tilde{f}'} = \eta_{\pi, \pi'}^* R_{\tilde{f}}$, and their projections on X coincide. We then conclude

$$A(f_*\nu) = A(\nu) + \text{ord}_E(\pi^*R_f) = A(\nu) + \nu(R_f).$$

□

Remark 4.27. Here we use in an essential way the hypothesis on the zero characteristic of the field. In fact the Jacobian formula is not valid, not even in the smooth case, for dominant maps between surfaces defined over positive characteristic fields (see [Example 9.6](#)).

Remark 4.28. For normalized semivaluations, the Jacobian formula (30) can be rewritten as

$$c(f, \nu)A(f_*\nu) = A(\nu) + \nu(R_f). \quad (32)$$

Notice that for any $x \in X$ so that $f(x) \neq y_0$, then (X, x) and $(Y, f(x))$ are both smooth, and R_f is given, locally at x , by the Weil divisor defined by the Jacobian determinant of $f : (X, x) \rightarrow (Y, f(x))$. In particular, the divisor R_f is effective whenever f is finite, but it may have negative components at the contracted curves, as the following example shows.

Example 4.29. Let (X, x_0) be any non log canonical singularity. Let $\pi : X_\pi \rightarrow (X, x_0)$ be any good desingularization of (X, x_0) , and let E be an exceptional prime for π so that $A(E) < 0$. By continuity of log discrepancy on dual graphs, for any $p \in E$ there exists a valuation $\nu \in \mathcal{V}_{X_\pi}$ centered at p such that $A(\pi_*\nu) < 0$. The Jacobian formula (30) applied to $\pi : (X_\pi, p) \rightarrow (X, x_0)$ at ν gives

$$\nu(R_\pi) = A(\pi_*\nu) - A(\nu) < 0,$$

where the last inequality follows from $A(\nu) > 0$ since X_{π_0} is smooth. In particular, R_π is not effective.

Non-effective Jacobian divisors are quite common, even for dynamical systems. Here we give some construction of selfmaps with some prescribed behavior.

Lemma 4.30. *Let $\mathcal{V} = \mathcal{V}_{\mathbb{C}^2}$ denote the valuative tree. Then for every divisorial valuation $\nu \in \mathcal{V}$, there exists a dominant superattracting germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ so that $f_{\bullet} \nu = \text{ord}_0$. The map f may be chosen both finite or non-finite.*

Proof. First, assume that $m_0(\nu) = 1$, where m_0 denotes the multiplicity at 0 of ν , see [26, p. 63]. We may find coordinates (x, y) at $0 \in \mathbb{C}^2$ so that ν belongs to the segment $[\text{ord}_0, \nu_x]$, where ν_x is the curve semivaluation associated to $\{x = 0\}$. Then there exists a monomial map $f_1(x, y) = (x^a y^b, x^c y^d)$, with $a, b, c, d \in \mathbb{N}$ and $ad - bc \neq 0$ (we may assume $ad - bc = \pm 1$), so that $(f_1)_{\bullet} \nu = \text{ord}_0$. We may pick the coefficients so that f_1 is not finite.

Assume now that $m_0(\nu) \geq 2$. Up to a linear change of coordinates, we may assume that ν belongs to the connected component U of $\mathcal{V} \setminus \{\text{ord}_0\}$ containing ν_x . Let $C = \{\phi(x, y) = 0\}$ be an irreducible curve so that the associated curve semivaluation ν_C satisfies $\nu < \nu_C$. Then the map $f_2(x, y) = (\phi(x, y), y)$ is order preserving on U , and sends ν_C to ν_x . Hence $(f_2)_{\bullet} \nu$ is a divisorial valuation (of multiplicity 1) which belongs to $[\text{ord}_0, \nu_x]$. The map $f = f_1 \circ f_2$ satisfies the desired properties.

If f is not finite, a small perturbation with an element of \mathfrak{m}^N for N big enough gives a finite map which preserves the property $f_{\bullet} \nu = \text{ord}_0$. \square

In the next proposition, we call *essential* divisorial valuation any divisorial valuation $\nu_E \in \mathcal{V}_X$ associated to an exceptional prime E in the minimal (possibly non good) resolution of (X, x_0) .

Proposition 4.31. *Let (X, x_0) be a normal surface singularity, $\nu \in \mathcal{V}_X$ be any divisorial valuation, and $\mu \in \mathcal{V}_X$ a non-essential divisorial valuation. Then there exists a (non-finite) superattracting germ $f : (X, x_0) \rightarrow (X, x_0)$ so that $f_{\bullet} \nu = \mu$.*

Proof. The hypothesis on μ assures that there exists a (not necessarily good) resolution $\pi : X_{\pi} \rightarrow (X, x_0)$ and a point $p \in \pi^{-1}(x_0)$ so that $\mu = \pi_{\bullet} \text{ord}_p$, where ord_p is the vanishing order at p (also called the multiplicity valuation at p). Embed the singularity (X, x_0) in an affine space $(\mathbb{C}^n, 0)$, and take a (generic) projection to $(\mathbb{C}^2, 0)$. Call $g : (X, x_0) \rightarrow (\mathbb{C}^2, 0)$ the composition. By Lemma 4.30 applied to the divisorial valuation $g_{\bullet} \nu$, there exists a superattracting germ $h : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ so that $h_{\bullet} g_{\bullet} \nu = \text{ord}_0$. Let $\sigma : (\mathbb{C}^2, 0) \rightarrow (X_{\pi}, p)$ be any local isomorphism sending 0 to p , so that $\sigma_{\bullet} \text{ord}_0 = \text{ord}_p$. Then the map $f = \pi \circ \sigma \circ h \circ g : (X, x_0) \rightarrow (X, x_0)$ satisfies

$$f_{\bullet} \nu = \pi_{\bullet} \sigma_{\bullet} h_{\bullet} g_{\bullet} \nu = \pi_{\bullet} \sigma_{\bullet} \text{ord}_0 = \pi_{\bullet} \text{ord}_p = \mu.$$

\square

Notice that the condition of μ being non-essential is sufficient, but not necessary. On the one hand, it is not always possible to send ν to an essential divisorial valuation μ . An easy example is given by a divisorial valuation $\mu = \nu_{E'}$ with E' a non-rational exceptional prime. If f_{\bullet} sends a divisorial valuation ν_E to μ , then by Proposition 4.7 f induces a non-constant map $E \rightarrow E'$, and $g(E) \geq g(E')$, where g denotes the genus.

On the other hand, essential valuations can be eigenvaluations, even for non-finite maps. See §9.3, §9.4, §9.5 for specific examples.

We conclude with some considerations on the effectiveness of the Jacobian divisor for non-finite selfmaps.

Remark 4.32. Let (X, x_0) be either non lc singularity, or (the finite quotient of) a cusp. Let $\nu \in \mathcal{V}_X$ be any divisorial valuation with positive log discrepancy $A(\nu) > 0$, and $\mu \in \mathcal{V}_X$ any non-essential divisorial valuation with non-positive log discrepancy $A(\mu) \leq 0$. In the first case, this is possible because the log discrepancy A is continuous on dual graphs, and by definition there exists an essential valuation with

negative log discrepancy. In the second case, the subset of \mathcal{V}_X where the log discrepancy is zero is not discrete, and we can find a non-essential divisorial valuation with zero log discrepancy. By [Proposition 4.31](#), there exists a superattracting germ $f : (X, x_0) \rightarrow (X, x_0)$ for which $f_\bullet \nu = \mu$. Then by the Jacobian formula [\(32\)](#), we get

$$\nu(R_f) = c(f, \nu)A(f_\bullet \nu) - A(\nu) = c(f, \nu)A(\mu) - A(\nu) < 0,$$

and R_f is not effective.

With similar arguments, one can prove that Jacobian divisors are always effective as far as the target space is *canonical*. We recall that a singularity (Y, y_0) is *canonical* (resp., *terminal*) if for any exceptional prime E in any good resolution, we have $\text{ord}_E(K_\varpi) = A(\text{div}_E) - 1 \geq 0$ (resp., > 0), where $\varpi : Y_\varpi \rightarrow (Y, y_0)$ is any good resolution.

Proposition 4.33. *Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ between normal surface singularities. If (Y, y_0) is canonical, then the jacobian divisor R_f is effective.*

Proof. Let C be an irreducible contracted critical curve, and ν_C the associated semivaluation. Let $\alpha_C C$ be the component of C in R_f . We want to show that $\alpha_C \geq 0$. We proceed as in the proof of [Proposition 4.10](#). Let $\pi : X_\pi \rightarrow (X, x_0)$ and $\varpi : Y_\varpi \rightarrow (Y, y_0)$ give a resolution of f with respect to $\{\nu_C\}$. The strict transform C_π of C intersects $\pi^{-1}(x_0)$ in a unique exceptional prime H , transversely at a point p . Up to taking higher models, we may also assume that π is a log resolution of \mathfrak{m}_X , and that the strict transform of any other irreducible component of $f^{-1}(y_0)$ doesn't pass through p . Set $b_H = m(C) =: m$. Take local coordinates (x, y) at p so that $H = \{x = 0\}$ and $C_\pi = \{y = 0\}$. For any $t \geq 0$, let μ_t be the monomial valuation at p of weights $\frac{1}{m}$ and t with respect to the coordinates (x, y) , and set $\nu_t = \pi_* \mu_t \in \mathcal{V}_X$. By direct computation, we have $A(\nu_t) = A(\nu_0) + t$, while $\nu_t(C) = t + \text{const}$. Taking the limit for $t \rightarrow +\infty$ of the Jacobian formula [\(32\)](#) divided by t on both sides, we get

$$\alpha_C = \lim_{t \rightarrow +\infty} \frac{\nu_t(R_f)}{t} = \lim_{t \rightarrow +\infty} \frac{c(f, \nu_t)A(f_\bullet \nu_t) - A(\nu_t)}{t} = c_A(f, \nu_C)A(f_\bullet \nu_C) - 1$$

Set $f_\bullet \nu_C = \nu_G$. By [Proposition 4.10](#), we get $c_A(f, \nu_C) = k_{C \rightarrow G} b_G$, and

$$\alpha_C = k_{C \rightarrow G} b_G A(\nu_G) - 1 = k_{C \rightarrow G} A(\text{div}_G) - 1 \geq A(\text{div}_G) - 1 \geq 0.$$

□

Remark 4.34. Notice that if (Y, y_0) is *terminal*, then every irreducible component of $f^{-1}(y_0)$ has a non-trivial contribution to R_f . This is not always the case for non-terminal singularities, see [§9.4](#) for an example.

In particular, not all curves contracted to y_0 belong to the support of R_f . Nevertheless, following the notations used in the proof of [Proposition 4.26](#), their strict transforms belong to the critical set of the lift \tilde{f} . In particular, contracted curves are necessarily holomorphic.

4.6. Critical skeleton

Here we study how the maps $\nu \mapsto c(f, \nu)$ and $\nu \mapsto \nu(R_f)$ behave. It turns out that in fact, they are locally constant outside a finite graph. In the smooth setting, such maps belong to the class of *tree potentials*, see [[29](#), Section 1.8]. To describe the analogous on valutive spaces associated to normal surface singularities (X, x_0) , we need to introduce a few objects. We already introduced in [§2.5](#) the skeleton \mathcal{S}_X of X , which is given by the embedded dual graph of any minimal good resolution of X ,

and the essential skeleton \mathcal{S}_X in §3.2. In the following we will need to consider other subgraphs of \mathcal{V}_X depending on \mathfrak{m}_X -primary ideals and finite sets of valuations.

Recall that for any \mathfrak{m}_X -primary ideal \mathfrak{a} , the associated b-divisor $Z(\mathfrak{a})$ is Cartier (and nef). In particular, there exists a finite set $\mathcal{R}(\mathfrak{a})$ of divisorial valuations so that $Z(\mathfrak{a}) = \sum_{\nu} a_{\nu} Z(\nu)$, with $a_{\nu} > 0$ for all $\nu \in \mathcal{R}(\mathfrak{a})$. The set $\mathcal{R}(\mathfrak{a})$ is called the set of *Rees valuations* of \mathfrak{a} .

Definition 4.35. Let (X, x_0) be a normal surface singularity, and $V \subset \mathcal{V}_X$ be a finite set of semivaluations. Denote by $r_X: \mathcal{V}_X \rightarrow \mathcal{S}_X$ the retraction map. We call the set $\mathcal{S}_X(V) := \mathcal{S}_X \cup \bigcup_{\nu \in V} [r_X \nu, \nu]$ the *skeleton generated by V* . If moreover \mathfrak{a} is a \mathfrak{m}_X -primary ideal, we refer to $\mathcal{S}_X(V \cup \mathcal{R}(\mathfrak{a}))$ as the *skeleton generated by V and adapted to \mathfrak{a}* . We denote it by $\mathcal{S}_{\mathfrak{a}}(V)$.

Recall that given two valuations $\nu, \mu \in \mathcal{V}_X$, we may compute the intersection of the associated b-divisors $Z(\mu) \cdot Z(\nu) \in [-\infty, 0)$. We now fix μ , and study how this intersection varies when ν varies in the valuative space \mathcal{V}_X .

Proposition 4.36. *Let (X, x_0) be a normal surface singularity, and $\mu \in \mathcal{V}_X$ be any normalized semivaluation. Then the map $g: \nu \mapsto Z(\nu) \cdot Z(\mu)$ is locally constant on $\mathcal{V}_X \setminus \mathcal{S}_{\mathfrak{m}_X}(\mu)$.*

Proof. Denote by $r: \mathcal{V}_X \rightarrow \mathcal{S}_{\mathfrak{m}_X}(\mu)$ the natural retraction. To avoid trivial cases, we may assume that $\nu \notin \mathcal{S}_{\mathfrak{m}_X}(\mu)$. In particular we want to prove that $Z(\nu) \cdot Z(\mu) = Z(r\nu) \cdot Z(\mu)$.

Notice that either $\nu = r\nu$, or ν and μ belong to different components of $\mathcal{V}_X \setminus \{r\nu\}$. Then by [Proposition 2.15](#), we have

$$(Z(r\nu) \cdot Z(\nu))(Z(r\nu) \cdot Z(\mu)) = (Z(\nu) \cdot Z(\mu))(Z(r\nu) \cdot Z(r\nu)).$$

By [Proposition 2.37](#) we have $r\nu \leq \nu$, and by [Lemma 2.33](#) we have $\beta(r\nu|\nu) = 1$, which written in terms of b-divisors gives

$$Z(r\nu) \cdot Z(r\nu) = Z(r\nu) \cdot Z(\nu).$$

Putting these last two equations together, we deduce the desired equality. \square

[Proposition 4.36](#) allows to study the properties of other functionals over \mathcal{V}_X . In particular, we will study the maps $\nu \mapsto \nu(\phi)$ for any $\phi \in R_X$, $\nu \mapsto \nu(R_f)$ for the Jacobian divisor R_f , and $\nu \mapsto c(f, \nu)$.

Let $\pi: X_{\pi} \rightarrow (X, x_0)$ be a good resolution, and $E \in \Gamma_{\pi}$ be an exceptional prime of π . Let $D \in \mathcal{D}(\pi)$ be the divisor associated to $\phi \circ \pi$, so that $\text{div}_E(\phi) = \text{ord}_E(\phi \circ \pi)$ is given by the coefficient of E in D . Since the non-exceptional part of D coincides with the strict transform C_{π} of the curve $C = \{\phi = 0\}$, and $D \cdot E = 0$ for all exceptional primes, we infer that

$$D = C_{\pi} - \sum_k d_k Z_{\pi}(\text{int}_{C_k}),$$

where we decomposed the curve C in its irreducible components C_k of multiplicity d_k .

To simplify notations, we set $Z_{\pi}(\text{int}_C) = Z_{\pi}(\text{int}_{\phi}) = \sum_k d_k Z_{\pi}(\text{int}_{C_k})$, and denote by $Z(\text{int}_C) = Z(\text{int}_{\phi})$ its associated b-divisor. Notice that in general the map int_C of local intersection with $C = \{\phi = 0\}$ fails to be a valuation (unless C is irreducible).

The coefficient of E in D is then given by $-\check{E} \cdot Z(\text{int}_C)$. Since divisorial valuations are dense in \mathcal{V}_X , by continuity we deduce the following result.

Proposition 4.37. *Let (X, x_0) be a normal surface singularity, $\phi \in R_X$ and $C = \{\phi = 0\}$. Then*

$$\nu(\phi) = -Z(\nu) \cdot Z(\text{int}_C) \tag{33}$$

for all $\nu \in \hat{\mathcal{V}}_X$.

In particular, $\nu \mapsto \nu(\phi)$ is locally constant on $\mathcal{V} \setminus \mathcal{S}_{\mathfrak{m}_X}(\{\nu_{C_k}\})$, where C_k varies among the irreducible components of C .

Similarly, for Jacobian divisors we get

Corollary 4.38. *Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant map between two normal surface singularities, and denote by R_f the Jacobian divisor of f . Then the map $\nu \mapsto \nu(R_f)$ is locally constant outside the skeleton \mathcal{S}_{R_f} adapted to \mathfrak{m}_X and generated by the curve valuations associated to irreducible components in the support of R_f .*

We now focus our attention on the map $\nu \mapsto c(f, \nu)$. First, we give an interpretation of the attraction rate $c(f, \nu)$ in the setting of b-divisors.

Proposition 4.39. *Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ between two normal surface singularities, and $\nu \in \mathcal{V}_X$. Then*

$$c(f, \nu) = -Z(\nu) \cdot \mathcal{E}(f^*Z(\mathfrak{m}_Y)). \quad (34)$$

Proof. Applying (21) and intersecting with $-Z(\mathfrak{m}_Y)$, we have

$$-f_*Z(\nu) \cdot Z(\mathfrak{m}_Y) = c(f, \nu) - \sum_{\nu_C \in \mathcal{C}_f} (-Z(\nu_C) \cdot Z(\nu))c_\alpha(f, \nu_C),$$

where we used the fact that $-Z(\nu') \cdot Z(\mathfrak{m}_Y) = 1$ for all normalized semivaluations $\nu' \in \mathcal{V}_Y$. By the projection formula, $-f_*Z(\nu) \cdot Z(\mathfrak{m}_Y) = -Z(\nu) \cdot f^*Z(\mathfrak{m}_Y)$. By (24) we have

$$\begin{aligned} -Z(\nu) \cdot \mathcal{R}(f^*Z(\mathfrak{m}_Y)) &= - \sum_{\nu_C \in \mathcal{C}_f} (-Z(f \cdot \nu_C) \cdot Z(\mathfrak{m}_Y))c_\alpha(f, \nu_C) \frac{-Z(C)}{m(C)} \cdot Z(\nu) \\ &= - \sum_{\nu_C \in \mathcal{C}_f} c_\alpha(f, \nu_C) (-Z(\nu_C) \cdot Z(\nu)), \end{aligned}$$

where we used the fact that $Z(C) \cdot Z = Z(\text{int}_C) \cdot Z$ for any exceptional b-divisor $Z \in \mathcal{E}(X)$. Equation (34) directly follows, recalling that $f^*Z(\mathfrak{m}_Y) = \mathcal{E}(f^*Z(\mathfrak{m}_Y)) + \mathcal{R}(f^*Z(\mathfrak{m}_Y))$. \square

Corollary 4.40. *Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a dominant map between two normal surface singularities. Set $V_f = \mathcal{C}_f \cup f_\bullet^{-1}(\mathcal{R}_Y)$, where \mathcal{C}_f is the set of contracted curve valuations, and $\mathcal{R}_Y = \mathcal{R}(\mathfrak{m}_Y)$ is the set of Rees valuations of the maximal ideal \mathfrak{m}_Y . Then the function $\nu \mapsto c(f, \nu)$ is locally constant on $\mathcal{V}_X \setminus \mathcal{S}_{c(f)}$, where $\mathcal{S}_{c(f)} = \mathcal{S}_{\mathfrak{m}_X}(V_f)$ is the skeleton generated by V_f and adapted to \mathfrak{m}_X .*

Proof. By Proposition 4.39, we may compute $c(f, \nu)$ as minus the intersection of $Z(\nu)$ with the exceptional part of $f^*(\mathfrak{m}_Y)$. Taking the exceptional part of (22), we have

$$\mathcal{E}(f^*Z(\mathfrak{m}_Y)) = \sum_{\nu \in V_f} a_\nu Z(\nu),$$

for suitable $a_\nu > 0$. We conclude by Proposition 4.36. \square

Notice that when $(Y, y_0) = (\mathbb{C}^2, 0)$ is smooth, then $Z(\mathfrak{m}) = Z(\text{ord}_0)$, where ord_0 is the multiplicity valuation at $0 \in \mathbb{C}^2$. Then $\mathcal{R}_{\mathbb{C}^2} = \{\text{ord}_0\}$, and Corollary 4.40 coincide with the characterization given by [29, Proposition 3.4].

The skeleta $\mathcal{S}_{c(f)}$ and \mathcal{S}_{R_f} will play an important role on the study of the dynamical features of f_\bullet . We will denote by $\mathcal{S}_f = \mathcal{S}_{c(f)} \cup \mathcal{S}_{R_f}$ their union, and refer to it as the *critical skeleton* of f . It is generated by a finite number of divisorial or curve semivaluations, and the functions $\nu \mapsto c(f, \nu)$ and $\nu \mapsto \nu(R_f)$ are both locally constant on $\mathcal{V}_X \setminus \mathcal{S}_f$.

4.7. Classification of valuative dynamics

Finally, we come now to a precise statement of [Theorem B](#), our classification of dynamics on the valuation spaces \mathcal{V}_X . The setup for the theorem the following: (X, x_0) is an irreducible normal surface germ, $f : (X, x_0) \rightarrow (X, x_0)$ is a dominant holomorphic map, and R_f is the Weil divisor on (X, x_0) defined by the vanishing of the Jacobian determinant of f , as discussed in [§4.5](#). In addition, we make the crucial assumption that f is *non-invertible*; note, everything discussed in this section up to now is valid without this assumption, but we will use it heavily in our proof of the classification. In fact for *invertible* germs, the action f_\bullet is always an isometry with respect to the angular distance ρ . Moreover, $A(f_\bullet \nu) = A(\nu)$ and $c(f, \nu) = 1$ for all valuations $\nu \in \mathcal{V}_X$.

In the statement, we will use the following terminology. A set $S \subseteq \mathcal{V}_X$ is said to be *totally invariant* if $f_\bullet^{-1}(S) = S$. This is stronger than usual notion of invariance $f_\bullet(S) \subseteq S$. If f_\bullet is surjective and S is totally invariant, then $f_\bullet(S) = S$.

Theorem 4.41. *For the dynamical system $f_\bullet : \mathcal{V}_X \rightarrow \mathcal{V}_X$, exactly one of the following statements holds.*

1. *The map f is not finite. In this case, there is a semivaluation $\nu_\star \in \mathcal{V}_X$ for which $f_\bullet^n \nu \rightarrow \nu_\star$ weakly as $n \rightarrow \infty$ for all $\nu \in \mathcal{V}_X$ of finite skewness. Obviously any such ν_\star is unique and fixed by f_\bullet . If ν_\star is of finite skewness, then in fact $f_\bullet^n \nu \rightarrow \nu_\star$ in the strong topology for each $\nu \in \mathcal{V}_X$ of finite skewness.*
2. *The map f is finite and $R_f \neq 0$. In this case, (X, x_0) must be a quotient singularity (or non-singular). There is a subset $I \subset \mathcal{V}_X$ which is homeomorphic to a closed interval or a point satisfying the following properties: (a) I is totally invariant, (b) I is fixed pointwise by f_\bullet^2 , and (c) for every $\nu \in \mathcal{V}_X$ of finite skewness there is a $\nu_\star \in I$ such that $f_\bullet^{2n} \nu \rightarrow \nu_\star$ weakly as $n \rightarrow \infty$. If this ν_\star itself has finite skewness, then in fact $f_\bullet^{2n} \nu \rightarrow \nu_\star$ in the strong topology.*
3. *The map f is finite and $R_f = 0$. In this case (X, x_0) is a lc but not lt singularity. The set $S := A^{-1}(0) \subset \mathcal{V}_X$ is totally invariant for f_\bullet , and $f_\bullet|_S$ is an isometry for the angular distance ρ . The set S is either a point (for simple elliptic singularities or their quotients), a segment (quotient-cusp singularities) or a circle (cusp singularities). Denote by $r_S : \mathcal{V}_X \rightarrow S$ the natural retraction. Then for every $\nu \in \mathcal{V}_X$ of finite skewness we have $\rho(f_\bullet^n \nu, f_\bullet^n r_S \nu) \rightarrow 0$ as $n \rightarrow \infty$.*

An important special case of [Theorem 4.41](#) was proved in the papers [\[35, 57\]](#), namely the case when $(X, x_0) = (\mathbb{C}^2, 0)$ is non-singular. More precisely, [\[35\]](#) proved the theorem for dominant holomorphic maps $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ which are *superattracting*, that is, for which the differential df_0 is nilpotent. If f is non-invertible but not superattracting, then df_0 has exactly one non-zero eigenvalue; in this case f is said to be *semi-superattracting*, and the theorem was proved in [\[57\]](#). Note, the only two statements that apply in the non-singular setting are the first two.

A strong dichotomy appears in the statement of [Theorem 4.41](#). When the map $f : (X, x_0) \rightarrow (X, x_0)$ is *non-finite*, we will use the contraction properties of f_\bullet with respect to the angular distance given by [Corollary 4.25](#). When the map f is *finite*, we will apply the following results, see Wahl [\[67\]](#), [\[25\]](#) for proofs.

