TORIC KATO MANIFOLDS

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Abstract. We introduce and study a special class of Kato manifolds, which we call *toric Kato manifolds*. Their construction stems from toric geometry, as their universal covers are open subsets of toric algebraic varieties of non-finite type. This generalizes previous constructions of Tsuchihashi and Oda, and in complex dimension 2, retrieves the properly blown-up Inoue surfaces. We study the topological and analytical properties of toric Kato manifolds and link certain invariants to natural combinatorial data coming from the toric construction. Moreover, we produce families of flat degenerations of any toric Kato manifold, which serve as an essential tool in computing their Hodge numbers. In the last part, we study the Hermitian geometry of Kato manifolds. We give a characterization result for the existence of locally conformally Kähler metrics on any Kato manifold. Finally, we prove that no Kato manifold carries balanced metrics and that a large class of toric Kato manifolds of complex dimension ≥ 3 do not support pluriclosed metrics.

1. Introduction

Kato manifolds are compact complex manifolds of non-Kähler type and were introduced by M. Kato in 1977 [Kat77] as manifolds containing a global spherical shell. More specifically, he showed that any such manifold, which we shall call henceforth a *Kato manifold*, is constructed in the following way. Let $\pi : \hat{\mathbb{B}} \to \mathbb{B}$ be a modification of the standard unit ball in $\mathbb{C}^n$ at finitely many points and let $\sigma : \overline{\mathbb{B}} \hookrightarrow \hat{\mathbb{B}}$ be a holomorphic embedding. Glue small neighborhoods of the two boundary components of $\hat{\mathbb{B}} \setminus \sigma(\mathbb{B})$ via the local biholomorphism $\sigma \circ \pi$. The resulting manifold $X(\pi, \sigma)$ is a compact complex manifold with infinite cyclic fundamental group. The couple (π, σ) is referred to as a *Kato data*.

A Kato manifold has an associated germ given by $F := \pi \circ \sigma : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Many of its properties are encoded in F, however the germ alone does not fully characterize the manifold, unless the complex dimension is 2. This comes from the fact that for $n = 2$, any modification is a sequence of blow-ups at points, while in higher dimension modifications can be much more complicated. As a result, Kato surfaces are much studied and fairly well understood (see for instance [Nak84], [Dlo84], [DOT03],[Tel05], [FP10] etc.), while in higher dimension very little is known.

One main motivation for studying Kato manifolds comes from the fact that a big part of them carry locally conformally Kähler metrics, but no Vaisman metrics [Bru11], [IOP20]. They constitute therefore an immense source of manifolds where one can study the interplay between the existence of special Hermitian metrics and different geometric and cohomological properties.

In [IOP20] we considered the simplest class of Kato manifolds, which corresponds to the case when $\pi : \hat{\mathbb{B}} \to \mathbb{B}$ is given by successive blow-ups at special points and σ is a standard

chart of a blow-up. We studied several of their analytical invariants, but did not have the tools to compute any Hodge number.

In the present paper, we introduce a much more general class, the *toric Kato manifolds*. They are determined by a toric Kato data, namely a smooth toric modification $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ at 0 and a chart $\sigma : \mathbb{C}^n \to \hat{\mathbb{C}}^n$ satisfying a natural $(\mathbb{C}^*)^n$-equivariance property. The corresponding germ has the form

$$F(z) = F_{\underline{\lambda},A}(z) := \underline{\lambda} z^A = (\lambda_1 z_1^{a_{11}} \cdots z_n^{a_{1n}}, \ldots, \lambda_n z_1^{a_{n1}} \cdots z_n^{a_{nn}}) \tag{1.1}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ and $A = (a_{jk}) \in \mathrm{GL}(n, \mathbb{Z})$.

Starting from the fan of $\hat{\mathbb{C}}^n$, viewed as a toric variety, and the embedding σ, we construct a natural infinite fan Σ_A endowed with a $\mathbb{Z}$-action. Then using the germ F, we define a $\mathbb{Z}$-invariant open set in the toric variety of non-finite type $\widetilde{X}_{\underline{\lambda},A} \subset X(\Sigma_A, \mathbb{Z}^n)$. Our starting point is the following description of toric Kato manifolds (see Theorem 4.2 for a more precise statement).

Theorem 1.1: *If X is any toric Kato manifold with germ $F_{\underline{\lambda},A}$, then we have a biholomorphism:*

$$X \cong \widetilde{X}_{\underline{\lambda},A}/\mathbb{Z}.$$

This second point of view of toric Kato manifolds generalizes the toric description of Inoue surfaces given by Oda [Oda78, Section 14], and a construction given by Tsuchihashi in [Tsu87] of a class of non-Kähler manifolds with infinite cyclic fundamental group. It allows for a better conceptual understanding of our manifolds, and in particular unveils new ways for computing different invariants using classical techniques from toric geometry.

For instance, denoting by a_j the number of j-dimensional cones of the fan of $\hat{\mathbb{C}}^n$, we have the following (see Theorem 6.1).

Theorem 1.2: *The n-dimensional toric Kato manifold X has the following Betti numbers:*

$$b_0(X) = b_1(X) = b_{2n-1}(X) = b_{2n}(X) = 1$$
$$b_{2j+1}(X) = 0, \quad 1 \leq j \leq n-2$$
$$b_{2j}(X) = -1 + \sum_{s=j}^{n} (-1)^{s-j} \binom{s}{j} \left(a_{n-s} + \binom{n}{s+1} \right), \quad 1 \leq j \leq n-1.$$

The matrix A in (1.1) has only non-negative coefficients. Consider the maximal sub-matrix P of A which is a permutation matrix. When $P = A$, the resulting toric Kato manifold is a primary Hopf manifold. When $P \in \mathrm{GL}(n-1, \mathbb{Z})$, the universal cover of the corresponding toric Kato manifold X is all of $X(\Sigma_A, \mathbb{Z}^n)$, and we call X of *parabolic type*. In all the other cases, $\widetilde{X}$ is a proper subset of $X(\Sigma_A, \mathbb{Z}^n)$ and we call X of *hyperbolic type*. Imitating Nakamura's classification of Inoue surfaces [Nak84] in terms of curves, we have the following characterization (see Theorem 5.2).

Theorem 1.3: *Let X be a toric Kato manifold.*

1. *X is a primary Hopf manifold if and only if any of its $(\mathbb{C}^*)^n$-invariant curves is elliptic;*
2. *X is of hyperbolic type if and only if any $(\mathbb{C}^*)^n$-invariant curve is rational;*

(3) X is of parabolic type if and only if X contains a unique $(\mathbb{C}^*)^n$-invariant elliptic curve, and at least one rational $(\mathbb{C}^*)^n$-invariant curve.

Concerning analytic invariants, we determine explicitly the canonical line bundle in terms of the maximal toric divisor and show that the Kodaira dimension of any toric Kato manifold is negative (Proposition 5.7). Also, we compute several Hodge numbers (see Proposition 8.4, Theorem 8.5 and Theorem 8.8).

Theorem 1.4: *Let X be a toric Kato manifold with divisor D induced by the exceptional divisor of π. Then one has $H^0(X, \Omega^p_X) = 0$ for any $p \geq 1$. If moreover X is of hyperbolic type, then one has:*

$$h^{0,0}(X) = h^{0,1}(X) = 1,\ h^{0,p}(X) = 0,\ p \geq 2$$
$$h^{1,p}(X) = 0,\ p \neq 1$$
$$h^{1,1}(X) = b_2 = \sharp\{\textit{irreducible components of } D\} > 0.$$

We note that the primary Hopf manifolds have the same Hodge numbers as above, and that our proof also works for parabolic type manifolds with $|\underline{\lambda}|$ small enough (see Remark 8.9). In particular, for all these cases one has $b_k = \sum_{p+q=k} h^{p,q}$ for $k \leq 2$. We suspect that all Kato manifolds should have the Hodge numbers of the above theorem, however showing this is beyond our means since much of our proof is based on the toric description of our manifolds.

A main tool for computing the Hodge numbers of toric Kato manifolds is developed in Section 7, where we exhibit two different types of flat toric degenerations of toric Kato manifolds. The first one generalizes a previous construction of Nakamura [Nak83] for surfaces. In this case we deform, in a smooth family, any toric Kato manifold to a singular space obtained by identifying two invariant hypersurfaces on a smooth compact toric variety. The second type of degeneration exists only in the hyperbolic case and extends a construction given by Tsuchihashi in [Tsu87]. In this case, the singular fiber is again given by the identification of two invariant hypersurfaces in a compact, possibly singular, toric variety. A special feature of this second family consists in the fact that all the smooth fibers are biholomorphic.

Finally, guided by the general principle of finding special Hermitian metrics on complex manifolds of non-Kähler type, we investigate the case of Kato manifolds. Special Hermitian metrics usually arise by imposing some power of the fundamental form Ω to be in the kernel of some differential operator. The existence of such metrics is far from being guaranteed on a generic compact complex manifold. As a consequence, constant effort has been put to find new specific examples and to unravel the different restrictions that the existence of such metrics might impose.

In this direction, our first result consists in a characterization for the existence of locally conformally Kähler (lcK) metrics (see Theorem 10.3).

Theorem 1.5: *Let $(\pi : \hat{\mathbb{B}} \to \mathbb{B}, \sigma : \overline{\mathbb{B}} \to \hat{\mathbb{B}})$ be a Kato data and let $X = X(\pi, \sigma)$ be the corresponding Kato manifold. The following are equivalent:*

(1) X admits a locally conformally Kähler metric;
(2) $\widetilde{X}$ admits a Kähler metric;
(3) $\hat{\mathbb{B}}^n$ admits a Kähler metric.

We should note here that in the case of a toric Kato manifold X, although the compact torus $\mathbb{T}^n$ acts on the universal cover $\widetilde{X}$, this action never descends to X if X is not Hopf. Therefore, X is not a toric manifold in any classical sense. On the other hand, if X admits an lcK metric, then $\widetilde{X}$ admits a Kähler metric with respect to which it becomes a toric Kähler manifold. In this manner, lcK toric Kato manifolds give interesting generalizations of the class of toric lcK manifolds.

Our last result is concerned with the non-existence of other special Hermitian metrics (see Section 10 for the definitions, Theorem 10.6 and Theorem 10.8).

Theorem 1.6:

(1) A Kato manifold admits no strongly Gauduchon metric, and in particular no balanced metric and no Hermitian symplectic metric.
(2) If X is a Kato manifold of dimension $n \geq 3$ satisfying $H^{1,2}_{\overline{\partial}}(X) = 0$, then it cannot be endowed with a pluriclosed metric. In particular, a toric Kato manifold of hyperbolic type does not admit pluriclosed metrics, unless it is a surface.

The paper is organized as follows. We begin by some necessary preliminaries on Kato manifolds. The first definition of toric Kato manifolds, together with the elementary but technical properties of toric Kato data, are given in Section 3. In the next section, we describe and prove the second characterization of toric Kato manifolds. Next, in Section 5, we study geometrical properties of toric Kato manifolds, such as the invariant complex submanifolds, or the different natural divisors. Section 6 is dedicated to the topological invariants, while Section 8, to the computation of the Hodge numbers. In Section 7, we describe the degenerations of toric Kato manifolds needed for the next section. In Section 9, we tackle the problem of the classification of toric Kato manifolds. Finally, Section 10 deals with the existence of special Hermitian metrics on a general Kato manifold. We conclude the section with an explicit family of examples of toric Kato manifolds in any complex dimension $n \geq 4$ that do not admit any lcK metrics.

2. Preliminaries on Kato manifolds

In the present paper, a Kato manifold will be a compact complex manifold obtained as follows [Kat77], [Dlo84]. We let $\mathbb{B} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n || z_1|^2 + \ldots + |z_n|^2 < 1\}$ be the standard open ball in $\mathbb{C}^n$. Consider a modification at 0 $\pi : \hat{\mathbb{B}} \to \mathbb{B}$ and a holomorphic embedding $\sigma : \overline{\mathbb{B}} \hookrightarrow \hat{\mathbb{B}}$. Glue the two boundaries of $\hat{\mathbb{B}} \setminus \sigma(\mathbb{B})$ via the real analytic CR-diffeomorphism $\gamma := \sigma \circ \pi$. The result is a compact complex manifold $X(\pi, \sigma)$, named here a *Kato manifold*. Any small neighborhood of the image of $\partial\hat{\mathbb{B}}$ in $X(\pi, \sigma)$ is a global spherical shell. We shall refer to the couple (π, σ) as *Kato data* and to $F := \pi \circ \sigma : (\mathbb{B}, 0) \to (\mathbb{B}, 0)$ as *the corresponding germ*.