Theorem 4.42 ([\[67\]](#), [\[25, Theorem B\]](#)). *Let $f : (X, x_0) \rightarrow (X, x_0)$ be a non-invertible germ. If f is finite, then (X, x_0) is log canonical. Moreover, (X, x_0) is log terminal if and only if $R_f \neq 0$.*

Remark 4.43. [Theorem 4.42](#) holds only over fields of characteristic zero. Favre's strategy to prove such a result is to use the Jacobian formula [\(30\)](#) (which does not hold in positive characteristic), and use the fact that the map f_\bullet is surjective for finite germs by [Proposition 4.4](#). Notice that in positive characteristic, any singularity (defined over \mathbb{F}_q) admits non-invertible finite germs (see [Example 9.7](#)).

The next two sections will be devoted to the proof of [Theorem 4.41](#) in the non-finite and in the finite cases.

5. Dynamics of non-finite germs

This section is devoted to the proof of [Theorem 4.41](#) in the non-finite case.

5.1. Construction of an eigenvaluation

The strong contraction property stated in [Theorem 4.24](#) will allow us to construct a unique (attracting) fixed valuation $\nu_\bullet \in \mathcal{V}_X$ for f_\bullet . An analogous construction in the smooth case can be found in [[29](#), Theorems 4.2 and 4.5].

We first need a few definitions.

Definition 5.1. Let (X, x_0) be a normal surface singularity, and $f : (X, x_0) \rightarrow (X, x_0)$ be a non-invertible dominant germ. A normalized semivaluation $\nu_\bullet \in \mathcal{V}_X$ is called *locally attracting* if it is fixed by f_\bullet and there exists a weak open set $U \subset \mathcal{V}_X$ containing ν_\bullet such that $f_\bullet^n \nu \rightarrow \nu_\bullet$ in the weak topology, for any $\nu \in U$ of finite skewness.

Remark 5.2. We will be interested in locally attracting semivaluations mainly when they are non-quasimonomial, which correspond to *ends* in the language of \mathbb{R} -trees. Similar definitions of local attracting behaviors were previously introduced in [[29](#), Section 4.1] for ends of the valuative tree. Transferring their definition to our setting, a non-quasimonomial semivaluation $\nu_\bullet \in \mathcal{V}_X \setminus \mathcal{V}_X^{\text{qm}}$ is a *strongly attracting end* if there exists a f_\bullet -invariant segment $I = [\nu_0, \nu_\bullet]$ with $\nu_0 < \nu_\bullet$ so that $f_\bullet \nu > \nu$ for all $\nu \in [\nu_0, \nu_\bullet)$. We will see that a strongly attracting end is locally attracting in the sense of [Definition 5.1](#). In fact, in the case of a strongly attracting end ν_\bullet , one gets the existence of a weak open neighborhood U of ν_\bullet so that $f_\bullet^n \nu \rightarrow \nu_\bullet$ for all $\nu \in U$ not necessarily of finite skewness.

The behavior of semivaluations with infinite skewness (in particular, of curve semivaluations) may be quite chaotic, see e.g. [[34](#)]. As an easy example, consider the map $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ given in coordinates by $f(x, y) = (x^2, xy)$. In this case f_\bullet admits a unique quasimonomial fixed point, the multiplicity ord_0 . On the one hand, the orbit of every valuation of finite skewness is attracted by ord_0 , and ord_0 is locally attracting. On the other hand, the curve semivaluation ν_{C_θ} associated to the curve $C_\theta = \{y = \theta x\}$ is fixed by f_\bullet for any $\theta \in \mathbb{C}$. It follows that there are no weak open neighborhoods U of ord_0 where $f_\bullet^n \nu \rightarrow \nu_\bullet$ weakly for all $\nu \in U$.

Definition 5.3. Let (X, x_0) be a normal surface singularity, and $f : (X, x_0) \rightarrow (X, x_0)$ be a non-invertible dominant germ. We say that a normalized semivaluation $\nu \in \mathcal{V}_X$ fixed by f_\bullet is an *eigenvaluation* for f if it is either a locally attracting end, or if it belongs to the (weak) closure of the set $\text{Fix}(f_\bullet) \cap \mathcal{V}_X^{\text{qm}}$ of fixed quasimonomial valuations.

Remark 5.4. Notice that, by definition, all quasimonomial valuations fixed by f_\bullet are eigenvaluations; nevertheless we could have fixed semivaluations that are not eigenvaluations. Consider for example the map $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ given by $(x, y) = (x, y^d)$, where $d \geq 2$. Denote by $\nu_{x,t}$ the monomial valuation of weights $(t, 1)$ with respect to the coordinates (x, y) . Then $f_\bullet \nu_{x,t} = \nu_{x,t/d}$ for any $t \geq d$. In particular, the curve semivaluation $\nu_x = \nu_{x,+\infty}$ is fixed, but it is not an eigenvaluation, since it is not locally attracting.

Quasimonomial valuations are not necessarily locally attracting, since they could belong to a segment or circle of fixed eigenvaluations, as indicated by [Theorem 4.41](#). Consider for example the map $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ given by $(x, y) = (x^d, y^d)$ for any $d \geq 2$. In this case the eigenvaluations are all elements of the segment $[\nu_x, \nu_y]$, since valuations in $I = (\nu_x, \nu_y)$ are fixed quasimonomial valuations, and ν_x and ν_y are fixed and belong to the weak closure of I .

Remark 5.5. From the study of the dynamics of f_\bullet that will lead to the proof of [Theorem 4.41](#) in this and the next sections, we may deduce several properties of the set of eigenvaluations of a given map f . In particular, we will show that if f admits a locally attracting eigenvaluation, then this is the unique eigenvaluation for f .

We can also deduce an alternative description of the set of eigenvaluations of a given map f . It is the weak closure of the set of semivaluations $\nu_\star \in \mathcal{V}_X$ admitting a weak open set $U \subseteq \mathcal{V}_X$ so that $\nu_\star \in \overline{U}$ and for any semivaluation $\nu \in U$ of finite skewness we have $f_\bullet^n \nu \rightarrow \nu_\star$ weakly as $n \rightarrow +\infty$.

Notice that when $\nu_\star \in U$, this is exactly the definition of locally attracting.

We will need the following (easy) fixed point theorem, see for example [\[21\]](#).

Lemma 5.6. *Let (S, ρ) be a compact metric space, and $F : S \rightarrow S$ be a continuous selfmap satisfying*

$$\rho(F(v), F(w)) < \rho(v, w) \quad \text{for all } v \neq w.$$

Then there exists a unique fixed point $\nu_\star \in S$, and $F^n(v) \rightarrow \nu_\star$ for all $v \in S$.

We deduce directly the following fixed point theorem for the maps induced by f_\bullet on skeleta.

Corollary 5.7. *Let (X, x_0) be a normal surface singularity, and $f : (X, x_0) \rightarrow (X, x_0)$ be a non-finite germ. For every good resolution $\pi : X_\pi \rightarrow (X, x_0)$, denote by $F_\pi = r_\pi \circ f_\bullet : \mathcal{S}_\pi \rightarrow \mathcal{S}_\pi$ be the induced map on the skeleton \mathcal{S}_π associated to π . Then F_π has a unique fixed point ν_π , and for all $\nu \in \mathcal{S}_\pi$ we have $F_\pi^n \nu \rightarrow \nu_\pi$.*

Proof. Let ν and μ be two different valuations in \mathcal{S}_π . Then we have

$$\rho(F_\pi(\nu), F_\pi(\mu)) = \rho(r_\pi f_\bullet \nu, r_\pi f_\bullet \mu) \leq \rho(f_\bullet \nu, f_\bullet \mu) < \rho(\nu, \mu),$$

where the first inequality follows from [Lemma 2.43](#), while the second inequality is given by [Theorem 4.24](#). We conclude by applying [Lemma 5.6](#) to F_π . \square

We can now proceed to construct eigenvaluation for non-finite germs.

Theorem 5.8. *Let (X, x_0) be a normal surface singularity, and let $f : (X, x_0) \rightarrow (X, x_0)$ be a non-finite germ. Then there exists a unique eigenvaluation for f .*

Proof. For any good resolution $\pi : X_\pi \rightarrow (X, x_0)$, we consider the map $F_\pi = r_\pi \circ f_\bullet : \mathcal{S}_\pi \rightarrow \mathcal{S}_\pi$ induced on the skeleton associated to π . By [Corollary 5.7](#), any F_π admits a unique fixed point $\nu_\pi \in \mathcal{S}_\pi$. The unicity guarantees that $\nu_\pi = r_\pi \nu_{\pi'}$ whenever π' dominates π .

Assume there is a π so that $f_\bullet \nu_\pi = \nu_\pi$. In this case $\nu_\star = \nu_\pi$ is a quasimonomial eigenvaluation. Assume by contradiction that μ_\star is another eigenvaluation for f . If μ_\star is quasimonomial, then there exists π' dominating π so that ν_\star and μ_\star belong to $\mathcal{S}_{\pi'}$. In this case $F_{\pi'}$ would have two fixed points ν_\star and μ_\star , a contradiction. Hence there are no other quasimonomial eigenvaluations, and the only other possibility for μ_\star is to be a locally attracting valuation. But in this case, there exists a good resolution π' dominating π so that $\mu = r_{\pi'} \mu_\star \neq \nu_\star$ is fixed by $F_{\pi'}$, a contradiction.

Assume now that $f_\bullet \nu_\pi \neq \nu_\pi$ for any good resolution π . In particular ν_π is divisorial for all π , since for irrational valuations we have $r_\pi^{-1}(\nu_\pi) = \{\nu_\pi\}$.

Fix any good resolution π_0 , and set $\nu_0 = \nu_{\pi_0}$. Recall that by [Proposition 2.38](#), the set $\mathcal{V}_{\pi_0, \nu_0}$ of valuations in \mathcal{V}_X whose retraction to \mathcal{S}_{π_0} is ν_0 is a tree rooted at ν_0 . By our assumption, $f_\bullet \nu_0$ is a divisorial valuation which does not belong to \mathcal{S}_{π_0} . Since $\nu_0 = F_{\pi_0}(\nu_0)$, we have $r_{\pi_0} f_\bullet \nu_0 = \nu_0$, hence $f_\bullet \nu_0 \in \mathcal{V}_{\pi_0, \nu_0}$. Let π_1 be any good resolution so that $f_\bullet \nu_0 \in \mathcal{S}_{\pi_1}$, set $\nu_1 = \nu_{\pi_1}$. Notice that $\nu_1 \in \mathcal{S}_{\pi_1} \setminus \mathcal{S}_{\pi_0} \subset \mathcal{V}_{\pi_0, \nu_0}$, and in particular $\nu_1 > \nu_0$.

Proceeding by induction, we construct a strictly increasing sequence of divisorial valuations $(\nu_n)_{n \in \mathbb{N}}$ such that $F_{\pi_n}(\nu_n) = \nu_n$ and $r_{\pi_n}(\nu_m) = \nu_n$ for any $m \geq n$. By construction, this sequence weakly converges to a non-quasimonomial semivaluation $\nu_* \in \mathcal{V}_X$. We get

$$f_\bullet \nu_* = \lim_{n \rightarrow +\infty} f_\bullet \nu_n \geq \lim_{n \rightarrow +\infty} \nu_n = \nu_*,$$

and $\nu_* = f_\bullet \nu_*$.

We now prove that ν_* is locally attracting, and hence an eigenvaluation. We recall that f_\bullet is order-preserving where $c(f, -)$ is locally constant. By [Corollary 4.40](#), this is the case outside the skeleton $\mathcal{S}_{c(f)}$ which contains \mathcal{S}_X , and it is generated by all critical curve valuations and a finite number of divisorial valuations. Since ν_* is either a infinitely-singular valuation or a curve semivaluation non-contracted by f , there exists a weakly-open neighborhood U of ν_* where $c(f, -)$ is constant. and where consequently f_\bullet is order-preserving. We may assume that $U = \{\nu \in \mathcal{S}_{\pi_0, \nu_0} \mid \nu > \nu_N\}$ for some N big enough. Again by [Proposition 2.38](#), \bar{U} is a tree rooted at ν_N , $f_\bullet \nu_N \geq \nu_N$ and $f_\bullet U \subset U$. We deduce that $f_\bullet|_U$ defines a regular tree map in the sense of [[29](#), §4], and ν_* is a strongly attracting end by [[29](#), Theorem 4.5]. Up to increasing N , we may assume that $I = [\nu_N, \nu_*]$ is f_\bullet -invariant and $f_\bullet^n \nu \rightarrow \nu_*$ for all $\nu \in I$. Since f_\bullet is order-preserving on U , we deduce that $f_\bullet^n \nu \rightarrow \nu_*$ for all $\nu \in U$, and ν_* is locally attracting. Finally, arguing as in the previous case, we deduce that ν_* is the unique eigenvaluation in this case as well. \square

5.2. Weak convergence

Recall that $\mathcal{V}_X^\alpha := \{\nu \in \mathcal{V}_X \mid \alpha(\nu) < +\infty\}$ denotes the set of normalized valuations with finite skewness.

Lemma 5.9. *Let (X, x_0) be a normal surface singularity, and $f : (X, x_0) \rightarrow (X, x_0)$ a dominant germ. Let S be any subset of \mathcal{V}_X^α . Then the set*

$$B_S^\alpha = \{\nu \in \mathcal{V}_X^\alpha \mid f_\bullet^n \nu \text{ strongly converges to } S\}$$

is closed in the strong topology.

Proof. Let $(\nu_k)_k$ be a sequence of valuations in B_S^α strongly converging to some valuation $\nu_\infty \in \mathcal{V}_X^\alpha$. For any $\varepsilon > 0$, let k be big enough so that $\rho(\nu_\infty, \nu_k) < \varepsilon/2$. Since $f_\bullet^n \nu_k$ converges to S , there exists N such that for all $n \geq N$, $\rho(f_\bullet^n \nu_k, S) < \varepsilon/2$. Then for all $n \geq N$, we have

$$\rho(f_\bullet^n \nu_\infty, S) \leq \rho(f_\bullet^n \nu_\infty, f_\bullet^n \nu_k) + \rho(f_\bullet^n \nu_k, S) \leq \rho(\nu_\infty, \nu_k) + \rho(f_\bullet^n \nu_k, S) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

\square

Theorem 5.10. *Let (X, x_0) be a normal surface singularity, and $f : (X, x_0) \rightarrow (X, x_0)$ a non-finite germ. Let ν_* be the unique eigenvaluation of f given by [Theorem 5.8](#). Assume that ν_* is not quasimonomial. Then for any normalized valuation $\nu \in \mathcal{V}_X^\alpha$ of finite skewness, we have $f_\bullet^n \nu \rightarrow \nu_*$ in the weak topology.*

Proof. We set $B^\alpha = \{\nu \in \mathcal{V}_X^\alpha \mid f_\bullet^n \nu \text{ weakly converges to } \nu_*\}$ the weak basin of attraction to ν_* . The strategy of the proof is to prove that B^α is both open and closed in the strong topology. Since \mathcal{V}_X^α is connected, this gives the statement.

Since ν_* is not quasimonomial, it is a weakly attracting end, and there exists a weakly open neighborhood U of ν_* so that $f_\bullet^n \nu \rightarrow \nu_*$ weakly for all $\nu \in U$. Note that we may pick U of the form $\{\nu \in \mathcal{V}_X \mid \nu > \mu\}$ for a suitable divisorial valuation μ .

The basin of attraction B^α is clearly a non-empty open set (in the weak and strong topologies), since it coincides with

$$B^\alpha = \mathcal{V}_X^\alpha \cap \bigcup_{n \in \mathbb{N}} f_\bullet^{-n}(U).$$

We now show that B^α is also closed in the strong topology.

Let $(\nu_k)_k$ be a sequence of valuations belonging to B^α and converging to some valuation $\nu_\infty \in \mathcal{V}_X^\alpha$.

Set $\varepsilon = \rho(\mu, f_\bullet \mu) > 0$, so that U contains a ε -neighborhood of $f_\bullet U \cap \mathcal{V}_X^\alpha$. Pick k big enough so that $\rho(\nu_k, \nu_\infty) < \varepsilon$. Since $f_\bullet^n \nu_k \rightarrow \nu_\star$ there exists $N \gg 1$ so that $f_\bullet^n \nu_k \in f_\bullet(U)$ for all $n \geq N$. By [Theorem 4.22](#), for all $n \in \mathbb{N}$ we have

$$\rho(f_\bullet^n \nu_k, f_\bullet^n \nu_\infty) \leq \rho(\nu_k, \nu_\infty) < \varepsilon.$$

In particular, $f_\bullet^n \nu_\infty$ belong to U for all $n \geq N$, and $\nu_\infty \in B^\alpha$. □

In the next section, we will deal with the case when the eigenvaluation ν_\star is quasimonomial. In this case, and more in general when ν_\star has finite skewness, we will show (see [Theorem 5.15](#)) that the orbit of all valuations with finite skewness strongly converge to ν_\star .

Remark 5.11. For all known applications, the weak convergence of orbits of valuations of finite skewness to ν_\star would be sufficient.

With this in mind, one could try to prove an analogous statement to the one of [Theorem 5.10](#) when the eigenvaluation ν_\star is quasimonomial.

Since the basin of attraction to ν_\star is closed by [Lemma 5.9](#), we only need to show that it is also open in \mathcal{V}_X^α . In some cases it is easy to show that, given a quasimonomial valuation $\nu \in \mathcal{V}_X$, the sequence $\nu_n = f_\bullet^n \nu$ weakly converges to ν_\star .

- The first trivial case is when there exists N so that $\nu_N = \nu_\star$, since in this case $\nu_n = \nu_\star$ for all $n \geq N$. In the next cases, we will assume that $\nu_n \neq \nu_\star$ for all n .
- If there exists a good resolution $\pi: X_\pi \rightarrow (X, x_0)$ so that $\nu_n \in \mathcal{S}_\pi$ for all n , then $\nu_n = f_\bullet^n(\nu) = F_\pi^n(\nu)$ for all $n \in \mathbb{N}$, where $F_\pi: \mathcal{S}_\pi \rightarrow \mathcal{S}_\pi$ is the map induced by f on the skeleton \mathcal{S}_π . By [Corollary 5.7](#), $\nu_n \rightarrow \nu_\star$ strongly, and hence weakly.
- Suppose that for any connected component U of $\mathcal{V}_X \setminus \{\nu_\star\}$, the set $I_U = \{n \in \mathbb{N} \mid \nu_n \in U\}$ is finite. Then again $\nu_n \rightarrow \nu_\star$ weakly. Recall that a weak open set is the union of connected components of the complement of a finite number of points in \mathcal{V}_X . In particular, any weak open connected neighborhood V of ν_\star contains all but finitely many connected components U_1, \dots, U_r of $\mathcal{V}_X \setminus \{\nu_\star\}$. Fixed any such V , take $N = N(V)$ big enough so that $I_{U_j} \subseteq \{0, \dots, N\}$ for all $j = 1, \dots, r$. Then ν_n belongs to V for all $n > N$, and $\nu_n \rightarrow \nu_\star$ weakly.

The situations described above are quite special, and do not cover all possible behaviors of orbits of a valuation ν . To deal with the general situation, even for weak convergence, we need to measure the speed of contraction towards the eigenvaluation through log discrepancies, using the Jacobian Formula ([32](#)).

5.3. Semi-superattracting germs

To prove the strong convergence of f_\bullet to eigenvaluations, for non-finite germs as well as for finite germs, we need to establish the *superattracting* behavior of f , i.e., that the orbits of points converge to x_0 faster than exponentially. In fact, when $(X, x_0) \cong (\mathbb{C}^2, 0)$ is a smooth point, the study of maps $f: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ differs according to the properties of the differential df_0 at 0. If f is non-invertible, there are essentially two cases: either df_0 is nilpotent (*superattracting case*), or it has a non-zero eigenvalue (*semi-superattracting case*).

Although this definition does not carry over the singular setting, its algebraic interpretation does.

Definition 5.12. Let (X, x_0) be a normal surface singularity, and $f : (X, x_0) \rightarrow (X, x_0)$ be a holomorphic germ. We say that f is *superattracting* if $(f^n)^* \mathfrak{m} \subseteq \mathfrak{m}^2$ for $n \in \mathbb{N}^*$ big enough.

Superattractivity can be rephrased in terms of the behavior of the maps $\nu \mapsto c(f^n, \nu)$ on \mathcal{V}_X .

Proposition 5.13. *A dominant selfmap $f : (X, x_0) \rightarrow (X, x_0)$ on a normal surface singularity is superattracting if and only if there exists $C > 1$ and n big enough so that $c(f^n, \nu) \geq C$ for all normalized semivaluations $\nu \in \mathcal{V}_X$.*

Proof. The direct implication is straightforward, since if $(f^n)^* \mathfrak{m} \subseteq \mathfrak{m}^2$, then for any normalized semivaluation $\nu \in \mathcal{V}_X$ we have

$$c(f^n, \nu) = \nu((f^n)^* \mathfrak{m}) \geq \nu(\mathfrak{m}^2) = 2\nu(\mathfrak{m}) = 2.$$

Conversely, assume there exists $C > 1$ so that $c(f, \nu) \geq C$ for all $\nu \in \mathcal{V}_X$ (achieved from the hypothesis up to replacing f by an iterate). Notice that

$$c(f^n, \nu) = \prod_{j=0}^{n-1} c(f, f_*^j \nu) \geq C^n.$$

Since $C > 1$, for any $k \geq 2$ there exists $n = n(k)$ so that $c(f^n, \nu) \geq k$ for all $\nu \in \mathcal{V}_X$. This implies that $(f^n)^* \mathfrak{m} \subseteq \overline{(f^n)^* \mathfrak{m}} \subseteq \overline{\mathfrak{m}^k}$, where, given an ideal \mathfrak{a} , we denote by $\overline{\mathfrak{a}}$ is integral closure (see, e.g., [40, Section 6.8]). By the Briançon-Skoda theorem (see [39, Theorem 4.13]), we have that $\overline{\mathfrak{m}^k} \subseteq \mathfrak{m}^2$ for k big enough, and we are done. \square

In the case of non-finite germs, being superattracting is the same as having $c(f, \nu_*) > 1$, where ν_* is the unique eigenvaluation of f given by Theorem 5.8.

In the smooth setting, the valuative dynamics of semi-superattracting germs $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ is similar to the one of superattracting germs, and in fact simpler. Since by definition the eigenvalues of df_0 are 0 and $\lambda \neq 0$, we have a “superstable” manifold D (associated to the eigenvalue 0) and a “relatively unstable” manifold C (associated to the eigenvalue $\lambda \neq 0$). In this case ν_C is the unique eigenvaluation of f , and all valuations (but at most ν_D) weakly-converge to ν_C (see [57]).

In the singular setting, a similar phenomenon happens.

Proposition 5.14. *Let (X, x_0) be a normal surface singularity, and $f : (X, x_0) \rightarrow (X, x_0)$ a non-finite germ. Let $\nu_* \in \mathcal{V}_X$ be the unique eigenvaluation of f given by Theorem 5.8. If $c(f, \nu_*) = 1$, then $\alpha(\nu_*) = +\infty$.*

Proof. Since $c(f, \nu_*) = 1$ and $f_* \nu_* = \nu_*$, (21) in this case gives

$$f_* Z(\nu_*) = Z(\nu_*) - \sum_{\nu_C \in \mathcal{C}_f} a_C Z(\nu_C)$$

for suitable $a_C > 0$. By intersecting with $-Z(\nu_*)$, we infer that

$$-Z(\nu_*) \cdot f_* Z(\nu_*) = \alpha(\nu_*) - a \tag{35}$$

for a suitable $a > 0$. Analogously, from (22) we get

$$f^* Z(\nu_*) = m(f, \nu_*) Z(\nu_*) + \sum_{\nu_\mu \in f_*^{-1}\{\nu_*\}} b_\mu Z(\nu_\mu) + \sum_{\nu_C \in \mathcal{C}_f} b_C Z(\nu_C)$$

for suitable $b_\mu, b_C > 0$, and by intersecting with $-Z(\nu_*)$ we get

$$-Z(\nu_*) \cdot f^* Z(\nu_*) = m(f, \nu_*) \alpha(\nu_*) + b \tag{36}$$

for a suitable $b > 0$ (notice that $Z(\nu_C) \cdot Z(\nu_*) < +\infty$, since ν_* cannot be a contracted curve semivaluation). By the projection formula, (35) and (36) coincide, and we have

$$\alpha(\nu_*) = m(f, \nu_*)\alpha(\nu_*) + a + b,$$

which implies $\alpha(\nu_*) = +\infty$. □

We will deal later with the situation of semi-superattracting finite germs (Theorem 6.3).

5.4. Strong convergence

We can now state and prove the strong convergence of valuations towards the eigenvaluation for non-finite germs.

Theorem 5.15. *Let (X, x_0) be a normal surface singularity, and $f : (X, x_0) \rightarrow (X, x_0)$ a non-finite germ. Let ν_* be the unique eigenvaluation of f given by Theorem 5.8, and assume that $\nu_* \in \mathcal{V}_X^\alpha$ has finite skewness. Then for any normalized valuation $\nu \in \mathcal{V}_X^\alpha$, we have $f_\bullet^n \nu \rightarrow \nu_*$ in the strong topology.*

Proof. Following the proof of Theorem 5.10, we only need to prove that there exists a strongly open neighborhood of ν_* in \mathcal{V}_X^α where the orbits strongly converge to ν_* . Notice that by Proposition 5.14, the attraction rate $c = c(f, \nu_*)$ is strictly bigger than 1. We split the proof according to the type of eigenvaluation ν_* .