The class of Kato manifolds introduced in [Kat77] is slightly more general, as π is allowed to be a modification at more than one point, so that the resulting manifolds are proper modifications of the ones we described above [Dlo84, Lemme 2.7, Part I]. However, for the present discussion this generality will not make any difference.

We denote by $q : \widetilde{X} \to X$ the universal cover of a Kato manifold $X = X(\pi, \sigma)$. The deck group Γ of q is canonically isomorphic to $\mathbb{Z}$, and we will denote also by γ the positive generator of Γ, which is indeed induced by the map $\sigma \circ \pi$.

We start by settling several general facts that will be needed in the paper.

Lemma 2.1: *Let $F : (\mathbb{B}, 0) \to (\mathbb{B}, 0)$ be a holomorphic germ with $F(\overline{\mathbb{B}}) \subset \mathbb{B}$. Then $\bigcap_{m\in\mathbb{N}} F^m(\mathbb{B}) = \{0\}$.*

Proof. By hypothesis there exists $0 < r < 1$ so that $F(\mathbb{B}) \subset \mathbb{B}_r$. By the Schwarz lemma [Sha92, Theorem 6], we have $||F(z)|| \leq r||z||$ for any $z \in \mathbb{B}$, hence $||F^m(z)|| \leq r^m$ for any $m > 0$ and any $z \in \mathbb{B}$. This implies that $\lim_{m\to\infty} \sup_{z\in\mathbb{B}} ||F^m(z)|| = 0$, and hence $\operatorname{diam}\left(\bigcap_{m\in\mathbb{N}} F^m(\mathbb{B})\right) = 0$. The conclusion then follows. ∎

Given a map $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, define its stable set:

$$W^s(F) = \{z \in \mathbb{C}^n \mid \lim_{m\to\infty} ||F^m(z)|| = 0\}.$$

Lemma 2.2: *For any holomorphic map $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with $F(\overline{\mathbb{B}}) \subset \mathbb{B}$, we have $W^s(F) = \bigcup_{m\in\mathbb{Z}} F^m(\mathbb{B})$, where for $m = -k < 0$, $F^{-k}(\mathbb{B})$ is the preimage of $\mathbb{B}$ via F^k.*

Proof. The previous lemma shows that $\mathbb{B} \subset W^s(\mathbb{B})$. Clearly $F^m(\mathbb{B}) \subset W^s(F)$ for any $m > 0$ and also for any $m < 0$. Conversely, if $z \in W^s(F)$ then there exists $m > 0$ so that $F^m(z) \in \mathbb{B}$, which shows the desired equality. ∎

Suppose that $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is given by a Kato data (π, σ), $F = \pi \circ \sigma$, so that π is a modification at 0. For any $k > 0$, let $H_k := F^{-k}(0)$ so that $H_k \subseteq H_{k+1}$ and put $H_\infty := \bigcup_{k>0} H_k$. Suppose that $H_\infty \subset \mathbb{C}^n$ is closed – this is the case for instance if $H_\infty = H_m$ for some $m > 0$. Let us set $\operatorname{Inv}(F) := \mathbb{C}^n \setminus H_\infty$ and $W^*(F) := W^s(F) \cap \operatorname{Inv}(F)$.

Proposition 2.3: *Let $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic map with $F(0) = 0$ given by a Kato data (π, σ), $F = \pi \circ \sigma$, so that π is a modification at 0 with exceptional divisor E. Suppose that each component of E induces a compact analytic subset of $X(\pi, \sigma)$ and let D be the divisor obtained by their formal sum. Then we have a biholomorphism $X(\pi, \sigma) \setminus D \cong W^*(F)/\langle F\rangle$.*

Proof. By hypothesis, $H_m = H_\infty$ for some $m > 0$. If we put $\mathbb{B}^* := \mathbb{B} \cap \operatorname{Inv}(F)$, then by Lemma 2.2 we have $W^*(F) = \bigcup_{m\in\mathbb{Z}} F^m(\mathbb{B}^*)$. Furthermore, by Lemma 2.1, we have:

$$\bigcup_{m\in\mathbb{Z}} \left(F^m(\mathbb{B}^*) \setminus F^{m+1}(\mathbb{B}^*)\right) = \bigcup_{m\in\mathbb{Z}} F^m(\mathbb{B}^*) \setminus \bigcap_{m\in\mathbb{Z}} F^m(\mathbb{B}^*) = W^*(F).$$

Now it is clear that $D \cap (\hat{\mathbb{B}} \setminus \sigma(\mathbb{B})) = \pi^{-1}(H_\infty) \cap (\hat{\mathbb{B}} \setminus \sigma(\mathbb{B}))$ and that $\pi : \left(\hat{\mathbb{B}} \setminus \sigma(\mathbb{B})\right) \setminus \pi^{-1}(H_\infty) \to \mathbb{B}^* \setminus F(\mathbb{B}^*)$ establishes a biholomorphism satisfying $\pi \circ \gamma = F \circ \pi$, where $\gamma = \sigma \circ \pi$. This implies in particular that the group $\langle F\rangle$ acts freely and properly on $W^*(F)$ and that π induces the desired isomorphism $X(\pi, \sigma) \setminus D \cong W^*(F)/\langle F\rangle$. ∎

Proposition 2.4: *Let $(\pi : \hat{\mathbb{B}} \to \mathbb{B}, \sigma : \mathbb{B} \to \hat{\mathbb{B}})$ be a Kato data with germ $F = \pi \circ \sigma$ and let $X = X(\pi, \sigma)$ be the corresponding Kato manifold. Then there exists a holomorphic open embedding $\varphi : \hat{\mathbb{B}} \setminus \{\sigma(0)\} \to \widetilde{X}$.*

Proof. We recall that $\widetilde{X} = \bigsqcup_{m\in\mathbb{Z}} W_m$, where $W_m = \hat{\mathbb{B}} \setminus \sigma(\mathbb{B})$ for any $m \in \mathbb{Z}$ and W_m is glued to W_{m+1} via $\gamma = \sigma \circ \pi$.

Let us put, for any $l \geq 1$, $\widetilde{X}_l := \bigsqcup_{1 \leq m \leq l} W_m \subset \widetilde{X}$. Also, following [IOP20, Section 1], denote by $(\pi_l : \hat{\mathbb{B}}^{(l)} \to \mathbb{B}, \sigma_l : \mathbb{B} \to \hat{\mathbb{B}}^{(l)})$ the Kato data obtained by composing (π, σ) with itself l times. Then it is easy to see that $\widetilde{X}_l = \hat{\mathbb{B}}^{(l)} \setminus \sigma_l(\mathbb{B})$.

If we denote by $\pi_{(l)} : \hat{\mathbb{B}}^{(l)} \to \hat{\mathbb{B}}^{(l-1)}$ the induced map, then using that $\pi_{(l)} \circ \sigma_l = \sigma_{l-1} \circ F$ we find that $\pi_{(l)}^{-1}$ gives an embedding of $\hat{\mathbb{B}}^{(l-1)} \setminus \sigma_{l-1}(F(\mathbb{B}))$ into $\hat{\mathbb{B}}^{(l)} \setminus \sigma_l(\mathbb{B})$. Therefore we find inductively, for any $l \geq 1$, open embeddings

$$\varphi_l : Q_l := \hat{\mathbb{B}} \setminus \sigma(F^l(\mathbb{B})) \to \hat{\mathbb{B}}^{(l)} \setminus \sigma_l(\mathbb{B}) \subset \widetilde{X}$$

which satisfy $\varphi_{l+1}|_{Q_l} = \varphi_l$. Now since $\bigcap_{l \geq 1} F^l(\mathbb{B}) = \{0\}$ by Lemma 2.1, this means that $\bigcup_{l \geq 1} Q_l = \hat{\mathbb{B}} \setminus \{\sigma(0)\}$, and thus the family $\{\varphi_l\}_{l \geq 1}$ naturally defines an open embedding $\varphi : \hat{\mathbb{B}} \setminus \{\sigma(0)\} \to \widetilde{X}$, which concludes the proof. ∎

3. Toric Kato manifolds

We refer the reader to [Ful93], [Oda78], [Oda88] or any other classical reference for the theory of toric algebraic varieties. Here, we will only fix notation.

We denote by $N := \mathbb{Z}^n$, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, and

$$\mathbb{T} := N \otimes_{\mathbb{Z}} \mathbb{S}^1 = (\mathbb{S}^1)^n \subset T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* = (\mathbb{C}^*)^n.$$

We recall that any toric algebraic variety X is a T_N-equivariant compactification of T_N, and its algebraic structure is encoded by a fan Σ with support in $N_{\mathbb{R}}$. We write $X = X(\Sigma, N)$. Traditionally, Σ is a finite collection of rational cones with some compatibility properties, so that X is covered by a finite number of affine open sets $(X_\tau)_{\tau \in \Sigma}$. However, in the present text, the fan Σ is allowed to be infinite, and then X is of non-finite type.

We denote by $|\Sigma| := \bigcup_{\tau \in \Sigma} \tau$ the support of Σ. We will let $\Sigma^{(k)} \subset \Sigma$ denote the subset of k-dimensional cones of Σ. Also, for a collection of vectors $v_j \in N_{\mathbb{R}}$, $j \in J$, we will denote by $\langle v_j \mid j \in J \rangle := \oplus_{j \in J} \mathbb{R}_{\geq 0} v_j$ the cone generated by it.

Given a cone $\tau \in \Sigma^{(k)}$, we recall the notation $\check{\tau} := \{l \in M_{\mathbb{R}} \mid \langle l, v \rangle \geq 0 \ \forall v \in \tau\}$, which is a k-codimensional cone in $M_{\mathbb{R}}$, $\tau^\perp := \{l \in M_{\mathbb{R}} \mid \langle l, v \rangle = 0 \ \forall v \in \tau\}$, which is a face of $\check{\tau}$, and $S_\tau := \check{\tau} \cap M$. Then one has

$$X_\tau = \mathrm{Spec}\, \mathbb{C}[S_\tau] = \mathrm{Hom}_{u.s.g.}(S_\tau, \mathbb{C})$$

where *u.s.g.* stands for unit semi-groups. We also recall [Oda78, Theorem 4.2] that there exists a one-to-one correspondence between cones $\tau \in \Sigma$ and T_N-orbits on $X(\Sigma, N)$, given by $\mathrm{orb}\, \tau = \mathrm{Hom}_{gr.}(\tau^\perp \cap M, \mathbb{C}^*)$, so that $\dim \tau = \mathrm{codim}\, \mathrm{orb}\, \tau$.

Finally, the map

$$\mathrm{ord} : \mathbb{C} \to \mathbb{R} \cup \{\infty\}, \quad z \mapsto -\log |z|$$

restricts to a group homomorphism from $\mathbb{C}^*$ onto $\mathbb{R}$, and induces a surjective map of fiber $\mathbb{T}$

$$\mathrm{ord} : T_N \to N_{\mathbb{R}}.$$

3.1. **Toric Kato data.** We let $z = (z_1, \ldots, z_n)$ denote the standard holomorphic coordinates on $\mathbb{C}^n$. In all that follows, $\mathbb{C}^n$ is viewed as a toric variety with the standard action of T_N. We will denote an element of the complex torus by $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in T_N$, and for a toric variety X and $x \in X$, we simply write $\underline{\lambda} x$ to denote the action of $\underline{\lambda}$ on x.

We will call a proper modification $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ at $0 \in \mathbb{C}^n$ *a toric modification* if $\hat{\mathbb{C}}^n$ is a T_N-toric variety and π is T_N-equivariant.

Definition 3.1: A Kato data $(\pi : \hat{\mathbb{B}} \to \mathbb{B}, \sigma : \mathbb{B} \to \hat{\mathbb{B}})$ is called a *toric Kato data* if the extension $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ is a smooth toric modification at 0 and there exists $\nu \in \mathrm{Aut}_{gr.}(T_N)$ so that $\sigma(\underline{\lambda} z) = \nu(\underline{\lambda})\sigma(z)$ for any $z \in \mathbb{B}$ and $\underline{\lambda} \in T_N$ for which this is defined. In this case, we say that σ is ν-equivariant. A Kato manifold will be called a *toric Kato manifold* if it admits a toric Kato data.