Case 1. The eigenvaluation ν_* is infinitely singular and has finite skewness. Assume first that ν_* has finite thinness as well. As in the proof of Theorem 5.8, we may find a connected weakly open set $U \subset \mathcal{V}_X$ where $c(f, \nu) = c$ and $\nu(R_f) = \varepsilon$ are constant, and f_\bullet is order preserving. We deduce by the Jacobian Formula (32) that

$$A(f_\bullet \nu) - A(\nu_*) = \frac{A(\nu) + \varepsilon}{c} - \frac{A(\nu_*) + \varepsilon}{c} = \frac{A(\nu) - A(\nu_*)}{c}.$$

By iterating this estimate, we get that the thinness distance between $f_\bullet^n \nu$ and ν_* decreases to 0 when n goes to $+\infty$, and we have strong convergence.

Assume now that ν_* has infinite thinness (but finite skewness). We can take the open neighborhood of ν_* as above of the form $U = \{\nu \in \mathcal{V} \mid \nu > \nu_0\}$ for a suitable ν_0 . Set $I = (\nu_0, \nu_*]$, and denote by $r_I : U \rightarrow I$ the natural retraction. Notice that I is f_\bullet -invariant, and for any $\mu \in I$ the orbit $f_\bullet^n \mu \rightarrow \nu_*$ in the strong topology. We conclude by remarking

$$A(f_\bullet^n \nu) - A(f_\bullet^n r_I \nu) = \frac{A(\nu) - A(r_I \nu)}{c^n} \rightarrow 0.$$

Case 2. The eigenvaluation ν_* is irrational. Let $I = [\nu_0, \nu_1]$ be an interval with divisorial endpoints, containing ν_* in its interior. Up to shrinking I , we may assume that f_\bullet is injective on I , and $\nu \mapsto c(f, \nu)$ and $\nu \mapsto \nu(R_f)$ are locally constant on $U(I) \setminus I$ (notice that such maps need not to be constant on I itself). Since $f_\bullet I$ is an interval containing ν_* , we have that $J = I \cap f_\bullet I$ is an interval containing ν_* in its interior, satisfying $f_\bullet(J) \subseteq I$. In particular, f_\bullet coincides with the map $F_\pi : \mathcal{S}_\pi \rightarrow \mathcal{S}_\pi$ for a suitable good resolution $\pi : X_\pi \rightarrow (X, x_0)$, as defined in the proof of Theorem 5.8. By Corollary 5.7, $f_\bullet^n \mu \rightarrow \nu_*$ for all $\mu \in J$.

We now want to show strong convergence of $f_\bullet^n \nu \rightarrow \nu_*$ for all $\nu \in U$, for a suitable strong open neighborhood of ν_* . Consider the set $K = f_\bullet^{-1}(J)$. It is made by a finite number of connected components, and denote by K_0 the one containing ν_* . Then K_0 is a finite union of segments with divisorial endpoints.

Up to shrinking J , we may assume that $K_0 \cap U(J) \subset J$. Set $U = U(J)$. By construction, $r_\pi f_\bullet^n \nu = f_\bullet^n r_\pi \nu$ for all $\nu \in U$. Since f_\bullet is order preserving on $U \setminus J$, f_\bullet is increasing on the interval $[r_\pi \nu, \nu]$. Moreover, $c(f, \nu) = c(f, r_\pi \nu)$ and $\nu(R_f) = r_\pi \nu(R_f)$ for all $\nu \in U$. Pick any $\nu \in U$ of finite thinness, and set $\mu = r_\pi \nu$. By the Jacobian formula (32), we infer that

$$A(f_\bullet^n \nu, f_\bullet^n \mu) = \frac{A(\nu) - A(\mu)}{c(f^n, \mu)} \rightarrow 0$$

when $n \rightarrow \infty$. While A could be constant on J (if J belongs to the essential skeleton \mathcal{S}_X of \mathcal{V}_X), A is not locally constant on $U \setminus J$, and we deduce the strong convergence of $f_\bullet^n \nu \rightarrow \nu_*$ for all $\nu \in U$ of finite thinness. Since $U \cap \{A(\nu) < +\infty\}$ is a strong neighborhood of ν_* , we are done.

Case 3. The eigenvaluation $\nu_* = \nu_{E_*}$ is divisorial. To construct a strongly open neighborhood of ν_* that belongs to the strong basin of attraction to ν_* , we follow the same strategy used in the smooth case, see [35, Lemma 3.9]. Let $\pi: X_\pi \rightarrow (X, x_0)$ be any good resolution that realizes E_* . Recall that for any p , the open set $U_\pi(p)$ represents a tangent vector $\vec{v}_p \in T_{\nu_*} \mathcal{V}_X$. For any such tangent vector, we want to find a positive number $\rho(\vec{v}_p)$ so that:

- for any $\nu \in U_\pi(p)$ so that $\rho(\nu, \nu_*) < \rho(\vec{v}_p)$, then $f_\bullet^n \nu \rightarrow \nu_*$ strongly;
- $\inf_{p \in E_*} \rho(\vec{v}_p) > 0$.

Let \mathcal{S}_f denote the critical skeleton, so that $\nu \mapsto c(f, \nu)$ and $\nu \mapsto \nu(R_f)$ are locally constant on $\mathcal{V}_X \setminus \mathcal{S}_f$. Let \mathcal{T} be the (finite) set of tangent directions \vec{v}_p so that $U_{\pi_0}(p) \cap \mathcal{S}_f \neq \emptyset$. Consider the sets \mathcal{T}_1 of tangent vectors \vec{v} so that $df_\bullet^n \vec{v} \notin \mathcal{T}$ for all $n \in \mathbb{N}$, and \mathcal{T}_2 of tangent vectors in \mathcal{T} which are df_\bullet -periodic. Any tangent vector \vec{v} is eventually mapped by df_\bullet to either \mathcal{T}_1 or \mathcal{T}_2 .

First, assume that $\vec{v}_p \in \mathcal{T}_1$, and pick any quasimonomial valuation $\nu \in U_\pi(p)$. Notice that $c(f, f_\bullet^n \nu) = c(f, \nu_*) =: c$ and $f_\bullet^n \nu(R_f) = \nu_*(R_f) =: \varepsilon$ for all $n \in \mathbb{N}$. As in *Case 1* we get

$$A(f_\bullet^n \nu) - A(\nu_*) = c^{-n}(A(\nu) - A(\nu_*)) \rightarrow 0$$

as $n \rightarrow +\infty$. Since the thinness function A is not constant on any segment in $\mathcal{V} \setminus \mathcal{S}_\pi$, this implies the strong convergence. We set in this case $\rho(\vec{v}_p) = +\infty$.

Assume now that $\vec{v}_p \in \mathcal{T}_2$, and pick again any quasimonomial valuation $\nu'' \in U_\pi(p)$. Consider the segment $[\nu_*, \nu'']$. Since \vec{v}_p is periodic, there exists $m \in \mathbb{N}^*$ and another segment $J = [\nu_*, \nu']$ with $\nu' \in (\nu_*, \nu'']$, so that $f_\bullet^m(J) \subseteq J$. Let $\pi': X_{\pi'} \rightarrow (X, x_0)$ be a good resolution dominating π and so that $J \subset \mathcal{S}_{\pi'}$. By construction, f_\bullet^m coincides with $F_{\pi'}^m$ on J , where $F_{\pi'}^m = r_{\pi'} \circ f_\bullet^m: \mathcal{S}_{\pi'} \rightarrow \mathcal{S}_{\pi'}$. By [Corollary 5.7](#), $f_\bullet^{nm} \mu \rightarrow \nu_*$ strongly for any $\mu \in J$. Repeating the same argument for $df_\bullet^k \vec{v}_p$ and $f_\bullet^k \nu''$ for all $k = 1, \dots, m-1$, we deduce that $f_\bullet^n \mu \rightarrow \nu_*$. Let U' denote the connected component of $\mathcal{V}_X \setminus \{\nu_*, \nu'\}$ intersecting J . Up to shrinking J , we may assume that $(U' \setminus J) \cap \mathcal{S}_f = \emptyset$. By the same argument used in the irrational case, we can deduce that $f_\bullet^n \nu \rightarrow \nu_*$ strongly for any $\nu \in U'$. In this case we set $\rho(\vec{v}_p) = \rho(\nu', \nu_*) > 0$.

Finally, assume that $\vec{v}_p \notin \mathcal{T}_1 \cup \mathcal{T}_2$. We define $\rho(\vec{v}_p)$ recursively as follows. Suppose that \vec{v}_p is such that $\rho(df_\bullet \vec{v}_p)$ has been defined. If \vec{v}_p is one of the (finite number of) tangent vectors in $\mathcal{T} \setminus \mathcal{T}_2$, then we just set $\rho(\vec{v}_p)$ to be a value $\rho > 0$ sufficiently small to have that for any $\nu \in U_\pi(p)$ so that $\rho(\nu, \nu_*) < \rho$, then $\rho(f_\bullet \nu, \nu_*) < \rho(df_\bullet \vec{v}_p)$. If $\vec{v}_p \notin \mathcal{T}$, then f_\bullet is order preserving on $U_\pi(p)$. It follows that $f_\bullet(U_\pi(p)) \subset \mathcal{V}_X \setminus \{\nu_*\}$, and by continuity, it belongs to the connected component associated to $df_\bullet \vec{v}_p$. We may define $\rho(\vec{v}_p) = \rho(df_\bullet \vec{v}_p)$. In fact, if $\nu \in U_\pi(p)$ and $\rho(\nu, \nu_*) \leq \rho$, by [Theorem 4.22](#) we have $\rho(f_\bullet \nu, \nu_*)$.

We have defined $\rho(\vec{v})$ for any $\vec{v} \in T_{\nu_*} \mathcal{V}_X$. Clearly by construction, we have that $\rho_* := \inf_{\vec{v}} \rho(\vec{v}) > 0$. In particular, the basin of attraction to ν_* with respect to the strong topology contains the strong open neighborhood $\{\nu \in \mathcal{V}_X \mid \rho(\nu, \nu_*) < \rho_*\}$. This completes the proof. \square

Remark 5.16. Notice that the weak convergence towards the eigenvaluation established by [Theorem 5.10](#) holds also over fields of positive characteristic. This will be enough for proving the existence of dynamically stable models for non-finite germs having a non-quasimonomial eigenvaluation even in positive characteristic.

6. Dynamics of non-invertible finite germs

This section is devoted to the proof of [Theorem 4.41](#) for finite germs. By [Theorem 4.42](#), (X, x_0) is log canonical (lc). We will analyze separately the cases when (X, x_0) is log terminal (lt, which are the quotient singularities), when (X, x_0) is not. In the latter case, we will show that f is necessarily superattracting ([Theorem 6.3](#)).

6.1. Quotient singularities

Suppose first that (X, x_0) is log terminal, i.e., a quotient singularity. It is isomorphic to the quotient $(\mathbb{C}^2, 0)/G$ of \mathbb{C}^2 by a finite subgroup G of $GL_2(\mathbb{C})$, acting freely on $\mathbb{C}^2 \setminus \{0\}$. We denote by $\text{pr}: \mathbb{C}^2 \rightarrow X$ the natural projection. Notice that when restricted to $\mathbb{C}^2 \setminus \{0\}$, the map pr gives the universal covering of $X \setminus \{x_0\}$.

Let now $f: (X, x_0) \rightarrow (X, x_0)$ be a non-invertible germ. While in the non-finite case, f cannot in general be lifted to a holomorphic germ $g: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ (see [§9.3](#)), it is always the case when f is finite.

Proposition 6.1. *Let $f: (X, x_0) \rightarrow (X, x_0)$ be a finite germ on a quotient singularity $(X, x_0) = (\mathbb{C}^2, 0)/G$. Let $\text{pr}: (\mathbb{C}^2, 0) \rightarrow (X, x_0)$ be the natural projection. Then there exists a finite germ $g: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ so that $f \circ \text{pr} = \text{pr} \circ g$.*

Proof. Since f is finite, $f^{-1}(x_0)$ is discrete, and there exists a neighborhood $U \subset X$ of x_0 such that $f: U \rightarrow f(U)$ satisfies $f^{-1}(x_0) = \{x_0\}$. The map $f \circ \text{pr}$, which sends $\text{pr}^{-1}(U) \setminus \{0\}$ to $f(U) \setminus \{x_0\}$, lifts to a holomorphic map $g: \text{pr}^{-1}(U) \setminus \{0\} \rightarrow \text{pr}^{-1}(f(U)) \setminus \{0\}$. By Hartog's theorem, g extends through 0 to a map defined on $\text{pr}^{-1}(U)$, which clearly satisfies the wanted properties. \square

The commutative diagram $f \circ \text{pr} = \text{pr} \circ g$ induces a commutative diagram on the action on valutive spaces, $f_\bullet \circ \text{pr}_\bullet = \text{pr}_\bullet \circ g_\bullet$. In particular pr_\bullet acts as a semi-conjugation between f_\bullet and g_\bullet .

By the result for non-invertible maps on smooth surfaces [[35](#), Theorem 3.1], we know that g_\bullet satisfies case 2 of [Theorem 4.41](#). Assume we are in the first case and let μ_* be the unique eigenvaluation for g_\bullet . Then it is easy to check that $\nu_* = \text{pr}_\bullet \mu_*$ is the unique eigenvaluation for f_\bullet , and the orbit of any quasimonomial valuation converges to ν_* , strongly as far as μ_* (and hence ν_*) has finite skewness. Assume we are in the second case, and let J be the segment of eigenvaluations for g_\bullet^2 . Then again $I = \text{pr}_\bullet J$ is a segment of eigenvaluations for f_\bullet^2 , and $f_\bullet^{2n} \nu \rightarrow I$ strongly for any quasimonomial valuation ν . This concludes the proof in this case.

6.2. Non-lt singularities

Recall that by [Theorem 4.42](#), for any non-invertible finite germ $f : (X, x_0) \rightarrow (X, x_0)$ on a lc non lt singularity, the Jacobian divisor R_f is trivial.

Lemma 6.2. *Let (X, x_0) be a lc non lt singularity, and $f : (X, x_0) \rightarrow (X, x_0)$ be a non-invertible finite germ. Then the set $S = \{\nu \in \mathcal{V}_X \mid A(\nu) = 0\}$ is totally invariant, and $f_\bullet|_S : S \rightarrow S$ is an isometry with respect to the angular distance ρ .*

Proof. The Jacobian formula [\(32\)](#) states in this case that $A(f_\bullet \nu) = \frac{A(\nu)}{c(f, \nu)}$. In particular, the set S is totally invariant by f_\bullet . By [§3.1](#), the set S is either a point, a segment or a circle. By [Theorem 4.22](#), the map f_\bullet is non-expanding for the angular distance. By [Proposition 4.4](#), $f_\bullet|_S$ is surjective. It follows that $f_\bullet|_S$ is an isometry. \square

As for non-finite germs, to prove strong convergence to S we need to show that f is necessarily superattracting. Notice that quotient singularities admit finite non-invertible germs which are not superattracting. Consider for example the cyclic group G generated by $\Phi(x, y) = (\zeta x, \zeta^q y)$, with ζ a primitive p -root of unity, and q coprime with p . Then the map $g(x, y) = (\lambda x, y^d)$ commutes with Φ as far as $d \equiv 1$ modulo p , and defines a semi-superattracting germ on the cyclic quotient singularity $(\mathbb{C}^2, 0)/G$. This cannot happen for other lc singularities.

Theorem 6.3. *Let $f : (X, x_0) \rightarrow (X, x_0)$ be a non-invertible finite germ on a normal surface singularity. Suppose that (X, x_0) is lc not lt. Then f is superattracting.*

Proof. Set $S = \{\nu \in \mathcal{V}_X \mid A(\nu) = 0\}$. By [Lemma 6.2](#), $f_\bullet|_S$ is an isometry with respect to ρ . By [§3.1](#), S is either a point, a segment, or a circle. Since $f_\bullet|_S$ is an isometry, it follows that it is either of finite order, or S is a circle and f_\bullet acts as an irrational rotation.

Let $\nu \in S$ be any valuation. Since S are totally invariant and $f|_S$ is a bijection, we get $f_\bullet \nu \in S$ and $f_\bullet^{-1}(f_\bullet \nu) = \{\nu\}$. By [\(21\)](#) and [\(22\)](#) we have

$$c(f, \nu)Z(f_\bullet \nu) \cdot Z(f_\bullet \nu) = Z(f_\bullet \nu) \cdot f_* Z(\nu) = f^* Z(f_\bullet \nu) \cdot Z(\nu) = e(f)Z(\nu) \cdot Z(\nu), \quad (37)$$

where $e(f)$ denotes the topological degree of f .

Assume that $f_\bullet|_S$ has finite order. Up to taking a suitable iterate, we may assume that f_\bullet acts as the identity on S . Suppose by contradiction that there exists a point $\nu \in S$ satisfying $c(f, \nu) = 1$. Then [\(37\)](#) applied to ν says that $e(f) = c(f, \nu) = 1$, which is a contradiction, since finite maps with topological degree 1 are invertible.

Assume now that $f_\bullet|_S$ is an irrational rotation on the circle S . Suppose by contradiction that there exists a $\nu \in S$ so that $c(f, f_\bullet^n \nu) = 1$ for all n . We may find a subsequence $(f_\bullet^{n_k} \nu)_k$ of the orbit of ν which converges to ν . Then [\(37\)](#) tells us that

$$e(f)^{n_k} Z(\nu) \cdot Z(\nu) = Z(f_\bullet^{n_k} \nu) \cdot Z(f_\bullet^{n_k} \nu) \rightarrow Z(\nu) \cdot Z(\nu),$$

and again $e(f) = 1$, a contradiction.

By continuity of $\nu \mapsto c(f^n, \nu)$ (with respect to the weak topology) and compactness of S , we infer that there exists $C > 1$ and $N \gg 1$ so that $c(f^N, \nu) \geq C$ for all $\nu \in S_\varepsilon$, where $S_\varepsilon = \{\nu \in \mathcal{V}_X \mid A(\nu) < \varepsilon\}$. In particular, $c(f^n, \nu) \rightarrow +\infty$ for all $\nu \in S_\varepsilon$. By the Jacobian formula [\(32\)](#), for any $s \in S_\varepsilon$ we have that

$$A(f_\bullet^n \nu) = \frac{A(\nu)}{c(f, \nu)} \rightarrow 0,$$

and S_ε is contained in the basin of attraction B_S^α to S (of valuations with finite skewness). Hence B_S^α is open in \mathcal{V}_X^α . But B_S^α is also closed in \mathcal{V}_X^α by [Lemma 5.9](#). By connectedness, $B_S^\alpha = \mathcal{V}_X^\alpha$.

In particular $c(f^n, \nu) \rightarrow \infty$ for any valuation of finite skewness, and f is superattracting. \square

As a consequence of the proof of [Theorem 6.3](#), we get strong convergence towards S .

Corollary 6.4. *Let (X, x_0) be a lc non lt singularity, and $f : (X, x_0) \rightarrow (X, x_0)$ be a non-invertible finite germ. Set $S = \{\nu \in \mathcal{V}_X \mid A(\nu) = 0\}$. Then for any valuation $\nu \in \mathcal{V}_X$ of finite skewness, we have $f_\bullet^n \nu \rightarrow S$ with respect to the strong topology.*

We have seen in [§3.1](#) that S can be either a point, a segment or a circle, according to whether (X, x_0) is (a quotient of) a simple elliptic singularity, a quotient-cusp singularity, or a cusp singularity.

In the first case, [Corollary 6.4](#) gives directly case 3 of the statement of [Theorem 4.41](#). In the second and third cases, we conclude by the following proposition.

Proposition 6.5. *Let $f : (X, x_0) \rightarrow (X, x_0)$ be a non-invertible finite germ. Let $S \subset \mathcal{V}_X$ be a (closed, connected) set of normalized semivaluations, totally invariant for f_\bullet . Denote by $r_S : \mathcal{V}_X \rightarrow S$ the retraction to S . Then for any valuation $\nu \in \mathcal{V}_X$ such that $f_\bullet^n \nu \rightarrow S$, we have $\rho(f_\bullet^n, f_\bullet^n r_S \nu) \rightarrow 0$.*

Proof. We claim that $r_S f_\bullet^n \nu = f_\bullet^n r_S \nu$ for any $n \in \mathbb{N}^*$. This implies the statement.

If $\nu \in S$, by invariance of S we also have $f_\bullet^n \nu \in S$, and there is nothing to prove. We assume that $\nu \notin S$, and set $\nu_* := r_S \nu \neq \nu$. Suppose by contradiction that there exists $n \in \mathbb{N}^*$ so that $r_S f_\bullet^n \nu \neq f_\bullet^n \nu_*$. Take a minimal n satisfying this condition. Up to replacing ν by $f_\bullet^{n-1} \nu$, we may assume that $n = 1$.

Consider the segment $[\nu_*, \nu]$. Notice that by hypothesis, $f_\bullet \nu$ and $f_\bullet \nu_*$ belong to different connected components of $\mathcal{V}_X \setminus S$. Since f_\bullet is continuous, there exists $\mu \in (\nu_*, \nu)$ so that $f_\bullet \mu \in S$. This contradicts the total invariance of S , since $\mu \notin S$. \square

6.3. Irrational rotations on cusp singularities

We conclude by showing that irrational rotations occur on any cusp singularity. The examples given are a generalization of [\[25, §2.5\]](#).

Proposition 6.6. *For any cusp singularity (X, x_0) , there exists a finite germ $f : (X, x_0) \rightarrow (X, x_0)$ so that the action of f_\bullet on the set S of normalized valuations of zero log discrepancy is conjugated to an irrational rotation.*

Proof. Any cusp singularity (X, x_0) is analytically isomorphic to one constructed as in [§A.1](#). It depends on a square-free positive integer $d \geq 2$, a totally positive element $\omega \in \mathbb{Q}(\sqrt{d})$, generating a rank 2 free \mathbb{Z} -module $N_\omega = \mathbb{Z} \oplus \omega\mathbb{Z}$, and a totally positive unit $\varepsilon = a + b\sqrt{d}$ so that $\varepsilon N_\omega = N_\omega$ (in particular, $\varepsilon^n \in N_\omega$ for all $n \in \mathbb{Z}$). Notice that a and b are not necessarily integers depending on d , but $2a$ and $2b$ are always positive integers.

We consider now the finite endomorphism f_α associated to a totally positive element $\alpha \in N_\omega \cap \varepsilon^{-1}N_\omega$ (see [§A.2](#)). The action of $(f_\alpha)_\bullet$ on the cycle $\mathcal{S}_X \cong \mathbb{R}/\mathbb{Z}$ is conjugated to the translation of

$$\beta = \frac{\log(\alpha/\alpha')}{2 \log \varepsilon}$$

modulo \mathbb{Z} .

To prove the statement, it suffice to find a $\alpha \in N_\omega \cap \varepsilon^{-1}N_\omega$ which is totally positive and so that $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Notice that $\beta \in \mathbb{Q}$ if and only if some power of α/α' is an integer multiple of a power of ε^2 . Write $\varepsilon = a + b\sqrt{d}$, and set consider $\alpha = p + b\sqrt{d}$, where $p \in \mathbb{N}$ is to determine. Notice that $\alpha = p - a + \varepsilon \in N_\omega$ and $\varepsilon\alpha = \varepsilon^2 + (p-a)\varepsilon \in N_\omega$. Moreover, α is totally positive whenever $p > a$, since in this case it is given by a positive linear combination of totally positive numbers. By direct computation,

$$\frac{\alpha}{\alpha'} = \frac{p + b\sqrt{d}}{p - b\sqrt{d}} = \frac{p^2 + db^2 + 2bp\sqrt{d}}{p^2 - db^2}.$$

Notice that for p big enough, $2|p^2 - db^2| > |2bp|$. It follows that for such p the number $\frac{\alpha}{\alpha'}$ is not integral in $\mathbb{Q}(\sqrt{d})$. The same holds for any power of α/α' . Since ε is integral in $\mathbb{Q}(\sqrt{d})$, so is any positive multiple of any power of ε . It follows that β is irrational, and we are done. \square

7. Algebraic stability

This section is devoted to proving [Theorem A](#). First, we introduce a definition to simplify the statement.

Definition 7.1. Let $f : (X, x_0) \rightarrow (X, x_0)$ be a dominant non-invertible germ. We say that a birational model $\pi : X_\pi \rightarrow (X, x_0)$ is a *geometrically stable* model for f if the lift $f_\pi : X_\pi \dashrightarrow X_\pi$ has the following property. For every exceptional prime E of X_π , $f_\pi^n(E)$ is an indeterminacy point of f_π for at most finitely many n .

Geometrical stability should be thought as the right concept of algebraic stability in the local setting. In fact, from geometrical stability one can recover easily algebraic stability.