Let $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ be a toric modification at 0, let Σ be the fan of $\mathbb{C}^n$ and let $\hat{\Sigma}$ be the fan of $\hat{\mathbb{C}}^n$. Note that $\hat{\Sigma}$ is uniquely determined by $\pi|_{\pi^{-1}(\mathbb{B})}$. Then $|\hat{\Sigma}| = |\Sigma| = (\mathbb{R}_{\geq 0})^n =: C_0$ [Oda88]. Moreover, since π is a biholomorphism outside 0, it follows that

$$|\hat{\Sigma}^{(1)} \setminus \Sigma^{(1)}| \subset \mathrm{Int}|\Sigma| \tag{3.1}$$

i.e. each new ray of $\hat{\Sigma}$ has only positive components. Conversely, any finite refinement with regular cones $\hat{\Sigma}$ of Σ satisfying (3.1) defines a toric modification of $\mathbb{C}^n$ at 0.

For every cone $\tau \in \hat{\Sigma}^{(n)}$, there exists a unique $\mathbb{Z}$-basis of N $\underline{e}^\tau := \{e_1^\tau, \ldots, e_n^\tau\}$, such that $\langle e_j^\tau \rangle \prec \tau$, $j = 1, \ldots, n$. If $\underline{f}^\tau := \{f_1^\tau, \ldots, f_n^\tau\}$ is the corresponding dual basis of M, then the map

$$S_\tau = \Big\{ \sum_{j=1}^n u_j f_j^\tau \mid u_1 \ldots, u_n \in \mathbb{N} \Big\} \to \mathbb{N}\langle e_1^*, \ldots, e_n^* \rangle, \quad \sum_{j=1}^n u_j f_j^\tau \mapsto \sum_{j=1}^n u_j e_j^* \tag{3.2}$$

gives rise to a holomorphic chart

$$\varphi_\tau : \mathbb{C}^n = \mathrm{Spec}\, \mathbb{C}[e_1^*, \ldots, e_n^*] \to \mathrm{Spec}\, \mathbb{C}[S_\tau] \subset \hat{\mathbb{C}}^n.$$

This chart is uniquely determined up to composing with a permutation of the coordinates. In what follows, we will call such a chart *a toric chart*.

Express the vectors of $\underline{e}^\tau$ in the standard basis $e_1, \ldots, e_n$ of N, and put $A_\tau := \begin{pmatrix} e_1^\tau & \cdots & e_n^\tau \end{pmatrix} \in \mathrm{GL}(n, \mathbb{Z})$. Then A_τ completely determines the chart φ_τ, so we will also use the notation $\varphi_\tau = \varphi_{A_\tau}$. Note that since $\tau \subset C_0$, A_τ has only non-negative coefficients. Since $A_\tau = (a_{kl})_{k,l}$ satisfies

$$e_j^* = \sum_{k=1}^n a_{jk} f_k^\tau, \quad 1 \leq j \leq n$$

it follows that the map $\pi \circ \varphi_\tau : \mathbb{C}^n \to \mathbb{C}^n$, which is induced by

$$\mathbb{N}\langle e_1^*, \ldots, e_n^* \rangle \to \mathbb{N}\langle e_1^*, \ldots, e_n^* \rangle, \quad e_j^* \mapsto \sum_{k=1}^n a_{jk} e_k^*$$

satisfies

$$\pi \circ \varphi_\tau(z) = z^{A_\tau} := (z_1^{a_{11}} \cdots z_n^{a_{1n}}, \ldots, z_1^{a_{n1}} \cdots z_n^{a_{nn}}), \quad z \in \mathbb{C}^n. \tag{3.3}$$

Using the equivariance of π, we immediately infer the equivariance relation:

$$\varphi_\tau(\underline{\lambda} z) = \underline{\lambda}^{A_\tau} \varphi_\tau(z), \quad \forall \underline{\lambda} \in T_N. \tag{3.4}$$

Any other toric chart φ'_τ for X_τ is given by $\varphi'_\tau = \varphi_\tau \circ \hat{s}$, where $s \in \mathcal{S}_n$ is a permutation and $\hat{s}(z) = (z_{s(1)}, \dots, z_{s(n)}) = z^{P_s}$, with $P_s = \begin{pmatrix} e_{s(1)} & \cdots & e_{s(n)} \end{pmatrix}^T$. Then the corresponding matrix for φ'_τ is given by $A'_\tau = A_\tau P_s$.

Lemma 3.2: *Let (π, σ) be a toric Kato data. Let $\hat{\Sigma}$ be the fan determined by π, and let $\tau \in \hat{\Sigma}^{(n)}$ be the cone representing the T_N-fixed point $\sigma(0) \in \hat{\mathbb{B}}$. Then there exists $\underline{\lambda}_0 \in T_N$ and a toric chart $\varphi_A : \mathbb{C}^n \to X_\tau$ so that $\sigma = \underline{\lambda}_0 \circ \varphi_A$. In particular, the germ $F := \pi \circ \sigma$ satisfies $F(z) = F_{\underline{\lambda}_0, A}(z) := \underline{\lambda}_0 z^A$ for all $z \in \mathbb{C}^n$.*

Proof. Note that $\mathrm{Aut}_{gr.}(T_N) \cong \mathrm{Aut}_{\mathbb{Z}}(N) \cong \mathrm{GL}(n, \mathbb{Z})$, and a matrix A determines the group automorphism $\underline{\lambda} \mapsto \underline{\lambda}^A$. Hence, by hypothesis, there exists $A \in \mathrm{GL}(n, \mathbb{Z})$ so that $\sigma(\underline{\lambda} z) = \underline{\lambda}^A \sigma(z)$, for all $\underline{\lambda} \in T_N$ and $z \in \mathbb{B}$ for which this is defined.

Let $B \in \mathrm{GL}(n, \mathbb{Z})$ be a matrix giving a toric chart $\varphi_B : \mathbb{C}^n \to X_\tau$. Then $f := \varphi_B^{-1} \circ \sigma : \mathbb{B} \to \mathbb{C}^n$ is a biholomorphism onto its image, and it satisfies, for any $\underline{\lambda} \in T_N \cap \mathbb{B}$:

$$f(\underline{\lambda}) = f(\underline{\lambda}_1(\underline{\lambda}\underline{\lambda}_1^{-1})) = \varphi_B^{-1}(\underline{\lambda}^A \underline{\lambda}_1^{-A} \sigma(\underline{\lambda}_1)) = \underline{\lambda}^{B^{-1}A} \underline{\lambda}_1^{-B^{-1}A} f(\underline{\lambda}_1)$$

where $\underline{\lambda}_1$ is some fixed element from $T_N \cap \mathbb{B}$. Thus, putting $\underline{\lambda}_2 := \underline{\lambda}_1^{-B^{-1}A} f(\underline{\lambda}_1) \in T_N$, we find that $f(z) = \underline{\lambda}_2 z^{B^{-1}A}$ on $\mathbb{B}$. Now since f has an holomorphic inverse, it is easy to check that $B^{-1}A$ is a permutation matrix P_s for some $s \in \mathcal{S}_n$. Therefore $A = BP_s$ is a toric matrix and $\sigma = \underline{\lambda}_0 \circ \varphi_A$ for $\underline{\lambda}_0 := \underline{\lambda}_2^B$, as expected. ∎

Lemma 3.3: *If (π, σ) is a toric Kato data with germ $F_{\underline{\lambda}_0, A}$ and $d > 1$ is a natural number, then there exist a toric Kato data (π^d, σ^d) naturally associated to (π, σ), with corresponding germ $F^d_{\underline{\lambda}_0, A}$. Moreover, $X(\pi^d, \sigma^d)$ is a cyclic unramified covering of $X(\pi, \sigma)$ of degree d.*

Proof. The Kato data (π^d, σ^d) is obtained by composing (π, σ) with itself d times, as it was described in [IOP20, Section 1], so that the resulting germ is the d-th power of the initial one. From the construction, it is clear that the resulting Kato data is again toric, and that the fan corresponding to π^d is given by

$$\hat{\Sigma}^d := \{A^k \tau \mid 0 \leq k \leq d-1,\ \tau \in \hat{\Sigma}_0\} \cup \{A^{d-1}\tau_A\}$$

where $\hat{\Sigma}_0 := \hat{\Sigma} \setminus \{\tau_A\}$, with $\hat{\Sigma}$ the fan of π and $\tau_A \in \hat{\Sigma}^{(n)}$ the cone representing $\sigma(0)$. Furthermore, by [IOP20, Lemma 1.7], $X(\pi^d, \sigma^d)$ is a cyclic unramified covering of $X(\pi, \sigma)$ of degree d. ∎

3.2. **Properties of toric Kato matrices.**

Definition 3.4: A matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ will be called a *toric Kato matrix* if there exists a toric modification $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ at 0 and a toric chart $\varphi_A : \mathbb{C}^n \to \hat{\mathbb{C}}^n$ with $\varphi_A(\underline{\lambda} z) = \underline{\lambda}^A \varphi_A(z)$ for all $z \in \mathbb{C}^n$ and $\underline{\lambda} \in T_N$.

Proposition 3.5: *A matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ is a toric Kato matrix if and only if its columns are either positive or standard vectors.*

Proof. The fact that a toric Kato matrix satisfies the desired properties is a direct consequence of (3.1). For the other implication, we need to prove that for any $A \in \mathrm{GL}(n, \mathbb{Z})$ with either positive or standard columns, there exists a smooth toric modification $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ at 0, such that the cone τ_A generated by the columns of A belongs to the corresponding fan $\hat{\Sigma}$ of $\hat{\mathbb{C}}^n$.

This can be achieved by first considering any simplicial fan containing the cone τ_A and whose rays are generated either by the standard vectors or positive vectors, and then regularizing it via barycentric subdivision (which adds only positive rays, see, e.g., [Ful93, p. 48]). ■

Proposition 3.5 shows that our setting is more general than the one considered in [IOP20]. In fact, toric modifications of $0 \in \mathbb{C}^n$ need not dominate the blow-up of the origin, as the following example shows.

Example 3.6: Consider the toric Kato matrix $A = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \in \mathrm{GL}(3,\mathbb{Z})$. Then the cone τ_A generated by the three columns of A intersects in its interior the cone generated by $e_1 = (1,0,0)$ and $e_1 + e_2 + e_3 = (1,1,1)$. Hence no toric modification $\pi : \hat{\mathbb{C}}^3 \to \mathbb{C}^3$ over 0 for which the fan $\hat{\Sigma}$ of $\hat{\mathbb{C}}^3$ contains the cone τ_A dominates the blow-up of the origin.

In Figure 1 we explicit two different realizations of the toric modification π containing τ_A.

In particular, the one on the right is obtained as a composition of three point blow-ups, followed by the blow-up of four equivariant lines, and by the contraction of the exceptional divisor associated to the ray generated by $(5,3,3)$. The last blow-up and contraction corresponds to a flip in the language of Minimal Model Program.

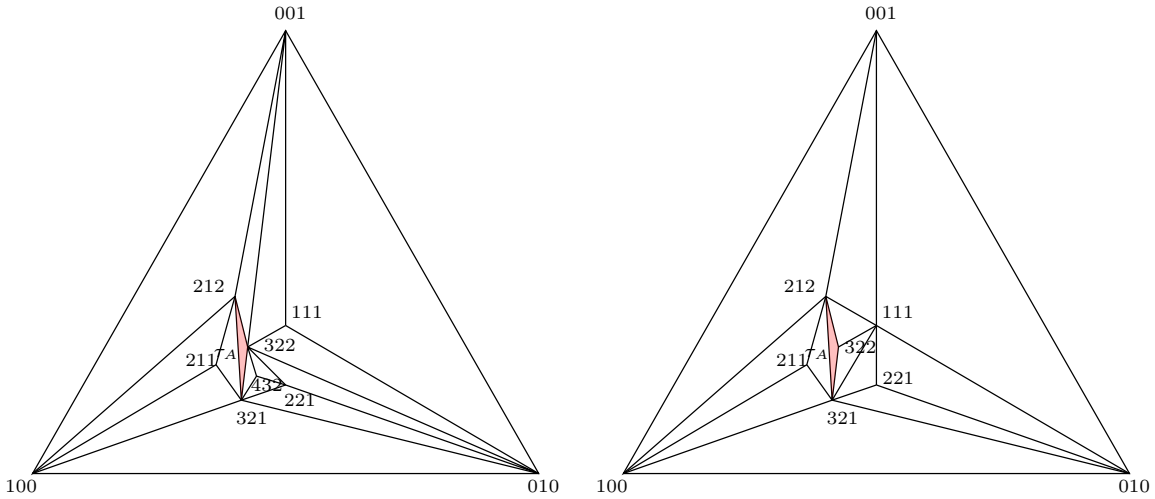

FIGURE 1. Toric Kato data with π not dominating the blow-up of the origin.