Lemma 7.2. Let $f : (X, x_0) \rightarrow (Y, y_0)$ and $g : (Y, y_0) \rightarrow (Z, z_0)$ be two dominant germs between normal surface singularities. Let $\pi : X_\pi \rightarrow (X, x_0)$, $\varpi : Y_\varpi \rightarrow (Y, y_0)$ and $\eta : Z_\eta \rightarrow (Z, z_0)$ be good resolutions. Set $\tilde{f} = \varpi^{-1} \circ f \circ \pi : X_\pi \dashrightarrow Y_\varpi$ and $\tilde{g} = \eta^{-1} \circ g \circ \varpi : Y_\varpi \dashrightarrow Z_\eta$. If for any exceptional prime $E \in \Gamma_\pi^*$ we have that $\tilde{f}(E)$ is not an indeterminacy point of \tilde{g} , then $(g \circ f)^* = f^* \circ g^*$.

The proof of [Lemma 7.2](#) is a straightforward adaptation of the arguments used in [[35](#), Lemma 5.1]. As an immediate consequence, we get:

Proposition 7.3. Let $f : (X, x_0) \rightarrow (X, x_0)$ be a dominant non-invertible germ. Let $\pi : X_\pi \rightarrow (X, x_0)$ be a proper birational map, and denote by $f_\pi : X_\pi \dashrightarrow X_\pi$ the lift of f to X_π . If π is a geometrically stable model for f , then there exists $N \in \mathbb{N}$ so that for any $n \geq N$, we have

$$(f_\pi^n)^* = (f_\pi^N)^*(f_\pi^*)^{n-N} \quad \text{and} \quad \mathcal{E}_\pi \circ (f_\pi^n)^* = \mathcal{E}_\pi \circ (f_\pi^N)^*(\mathcal{E}_\pi \circ f_\pi^*)^{n-N}.$$

[Theorem A](#) can be reformulated as the existence of geometrically stable models, dominating any given model.

Theorem 7.4. Let (X, x_0) be an irreducible germ of a normal complex surface at a point $x_0 \in X$, and let $f : (X, x_0) \rightarrow (X, x_0)$ be a dominant non-invertible holomorphic map. Assume that f is not a finite germ at a cusp singularity inducing an irrational rotation. Then for any modification $\pi : X_\pi \rightarrow (X, x_0)$, one can find another modification $\pi' : X_{\pi'} \rightarrow (X, x_0)$ dominating π for which f is geometrically stable. In general $X_{\pi'}$ may have cyclic quotient singularities. It can be taken smooth up to replacing f by an iterate.

Notice that the singular surfaces $X_{\pi'}$ will only have (cyclic) quotient singularities. Such surfaces can be also described with the formalism of *orbifolds* (also called *V-manifolds*). Hence [Theorem 7.4](#) could be also restated as a result on the existence of geometrically stable orbifolds.

Finally, in view of [Proposition 7.3](#), we clearly have that [Theorem A](#) implies [Theorem C](#).

7.1. Existence of geometrically stable models

This section is dedicated to proving [Theorem 7.4](#).

We split the proof of [Theorem 7.4](#) according to the dynamics of f_\bullet (see [Theorem 4.41](#)):

1. f_\bullet admits a unique eigenvaluation ν_* , which is not quasimonomial; every valuation $\nu \in \mathcal{V}_X^\alpha$ weakly converges to ν_* .

2. f_\bullet admits a unique eigenvaluation ν_* , which is irrational; every valuation $\nu \in \mathcal{V}_X^\alpha$ strongly converges to ν_* .
3. There exists $S \subseteq \mathcal{V}_X$ which is either a divisorial point, a segment with divisorial or curve endpoints, or a circle, so that $f_\bullet^k|_S = \text{id}_S$, and every valuation $\nu \in \mathcal{V}_X^\alpha$ converges strongly to S .

Case 1. f_\bullet admits a unique eigenvaluation ν_* , which is not quasimonomial.

Let U be a weak open neighborhood of ν_* which avoids the critical skeleton \mathcal{S}_f where either $\nu \mapsto c(f, \nu)$ or $\nu \mapsto \nu(R_f)$ are not locally constant. As we have seen in the proof of [Theorem 5.15](#), this implies that $f_\bullet(U) \subseteq U$ and $f_\bullet^n \nu \rightarrow \nu_*$ for any $\nu \in U$. In fact, we may take U of the form $U = U(\nu_0) = \{\nu \in \mathcal{V}_X \mid \nu > \nu_0\}$, where ν_0 is any valuation such that $\nu_0 < \nu_*$ and sufficiently close to ν_* .

For any modification $\pi: X_\pi \rightarrow (X, x_0)$, consider another modification $\pi': X_{\pi'} \rightarrow (X, x_0)$ dominating π and so that there exists an exceptional prime E in $X_{\pi'}$ so that $\nu_E \geq \nu_0$.

We may assume (by taking the exceptional prime E closer to ν_*) that the center of ν_* in $X_{\pi'}$ is a free point p in E . Then $U(\nu_E) = U_{\pi'}(p)$ is a neighborhood of ν_* which is f_\bullet -invariant.

By [Proposition 4.12](#), the lift $f_{\pi'}: X_{\pi'} \rightarrow X_{\pi'}$ defines a holomorphic fixed point at p . By [Theorem 5.10](#), for any exceptional prime D of $X_{\pi'}$, there exists N so that $f_\bullet^n(\nu_D) \in U$ for all $n \geq N$. Hence $f_{\pi'}^n(D) = p$ for all $n \in \mathbb{N}$, and the model π' is geometrically stable.

Notice that in this case $X_{\pi'}$ is smooth.

Case 2. f_\bullet admits a unique eigenvaluation ν_* , which is irrational.

As seen in the proof of [Theorem 5.15](#) for non-finite germs, or in [§6.1](#) for finite germs, we may find a closed interval J containing ν_* such that $f_\bullet J \subset J$, and $f_\bullet U(J) \subseteq U(J)$. We may assume that J has divisorial endpoints ν_E and ν_F . Let $\pi'': X_{\pi''} \rightarrow (X, x_0)$ be a good resolution dominating π so that $\nu_* \in \mathcal{S}_{\pi''}$, and ν_E and ν_F are realized in $X_{\pi''}$. Notice that this implies $J \subset \mathcal{S}_{\pi''}$. Let $\eta_{\tilde{\pi}\pi''}: X_{\pi''} \rightarrow X_{\tilde{\pi}}$ be the contraction of the (possibly empty) set of exceptional primes D in $X_{\pi''}$ for which ν_D belongs to the open segment $\mathring{J} =]\nu_E, \nu_F[$, and denote by $\tilde{\pi}: X_{\tilde{\pi}} \rightarrow (X, x_0)$ the induced modification. By our choice of J , the modification $\tilde{\pi}$ still dominates π . The image through $\eta_{\tilde{\pi}\pi''}$ of the contracted divisor is a (possibly singular) point $\tilde{p} \in \tilde{\pi}^{-1}(x_0)$, and $U(J) = U_{\tilde{\pi}}(\tilde{p})$. Up to taking a smaller J , we may assume that the contracted divisor is a chain of rational curves, so that \tilde{p} is either a smooth point or a cyclic quotient singularity.

By [Proposition 4.12](#), the lift $f_{\tilde{\pi}}: X_{\tilde{\pi}} \dashrightarrow X_{\tilde{\pi}}$ has a holomorphic fixed point at \tilde{p} . By [Theorem 5.10](#), for any exceptional prime D , there exists N so that $f_\bullet^n(\nu_D) \in U(J)$ for all $n \geq N$. Hence $f_{\tilde{\pi}}^n(D) = \tilde{p}$ for all $n \in \mathbb{N}$, and the model $\tilde{\pi}$ is geometrically stable.

Case 3. The set S of eigenvaluations for the iterates of f contains at least a divisorial valuation.

We split this case according to the geometry of S (which is either a point, a segment, or a circle). Before proceeding, we need to introduce some definitions and small lemmas.

Suppose we have a good resolution $\pi_0: X_{\pi_0} \rightarrow (X, x_0)$ so that the set $\mathcal{S}_{\pi_0}^*$ of divisorial valuations realized by π_0 contains at least the orbit of a divisorial valuation $\nu_{E_0} \in S$. Notice that by construction, the whole orbit belong to S . Set $\nu_{E_i} = f_\bullet^i \nu_{E_0}$ for all $i = 1, \dots, k$, where $E_k = E_0$ (we allow also other exceptional primes to coincide, so k needs not to be the minimal period of ν_{E_0}).

We recall (see [§2.4](#)) that to each point $p \in E_i$ correspond a tangent vector \vec{v}_p^i at ν_{E_i} . Moreover, any such tangent vector is associated to a connected component of $\mathcal{V}_X \setminus \{\nu_{E_i}\}$, that we denote by $U_i(p)$. Notice that different tangent vectors may be associated to the same connected component, notably when S is a circle.

Definition 7.5. We say that a point p is *parallel to S* if $U_i(p) \cap S \neq \emptyset$.

Remark 7.6. Notice that if $p \in E_i \cap E_j$, then the segment $]v_{E_i}, v_{E_j}[p$ belong to $S \cap U_i(p) \cap U_j(p)$. We deduce two properties. Firstly, being parallel depend only on p and not on the exceptional prime E_i that contains it, so the definition is well posed. Secondly, points that are non-parallel to S belong to a unique exceptional prime among the $\{E_i, i = 0, \dots, k-1\}$.

In fact, if we denote by $S_{\pi_0}^* = \{v_E \in S \mid E \in \Gamma_{\pi_0}^*\}$ the set of divisorial valuations on S realized by π_0 , and by $D(S_{\pi_0})$ the union of exceptional primes $E \subset X_{\pi_0}$ so that $v_E \in S_{\pi_0}^*$, then the points parallel to S are exactly the singular points of $D(S_{\pi_0})$. In particular, if S consists of a single divisorial valuation v_{E_*} , all points of E_* are not parallel to S .

Let now p^0 be a point non parallel to S . It belongs to a unique exceptional prime $E^0 \subset \pi_0^{-1}(x_0)$. Let $\pi_1: X_{\pi_1} \rightarrow (X, x_0)$ be a modification dominating π_0 , so that $\eta = \eta_{\pi_0\pi_1}: X_{\pi_1} \rightarrow X_{\pi_0}$ is an isomorphism over $X_{\pi_0} \setminus \{p^0\}$. The strict transform of E^0 by η intersects $\eta^{-1}(p^0)$ in a unique point, which we denote by p^1 . In fact $p^0 \in X_{\pi_0}$ and $p^1 \in X_{\pi_1}$ correspond to the same tangent vector \vec{v}_p^i at v_{E_i} . To ease notations, we write $p^0 = p^1 = p$, having in mind this interpretation through tangent vectors.

Moreover, for any point $p \in E$, we denote by $f_\pi(p)$ the image of p though $f_\pi|_E: E \rightarrow E$, which is well defined even if $p \in \text{Ind}(f_\pi)$.

We now need a lemma, identical in spirit to the analogous [30, Lemma 4.6] and [35, Lemma 5.4]. Here the proof is simpler with respect to the cited papers, since we allow the creation of cyclic quotient singularities. As usual, for any modification $\pi: X_\pi \rightarrow (X, x_0)$, we denote by $f_\pi: X_\pi \dashrightarrow X_\pi$ the lift of f to X_π .

Lemma 7.7. *Let $f: (X, x_0) \rightarrow (X, x_0)$ be a dominant germ satisfying condition 3. Consider a periodic divisorial valuation $v_{E_0} \in S$, and let $\pi_0: X_{\pi_0} \rightarrow (X, x_0)$ be any good resolution realizing $v_{E_i} = f_\bullet^i v_{E_0}$ for all $i \in \mathbb{N}$. Let $p_0 \in E_0$ be any periodic point for f_{π_0} satisfying the condition $p_i = f_{\pi_0}^i(p_0)$ is not parallel to S for any $i \in \mathbb{N}$. Then there exists a modification $\pi_1: X_{\pi_1} \rightarrow (X, x_0)$ dominating π_0 , obtained as modifications over the orbit of p_0 , so that f_{π_1} is regular along the orbit of \tilde{p}_0 . Moreover, π_1 can be chosen so that X_{π_1} has at most cyclic quotient singularities.*

Proof. Set $r \in \mathbb{N}^*$ the period of p_0 . Notice that since p_i is not parallel to S for all i , it belongs to a unique component in the orbit of E_0 , which is E_i by construction. Let v_0 be any divisorial valuation on the tangent direction associated to p_0 , and set $v_i = f_\bullet^i v_0$ for $i = 1, \dots, r$. By taking v_0 sufficiently close to v_{E_0} , we may assume that v_i is on the tangent direction associated to p_i at v_{E_i} for all $i = 0, \dots, r$. Denote by J_i the segment $]v_{E_i}, v_i]_{p_i}$ for all $i = 0, \dots, r$. Again by taking v_0 sufficiently close to v_* , we may assume that $U(J_i) \cap \mathcal{S} \subseteq J_i$. It follows that $f_\bullet(U(J_i)) \subseteq U(J_{i+1})$ for all $i = 0, \dots, r-1$. Moreover, we may also assume that J_i does not contain in its relative interior any divisorial valuation associated to non-rational exceptional primes. Consider a good resolution π_2 dominating π_0 and realizing v_0, \dots, v_{r-1} . Now let $\eta_{\tilde{\pi}\pi_2}: X_{\pi_2} \rightarrow X_{\tilde{\pi}}$ be the contraction of all prime divisors D whose associated valuation v_D belongs to $]v_{E_i}, v_i]_{p_i}$ for $i = 0, \dots, r-1$.

The induced modification $\tilde{\pi}: X_{\tilde{\pi}} \rightarrow (X, x_0)$ has (at most) r cyclic quotient singularities $\tilde{p}_0, \dots, \tilde{p}_{r-1}$, corresponding to the contraction of the chains of rational curves belonging to J_0, \dots, J_{r-1} . By construction, $U(J_i) = U_{\tilde{\pi}}(\tilde{p}_i)$, and by Proposition 4.12, the action of the lift $f_{\tilde{\pi}}: X_{\tilde{\pi}} \dashrightarrow X_{\tilde{\pi}}$ is regular at the points \tilde{p}_i , $i = 0, \dots, r-1$. \square

We now come back to the proof of Theorem 7.4.

Case 3a. $S = \{v_*\}$ is a unique divisorial eigenvaluation.

Let $\pi: X_\pi \rightarrow (X, x_0)$ be any modification, and consider a good resolution $\pi_0: X_{\pi_0} \rightarrow (X, x_0)$ dominating π and so that $v_* = v_{E_*}$ is realized by π_0 . Let U be any weak open neighborhood of v_* that does not

contain any other divisorial valuation realized by π_0 . By construction, for any exceptional prime D in X_{π_0} , there exists N so that $f_{\bullet}^n \nu_D \in U$ for all $n \geq N$.

If there exists M so that $f_{\bullet}^M \nu_D = \nu_*$, then we get that $f_{\pi}(D) = E_*$ for all $n \geq M$, and D satisfies the geometrical stability condition. Assume this is not the case. Then for any $n \geq N$, $f_{\bullet}^n \nu_D$ belongs to the connected component of $\mathcal{V}_X \setminus \Gamma_{\pi_1}^*$ associated to some tangent direction \vec{v}_n corresponding to a point $p_n \in E_*$. For n big enough, $\vec{v}_{n+1} = df_{\bullet} \vec{v}_n$. If the orbit \vec{v}_n is infinite, it will avoid indeterminacy points after a finite number of iterates, and the geometrical stability condition is satisfied. Assume the orbit is finite, and up to replacing ν_D by $f_{\bullet}^n \nu_D$ for n big enough, we may assume that p_n is periodic of some period r . By applying [Lemma 7.7](#) to the periodic point p_n , we may find another modification $\pi_1: X_{\pi_1} \rightarrow (X, x_0)$ dominating π_0 so that the orbit of \tilde{p}_n does not meet $\text{Ind}(f_{\pi_1})$. In particular D satisfies the geometrical stability condition.

Notice that since E_* is compact, the number of indeterminacy points of f_{π_0} in E_* is finite. Moreover, the modification π_1 is an isomorphism outside of the orbit of p_n . In particular, the number of indeterminacy points of f_{π_1} on E_* is strictly smaller than the one for f_{π_0} . By applying this argument recursively on the number of periodic orbits in E_* meeting indeterminacy points, we find a model $\pi': X_{\pi'} \rightarrow (X, x_0)$ where $f_{\pi'}$ acts regularly along all periodic orbits in E_* . The model π' is geometrically stable for f .

Case 3b. S is either a (non-trivial) segment or a circle of eigenvaluations for f . Let $\pi: X_{\pi} \rightarrow (X, x_0)$ be any modification, and let S_{π}^* be the set of all divisorial valuations in S realized by π . Let $\pi_0: X_{\pi_0} \rightarrow (X, x_0)$ be any good resolution dominating π and realizing all divisorial valuations in S_{π}^* , plus possibly the divisorial endpoints of S if it is a segment. Up to taking an even higher model, we may assume that $S_{\pi_0}^*$ is not empty.

For any divisorial valuation $\nu_E \in S_{\pi_0}^*$, the exceptional prime E is invariant by f_{π_0} . By applying recursively [Lemma 7.7](#), we find a modification $\pi_1: X_{\pi_1} \rightarrow (X, x_0)$ dominating π_0 , with at most cyclic quotient singularities, so that f_{π_1} is regular along all periodic orbits which are not parallel to S . Now, let $\nu_E, \nu_F \in S_{\pi_1}^*$ be any two divisorial valuations realized by π_1 for which $E \cap F = \{p\}$ in X_{π_1} . By construction, p is a smooth point, and by total invariance of S , we have that $f_{\bullet}(U_{\pi_1}(p)) \subseteq U_{\pi_1}(p)$. In particular, f_{π_1} is regular at p for all $p \in \mathcal{V}_X$. If S has a curve endpoint, say ν_C , then an analogous property holds for the point of intersection between the strict transform of C and the exceptional divisor of π_1 . We deduce that, if E is the union of all exceptional primes associated to valuations in $S_{\pi_1}^*$, then f_{π_1} is regular along every periodic orbit in E . Since $f_{\bullet}^n \nu \rightarrow S$ for any $\nu \in \mathcal{V}_X^{\alpha}$, we have that for any exceptional prime $D \in X_{\pi_1}$, $f_{\pi_1}^n(D)$ is either a component E so that $\nu_E \in S$, or $f_{\pi_1}^n(D) = p_n$ is a point in E , for n big enough. Notice that if there is N so that $f_{\pi_1}^N(D) = E$, then $f_{\pi_1}^n(D) = E$ for all $n \geq N$, and the geometrical stability condition holds. If this is not the case, then either p_n has infinite orbit, and it will avoid the indeterminacy points of f_{π_1} for n big enough, or p_n has finite orbit, and again it will avoid indeterminacy points as far as it enters the periodic cycle in E . Hence π_1 is geometrically stable.

Case 3c. S is a (non-trivial) segment or a circle of eigenvaluations for f^k , $k \geq 2$, but not for f .

Let $\pi_0: X_{\pi_0} \rightarrow (X, x_0)$ be any good resolution dominating π and realizing all divisorial valuations in the set V given by $\bigcup_{i=0}^{k-1} f_{\bullet}^i S_{\pi}^*$, plus possibly the divisorial endpoints of S when it is a segment. Again by [Lemma 7.7](#), we may find a modification $\pi_1: X_{\pi_1} \rightarrow (X, x_0)$ dominating π_0 and so that f_{π_1} is regular along any periodic orbit of $E = \bigcup_{\nu_D \in V} D$ not parallel to S . In this case, there may be points $p \in E$, parallel to S , where f_{π_1} is not regular. To fix this, we consider the contraction $\eta_{\tilde{\pi}\pi_1}: X_{\pi_1} \rightarrow X_{\tilde{\pi}}$ of every exceptional prime $D \in S_{\pi_1}^* \setminus V$. These primes are organized in chains of rational curves between consecutive divisorial elements of V . In particular, the induced modification $\tilde{\pi}: X_{\tilde{\pi}} \rightarrow (X, x_0)$ has at most cyclic quotient singularities (the ones given by π_1 , plus the ones obtained by the contraction

$\eta_{\tilde{\pi}\pi_1}$). Notice that by construction, $\tilde{\pi}$ dominates π_0 , and hence π . By invariance of V under the action of f_\bullet , and since $f_\bullet|_S$ is totally invariant, we deduce that the action of $f_{\tilde{\pi}}$ is regular along every periodic orbit of E . As before, we deduce that $\tilde{\pi}$ is a geometrically stable model for f .

7.2. Smoothness of geometrically stable models

In the previous section, we constructed geometrically stable models which have cyclic quotient singularities. While this result is quite natural, and sufficient for all known applications, one could be interested in knowing when we can construct *smooth* geometrically stable models. Here we collect a few techniques and remarks to get such smooth models.

First, notice that the proof of [Theorem 7.4](#) produces already a smooth geometrically stable model when f falls into *Case 1*, i.e., it admits a unique eigenvaluation, which is non-quasimonomial. For the other cases, we first need some preliminary lemmas.

Consider a dominant germ $f: (X, x_0) \rightarrow (Y, y_0)$, and its induced map $f_\bullet: \mathcal{V}_X \rightarrow \mathcal{V}_Y$ on valutive spaces. If we are interested in the local behavior of this map on monomial valuations at some infinitely-near point p , it is natural to consider the action with respect to the monomial weights.

Lemma 7.8. *Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a dominant germ between normal surface singularities, and let $\nu \in \mathcal{V}_X$ be any quasimonomial valuation. Then there exists good resolutions $\pi: X_\pi \rightarrow (X, x_0)$ and $\varpi: Y_\varpi \rightarrow (Y, y_0)$ and infinitely near points $p \in \pi^{-1}(x_0)$, $q \in \varpi^{-1}(y_0)$, so that:*

- ν is a monomial valuation at p .
- For all $r, s \geq 0$ not both zero, the monomial valuation $\nu_{r,s}$ at p is sent to a monomial valuation $\nu_{r',s'} = f_\bullet \nu_{r,s}$ at q .
- There exists an invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with non-negative integer entries, so that $(r', s') = (ar + bs, cr + ds)$.

Proof. Take any good resolution π so that ν is monomial at a certain point in $\pi^{-1}(x_0)$. By [[15](#), Theorem 3.2], there exists good resolutions π and ϖ and infinitely-near points p and q as in the statement, so that the map $\tilde{f} = \varpi^{-1} \circ f \circ \pi: X_\pi \rightarrow Y_\varpi$ is regular, sends p to q , and can be written as

$$\tilde{f}(x, y) = (x^a y^b u(x, y), x^c y^d v(x, y)),$$

where (x, y) are local coordinates at p adapted to $\pi^{-1}(x_0)$, the coordinates in the target space are also centered at q and adapted to $\varpi^{-1}(y_0)$, and u and v are suitable holomorphic functions not vanishing at p . The statement easily follows from a direct computation. \square

For selfmaps, we need to express this behavior on monomial weights with the respect to the same monomial weights at the source and at the target, or equivalently, we need to impose $p = q$.

Lemma 7.9. *Let $f: (X, x_0) \rightarrow (X, x_0)$ be a dominant germ, and let $\nu_\star \in \mathcal{V}_X$ be any quasimonomial eigenvaluation. Assume there are intervals $I \subset \mathcal{V}_X$ containing ν_\star , as small as wanted, that are f_\bullet -invariant. Let $\pi: X_\pi \rightarrow (X, x_0)$ be any good resolution so that all valuations in I are monomial at a suitable point $p \in \pi^{-1}(x_0)$. Denote by (r_\star, s_\star) the weights of ν_\star as a monomial valuation at p . Then the following properties hold.*

- For all $r, s \geq 0$ so that s/r is sufficiently close to s_\star/r_\star , the monomial valuation $\nu_{r,s}$ at p is sent to a monomial valuation $\nu_{r',s'} = f_\bullet \nu_{r,s}$ at p .

- There exists an invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with non-negative integer entries, so that $(r', s') = (ar + bs, cr + ds)$ for all (r, s) with s/r sufficiently close to s_*/r_* .

Proof. Up to taking an higher good resolution, we may assume that $\{p\} = E \cap F$ for two exceptional primes E, F in X_{π_1} . The weights (r', s') may be computed using b-divisors, as $r' = Z(f_* \nu) \cdot Z(E)$ and $s' = Z(f_* \nu) \cdot Z(F)$, where $Z(E)$ denotes the Cartier b-divisor determined by E in the good resolution π_1 .