Let $(\pi, \sigma = \underline{\lambda}_0 \circ \varphi_A)$ be a toric Kato data, and let $\tau_A \in \hat{\Sigma}^{(n)}$ be the cone generated by the columns of A, or equivalently, the cone corresponding to the fixed point $\sigma(0)$. Then A is a positive matrix if and only if $\tau_A \subset \mathrm{Int}C_0$. In general, we have:

Lemma 3.7: *Let $A = (A_1 \;\; A_2 \;\; \cdots \;\; A_n)$ be a toric Kato matrix. There exists a unique maximal subset $P(A) \subset \{1,\dots,n\}$ satisfying that there exists a permutation $s : P(A) \to P(A)$ so that $A_j = e_{s(j)}$ for any $j \in P(A)$. Furthermore, there exists $m_0 \geq 1$ so that for any $m \geq m_0$ and for any $j \in \{1,\dots,n\} \setminus P(A)$, the j-th column of A^m has strictly positive components.*

Proof. Let us define

$$S_1(A) := \{j \in \{1,\dots,n\} \mid A_j \in \{e_1,\dots,e_n\}\}$$

and let $k : S_1(A) \to \{1, \dots, n\}$ be the injective map satisfying $A_j = e_{k(j)}$ for all $j \in S_1(A)$. Moreover, define inductively, for all $m \geq 2$, $S_m(A) := S_{m-1}(A) \cap k^{-1}(S_{m-1}(A))$. Then one has:

$$\dots \subseteq S_m(A) \subseteq S_{m-1}(A) \subseteq \dots \subseteq S_1(A)$$

and so there exists $m_0 \geq 1$ with $S_m(A) = S_{m_0}(A)$ for all $m \geq m_0$. One clearly has then $P(A) = S_{m_0}(A)$ and $s = k|_{P(A)}$.

On the other hand, it is easy to check that $S_m(A) = S_1(A^m)$ for all $m \geq 1$. Moreover, by eq. (3.1), a j-th column of A^m has only positive components exactly when $j \notin S_1(A^m)$, from which the conclusion follows. ∎

For any matrix $A = (a_{kl})_{1 \leq k,l \leq n} \in \mathrm{GL}(n, \mathbb{Z})$ and $J \subset \{1, \dots, n\}$, we denote by $J^c = \{1, \dots, n\} \setminus J$ and we put $A_J := (a_{kl})_{k,l \in J} \in \mathcal{M}(|J|, \mathbb{Z})$, which is the matrix obtained from A by erasing the lines and columns prescribed by J^c. We also use the notation $\tau_J \in \Sigma^{(|J|)}$ for the cone generated by e_j with $j \in J$ and we put $N_J := \mathrm{span}_{\mathbb{Z}}\{e_j | j \in J\} \subset N$, respectively $T_J := N_J \otimes_{\mathbb{Z}} \mathbb{C}^*$, $\mathbb{C}_J = N_J \otimes \mathbb{C}$ etc. Finally, we denote by p_J the natural projection from N to N_J, as well as its $\mathbb{R}$ or $\mathbb{C}$-linear extension, and for $v \in \mathbb{C}^n$, we also write $v_J := p_J(v)$.

Lemma 3.8: *Let $A = (a_{kl})_{1 \leq k,l \leq n}$ be a toric Kato matrix and let $J \subseteq P(A)$ be any subset with $|J| < n - 1$ which is fixed by the permutation s given in Lemma 3.7. Then A_{J^c} is again a toric Kato matrix.*

Proof. Let $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ be a toric modification at 0 so that A gives a toric chart in $\hat{\mathbb{C}}^n$. Let $\hat{\mathbb{C}}_{J^c} := \overline{\mathrm{orb}\, \tau_J} \subset \hat{\mathbb{C}}^n$ be the strict transform of $\mathbb{C}_{J^c} = \overline{\mathrm{orb}\, \tau_J} \subset \mathbb{C}^n$ via π. Note that since A preserves N_J, it follows that also the morphism of T_N given by $\nu_A(\underline{\lambda}) = \underline{\lambda}^A$ fixes T_J. Since $\hat{\mathbb{C}}_{J^c}$ is uniquely determined as the subset of $\hat{\mathbb{C}}^n$ on which T_J acts trivially, by the ν_A-equivariance of φ_A it follows that $\varphi_A(\mathbb{C}_{J^c}) \subset \hat{\mathbb{C}}_{J^c}$. Moreover, by the definition (3.2) of φ_A we find that $\varphi_A|_{\mathbb{C}_{J^c}} = \varphi_{A_{J^c}}$. Hence A_{J^c} gives a toric chart in the toric modification $\pi|_{\hat{\mathbb{C}}_{J^c}} : \hat{\mathbb{C}}_{J^c} \to \mathbb{C}_{J^c}$. ∎

Lemma 3.9: *Let $A = (a_{kl})_{1 \leq k,l \leq n} \in \mathrm{GL}(n, \mathbb{Z})$ be a toric Kato matrix with $|P(A)| < n - 1$. Then A has a simple real eigenvalue $\alpha > 1$ so that for any other $\alpha \neq \beta \in \mathrm{Spec}(A)$, we have $|\beta| < \alpha$. Moreover, A admits a Perron eigenvector f_A with $Af_A = \alpha f_A$ so that $\langle e_j^*, f_A \rangle > 0$ for all $1 \leq j \leq n$.*

Proof. Let s be the permutation of $P(A)$ given by Lemma 3.7 and let $d = \mathrm{ord}\, s$. Then for $m = m_0 d$ big enough, $(A^m)_{P(A)^c} = \mathsf{id}$ and $B := A^m_{P(A)}$ is a positive matrix. After eventually replacing A by some of its power, we suppose that this happens for $m = 1$.