We make the computation for r' , the one for s' being completely analogous. By (21), we have

$$r' = Z(f_* \nu) \cdot Z(E) = f_* Z(\nu) \cdot Z(E) + \sum_{\nu_C \in \mathcal{C}_f} (-Z(\nu_C) \cdot Z(\nu)) c_a(f, \nu_C) Z(f_* \nu_C) \cdot Z(E).$$

Up to shrinking I and taking a higher good resolution π , we may assume that for all contracted curve valuations $\nu_C \in \mathcal{C}_f$, $f_* \nu_C$ does not belong to $[\nu_E, \nu_F]_p$. This implies that $Z(f_* \nu_C) \cdot Z(E) = 0$ for all $\nu_C \in \mathcal{C}_f$. Hence

$$r' = f_* Z(\nu) \cdot Z(E) = Z(\nu) \cdot f^* Z(E) = \sum_{D' \in \Gamma_{\pi'}^*} k_{D' \rightarrow E} Z_{\pi'}(\nu) \cdot D' + \sum_{C \in \text{Cc}_f} k_{C \rightarrow E} Z_{\pi'}(\nu) \cdot C,$$

where $\pi': X_{\pi_2} \rightarrow (X, x_0)$ is a good resolution dominating π and so that π' and π give a resolution of f with respect to \mathcal{C}_f . Notice that by our assumption on $f_* \nu_C$ not belonging to $[\nu_E, \nu_F]_p$, we infer that $k_{C \rightarrow E} = 0$ for all $C \in \text{Cc}_f$. We hence have

$$r' = \sum_{D' \in \Gamma_{\pi'}^*} k_{D' \rightarrow E} \eta^* Z_{\pi}(\nu) \cdot D' = \sum_{D' \in \Gamma_{\pi'}^*} k_{D' \rightarrow E_1} Z_{\pi}(\nu) \cdot \eta_* D' = \sum_{D \in \Gamma_{\pi}^*} k_{\eta^* D \rightarrow E} Z_{\pi}(\nu) \cdot D,$$

where $\eta = \eta_{\pi\pi'}$ and $\eta^* D$ denotes the strict transform of D through η . It follows that $r' = ar + bs + h$, where $a = k_{\eta^* E_1 \rightarrow E_1}$ and $b = k_{\eta^* F_1 \rightarrow E_1}$ are non-negative integers, and h is some constant. We proceed analogously F , obtaining $s' = cr + ds + k$ with analogous properties for c, d, k . We conclude by noticing that $h = k = 0$ when $s/r \in I$, and the matrix M is invertible, since we know that for such (r, s) we have $f_* \nu_{r,s} = \nu_{r',s'}$, and the map f_* cannot be locally constant since f is dominant, see Proposition 4.4(2). \square

Assume now we are in *Case 2* of the proof of Theorem 7.4, and f admits a unique eigenvaluation ν_* , which is irrational. To get a smooth geometrically stable model, we need to find an interval J' so that $f_* J' \subseteq J'$ of the form $J' = [\nu_{E'}, \nu_{F'}]$, where E' and F' are two exceptional primes which intersect at a point in a suitable good resolution.

Consider J as in the proof of Theorem 7.4. By taking J small enough, we may assume that $J \subset U = U_{\pi_0}(p_0)$ for a suitable (smooth) point p_0 in the exceptional divisor of a good resolution π_0 dominating π . Notice that the closure of U is isomorphic to the space of normalized valuations \mathcal{V}_{p_0} at $(X_{\pi_0}, p_0) \cong (\mathbb{C}^2, 0)$.

We may find a good resolution $\pi_1: X_{\pi_1} \rightarrow (X, x_0)$ dominating π_0 so that there exists two exceptional primes E_1, F_1 in X_{π_1} intersecting in a smooth point p_1 and so that $J \subseteq [\nu_{E_1}, \nu_{F_1}]_{p_1}$. Pick coordinates (x, y) at p_1 so that $E_1 = \{x = 0\}$ and $F_1 = \{y = 0\}$. Denote by $\nu_{r,s}$ the monomial valuation at p_1 of weights (r, s) with respect to the coordinates (x, y) . Notice that $\nu_{r,s}$ is normalized in $\hat{\mathcal{V}}_{p_0}$ the set of centered valuations at p_0 as long as $b_{E_1}^0 r + b_{F_1}^0 s = 1$, where b^0 denotes the general multiplicity in $\hat{\mathcal{V}}_{p_0}$. Moreover, J is given by such normalized valuations so that $s/r \in I$ for a suitable interval $I \subset [0, \infty]$. Since $f_* J \subset J$, then for any (r, s) with $s/r \in I$, we have $f_* \nu_{r,s} = \nu_{r',s'}$ for suitable (r', s') with $s'/r' \in I$. Notice that the irrational eigenvaluation $\nu_* \in J$ also satisfies $\nu_* = \nu_{r_*,s_*}$ and $f_* \nu_* = \nu_{r'_*,s'_*}$ is so that $s'_*/r'_* = s_*/r_* \in I \setminus \mathbb{Q}$.

By Lemma 7.9, for all (r, s) so that $s/r \in I$ is close enough to s_*/r_* , we have $(r', s') = (ar + bs, cr + ds) =: M(r, s)$, with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an invertible matrix with non-negative integer coefficients.

Now we can argue as in [29, p.330]. By [29, Lemma 5.7], there exists arbitrarily large integers r_0, s_0, r_1, s_1 such that $\frac{s_0}{r_0} < \frac{s_*}{r_*} < \frac{s_1}{r_1}$, $s_1 r_0 - s_0 r_1 = 1$, and M maps the interval $\left] \frac{s_0}{r_0}, \frac{s_1}{r_1} \right[$ into itself. This says exactly that there exists a toric modification (as high as desired) η_2 over p_1 , and exceptional primes E_2, F_2 which intersect at a smooth point p_2 , so that the segment $] \nu_{E_2}, \nu_{F_2}[_{p_2}$ is invariant by the action of f_\bullet . The toric modification is given in local charts at p_2 and p_1 by the monomial map associated to the matrix $\begin{pmatrix} s_1 & s_0 \\ r_1 & r_0 \end{pmatrix}$ for suitable coordinates adapted to E_2, F_2 and E_1, F_1 .

Assume we are in *Case 3c*, i.e., f admits a segment or a circle of eigenvaluations for f^k , $k \geq 2$, but not for f . In this case smooth geometrically stable models don't exist, not even when (X, x_0) is smooth (see [35]). But the k -th iterate of f falls into *Case 3b*, and we may try to construct a smooth geometrically stable model for f^k .

Assume hence we are in the remaining *Cases 3a* or *3b*. In these cases, the only cyclic quotient singularities which appear in the construction given by the proof of [Theorem 7.4](#) are given by applying [Lemma 7.7](#). In the smooth case, one can prove a similar result, without creating cyclic quotient singularities. This proves the existence of smooth geometrically stable models in these cases. The key point of the argument is a statement on the speed of convergence of $f_\bullet^n \nu$ to a (divisorial) eigenvaluation ν_* , when ν belongs to a small segment I with endpoints $\nu_* = \nu_{E_*}$ and some divisorial valuation ν_F . In fact, take any sufficiently small segment $I = [\nu_{E_*}, \nu_F]$ so that E_* and F intersect at a smooth point p of some good resolution π , so that we may parameterize I with respect to the monomial parameterization $w(t)$, $t \in [0, 1]$. Assume $f_\bullet I \subset I$; then by [Lemma 7.9](#), $f_\bullet(w(t)) = w(h(t))$, where h is a suitable Möbius map.

By [Proposition 2.34](#), $t \mapsto \alpha(w(t))$ is a polynomial mapping of degree ≤ 2 . If its degree is exactly 1, then $\alpha(f_\bullet(w(t))) = H(\alpha(w(t)))$, with H again a Möbius map (with non-negative integer coefficients).

The easy [\[29, Lemma 5.5\]](#) states that if H is such a map and t_* is a *strictly positive* fixed point for H , then either $|H'(t_*)| < 1$, or H has finite order. This is exactly the technical statement we need to ensure the smooth analogous of [Lemma 7.7](#).

As we have seen above, this same property holds when ν_* is the unique irrational eigenvaluation for f . By [Proposition 2.38](#), the desired property on α holds as far as there exists a log resolution π of m_X so that the tangent direction at ν_* corresponding to ν_F is not parallel to \mathcal{S}_π . Notice that if we work with respect to the monomial parameterization, we always get a Möbius map h which has the desired properties, but in this case the weight corresponding to the fixed point ν_{E_*} is zero, and [\[29, Lemma 5-5\]](#) does not apply.

To deal with tangent directions which lie in \mathcal{S}_π , we would need to prove a strong contraction property with respect to another weight function different from α . Although we believe that this strong contraction property still holds in general, we are unable to prove it and leave it as an open question.

8. Attraction rates

Let $f : (X, x_0) \rightarrow (X, x_0)$ be a dominant germ on a normal surface singularity (X, x_0) . Here we study for any (normalized) valuation $\nu \in \mathcal{V}_X$ (of finite skewness), the sequence $c(f^n, \nu)$ of attraction rates, and its asymptotic behavior the *first dynamical degree* (see below).

8.1. First dynamical degree

Before defining the first dynamical degree, we need to describe a few properties of the sequence of attraction rates.

Proposition 8.1. *Let $f : (X, x_0) \rightarrow (X, x_0)$ be a dominant germ on a normal surface singularity. Let $\nu \in \mathcal{V}_X^\alpha$ be any normalized valuation of finite skewness. Then $c_\infty(f, \nu) = \lim_{n \rightarrow \infty} \sqrt[n]{c(f^n, \nu)}$ exists, belongs to $[1, +\infty)$, and it does not depend on ν .*

Proof. We first assume that the limit exists for a suitable normalized valuation $\nu \in \mathcal{V}_X^\alpha$ of finite skewness. Take any other valuation $\mu \in \mathcal{V}_X^\alpha$. By [Corollary 2.16](#), $\beta(\nu|\mu)$ and $\beta(\mu|\nu)$ are finite. Assume for the moment that f is finite. Then

$$c(f^n, \nu) = \nu((f^n)^* \mathfrak{m}) \leq \beta(\nu|\mu) \mu((f^n)^* \mathfrak{m}),$$

and we deduce $c_\infty(f, \nu) \leq \liminf_n \sqrt[n]{c(f^n, \mu)}$ by taking the inferior limit for $n \rightarrow \infty$ of the n -th root of this inequality. In the general case, we may consider $\alpha_k = (f^n)^* \mathfrak{m} + \mathfrak{m}^k$. Then $\nu(\alpha_k)$ is an increasing sequence converging to $\nu((f^n)^* \mathfrak{m})$, the same for μ , and we deduce the same estimate obtained previously for the finite case. By repeating the argument by interchanging the roles of ν and μ , and taking the superior limit, we deduce $c_\infty(f, \nu) \geq \limsup_n \sqrt[n]{c(f^n, \mu)}$. Hence the limit $c_\infty(f, \mu)$ exists and coincide with $c_\infty(f, \nu)$.

We now prove that there exists $\nu \in \mathcal{V}_X^\alpha$ so that $\sqrt[n]{c(f^n, \mu)}$ converges. By taking the logarithm, we want to study limit of the sequence

$$\frac{1}{n} \log c(f^n, \nu) = \frac{1}{n} \sum_{j=0}^{n-1} \log c(f, f_\bullet^j \nu).$$

If the sequence $\log c(f, f_\bullet^j \nu)$ converges to some value $c \in \mathbb{R}$, then its associated sequence of Cesaro means $\frac{1}{n} \log c(f^n, \nu)$ does as well. If $f_\bullet^j \nu$ converges (weakly) to an eigenvaluation ν_* , then by continuity $c(f_\bullet^j \nu)$ converges to $c(f, \nu_*)$, and we are done. It may be that f does not admit eigenvaluations, exactly when (X, x_0) is a cusp singularity, and f_\bullet acts as a rotation on the circle \mathcal{S}_X . If the rotation is rational, then there exists $k \in \mathbb{N}^*$ so that f_\bullet^k is the identity on \mathcal{S}_X , and $c(f, f_\bullet^{hk+i} \nu)$ converges to some constant c for all $i \in \mathbb{N}$. It follows that the sequence $\log c(f^j, \nu)$ converges to c . If the rotation is irrational, assume that $\nu \in \mathcal{S}_X$. Then by the ergodic theorem,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log c(f, f_\bullet^j \nu) = \int_{\mathcal{S}_X} \log c(f, \nu) d\rho(\nu),$$

where $\rho(\nu)$ is the ergodic measure on the circle \mathcal{S}_X . □

Definition 8.2. Let $f : (X, x_0) \rightarrow (X, x_0)$ be a dominant germ on a normal surface singularity. The *first dynamical degree* of f is $c_\infty(f) = c_\infty(f, \nu)$ for any valuation $\nu \in \mathcal{V}_X^\alpha$ of finite skewness.

The dynamical degree describes the speed of convergence of the orbits of X towards x_0 . It is a fundamental invariant of (bimeromorphic) conjugacy for global dynamics as well as in our setting, see e.g. [\[17\]](#). We now show that the first dynamical degree is always a quadratic integer in our setting, generalizing [\[29, Theorem A\]](#) to normal surface singularities. Our contribution mainly lies in the study of the case of rotations on cusps singularities.

Theorem 8.3. *Let $f : (X, x_0) \rightarrow (X, x_0)$ be a dominant germ on a normal surface singularity. Then the first dynamical degree $c_\infty(f)$ is a quadratic integer.*

Proof. First, notice that if ν_* is an eigenvaluation, then $c_\infty(f) = c(f, \nu_*)$.

When f admits a divisorial eigenvaluation $\nu_* = \nu_{E_*}$, by [Proposition 4.7](#) we get $c(f, \nu_{E_*}) = k_{E_* \rightarrow E_*} \in \mathbb{N}^*$. The situation is analogous when f admits a curve eigenvaluation $\nu_* = \nu_{C_*}$, since by [Proposition 4.8](#) $c(f, \nu_*) = e_{C \rightarrow C}$ is an integer.

Assume now that f admits an irrational eigenvaluation. Then there exists a good resolution $\pi: X_\pi \rightarrow (X, x_0)$ and a point p in the intersection of two exceptional primes of X_π so that the lift of f to X_π is a holomorphic monomial map, associated to a suitable 2×2 matrix M . It follows that $c(f, \nu_*)$ is a quadratic integer, since it satisfies $c(f, \nu_*)^2 - \text{tr}M c(f, \nu_*) + \det M = 0$. We could also argue by applying [Lemma 7.9](#) and noticing that $c(f, w(t))$ varies affinely with respect to the monomial weight.

The case of an infinitely singular eigenvaluation can be treated similarly, see [[29](#), Theorem 5.1], to show that $c(f, \nu_*)$ is an integer.

The remaining case, when f does not admit eigenvaluations, consists of finite maps on cusp singularities, which induce a rotation on the circle \mathcal{S}_X .

Let (X, x_0) be a cusp singularity, that we may assume is given by the toric construction of [§A.1](#). By [Proposition A.4](#), the action of f_\bullet on \mathcal{S}_X^* corresponds to the action of a linear map g_α on $\mathcal{C} = (\mathbb{R}_+^*)^2$, where $\alpha \in K = \mathbb{Q}(\sqrt{d})$ is such that $Q(\alpha) = \alpha\alpha' \in \mathbb{N}^*$.

We claim that $c_\infty(f) = \sqrt{Q(\alpha)} =: q(\alpha)$, which is in particular a quadratic integer. In fact, given a ray generated by $\nu \in \mathcal{C}_0$, its image $L\nu$ by the linear action L induced by g_α satisfies $q(L\nu) = q(\alpha)q(\nu)$. Denote by $b(\nu)$ its norm, i.e., the value the valuation associated to ν takes on the maximal ideal. The attraction rate is then given by $c(f, \nu) = q(\alpha)b(\nu)/b(L\nu)$. By iterating, $c(f^n, \nu) = q(\alpha)^n b(\nu)/b(L^n\nu)$. Now, by compactity, $b(\nu)$ is bounded. It follows that $\lim_{n \rightarrow +\infty} \sqrt[n]{c(f^n, \nu)} = q(\alpha)$. \square

8.2. Recursion relations for the sequence of attraction rates

In the previous section we have showed that $c_\infty(f)$ is a quadratic integer. We now focus our attention on the sequence $c(f^n, \nu)$ for a given normalized valuation $\nu \in \mathcal{V}_X^\alpha$ of finite skewness. In the smooth setting, this sequence always satisfies eventually a linear recursion relation with integer coefficients, see [[35](#), Theorem A]. Here we show that the same property holds in the singular case, with a very specific exception, that will be dealt with in the next section. In fact, here we assume we are not in the following situation: (X, x_0) is a cusp singularity, f is a finite non-invertible germ on (X, x_0) , which induces an irrational rotation on the skeleton \mathcal{S}_X . Our assumption is equivalent to the fact that f^k admits an eigenvaluation for some $k \in \mathbb{N}^*$, and by [Theorem 7.4](#), it implies the existence of geometrically stable models. We will see in the next section that the existence of such models is actually equivalent to admitting an eigenvaluation for an iterate of f .

We first prove that the existence of a geometrically stable model, together with the interpretation of the attraction rate $c(f, \nu)$ in terms of intersections of suitable b-divisors established by [Proposition 4.39](#), implies the existence of the desired recurrence relation for the sequence of attraction rates. This result is a straightforward generalization of [[35](#), Corollary 5.5].

Corollary 8.4. *Let $f: (X, x_0) \rightarrow (X, x_0)$ be a dominant non-invertible germ at a normal surface singularity. Assume we are in the hypotheses of [Theorem A](#). Then for any quasimonomial valuation $\nu \in \mathcal{V}_X$, the sequence $(c(f^n, \nu))_n$ eventually satisfies a linear recursion relation with integer coefficients.*

Proof. By regularity of f_\bullet , we may assume that $\nu = \nu_E$ is divisorial. Let $\pi: X_\pi \rightarrow (X, x_0)$ be a good resolution high enough to have E as an exceptional prime. By [Theorem A](#), up to taking a higher model, we may assume that π is geometrically stable for f . Denote by $f_\pi = \pi^{-1} \circ f \circ \pi: X_\pi \dashrightarrow X_\pi$ the lift of f to X_π . By [Proposition 7.3](#), there exists $N \in \mathbb{N}$ so that for any $n \geq N$, we have

$$\mathcal{E}_\pi \circ (f_\pi^n)^* = \mathcal{E}_\pi \circ (f_\pi^N)^* (\mathcal{E}_\pi \circ f_\pi^*)^{n-N}.$$

By [Proposition 4.39](#), $c(f, \nu) = -Z_\pi(\nu) \cdot \mathcal{E}_\pi \circ f_\pi^* Z_\pi(\mathfrak{m}_X)$. Since the map $\mathcal{E}_\pi \circ f_\pi^*: \mathcal{E}(\pi) \rightarrow \mathcal{E}(\pi)$ is \mathbb{Z} -linear, by the Cayley-Hamilton theorem we have that $\mathcal{E}_\pi \circ f_\pi^*$ satisfies a monic polynomial $t^m + a_1 t^{m-1} + \dots + a_m$

with integer coefficients. If $n \geq N + m$, this gives

$$\begin{aligned}
c(f^n, \nu) &= -Z_\pi(\nu) \cdot \mathcal{E}_\pi \circ (f_\pi^n)^* Z_\pi(\mathfrak{m}_X) \\
&= -Z_\pi(\nu) \cdot \mathcal{E}_\pi \circ (f_\pi^N)^* (\mathcal{E}_\pi \circ (f_\pi)^*)^{n-N} Z_\pi(\mathfrak{m}_X) \\
&= -\sum_{j=1}^m -a_j Z_\pi(\nu) \cdot \mathcal{E}_\pi \circ (f_\pi^N)^* (\mathcal{E}_\pi \circ (f_\pi)^*)^{n-N-j} Z_\pi(\mathfrak{m}_X) \\
&= -\sum_{j=1}^m -a_j c(f^{n-j}, \nu).
\end{aligned}$$

This is the desired linear recursion relation eventually satisfied by the sequence of attraction rates. \square

To prove [Theorem D](#), we need to prove this recurrence relation has order at most 2, at least for a suitable iterate of f . The argument used in [[35](#), Theorem 6.1] equally works in our setting. Here we outline the proof, details are left to the reader. Consider a normalized valuation $\nu \in \mathcal{S}_X^\alpha$ of finite skewness. Since we are working in the case of existence of eigenvaluations for f^k , by [Theorem 4.41](#) there exists ν_* an eigenvaluation for f^k so that $f_\bullet^{nk} \nu \rightarrow \nu_*$ when $n \rightarrow \infty$. The convergence is strong as far as $\alpha(\nu_*) < +\infty$. Up to replacing f by f^k , we may always assume $k = 1$, and ν_* is an eigenvaluation.

Case 1. The eigenvaluation ν_* is not quasimonomial.

In this case, we have seen that $c(f, \nu) = c_\infty(f)$ is a constant integer for all ν in a suitable (weak open) neighborhood U of ν_* . By [Theorem 4.41](#) $f_\bullet^n \nu \rightarrow \nu_*$ (with respect to the weak topology) for all $\nu \in \mathcal{V}_X^\alpha$. We deduce that $c(f^{n+1}, \nu) = c(f, f_\bullet^n \nu) c(f^n, \nu) = c_\infty(f) c(f^n, \nu)$ for all $n \geq N$ so that $f_\bullet^N \nu \in U$.

Case 2. The eigenvaluation ν_* is irrational.

In this case, we have seen that there exists a log resolution $\pi: X_\pi \rightarrow (X, x_0)$ of the maximal ideal, and a point $p \in \pi^{-1}(x_0)$ in the intersection of two exceptional primes E and F , so that $c(f, \nu_{r,s})$ is linear with respect to the weights (r, s) , where $\nu_{r,s}$ is the monomial valuation at p of weights (r, s) , with respect to coordinates adapted to $E \cup F$. Up to shrinking the segment $I = [\nu_E, \nu_F]_p$, by [Lemma 7.9](#), we may assume that f_\bullet acts a linear map M with non-negative coefficients on the weights (r, s) . In particular, let n is big enough so that $f_\bullet^n \nu$ belongs to $U(I)$. Up to shrinking I if necessary, we may also assume that $c(f, \nu)$ is locally constant on $U(I) \setminus I$, so that $c(f, f_\bullet^n \nu) = c(f, r_i f_\bullet^n \nu)$. In this case (see also [[35](#), Lemma 6.2]), we get the recursion relation of order 2

$$c(f^{n+2}, \nu) = \text{tr}(M)c(f^{n+1}, \nu) - \det(M)c(f^n, \nu).$$

Case 3. The eigenvaluation $\nu_* = \nu_{E_*}$ is divisorial.

If $f_\bullet^n \nu = \nu_*$ for some n , $c(f^n, \nu)$ clearly eventually satisfies a linear integral recursion relation of order 1. Assume this is not the case. We can then associate to $f_\bullet^n \nu$ the tangent vector \vec{v}_n at ν_* towards $f_\bullet^n \nu$. Moreover, we have that $\vec{v}_{n+1} = d(f_\bullet)_{\nu_*} \vec{v}_n$ for n big enough. Let \mathcal{S}_f be the critical skeleton of f , and denote by S the set of tangent vectors at ν_* parallel to S . If $\vec{v}_n \notin S$ for all n big enough, then $c(f, f_\bullet^n \nu)$ is constant for such n , and $c(f^n, \nu)$ satisfies a recursion relation of order 1. If \vec{v}_n belongs to S for infinitely many n , then the orbit \vec{v}_n is preperiodic, and can be considered fixed if we replace ν by $f_\bullet^N \nu$ and f by a suitable iterate. In this case we may conclude as before, and find a recursion relation of order 2.

8.3. Finite germs on cusp singularities

This section is devoted to study the sequence of attraction rates for finite germs on cusp singularities. The only situation that is not covered by [Theorem D](#) is when f_\bullet acts on the circle $S = \{\nu \in \mathcal{V}_X \mid A(\nu) = 0\}$ as an irrational rotation. We shall show that in this case, no linear recursion relations are satisfied.

Proposition 8.5. *Let $f : (X, x_0) \rightarrow (X, x_0)$ be a superattracting germ on a cusp which induces an irrational rotation on S . Then $c(f, -)$ cannot be constant on S .*

Proof. Suppose for contradiction that $c(f, -) \equiv M$ on S . That is to say, for any ν on S , we have

$$M = c(f, \nu) = -\lim_{\pi} Z_{\pi}(\nu) \cdot f^* Z_{\pi}(\mathfrak{m}).$$

Let $\pi_1 : X_{\pi_1} \rightarrow (X, 0)$ be a log resolution of \mathfrak{m} . Then we can write

$$Z_{\pi_1}(\mathfrak{m}) = \sum_{i \in I} \lambda_i \check{E}_i + \sum_{j \in J} \mu_j \check{F}_j,$$

where $\lambda_i, \mu_j > 0$ for each i and j , $\nu_{E_i} \in S$ and $\nu_{F_j} \notin S$.

Take any exceptional prime $E \subset X_{\pi_1}$ so that $\nu_E \in S$. By [Proposition 4.14](#), it is of the form $f^* \check{E} = e_{G \rightarrow E} \check{G}$ for some $\nu_G \in S$. Since f_\bullet is an irrational rotation on S , up to taking a suitable iterate, we can assume that G does not appear as an exceptional prime in X_{π_1} . In particular, the center of ν_G in X_{π_1} must be a closed point p lying in the intersection of two exceptional primes D_1 and D_2 which belong to the cycle.

Blow up p in X_{π_1} , thus obtaining a modification $\pi_2 : X_{\pi_2} \rightarrow (X, x_0)$ and a new exceptional prime H . Let $\pi_3 : X_{\pi_3} \rightarrow X_{\pi_2}$ be a modification dominating π_2 , large enough to contain all the primes appearing in $f^* Z_{\pi_1}(\mathfrak{m})$. Set $\tilde{\eta} = \pi_2^{-1} \circ \pi_3$. Then we get

$$M = c(f, \nu_H) = -Z_{\pi_3}(\nu_H) \cdot f^* Z_{\pi_1}(\mathfrak{m}) = -\frac{1}{b_H} \tilde{\eta}^* \check{H} \cdot f^* Z_{\pi_1}(\mathfrak{m}) = -\frac{1}{b_H} \check{H} \cdot \tilde{\eta}_* f^* Z_{\pi_1}(\mathfrak{m}).$$

Now we use the fact that $\check{H} = \check{D}_1 + \check{D}_2 - H$ in X_{π_2} to get

$$\begin{aligned} M &= -\frac{1}{b_H} \check{D}_1 \cdot \eta_* f^* Z_{\pi_1}(\mathfrak{m}) - \frac{1}{b_H} \check{D}_2 \cdot \eta_* f^* Z_{\pi_1}(\mathfrak{m}) + \frac{1}{b_H} H \cdot \eta_* f^* Z_{\pi_1}(\mathfrak{m}) \\ &= -\frac{1}{b_H} \eta^* \check{D}_1 \cdot f^* Z_{\pi_1}(\mathfrak{m}) - \frac{1}{b_H} \eta^* \check{D}_2 \cdot f^* Z_{\pi_1}(\mathfrak{m}) + \frac{1}{b_H} H \cdot \eta_* f^* Z_{\pi_1}(\mathfrak{m}) \\ &= \frac{1}{b_H} b_{D_1} c(f, \nu_{D_1}) + \frac{1}{b_H} b_{D_2} c(f, \nu_{D_2}) + \frac{1}{b_H} H \cdot \eta_* f^* Z_{\pi_1}(\mathfrak{m}) \\ &= \frac{b_{D_1} + b_{D_2}}{b_H} M + \frac{1}{b_H} H \cdot \eta_* f^* Z_{\pi_1}(\mathfrak{m}) \\ &= M + \frac{1}{b_H} H \cdot \eta_* f^* Z_{\pi_1}(\mathfrak{m}) \end{aligned}$$

We conclude that $H \cdot \eta_* f^* Z_{\pi_1}(\mathfrak{m}) = 0$. We claim that this gives a contradiction, since by construction, this intersection number should be positive.

Assume first that I is not empty, and take $E = E_1$. Then

$$H \cdot \eta_* f^* Z_{\pi_1}(\mathfrak{m}) = H \cdot \eta_* f^* \check{E}_1 + \sum_{i>1} H \cdot \eta_* f^* \check{E}_i + \sum_j H \cdot \eta_* f^* \check{F}_j,$$

and each of the terms in the two summations is nonnegative. The first term can be computed by

$$H \cdot \eta_* f^* \check{E}_1 = e_{G \rightarrow E_1} H \cdot \eta_* \check{G},$$

and, since the valuation ν_G is centered in H (by our choice of H), it follows that the intersection $H \cdot \eta_* \check{G}$ is positive.

Assume now that I is empty. Set E to be the exceptional prime in the cycle corresponding to the retraction to the cycle of the divisorial valuation ν_{F_1} . In this case

$$H \cdot \eta_* f^* Z_{\pi_1}(\mathfrak{m}) = H \cdot \eta_* f^* \check{F}_1 + \sum_{j>1} H \cdot \eta_* f^* \check{F}_j.$$

Again each of the terms in the summation is nonnegative, and the first one is positive, since the center of any preimage of ν_{F_1} in X_{π_2} is H . This completes the proof. \square

Using an argument due to [37], we show that for irrational rotations on cusps, $c(f^n, \nu)$ does not satisfy any linear recursion relation. As a corollary, such maps do not admit geometrically stable models.

Proposition 8.6. *Let $f : (X, x_0) \rightarrow (X, x_0)$ be a superattracting germ on a cusp which induces an irrational rotation on the cycle. Let $\nu \in \mathcal{V}_X^\alpha$ be any valuation of finite skewness. Then the sequence $c(f^n, \nu)$ does not satisfy any linear recursion relation.*

Proof. We may assume that (X, x_0) is constructed as in §A.1. The construction depends on a lattice N_ω , and a totally positive unit ε leaving N_ω invariant, which induces a linear map g_ε leaving the open cone $\mathcal{C} = (\mathbb{R}_+^*)^2$ invariant. By Remark A.1, valuations in \mathcal{S}_X^* are in 1-to-1 correspondence with $\mathcal{C}_\varepsilon = \mathcal{C}/\langle g_\varepsilon \rangle$. Moreover, the function $\text{ev}_\mathfrak{m} : \mathcal{S}_X^* \rightarrow \mathbb{R}_+^*$ given by $\text{ev}_\mathfrak{m}(\nu) = \nu(\mathfrak{m})$ lifts to a continuous, g_ε -equivariant map on \mathcal{C} , which is piecewise linear and in fact made up of linear maps on sectors. By Proposition A.4, the action of f_* on \mathcal{S}_X^* lifts to a linear map $\Lambda = g_\alpha : \mathcal{C} \rightarrow \mathcal{C}$.

Suppose at first that ν belongs to the circle \mathcal{S}_X . We have seen that ν corresponds to a point $[p] \in \mathcal{C}_\varepsilon$. Take any representative $p \in \mathcal{C}$. Then $c(f^n, \nu) = (f_*^n \nu)(\mathfrak{m}) = \text{ev}_\mathfrak{m}(\Lambda^n p)$. Define $c_n := c(f^n, \nu)$. Now, let L be one of the linear maps making up $\text{ev}_\mathfrak{m}$, and define $d_n = L(\Lambda^n p)$. Note that d_n satisfies a linear recursion relation. Suppose that c_n also eventually satisfies a recursion relation. Then so would the sequence $c_n - d_n$. By Skolem-Mahler-Lech's theorem, we conclude that $\{n : c_n = d_n\}$ consists of a finite union of arithmetic progressions. Since f gives an irrational rotation, the only way this can happen is if $\text{ev}_\mathfrak{m} \equiv L$, which implies $c(f, \cdot)$ is constant on the cycle, a contradiction by Proposition 8.5.

Let now take any valuation $\nu \in \mathcal{V}_X^\alpha$. Notice that $c(f, \cdot)$ is locally constant outside the (finite) skeleton $\mathcal{S}_{c(f)}$. In particular, there is only a finite number of tangent vectors at divisorial valuations in \mathcal{S}_X along which $c(f, \cdot)$ is not locally constant. Since f_\bullet acts as an irrational translation on \mathcal{S}_X and $r_X f_\bullet^n \nu = f_\bullet^n r_X \nu$, we avoid these directions after a finite number of iterates. We deduce that $c(f^n, \nu)$ does not eventually satisfy any linear recursion relation for any valuation ν of finite skewness. \square

9. Examples and remarks

9.1. A finite map at a smooth point

Consider a smooth point $(X, x_0) = (\mathbb{C}^2, 0)$. In this case the space of normalized valuations is called the *valuative tree* \mathcal{V} . It is a complete real tree, rooted at the multiplicity valuation ord_0 , which is the divisorial valuation associated to the exceptional prime $E_0 = \pi_0^{-1}(0)$, where $\pi_0 : X_{\pi_0} \rightarrow (\mathbb{C}^2, 0)$ is the

blow-up of the origin. Notice that π_0 is also a log resolution of the maximal ideal, and $Z_{\pi_0}(\mathfrak{m}) = Z_{\pi_0}(\text{ord}_0) = \check{E}_0 = -E_0$. For any $k \in \mathbb{N}$, let $f_k: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be the dominant germ

$$f_k(x, y) = (x^k(y + x^3), x^2y).$$

Notice that f_k is non-invertible, and finite if and only if $k = 0$. For any irreducible $\phi \in R = \hat{\mathcal{O}}_{\mathbb{C}^2, 0}$, we denote by ν_ϕ the associated curve semivaluation. For any $t \in [1, \infty]$, we denote by $\nu_{\phi, t}$ the unique normalized valuation in $[\text{ord}_0, \nu_\phi]$ with skewness t . Consider first the case $k = 0$ and write $f = f_0$. By direct computation, we notice that $f_\bullet^{-1}(\text{ord}_0) = \{\nu_{y+x^3, 5}\}$. By [Corollary 4.40](#), the attraction rate $c(f, \nu)$ is locally constant outside $\mathcal{T}_{c(f)} = [\text{ord}_0, \nu_{y+x^3, 5}]$. Moreover,

$$f_* \nu_{x, t} = \nu_{y, 1+2t}, \quad f_* \nu_{y, t} = \begin{cases} t \nu_{y, 1+\frac{2}{t}} & \text{if } t \in [1, 3], \\ 3 \nu_{y, \frac{2+t}{3}} & \text{if } t \in [3, +\infty], \end{cases} \quad f_* \nu_{y+x^3, t} = \begin{cases} t \nu_{y, \frac{5}{t}} & \text{if } t \in [3, 5], \\ 5 \nu_{x, \frac{t}{5}} & \text{if } t \in [5, +\infty]. \end{cases}$$

In particular, f admits a unique divisorial eigenvaluation $\nu_* = \nu_{y, 2}$, and $c_\infty(f) = c(f, \nu_*) = 2$. It is easy to check (see the similar computation on [§9.2](#)) that the sequence of attraction rates $c_n = c(f^n, \text{ord}_0)$ satisfies the recursion relation

$$c_0 = 1, c_1 = 1, c_{n+2} = c_{n+1} + 2c_n, \forall n \in \mathbb{N}.$$

Notice that the sequence $c'_n = c(f^n, \nu_{y+x^3, 5})$ satisfies $c'_{n+1} = c'_1 c_n$, and hence satisfies the same recursion relation for $n \geq 1$, while $c'_0 = 1$, $c'_1 = 5$, $c'_2 = 5$, so in particular the recursion relation does not hold for $n = 0$.

Remark 9.1. We can also compute the multiplicity of f at any valuation on $[\nu_x, \nu_y] \cup [\nu_{y, 3}, \nu_{y+x^3}]$. We get

$$m(f, \nu) = \begin{cases} 5 & \text{if } \nu \in [\nu_{y, 3}, \nu_{y+x^3}], \\ 3 & \text{if } \nu \in]\nu_{y, 3}, \nu_y], \\ 2 & \text{if } \nu \in]\nu_{y, 3}, \nu_x]. \end{cases}$$

By [Theorem 4.18](#), the map $M(f, \nu') = \sum_{f_\bullet \nu = \nu'} m(f, \nu)$ is the constant $5 = e(f)$ the topological degree of f .

The values of $m(f, \nu)$ are quite difficult to control in general. In fact, at any branch point (i.e., divisorial valuation) ν_E , the value of (f, ν) on branches at ν_E depend on the values of k_E and e_E . For example, one can check that if ν_E is the divisorial valuation associated to $\nu_{y+x^3, 5}$, we get $k_E = 5$ and $e_E = 1$. In particular, if we denote by E_0 the exceptional prime obtained by blowing up the origin, the tangent map $df_\bullet: T_{\nu_E} \mathcal{V} \rightarrow T_{\nu_{E_0}} \mathcal{V}$ has degree 1, and $m(f, \nu) = 5$ for valuations ν along any branches, close enough to ν_E .

We summarize in [Figure 3](#) the dynamics of f_\bullet . We denote by black dots curve semivaluations, and by white dots divisorial valuations.

The model π_0 is not algebraically stable for f . In fact, denoting by $p_0 \in E_0$ the point associated to the tangent vector at ν_{E_0} towards ν_y , we have $\text{Ind}(f_{\pi_0}) = \{p_0\}$, and $f_{\pi_0}(E_0) = p_0$. Consider $\pi_1: X_{\pi_1} \rightarrow (\mathbb{C}^2, 0)$, obtained from π_0 by blowing up p_0 . Denote by E_1 the new exceptional prime. We have $\nu_{E_1} = \nu_*$, hence f_{π_1} acts on E_1 as the rational selfmap $h: \zeta \mapsto \zeta^{-1}$, where ζ corresponds to the tangent vector at ν_* towards $\nu_{y-\zeta x^2}$. We notice that the point p_1 corresponding to $\zeta = 0$ is of indeterminacy for f_{π_1} , and it is periodic for h . We blow up p_1 , obtaining another model $\pi_2: X_{\pi_2} \rightarrow (\mathbb{C}^2, 0)$. Although f_{π_2} has still indeterminacy points (corresponding to the tangent vectors at $\nu_{y, 3}$ towards ν_{y+x^3} and ν_y), it acts regularly on E_1 , and the model is algebraically stable. [Figure 4](#) resumes the situation, where black dots are regular points, and white dots are indeterminacy points.

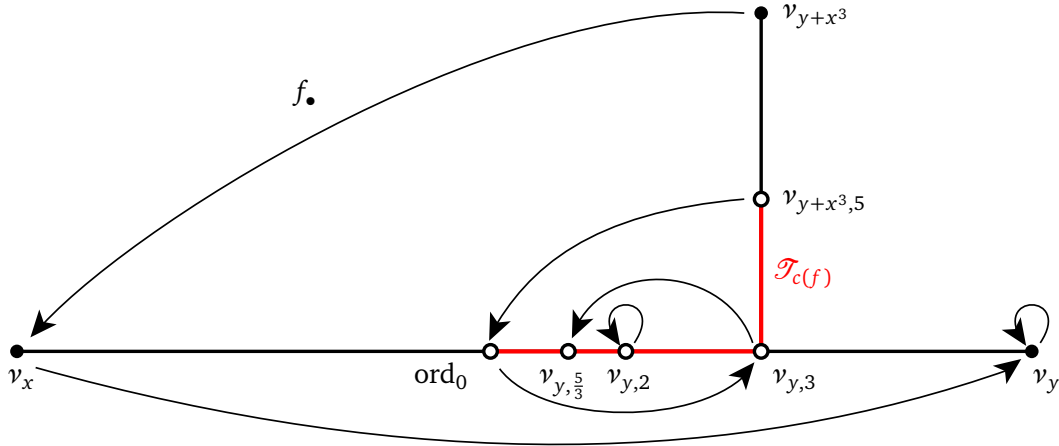
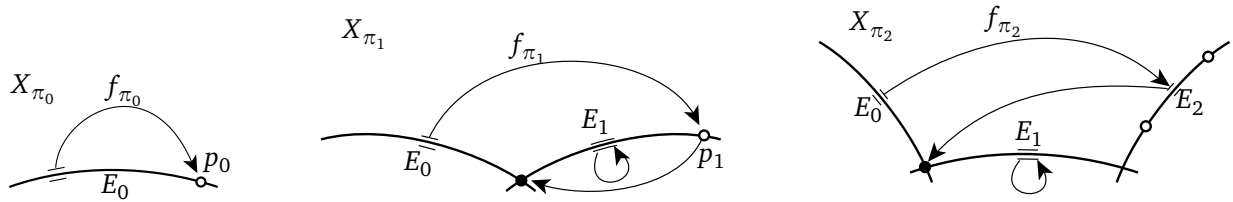
Figure 3: Action of f . (§9.1).

Figure 4: Construction of an algebraically stable model (§9.1).

9.2. A non-finite map at a smooth point

We now study the case $k = 1$ of the previous section. In this case, $\mathcal{C}_f = \{v_x\}$, and $f_\bullet^{-1}(\text{ord}_0) = \{v_{y+x^3, 4}\}$, so $\mathcal{T}_{c(f)} = [v_x, v_{y+x^3, 4}]$. Moreover,

$$f_\bullet v_{x,t} = v_{y, \frac{2t+1}{t+1}}, \quad f_\bullet v_{y,t} = \begin{cases} v_{y, \frac{t+2}{t+1}} & \text{if } t \in [1, 3], \\ v_{y, \frac{t+2}{4}} & \text{if } t \in [3, +\infty], \end{cases} \quad f_\bullet v_{y+x^3,t} = \begin{cases} v_{y, \frac{5}{t+1}} & \text{if } t \in [3, 4], \\ v_{x, \frac{t+1}{5}} & \text{if } t \in [4, +\infty]. \end{cases}$$

$$c(f, v_{x,t}) = (t+1), \quad c(f, v_{y,t}) = \begin{cases} (t+1) & \text{if } t \in [1, 3], \\ 4 & \text{if } t \in [3, +\infty], \end{cases} \quad c(f, v_{y+x^3,t}) = \begin{cases} (t+1) & \text{if } t \in [3, 4], \\ 5 & \text{if } t \in [4, +\infty]. \end{cases}$$

In this case the unique eigenvaluation given by [Theorem 5.8](#) is the irrational valuation $v_\star = v_{y, \sqrt{2}}$, and $c_\infty(f) = c(f, v_\star) = 1 + \sqrt{2}$. Notice also that $f_\bullet v_x = v_{y, 2}$. Consider the sequence of attraction rates $c_n = c(f^n, \text{ord}_0)$. Set $t_n = \alpha(f_\bullet^n \text{ord}_0)$. Then for any $n \in \mathbb{N}$ we get

$$\begin{aligned} c_{n+2} &= c(f^{n+2}, \text{ord}_0) = c(f^n, \text{ord}_0) c(f, f_\bullet^n \text{ord}_0) c(f, f_\bullet^{n+1} \text{ord}_0) \\ &= c_n (t_n + 1) (t_{n+1} + 1) = c_n (t_n + 1) \left(\frac{t_n + 2}{t_n + 1} + 1 \right) = c_n (2t_n + 3) = 2c_n (t_n + 1) + c_n \\ &= 2c_{n+1} + c_n. \end{aligned}$$

Remark 9.2. As in the previous example, we can compute the multiplicity of f at any valuation on

$[\nu_x, \nu_y] \cup [\nu_{y,3}, \nu_{y+x^3}]$, and we get

$$m(f, \nu) = \begin{cases} 5 & \text{if } \nu \in [\nu_{y,3}, \nu_{y+x^3}], \\ 4 & \text{if } \nu \in]\nu_{y,3}, \nu_y], \\ 1 & \text{if } \nu \in]\nu_{y,3}, \nu_x]. \end{cases}$$

In this case, the map $M(f, \nu') = \sum_{f \bullet \nu = \nu'} m(f, \nu)$ is locally constant on $\mathcal{V} \setminus \{\nu_{y,2}\}$. In fact, it is equal to $5 = e(f)$ in the connected component containing $\nu_{y,t}$ with $t < 2$ (and on $\nu_{y,2}$ itself), while it is equal to 4 on the other connected components.

If we focus on $\nu_E = \nu_{y,4}$, by direct computation we get $k_E = 2$ and $e_E = 2$. One can compute in this case that on all branches at ν_E (but the ones associated to the directions towards ν_x and ν_y), the multiplicity $m(f, \nu) = 2$ for ν close enough to ν_E .

We summarize in [Figure 5](#) the dynamics of f_\bullet . Here the square dot denotes irrational valuations.

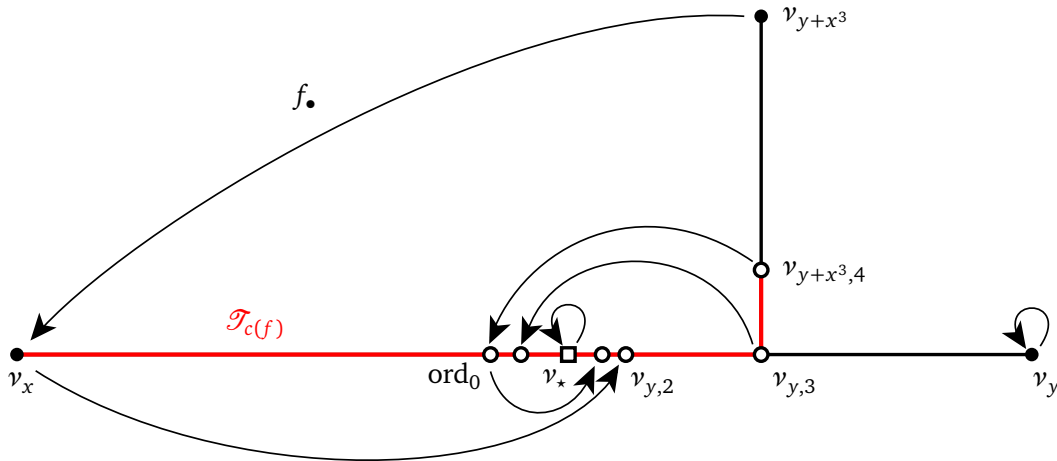


Figure 5: Action of f_\bullet . (§9.2).

As in the previous example, the model π_0 is not geometrically stable for f , since again $f_{\pi_0}(E_0) = p_0 \in \text{Ind}(f_{\pi_0})$. Notice that we also have $f_{\pi_0}(C_x) = p_0$, where C_x is the strict transform of the curve $\{x = 0\}$ to X_{π_0} . In this case, it suffices to blow-up p_0 , to get a geometrically stable model π_1 . We get a new exceptional prime E_1 , intersecting the strict transform of E_0 at a point p_1 . We easily notice that p_1 is a fixed point for f_{π_1} , and the orbit of any exceptional prime through f_{π_1} will eventually go to p_1 . Notice also that since $f_\bullet \nu_{x,t} \rightarrow \nu_{y,2}$ when $t \rightarrow +\infty$, from the tangent direction at $\nu_{y,2}$ associated to p_1 , then $f_{\pi_1}(C_x, q) \rightarrow (E_1, p_1)$. See [Figure 6](#) for a description of the actions of f_{π_0} and f_{π_1} .

We refer to [\[29, 35\]](#) for further examples on the smooth case.

9.3. A quotient singularity

Let (X, x_0) be the singularity defined by $X = \{(x, y, z) \in \mathbb{C}^3 \mid xy = z^2\}$ at the origin $x_0 = (0, 0, 0)$. This is the simplest cyclic quotient singularity, usually denoted by A_2 . Its minimal good resolution, which is also a log resolution of the maximal ideal, has a unique rational exceptional prime E_0 , and self-intersection $E_0^2 = -2$. This singularity can be obtained by quotient of $(\mathbb{C}^2, 0)$ by the action of $(u, v) \mapsto (-u, -v)$. Denote by $pr: (\mathbb{C}^2, 0) \rightarrow (X, x_0)$ the natural projection, given by $pr(u, v) = (u^2, v^2, uv)$.

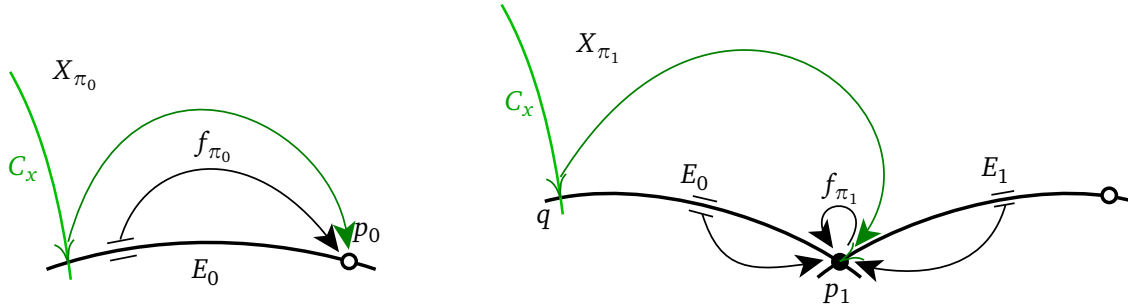


Figure 6: Construction of a geometrically stable model (§9.2).

Consider the map $f(x, y, z) = (x^2z, y^2z, xyz)$. The maps f leaves X invariant, so it defines a germ at (X, x_0) . Moreover, the curves $C_x = \{x = z = 0\}$ and $C_y = \{y = z = 0\}$ are contracted by f , which is hence non-finite.

Notice that $f \circ pr(u, v) = (u^5v, uv^5, u^3v^3)$. In particular there does not exist a holomorphic map $g: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ so that $f \circ pr = pr \circ g$.

An embedded resolution of (X, x_0) is obtained by a single blow-up π_0 of the origin. Pick coordinates so that $\pi_0(x_0, y_0, z_0) = (x_0, x_0y_0, x_0z_0)$. Then $X_{\pi_0} = \{y_0 = z_0^2\}$, and (x_0, z_0) are local parameters of X_{π_0} . By direct computation, we find that $E_0 = \{x_0 = y_0 - z_0^2 = 0\}$ is invariant by the action of f_{π_0} , and hence ν_{E_0} is the unique eigenvaluation given by Theorem 5.8. The map f_{π_0} restricted to E_0 corresponds to the map $z_0 \mapsto z_0^2$. In particular, $e_{E_0 \rightarrow E_0} = 2$, while we can compute $k_{E_0 \rightarrow E_0} = c(f, \nu_{E_0}) = c_\infty(f) = 3$. For any $t \in [0, +\infty]$ and $z_0 \in \mathbb{C}$, denote by $\mu_{z_0, t}$ the monomial valuation at the point $(0, z_0^2, z_0) \in E_0$ with weights 1 and t with respect to the parameters (x_0, z_0) . Notice that the valuation $\nu_{z_0, t} = (\pi_0)_* \mu_{z_0, t}$ is normalized. By interchanging the coordinates x and y , we recover the last point in E_0 , associated to the parameter $z_0 = \infty \in \mathbb{P}^1$. By direct computation, we get

$$f_* \nu_{0, t} = (3+t) \nu_{0, \frac{2t}{3+t}}, \quad f_* \nu_{z_0, t} = 3 \nu_{z_0^2, \frac{t}{3}}, \quad f_* \nu_{\infty, t} = (3+t) \nu_{\infty, \frac{2t}{3+t}}.$$

In particular we get $f_* \nu_{0, \infty} = \nu_{0, 2}$, and analogously $f_* \nu_{\infty, \infty} = \nu_{\infty, 2}$. The Jacobian divisor can be computed directly, and we obtain $R_f = 2C_x + 2C_y$.

9.4. A simple elliptic singularity

Let (X, x_0) be the singularity defined by $X = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^6 = 0\}$ at $x_0 = (0, 0, 0)$. One can show that (X, x_0) is a simple elliptic singularity: its minimal good resolution $\pi_0: X_{\pi_0} \rightarrow (X, x_0)$ has a unique exceptional prime E_0 of genus 1, and self-intersection $E_0^2 = -1$.

One can check that π_0 is not a log resolution of the maximal ideal \mathfrak{m} . To obtain it, we need to blow up a point $p_0 \in E_0$, obtaining a new exceptional prime E_1 of genus 0, that intersect transversely the strict transform of E_0 . In this model, the self intersection of E_1 is -1 , and the self intersection of the strict transform of E_0 is -2 . One can also check that $b_{E_0} = 1$ and $b_{E_1} = 2$. It follows that $A(\nu_{E_0}) = 0$, and $A(\nu_{E_1}) = \frac{1}{2}$.

Consider the map $f: (X, x_0) \rightarrow (X, x_0)$ given by

$$f(x, y, z) = (xy^3z^3, y^3z^2, yz^2).$$

The map f defines a non-finite germ at x_0 . In fact, f contracts three irreducible curves: $C_y^+ = \{y = x - iz^3 = 0\}$, $C_y^- = \{y = x + iz^3 = 0\}$, and $C_z = \{z = x^2 + y^3 = 0\}$. One can check by direct

computation that the unique eigenvaluation given by [Theorem 5.8](#) is the divisorial valuation associated to E_0 . The action on the tangent space attached to ν_{E_0} is the identity. Since the critical points of f in \mathbb{C}^3 are contained in $\{yz = 0\}$, the jacobian divisor takes the form $R_f = a_y^+ C_y^+ + a_y^- C_y^- + a_z C_z$. A direct computation shows that $a_y^+ = a_y^- = a_z = 0$. In particular R_f is trivial even though f is non-finite.

9.5. Quasihomogeneous singularities

The previous example can be easily generalized to *quasihomogeneous* singularities (also called *weighted homogeneous* singularities). Up to isomorphisms, they are constructed as follows. Let $\omega = (\omega_1, \dots, \omega_n)$ be a vector of positive integers (we may assume without common factors). Consider a finite family of polynomials $P_j \in \mathbb{C}[x_1, \dots, x_n]$ which are homogeneous with respect to the weight ω , i.e., there exists $d_j \in \mathbb{N}^*$ so that $P_j(\lambda^{\omega_1} x_1, \dots, \lambda^{\omega_n} x_n) = \lambda^{d_j} P_j(x_1, \dots, x_n)$ for all $\lambda \in \mathbb{C}$. The common zero locus of this family of polynomials defines a quasihomogeneous singularity at $x_0 = (0, \dots, 0)$. Quasihomogeneous singularities can be also described as finite quotients of *cone singularities*. They are obtained by contracting the zero section of a negative degree line bundle $L \rightarrow E$ over a compact curve E . We refer to [\[66\]](#) for further details on quasihomogeneous singularities.

For example, take any $p, q \geq 2$ coprime, we consider the surface $X = \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^{pq} = 0\}$ at its singular point $x_0 = (0, 0, 0)$. This is a quasihomogeneous singularity with respect to the weight $(q, p, 1)$. By direct computation, (X, x_0) is a cone singularity given by a line bundle of degree -1 over a curve E_0 of genus $(p-1)(q-1)/2$. The minimal good resolution of (X, x_0) is not a log resolution of the maximal ideal. The latter is obtained by blowing-up $p-1$ times the intersection between the exceptional divisor and the strict transform of the curve $\{z = 0\}$.

Come back to the general case of a quasihomogeneous singularity $(X, 0) \subset (\mathbb{C}^n, 0)$ with respect to some weight $\omega = (\omega_1, \dots, \omega_n)$ on the coordinates $x = (x_1, \dots, x_n)$. Let $\phi \in \mathcal{O}_{X,0}$ be any holomorphic function at $0 \in X$, and $\sigma: (X, 0) \rightarrow (X, 0)$ any automorphism (see [\[52, 31\]](#) for constructions of automorphisms over quasihomogeneous singularities). We may consider the germ $f = \phi^\omega \sigma$, defined in coordinates by

$$f(x) = (\phi^{\omega_1} \sigma_1(x), \dots, \phi^{\omega_n} \sigma_n(x)),$$

where $(\sigma_1, \dots, \sigma_n)$ are the coordinates of σ . Then f defines an endomorphism of $(X, 0)$, which is non-invertible as long as ϕ is not a unit. Notice also that the curve $\{\phi = 0\}$ is contracted to 0, so f is non-finite (when non-invertible).

In the cone case described above, the map f leaves invariant the curve E_0 in the minimal good resolution, hence ν_{E_0} is the unique eigenvaluation given by [Theorem 5.8](#).

9.6. A non-finite map on a cusp singularity

Consider the cusp singularity defined by $X = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^9 - xyz = 0\}$ at $x_0 = (0, 0, 0)$. The exceptional divisor of its minimal good resolution $\pi_0: X_{\pi_0} \rightarrow (X, x_0)$ is a cycle of three rational curves E_1, E_2, E_3 of self-intersection $-2, -2$ and -3 respectively. Moreover $b_{E_j} = 1$ for $j = 1, 2, 3$. By direct computation, one can check that

$$Z_{\pi_0}(\nu_{E_1}) = -\left(\frac{5}{3}E_1 + \frac{4}{3}E_2 + E_3\right), \quad Z_{\pi_0}(\nu_{E_2}) = -\left(\frac{4}{3}E_1 + \frac{5}{3}E_2 + E_3\right), \quad Z_{\pi_0}(\nu_{E_3}) = -(E_1 + E_2 + E_3).$$

In particular by [Proposition 2.32](#), $\nu_{E_3} \leq \nu_{E_i}$ for $i = 1, 2$, while E_1 and E_2 (and any two normalized monomial valuation at $E_1 \cap E_2$) are not comparable.

The minimal log resolution $\pi_1: X_{\pi_1} \rightarrow (X, x_0)$ of \mathfrak{m} is obtained from π_0 by blowing up a free point in E_3 . We denote by E'_1, E'_2, E'_3 the strict transforms of E_1, E_2, E_3 , and by E'_4 the new exceptional prime. One can check that $b_{E'_4} = 2$, and $Z_{\pi_1}(\mathfrak{m}_X) = \check{E}'_4 = -(E'_1 + E'_2 + E'_3 + 2E'_4)$.

We now construct a non-finite germ, similar to the construction used in [Proposition 4.31](#).

Set $p_0 = E'_1 \cap E'_2$, $p_1 = E'_1 \cap E'_3$, $p_2 = E'_2 \cap E'_3$, and $p_3 = E'_3 \cap E'_4$. For any $j = 0, \dots, 3$, we let $a_j < b_j$ so that $p_j = E'_{a_j} \cap E'_{b_j}$. Computing directly π_1 as an embedded resolution of $(X, x_0) \subset (\mathbb{C}^3, 0)$, we may describe π in coordinates at the points p_0, p_1, p_2, p_3 . For example, we may chose local coordinate (x_0, y_0, z_0) at p_0 , so that

$$X_{\pi_1} = \{z_0(1 + y_0^3) = x_0(y_0 - x_0)\}, \quad \pi_1(x_0, y_0, z_0) = (x_0 z_0^4, y_0 z_0^3, z_0).$$

$$E_1 = \{z_0 = x_0 = 0\}, \quad E_2 = \{z_0 = y_0 - x_0 = 0\},$$

We now consider the map $f : (X, x_0) \rightarrow (X, x_0)$ obtained as the composition $f = \pi_1 \circ \sigma \circ g$, where $g : (X, x_0) \rightarrow (\mathbb{C}^2, 0)$ is the projection to the first two coordinates, $\sigma : (\mathbb{C}^2, 0) \rightarrow (X_{\pi_1}, p_0)$ is the automorphism given by $\sigma(x, y) = (x, y, x(y - x)(1 + y^3)^{-1})$ with respect to the coordinates (x_0, y_0, z_0) at p_0 . In particular we can write f in coordinates as

$$f(x, y, z) = \left(\frac{x^5(y - x)^4}{(1 + y^3)^4}, \frac{x^3 y(y - x)^3}{(1 + y^3)^3}, \frac{x(y - x)}{1 + y^3} \right).$$

Notice that f is non-finite, since it contracts to 0 the curve $X \cap \{x(y - x) = 0\}$.

Denote by $\nu_{r,s}^j \in \mathcal{V}_X^j$ be the monomial valuation at p_j of weights (r, s) with respect to coordinates adapted to $\pi_1^{-1}(x_0)$, so that $\mu_{1,0}^j = \text{ord}_{E'_{a_j}}$ and $\mu_{0,1}^j = \text{ord}_{E'_{b_j}}$. Notice that $\nu_{r,s}^j$ is normalized as far as $r + s = 1$ when $j = 0, 1, 2$, while $\nu_{r,s}^3$ is normalized as far as $r + 2s = 1$. We first study the dynamics of f_* on the circle \mathcal{S}_X . By direct computation, we get

$$f_* \nu_{r,s}^0 = \begin{cases} \nu_{5r+4s, 4r+3s}^0 & \text{if } r \leq s, \\ \nu_{5r+4s, 3r+4s}^0 & \text{if } r \geq s, \end{cases} \quad f_* \nu_{r,s}^1 = \nu_{5r+3s, 3r+2s}^0, \quad f_* \nu_{r,s}^2 = \nu_{4r+3s, 3r+2s}^0.$$

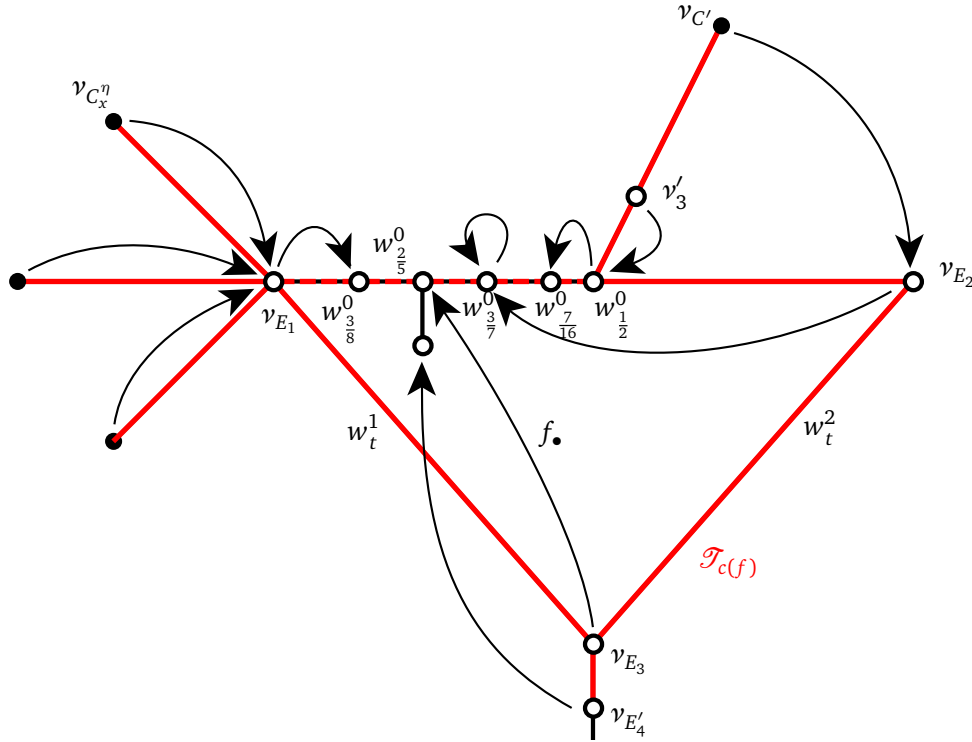
With respect to the monomial parameterization $w^j : [0, 1] \rightarrow [\nu_{E_{a_j}}, \nu_{E_{b_j}}]_{p_j}$ given by $w(t) = \nu_{1-t,t}^j$, we get

$$f_* w^0(t) = \begin{cases} w^0\left(\frac{4-t}{9-2t}\right) & \text{if } t \geq \frac{1}{2}, \\ w^0\left(\frac{3+t}{8}\right) & \text{if } t \leq \frac{1}{2}. \end{cases} \quad f_* w^1(t) = w^0\left(\frac{3-t}{8-3t}\right), \quad f_* w^2(t) = w^0\left(\frac{3-t}{7-2t}\right).$$

$$c(f, w^0(t)) = \begin{cases} 9-2t & \text{if } t \geq \frac{1}{2}, \\ 8 & \text{if } t \leq \frac{1}{2}. \end{cases} \quad c(f, w^1(t)) = 8-3t, \quad c(f, w^2(t)) = 7-2t.$$

In particular, $f_* \nu_{E_1} = w^0(3/8)$, $f_* \nu_{E_2} = w(3/7)$, $f_* \nu_{E_3} = w^0(2/5)$. Notice also that $f_* w^0(3/7) = w^0(3/7) =: \nu_*$ is the unique eigenvaluation given by [Theorem 5.8](#), and $c_\infty(f) = c(f, \nu_*) = 8$. Notice also that the Möbius maps appearing in the expressions of $f_* w^j(t)$ are all monotone, and $f_* w^0(1/2) = w^0(7/16)$. Similar computations show that the segment $[\nu_{E_3}, \nu_{E'_4}]_{p_3}$ is sent to a segment $[w^0(2/5), \nu_{E''_5}]$, where E''_5 is the exceptional prime obtained by blowing up a suitable free point in the exceptional prime associated to $w^0(2/5)$. Finally, $c(f, w^3(t)) = 5-t$. Notice that $c(f, \nu)$ is locally constant on a weak open neighborhood of ν_* . It follows that for any valuation $\nu_0 \in \mathcal{V}_X^\alpha$, the sequence $c_n = c(f^n, \nu_0)$ eventually satisfy the recursion relation $c_{n+1} = 8c_n$.

We now study the behavior of f_* towards contracted curve semivaluations in \mathcal{C}_f . It consists of 4 curve semivaluations, associated to the irreducible curves $C_x^\eta = \{x = y - \eta z^3 = 0\}$ for any η satisfying $\eta^3 = -1$, and $C' = \{y - x = 0\} \cap X$. Notice that with respect to the coordinates (x_0, y_0, z_0) at p_0 , C_x^η lifts to the curve $\tilde{C}_x^\eta = \{x_0 = y_0 - \eta = 0\}$. It intersects the exceptional prime E_1 transversely at the point $p_\eta = (0, \eta, 0)$. At $p_\eta \in X_{\pi_1}$ local parameters are given by $(y_0 - \eta, z_0)$. The monomial valuation


 Figure 7: Action of f_\bullet . (§9.6).

ν_t^η of weights t and 1 with respect to $y_0 - \eta$ and z_0 is normalized for all $t \in [0, +\infty]$, and by direct computation we get

$$f_* \nu_t^\eta = \nu_{t+5,3}^0 = (8+t)w^0 \left(\frac{3}{8+t} \right).$$

In particular $f_\bullet \nu_{C_x^\eta} = \nu_{E_1}$.

We turn our attention to the curve C' , which lifts to the curve $\{y_0 = x_0 z_0\} \cap X_{\pi_1}$. In particular, it intersects $\pi_1^{-1}(x_0)$ in p_0 , transversely to E_1' and E_2' . Denote by μ_t' the monomial valuation at p_0 of weights 1 and t with respect to the coordinates x_0 and $y_0 - x_0 z_0$. It is easy to check that

$$\mu_t' = (1+t)w^0 \left(\frac{t}{1+t} \right) \text{ for } t \leq 1, \quad \mu_t'(m) = 2 \text{ for } t \geq 1.$$

Let ν_t' be the normalized valuation proportional to μ_t' . By direct computations, we get

$$f_* \mu_t' = \begin{cases} \nu_{5+4t,3+4t}^0 & \text{if } t \leq 1, \\ \nu_{9,6+t}^0 & \text{if } t \geq 1, \end{cases} \quad \text{and we deduce} \quad f_* \nu_t' = \begin{cases} 8w^0 \left(\frac{3+4t}{8(1+t)} \right) & \text{if } t \leq 1, \\ \frac{15+t}{2} w^0 \left(\frac{6+t}{15+t} \right) & \text{if } t \geq 1. \end{cases}$$

Notice in particular that $f_\bullet \nu_3' = w^0(1/2)$. Figure 7 describes the action of f_\bullet on \mathcal{V}_X , while Figure 8 describes the action induced by f on the minimal resolution X_{π_0} , the minimal log resolution X_{π_1} of m , and the minimal log resolution X_{π_2} of m which also is an embedded resolution of $f^{-1}(x_0)$. Notice that all three models are geometrically stable in this case.

9.7. A finite map on a cusp singularity

This example was proposed in [25]. Let (X, x_0) be the cusp singularity whose minimal desingularization is a cycle of two rational curves, of self-intersections -4 and -2 . It can be obtained as the hypersurface

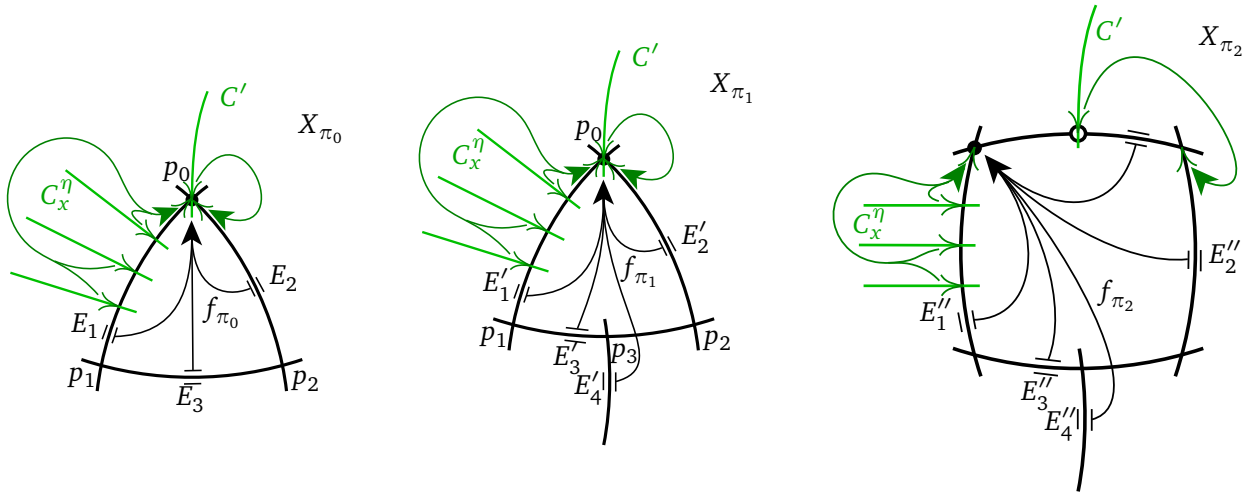


Figure 8: Examples of actions induced by f on modified spaces (§9.6).

singularity $X = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^4 + z^6 = xyz\}$. Here we consider the toric construction of this cusp, as described in §A.1. Set $\omega = [[4, 2]]$ as in (40). By direct computation, $\omega = 2 + \sqrt{2}$. We follow the notations of §A.1, and set $N_\omega = \mathbb{Z} \oplus \omega\mathbb{Z}$. The ring $\sigma^>$ of positive units is generated by $1 + \sqrt{2}$, while the ring σ^+ of totally positive units is generated by its square $3 + 2\sqrt{2}$. Notice that the lattice N_ω is preserved, and $\varepsilon_\omega = 3 + 2\sqrt{2}$ generates σ^+ . The cusp (X, x_0) is then constructed by choosing $\varepsilon = \varepsilon_\omega$ in the construction given in §A.1. Recall that by Remark A.1, the vectors $(1, 0)$ and $(0, 1)$ in the basis $\{1, \omega\}$ correspond to the exceptional primes E_0 and E_1 in the cycle, whose selfintersections are -4 and -2 respectively. Notice that a basis of a lattice N_ω is given by $\{1, \sqrt{2}\}$. With respect to this basis, E_0 and E_1 correspond to the vectors $e_0 = (1, 0)$ and $e_1 = (2, 1)$. The exceptional primes $E_n, n \in \mathbb{Z}$ that appear in the universal covering X_∞ of (X, x_0) are associated to e_n , constructed inductively so that $e_{n+2} = Ae_n$ for all n , where A is the matrix $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ representing the multiplication by ε .

Consider now the matrix $B = \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$, representing the multiplication by $\alpha = 3 + \sqrt{2}$. The action of B induces a finite endomorphism $f : (X, x_0) \rightarrow (X, x_0)$ on the cusp, of topological degree $e(f) = 7 = \det B$ and first dynamical degree $c_\infty(f) = \sqrt{7}$. Using the correspondence described in Remark A.1, we can study the action induced by f on the cycle.

Any given vector $v \in \mathcal{C}$ belongs to at least one (and at most two) face of the fan defining X_∞ , generated by some (e_n, e_{n+1}) . So we can write $v = re_n + se_{n+1}$ for suitable $r, s \geq 0$ (not both 0). We may do the same for $w = Bv$, which will belong to some other face generated by (e_m, e_{m+1}) . We set $w = r'e_m + r'e_{m+1}$.

We denote by $\nu = \nu_{r,s}$ the monomial valuation at $E_n \cap E_{n+1}$ corresponding to $v = re_n + se_{n+1}$ (modulo the action of g_ε). The normalization with respect to the maximal ideal corresponds here to $\nu(\mathfrak{m}) = r + s$. We denote by ν_t the normalized valuation corresponding to $(1-t)e_0 + te_1$, and by μ_t the monomial valuation corresponding to $(1-t)e_1 + te_2$. Notice that $\nu_1 = \mu_0$ and $\nu_0 = \mu_1$. In fact, $t \mapsto \nu_t$ and $t \mapsto \mu_t$ are the monomial parameterizations of $[\nu_{E_0}, \nu_{E_1}]_{p_0}$ and $[\nu_{E_1}, \nu_{E_0}]_{p_1}$ respectively, where p_0 and p_1 are the two points of intersections between the two exceptional primes E_0 and E_1 in the minimal resolution

of (X, x_0) . By direct computation, we get:

$$c(f, \nu_t) = t + 2, \quad f \bullet \nu_t = \begin{cases} \nu_{\frac{1+4t}{2+t}} & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \mu_{\frac{3t-1}{2+t}} & \text{if } \frac{1}{3} \leq t \leq 1, \end{cases}$$

$$c(f, \mu_t) = \begin{cases} t + 3 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 5 - 3t & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \quad f \bullet \mu_t = \begin{cases} \mu_{\frac{2+3t}{3+t}} & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \nu_{\frac{2t-1}{5-3t}} & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The parameterization γ of rays of \mathcal{C} induces a parameterization $\gamma_X: \mathbb{R}/\mathbb{Z} \rightarrow \mathcal{S}_X$ (see §A.2). Explicitly, ν_t correspond to the vector $(1-t)e_0 + te_1$, which correspond to the real $a(t) = (1-t) + t(2 + \sqrt{2})$. Notice that $Q(a(t)) = a(t)a'(t) = (1+t)^2 - 2t^2$, where a' denotes the extension of the conjugation in $K = \mathbb{Q}(\sqrt{2})$ to \mathbb{R} . Set $q = \sqrt{Q}$. Then $\nu_t = \gamma_X(u(t))$, where

$$u(t) = \frac{\log(a(t)) - \log(q(a(t)))}{2 \log \varepsilon}.$$

Analogous computations can be done for μ_t . With respect to this parameterization, $f \bullet$ acts as an irrational rotation, by the value $\beta \in \mathbb{R}/\mathbb{Z}$ given by

$$\beta = \frac{\log(\alpha) - \log(q(\alpha))}{2 \log \varepsilon} = \frac{\log(3 + \sqrt{2}) - \log(\sqrt{7})}{2 \log(3 + 2\sqrt{2})}.$$

9.8. Different normalizations

In the whole paper, to study the dynamics induced by a holomorphic germ $f: (X, x_0) \rightarrow (X, x_0)$ on the set of (finite) semivaluations $\hat{\mathcal{V}}_X^*$, we fixed a normalization, by considering the value they take on the maximal ideal \mathfrak{m} . While being a quite natural choice, this is only one of the possible normalizations.

The normalization with respect to the maximal ideal can be reformulated as follows. For any valuation $\nu \in \hat{\mathcal{V}}_X^*$, we consider its “norm”

$$\|\nu\|_Z = -Z(\nu) \cdot Z, \quad (38)$$

where $Z = Z(\mathfrak{m})$ is the b-divisor associated to \mathfrak{m} . In this case, the intersection (38) is always well defined and finite, since Z is a Cartier b-divisor, and this intersection coincides with $-Z_\pi(\nu) \cdot Z_\pi(\mathfrak{m}_X)$, where π is a log-resolution of \mathfrak{m} . But (38) makes sense for any nef b-divisor Z , for which we get $-Z(\nu) \cdot Z \in (0, +\infty]$ for all $\nu \in \hat{\mathcal{V}}_X^*$. As meaningful examples, we can take $Z = Z(\mathfrak{a})$ for any (\mathfrak{m} -primary) ideal \mathfrak{a} , or $Z = Z(\mu)$ for any semivaluation $\mu \in \hat{\mathcal{V}}_X^*$.

Notice that if Z is not Cartier, the value $-Z(\nu) \cdot Z$ could be $+\infty$. For example, consider the valutive space $\hat{\mathcal{V}}^*$ associated to a smooth point $(\mathbb{C}^2, 0)$, and its normalization with respect to the value at some coordinate x . In this case $\text{int}_x(x) = +\infty$, and the normalization is not well defined here. Nevertheless the normalization on quasimonomial valuations can be extended by continuity, and in this case it would give the monomial valuation of weights $(1, 0)$ with respect to the coordinates (x, y) . See [26] for an extensive literature on different normalizations on the valutive tree.

In fact, one should think of \mathcal{V}_X as the quotient of $\hat{\mathcal{V}}_X^*$ by the natural multiplicative action of $\mathbb{R}_+^* = (0, +\infty)$. The normalized space $\mathcal{V}_X[Z] = \{\nu \in \hat{\mathcal{V}}_X^* \mid \|\nu\|_Z = 1\}$ would then correspond to a section of the natural projection $\hat{\mathcal{V}}_X^* \rightarrow \mathcal{V}_X$.

All results obtained in this paper can be reformulated more generally for any choice of normalizations as above. As we have seen, the skewness α , the thinness A , and the attraction rate $c(f, -)$ depend on the normalizations chosen. Nevertheless, one can easily pass from one normalization to the other by homogeneity. Notice finally that the angular distance does not depend on the normalization, and hence it defines a distance on the intrinsic space \mathcal{V}_X more than on the normalized space $\mathcal{V}_X[\mathfrak{m}]$.

9.9. Automorphisms

The valuative analysis described for non-invertible germs can be worked out for invertible germs as well. For an automorphism $f : (X, x_0) \rightarrow (X, x_0)$, one can easily check that $f_\bullet = f_*$ acts as an isomorphism with respect to the angular distance, and preserves log discrepancies.

In particular, for smooth points $(X, x_0) = (\mathbb{C}^2, 0)$ this implies that ord_0 is a totally invariant valuation. In general the sets $\text{Fix}(f_*)$ and $\text{Per}(f_*)$ of fixed and periodic valuations is quite huge, and contains enough information to allow to distinguish different conjugacy classes. As an example, for any $\lambda \neq 0$ and $u \in \mathbb{N}^*$, consider

$$f(x, y) = (\lambda x, \lambda^u y + \varepsilon x^u). \quad (39)$$

When $|\lambda| < 1$, this is the Poincaré-Dulac normal form of contracting invertible automorphisms which are resonant, see [62, 56, 6]. Denote by $\nu_{y,t}$ the monomial valuation of weights 1 and t with respect to the coordinates (x, y) . It is easy to check that $f_* \nu_{y,u} = \nu_{y,u}$, and the action of f on the tangent space is given in suitable coordinates by $\zeta \mapsto \zeta + \varepsilon \lambda^{-u}$. By further computations, one can notice that $\text{Fix}(f_*) = [\nu_x, \nu_{y,u}]$ as far as $\varepsilon \neq 0$, while $\text{Fix}(f_*)$ contains all segments $[\text{ord}_0, \nu_{y+\theta x^u}]$ for all $\theta \in \mathbb{C}$ when $\varepsilon = 0$.

For singular spaces, studying finite order automorphisms $g : (X, x_0) \rightarrow (X, x_0)$ (for which $\text{Fix}(g) = \{x_0\}$) and their action on the valuative space allows to relate the valuative spaces of a singularity and of its quotient by the action of g . We have seen examples of this phenomenon in §6.1.

9.10. Positive characteristic

All along the paper, we considered maps and singularities over the complex numbers. Here we comment on the more general situation where we replace \mathbb{C} by any field \mathbb{K} . First, remark that we can always work on the algebraic closure of a given field. We will hence assume that \mathbb{K} is algebraically closed. As commented in the introduction, all of the results in this paper hold over any field \mathbb{K} of zero characteristic, as we never use the archimedean structure of the complex numbers.

Consider now an algebraically closed field \mathbb{K} of characteristic $p > 0$. The contents of §2 still hold in this setting, as well the contents of §4, but for the Jacobian formula of §4.5. In particular, Theorem 5.10, and all its corollaries on the existence of geometrically stable models in §7 and recursion relations for the sequence of attraction rates in §8, still hold for *non-finite* germs over fields \mathbb{K} of positive characteristic having a non-quasimonomial eigenvaluation.

While we believe that for non-finite germs Theorem 5.15 should also hold in characteristic p , for finite germs new phenomena arise, due to the presence of *wild ramifications*.

Example 9.3. Consider the map $F : (\mathbb{A}_{\mathbb{K}}^2, 0) \rightarrow (\mathbb{A}_{\mathbb{K}}^2, 0)$ given by

$$F(x, y) = (x^p, y^p).$$

Notice that the jacobian determinant J_F of F vanishes identically, even though the map F is dominant. It is easy to show that the set of eigenvaluations for F is given by the set of valuations whose center in X_π belongs to $X_\pi(\mathbb{F}_p)$ for any modification $\pi : X_\pi \rightarrow (\mathbb{C}^2, 0)$. Similarly, eigenvaluations for F^k corresponds to valuations whose center belongs to $X_\pi(\mathbb{F}_{p^k})$. This is in sharp contrast with the situation described by Theorem 4.41. This phenomenon is given by the fact that for any divisorial eigenvaluation ν_E , the map F acts on E as the Frobenius map $\zeta \mapsto \zeta^p$.

Notice that in the non-finite case, Frobenius maps may still appear, but not for the self-action at an exceptional prime associated to a (totally invariant) divisorial eigenvaluation, since it would contradict the uniqueness of the eigenvaluation established by Theorem 5.8. Ramification indices may increase with respect to the characteristic zero case, as the following example shows.

Example 9.4. Consider the map $f : (\mathbb{A}_{\mathbb{K}}^2, 0) \rightarrow (\mathbb{A}_{\mathbb{K}}^2, 0)$ given by

$$f(x, y) = (x^p, x^{p-1}y).$$

As in the previous example, the jacobian determinant J_f vanishes identically. In this case however, we still have a unique eigenvaluation ord_0 , and any other valuation $\nu \in \mathcal{V}^\alpha$ converges to ord_0 . Denote by $\nu_{y,t}$ the monomial valuation at 0 of weights 1 and t with respect to coordinates (x, y) . We have $f_* \nu_{y,2} = p \nu_{y,1+1/p}$. If we denote by E and F the exceptional primes associated to $\nu_{y,2}$ and $\nu_{y,1+1/p}$ respectively, than f induces the Frobenius map $\zeta \mapsto \zeta^p$. For any $t \geq 2$ and $\zeta \in \mathbb{K}^*$, denote by $\nu_{\zeta,t}$ the valuation in $[\text{ord}_0, \nu_{y-\zeta x^2}]$ whose skewness is t . Then $f_* \nu_{\zeta,t}$ is the valuations in $[\text{ord}_0, \nu_{y^p-\zeta^p x^{p+1}}]$ whose skewness is $h(t) = (p-1+t)/p$. If f is seen as an action on $(\mathbb{C}^2, 0)$, the same holds, but with $h_{\mathbb{C}}(t) = ((p+2)(p-1)+t)/p^2$ replacing $h(t)$.

[Example 9.3](#) can be easily generalized. For example, for any germ $\sigma : (\mathbb{A}_{\mathbb{K}}^2, 0) \rightarrow (\mathbb{A}_{\mathbb{K}}^2, 0)$ and any $k \in \mathbb{N}^*$, we may consider maps of the form $f = \sigma \circ F^k$. Notice that, unlike the case of dimension one, we cannot split any germ f in such a way so that the Jacobian of σ is not vanishing (see [Example 9.4](#)).

Example 9.5. Let $u \in \mathbb{N}^*$, $\varepsilon \in \mathbb{K}$, and consider the map $f : (\mathbb{A}_{\mathbb{K}}^2, 0) \rightarrow (\mathbb{A}_{\mathbb{K}}^2, 0)$ given by

$$f(x, y) = (x^p, y^p + \varepsilon x^{pu}).$$

This map is obtained as the composition $\sigma \circ F$, where σ is of the form (39). In this case, one can easily notice that $[\nu_x, \nu_{y,u}]$ is fixed by f_* . The action of f on the exceptional prime associated to $\nu_{y,u}$ is given by $h: \zeta \mapsto \zeta^p + \varepsilon$. Its properties depend on whether ε belongs to \mathbb{F}_p (or its algebraic closure $\overline{\mathbb{F}_p}$) or not.

Notice that $h^n(\zeta) = \zeta^{p^n} + \sum_{k=0}^{n-1} \varepsilon^{p^k}$. If $\varepsilon = 0$, we have again the Frobenius map described in [Example 9.3](#).

If $\varepsilon \in \mathbb{F}_p^*$, then h^p is the p -th iterate of the Frobenius map. Analogously, if $\varepsilon \in \overline{\mathbb{F}_p}$, then some iterate of h and the Frobenius map coincide. If $\varepsilon \notin \overline{\mathbb{F}_p}$, then we have no periodic directions associated to points over $\overline{\mathbb{F}_p}$.

In particular, the study of the structure of $\text{Fix}(f_*)$ becomes quite intricate, and depends on the properties of the field \mathbb{K} .

Notice that even though the Jacobian formula (30) doesn't hold in positive characteristic, it remains as the inequality

$$A(f_* \nu) \leq A(\nu) + \nu(R_f),$$

even when the Jacobian determinant doesn't vanish identically.

Example 9.6. Let $u \in \mathbb{N}^*$, $\varepsilon \in \mathbb{K}$, and consider the map $f : (\mathbb{A}_{\mathbb{K}}^2, 0) \rightarrow (\mathbb{A}_{\mathbb{K}}^2, 0)$ given by

$$f(x, y) = (x^p(1+x), y^p(1+y)).$$

In this case, the jacobian determinant $J_f = x^p y^p$ does not vanish, but it has an higher multiplicity than expected. In particular,

$$2p = A(p \text{ord}_0) = A(f_* \text{ord}_0) < A(\text{ord}_0) + \text{ord}_0(J_f) = 2 + 2p.$$

The proof of [Theorem 4.42](#) relies on the Jacobian formula, and hence it does not remain valid in positive characteristic. In fact, any singularity defined over \mathbb{F}_q admits non-invertible finite self-maps.

Example 9.7. Let (X, x_0) be a singularity over a field \mathbb{K} of characteristic $p > 0$. Suppose that (X, x_0) is embedded in an affine space $\mathbb{A}_{\mathbb{K}}^m$, and obtained as the vanishing locus of some polynomials (or more generally analytic functions) with coefficients in \mathbb{F}_{p^s} for some $s \in \mathbb{N}^*$. Consider the map $f : \mathbb{A}_{\mathbb{K}}^m \rightarrow \mathbb{A}_{\mathbb{K}}^m$ given by $(x_1, \dots, x_m) = (x_1^p, \dots, x_m^p)$. Then f^s induces a finite non-invertible self-map on (X, x_0) .

The conclusions of [Theorem 4.42](#) remain valid as far as the Jacobian formula holds, while they are clearly false when the Jacobian determinant identically vanishes. As showed by [Example 9.6](#), some mixed behavior may occur. We believe it could be interesting to investigate which classes of singularities admit non-invertible finite self-maps whose Jacobian determinant doesn't identically vanish. We refer to this case as maps admitting a Jacobian divisor, which can be constructed by considering the Jacobian determinant on $X \setminus \{x_0\}$, which is defined since f is finite.

Question 9.8. Let (X, x_0) be a normal surface singularity over an algebraically closed field \mathbb{K} of positive characteristic. Assume there exists a finite germ $f : (X, x_0) \rightarrow (X, x_0)$ which is non-invertible, and admits a Jacobian divisor. Does this give obstructions for the geometry of (X, x_0) ?

A. Cusp singularities

A.1. Arithmetic construction of cusp singularities

We describe here the arithmetic construction of cusps singularities, see e.g. [[55](#), Section 4.1]. Let $r \in \mathbb{N}^*$ and k_0, \dots, k_{r-1} be a finite sequence of integer numbers ≥ 2 , not all equal to 2. Set $k_{nr+j} := k_j$ for all $n \in \mathbb{Z}$, so that $(k_j)_{j \in \mathbb{Z}}$ defines a periodic sequence of integers ≥ 2 (not all equal to 2), that we may assume of exact period r . We denote by $\omega = [[k_0, k_1, \dots]] = \overline{[[k_0, \dots, k_{r-1}]]}$ the modified continued fraction

$$\omega = k_0 - \frac{1}{k_1 - \frac{1}{\ddots}}. \quad (40)$$

Then ω is a quadratic irrational number, $\omega = a + b\sqrt{d}$, where $a, b \in \mathbb{Q}_+^*$ are positive rational numbers and $d \in \mathbb{N}$ is a positive and square-free integer. Moreover $\omega > 1 > \omega' > 0$, where ω' is the conjugate of ω in $\mathbb{Q}(\sqrt{d}) =: K$. Set now

- $N = N_\omega := \mathbb{Z} \oplus \omega\mathbb{Z}$ the lattice of rank 2 in $\mathbb{Q}(\sqrt{d})$ generated by 1 and ω ;
- $\mathfrak{o}^>$ the group of positive units of K , i.e., (algebraic) integers $u \in K$ which are invertible and satisfying $u > 0$;
- \mathfrak{o}^+ the *totally positive*, i.e., elements $u \in \mathfrak{o}^>$ satisfying $u' > 0$;
- $\mathfrak{o}_\omega^>$ and \mathfrak{o}_ω^+ the group of positive and totally positive units u satisfying $uN_\omega = N_\omega$.

By Dirichlet's unit theorem, $\mathfrak{o}^>$ and \mathfrak{o}^+ are infinite cyclic groups, and \mathfrak{o}^+ has index either 1 or 2 in $\mathfrak{o}^>$. The same property holds for unities preserving the lattice N_ω . Let ε_ω be the generator of \mathfrak{o}_ω^+ satisfying $\varepsilon_\omega > 1$.

The embedding $\Phi : K \rightarrow \mathbb{R}^2$ given by $\Phi(\xi) = (\xi, \xi')$ induces a canonical isomorphism $K \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow \mathbb{R}^2$. Set $e_0 = (1, 0)$, $e_1 = (0, 1)$, and recursively e_n for all $n \in \mathbb{Z}$ by imposing $e_{n-1} + e_{n+1} = k_n e_n$. Let $L_\omega : \mathbb{Q}^2 \rightarrow K$ be defined by $(x, y) \mapsto x + \omega y$, and set $\Phi_\omega = \Phi \circ L_\omega$. One can check that $L_\omega(e_n)$ are totally positive, and hence for all $n \in \mathbb{Z}$, we have $\Phi_\omega(e_n) \in (\mathbb{R}_+^*)^2 =: \mathcal{C}$. Denote by Δ boundary of the convex hull of $\Phi(N_\omega) \cap \mathcal{C}$. Then $\Phi(N_\omega) \cap \Delta = \{\Phi_\omega(e_n), n \in \mathbb{Z}\}$. The fan \mathcal{F} consisting of the 0-face $(0, 0)$, the

1-faces $\mathbb{R}_+\Phi_\omega(e_n)$ and the 2-faces $\mathbb{R}_+\Phi_\omega(e_n) + \mathbb{R}_+\Phi_\omega(e_{n+1})$ for all $n \in \mathbb{Z}$ is a regular fan of the open cone $\mathcal{C}_0 = \{0\} \cup \mathcal{C}$. We call X_∞ the (non-compact) toric surface associated to \mathcal{F} . It contains an infinite chain of compact rational curves $(E_n)_{n \in \mathbb{Z}}$, with selfintersections $(E_n)^2 = -k_n$.

For any element $\varepsilon \in \mathfrak{o}_\omega^+$, consider the map $g_\varepsilon: \mathcal{C}_0 \rightarrow \mathcal{C}_0$ defined by $g_\varepsilon(x, y) = (\varepsilon x, \varepsilon' y)$. Since $\varepsilon N_\omega = N_\omega$, the map g_ε leaves $\Phi(N_\omega)$ invariant, and induces an automorphism $G_\varepsilon: X_\infty \rightarrow X_\infty$ (the inverse given by $G_{\varepsilon'}$) which (as far as $\varepsilon \neq 1$), acts freely and properly discontinuously. Hence the quotient $X_\varepsilon = X_\infty / \langle G_\varepsilon \rangle$ is a complex surface, which has a cycle of rational curves. In fact, assume that $\varepsilon = \varepsilon_\omega^s$, $s \in \mathbb{N}^*$. Up to refining the fan \mathcal{F} by blowing up points in $E_{n+r} \cap E_{n+1+r}$ for all $k \in \mathbb{Z}$, we may assume that $rs \geq 2$. Then the cycle of rational curves has rs components, of self-intersections $-k_0, \dots, -k_{rs-1}$, i.e., s copies of the starting sequence $-k_0, \dots, -k_{r-1}$. We may contract this cycle in X_ε , obtaining a cusp singularity (X, x_0) .

By a result of Laufer's [47], cusps singularities are taut, meaning that the dual graph, together with selfintersections of components, determine the analytical type of a cusp singularity. In particular, all cusps singularities can be constructed as above, up to isomorphisms.

Remark A.1. Notice that $\Phi_\omega: \mathbb{Q}^2 \rightarrow \mathbb{R}^2$ is \mathbb{Q} -linear, and it can be extended by continuity to a \mathbb{R} -linear map $\Phi_\omega: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Analogously, consider the quadratic form $Q(\alpha) = \alpha\alpha'$ on K . The map $Q_\omega = L_\omega \circ Q: \mathbb{Q}^2 \rightarrow \mathbb{R}$ can be extended to a continuous map $Q_\omega: \mathbb{R}^2 \rightarrow \mathbb{R}$. Notice that $Q_\omega(x, y) > 0$ as far as $\Phi_\omega(x, y) \in \mathcal{C}$.

If we denote by \mathcal{S}_X the skeleton of (X, x_0) , and by $\hat{\mathcal{S}}_X^*$ the cylinder $\mathcal{S}_X \times \mathbb{R}_+^*$, then there is a natural isomorphism $\tau: \hat{\mathcal{S}}_X \rightarrow \mathcal{C} / \langle g_\varepsilon \rangle =: \mathcal{C}_\varepsilon$. Denote by $[\cdot]$ the natural projection $\mathcal{C} \rightarrow \mathcal{C}_\varepsilon$. Then τ has the following properties :

- For any exceptional prime E_n , $n = 0, \dots, rs - 1$, $\tau(\text{ord}_E) = [\Phi_\omega(e_n)]$.
- For any valuation $\nu \in \hat{\mathcal{S}}_X^*$ and any $\lambda > 0$, we have $\tau(\lambda\nu) = \lambda\tau(\nu)$.
- For any intersection point p_n of two exceptional primes E_n and E_{n+1} , the monomial valuation $\nu_{r,s}$ at p_n satisfying $Z_{\pi_0}(\nu_{r,s}) = r\check{E}_n + s\check{E}_{n+1}$ is sent by τ to $[\Phi_\omega(re_n + se_{n+1})]$.

Notice that two valuations $\nu, \mu \in \hat{\mathcal{S}}_X^*$ are proportional if and only if $\tau(\nu)$ and $\tau(\mu)$ belong to the same ray of \mathcal{C}_0 . A normalization of valuations in $\hat{\mathcal{S}}_X^*$ corresponds to taking a non-vanishing section of the set of rays of \mathcal{C}_0 . We can take for example the section given by $Q \equiv 1$. Notice that since $Q(\varepsilon) = 1$, this section is g_ε -invariant, and it induces a section of the rays in \mathcal{C}_ε .

A.2. Finite endomorphisms

We now describe an arithmetic construction which produces finite endomorphisms of cusps (X, x_0) constructed as above (see [25, §2.5]).

We denote by \mathfrak{c}_ω^+ the set of (totally) positive elements $\alpha \in K$ so that $\alpha N_\omega \subseteq N_\omega$. For any $\alpha \in \mathfrak{c}_\omega^+$, the map $g_\alpha: \mathcal{C}_0 \rightarrow \mathcal{C}_0$ induces a morphism $G_\alpha: X_\infty \rightarrow X_\infty$. This map will have in general indeterminacy points, and it is not an automorphism as far as α is not a unit. Since g_α commutes with g_ε , G_α induces a map on the quotient $X_\varepsilon = X_\infty / \langle G_\varepsilon \rangle$. By contracting the cycle of rational curves, we obtain a finite endomorphism $f_\alpha: (X, x_0) \rightarrow (X, x_0)$.

Remark A.2. The fact that $\alpha N_\omega \subseteq N_\omega$ guarantees that f_α can be expressed as a formal (rational) map (with respect to suitable coordinates), while the fact that the cone \mathcal{C} is invariant assures that f_α defines a holomorphic map at x_0 . Notice also that α needs to be an integer in K . In fact, since α leaves N_ω invariant, we have in particular $\alpha^n \in N_\omega$ for all n . If α is not an integer, then $Q(\alpha^n)$ would have unbounded denominators, in contradiction with belonging to N_ω . Finally, we notice that f_α is finite, of topological degree $Q(\alpha) = \alpha\alpha' \in \mathbb{N}^*$.

The map $\gamma(t) = (\varepsilon^{2t}, 1) \cdot \mathbb{R}_+$, where $t \in \mathbb{R}$, gives a parameterization of the rays in \mathcal{C}_0 (see [25, p. 413]). The action of g_ε on the rays of \mathcal{C}_0 corresponds to the translation by 1 on \mathbb{R} . By Remark A.1 we deduce that the rays in \mathcal{C}_0 , quotiented by the action of g_ε , are in 1-to-1 correspondence with the skeleton \mathcal{S}_X . In fact, the normalization of valuations, which corresponds to taking a section of $\hat{\mathcal{S}}_X^*$, here correspond to taking a non-vanishing section of the rays of \mathcal{C}_0 which is g_ε -invariant. We deduce that γ induces a parameterization $\gamma_X: \mathbb{R}/\mathbb{Z} \rightarrow \mathcal{S}_X$.

The action of g_α corresponds, with respect to this parameterization, to a translation by the value

$$\beta = \frac{\log \alpha - \log \alpha'}{2 \log \varepsilon}.$$

This construction of finite germs f_α on cusp singularities is quite general. In fact, for any finite germ f , there exists a f_α constructed as above so that the actions of f_\bullet and $(f_\alpha)_\bullet$ coincide on the skeleton \mathcal{S}_X . To prove this result, we first need a lemma.

Lemma A.3. *Let (X, x_0) be a cusp singularity constructed as in §A.1. Then there exists a constant $K > 0$ so that, for all $t, u \in \mathbb{R}/\mathbb{Z}$, we have that*

$$\rho(\gamma_X(t), \gamma_X(u)) = K d_{\mathbb{Z}}(t, u),$$

where $d_{\mathbb{Z}}$ denotes the distance induced by the euclidean distance on \mathbb{R}/\mathbb{Z} .

Proof. For any $\alpha \in N_\omega$, consider the finite germ $f_\alpha: (X, x_0) \rightarrow (X, x_0)$. We just need to consider a α which gives an irrational rotation on \mathcal{S}_X , whose existence is guaranteed by Proposition 6.6. Since f_α acts as an isometry on \mathcal{S}_X , the measure induced by ρ is the Haar measure of \mathcal{S}_X . The action induced by f_α on \mathbb{R}/\mathbb{Z} is a translation by an irrational number β , and $d_{\mathbb{Z}}$ is the Haar measure of \mathbb{R}/\mathbb{Z} . We conclude by uniqueness of the Haar measure up to multiplicative constants. \square

Proposition A.4. *Let $f: (X, x_0) \rightarrow (X, x_0)$ be a finite germ on a cusp singularity (X, x_0) constructed as in §A.1. Suppose that f_\bullet preserves the orientation of the circle \mathcal{S}_X (it may be always assumed by replacing f by its second iterate). Then there exists $\alpha \in N_\omega$ so that the action of f_* on $\hat{\mathcal{S}}_X^*$ corresponds to the action of g_α on \mathcal{C} .*

Proof. By Lemma A.3, the action of f_\bullet corresponds, through the parameterization given by γ_X , to the translation by some $\beta \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ be so that $\varepsilon^{2\beta} = \alpha^2/Q(\alpha)$. Notice that this defines α up to positive multiplicative constants.

By Lemma 7.8, we know that f_* acts on \mathcal{C} as a piecewise linear map with integer coefficients. This implies that, up to rescaling by a suitable multiplicative constant, f_* acts as the linear map g_α . Since f is holomorphic at x_0 , we must have $g_\alpha N_\omega \subseteq N_\omega$, and $g_\alpha \mathcal{C} \subset \mathcal{C}$, i.e., α is totally positive. \square

Proposition A.4 states that for any finite germ f on a cusp singularity for which the action on the circle \mathcal{S}_X is a rotation, then this action coincides with the one of a finite germ f_α constructed arithmetically as above. Of course, in general f and f_α do not coincide. In fact, any singularity (X, x_0) admits lots of non-commuting automorphisms, coming from the time-1 flow of vector fields tangent to the singularity (see [52]). We may compose f_α by any such automorphism of (X, x_0) , obtaining another germ f . Notice that this operation does not change the action induced on \mathcal{S}_X . In particular there exists infinitely many different finite germs whose action on \mathcal{S}_X coincide. Nevertheless, we may wonder if all such germs are analytically conjugated one to another.

Question A.5. *Is any finite germ $f: (X, x_0) \rightarrow (X, x_0)$ on a cusp singularity (X, x_0) , whose action preserves the orientation of the circle \mathcal{S}_X , analytically conjugated to a finite germ of the form f_α , for some $\alpha \in N_\omega$?*

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