By the Perron-Frobenius theorem, B has a simple real eigenvalue α and a Perron eigenvector f_B with positive components. Since $|\det(B)| = 1$ and $|P(A)^c| > 1$, we have $\alpha > 1$.

For any $\beta \in \mathbb{C}$ and $k \times k$ matrix M, denote by $V_M(\beta) \subset \mathbb{C}^k$ the generalized eigenspace of M for the eigenvalue β. Put $P = P(A)$ and write $p := p_{P^c} : \mathbb{C}^n \to \mathbb{C}_{P^c}$ for the natural projection. Then p is compatible with the splitting

$$\mathbb{C}^n = \bigoplus_{\beta \in \mathbb{C}} V_A(\beta) \to \bigoplus_{\beta \in \mathbb{C}} V_B(\beta) = \mathbb{C}_{P^c}$$

and $\mathrm{Spec}(A) = \mathrm{Spec}(B) \cup \mathrm{Spec}(A_P)$, where $\mathrm{Spec}(A_P) = \{1\}$. In particular, we infer that $\alpha \in \mathrm{Spec}(A)$ is a simple eigenvalue and there exists an eigenvector $f_A \in \mathbb{C}^n$ with $p(f_A) = f_B$.

For any $k \in P$, we find:

$$\langle e_k^*, f_A \rangle = \frac{1}{\alpha - 1} \sum_{j \in P^c} a_{kj} \langle e_j^*, f_B \rangle > 0$$

from which the conclusion follows. ∎

For a toric Kato matrix A with $|P(A)| < n-1$, let f_A be a Perron vector for A with positive components. Then A^T acting on $(\mathbb{C}^n)^*$ also has a Perron vector f_A^* with $\langle f_A^*, f_A \rangle = 1$. We note that since ± 1 are the only possible rational eigenvalues of A, f_A^* cannot be chosen rational. We also observe that since $\mathbb{C}_{P(A)}$ is spanned by eigenvectors of A with eigenvalue different from α, for any $k \in P(A)$ we have $\langle f_A^*, e_k \rangle = 0$, while for $k \in P(A)^c$, we have $\langle f^*, e_k \rangle > 0$. Define the half-space:

$$H(A) := \{v \in N_{\mathbb{R}} \mid \langle f_A^*, v \rangle > 0\} \subset N_{\mathbb{R}}.$$

Also, for a toric Kato matrix with $P(A)^c = \{j\}$, put

$$H(A) := \{v \in N_{\mathbb{R}} \mid \langle e_j^*, v \rangle > 0\} \subset N_{\mathbb{R}}.$$

3.3. **Toric Kato germs.** Note that for any toric Kato germ $F = F_{\underline{\lambda}_0, A}$, if $m_0 > 0$ is so that A^{m_0} has strictly positive components on the columns $j \in P(A)^c$, by Lemma 3.7 we have

$$H_\infty = \bigcup_{m>0} F^{-m}(0) = F^{-m_0}(0) = \bigcup_{j \in P(A)^c} \mathbb{C}_{\{j\}^c}$$

$$\mathrm{Inv}(F) = \{z \in \mathbb{C}^n \mid z_j \neq 0,\ j \in P(A)^c\} = \mathbb{C}_{P(A)} \times T_{P(A)^c} \supseteq T_N.$$

Let us put $W_T(F) := W^s(F) \cap T_N \subseteq W^*(F)$. Then we have the following:

Lemma 3.10: *Let $F = F_{\underline{\lambda}_0, A}$ be a toric Kato germ. Let $B := A_{P(A)^c}$, $\underline{\lambda}' := (\underline{\lambda}_0)_{P(A)^c}$ and let $F' = F_{\underline{\lambda}', B}$ be the induced toric Kato germ acting on $T_{P(A)^c}$. Then we have a biholomorphism:*

$$W_T(F) = T_{P(A)} \times W_T(F').$$

In particular, if $|P(A)| = n - 1$ then $W_T(F) = T_N$.

Proof. As $W_T(F) = W_T(F^m)$ for any $m \in \mathbb{N}$, we can suppose without loss of generality that $A_{P(A)} = \mathsf{id}$. Let us denote by $p := p_{P(A)^c}$ and by $p' := p_{P(A)}$ the two projections. Since $p \circ F = F' \circ p$, it is clear that $p(W_T(F)) \subseteq W_T(F')$. Denote by $\underline{\lambda}'' := (\underline{\lambda}_0)_{P(A)}$. Then we have

$$(F^m(z))_{P(A)} = (\underline{\lambda}'')^m z_{P(A)} z_{P(A)^c}^{Q_m}, \quad m \in \mathbb{N} \tag{3.5}$$

where Q_m is an integer valued matrix determined by A, whose coefficients are each at least m. From this, it is clear that the fiber of $p : W_T(F) \to p(W_T(F)) \subset W_T(F')$ is $T_{P(A)}$. It remains to show that this restriction of p is surjective.

Let $w \in W_T(F')$ and let $0 < \epsilon < 1$. Since $w \in p(W_T(F)) \Leftrightarrow (F')^m(w) \in p(W_T(W))\ \forall m \in \mathbb{N}$, we can suppose that w satisfies $|\lambda_j| \prod_{k \in P(A)^c} |w_k| < \epsilon$ for each $j \in P(A)$. Then (3.5) implies that for any $z \in T_N$ with $p(z) = w$ and each $j \in P(A)$, $m \in \mathbb{N}$, we have $|F^m(z)_j| < |z_j| \epsilon^m$. Thus every such z is in $W_T(F)$ and so $w \in p(W_T(F))$.

Finally, if $P(A)^c = \{j\}$, then $F'(w) = \underline{\lambda}' w$ with $|\underline{\lambda}'| < 1$, from which it follows that $W_T(F') = \mathbb{C}^*$ and so $W_T(F) = T_N$. ∎

4. A different construction of toric Kato manifolds

In this section, we wish to give a different realization of toric Kato manifolds using toric geometry. In particular, we will see that the universal cover of a toric Kato manifold is an open subset of a toric variety of non-finite type. The construction generalizes the one did by Tsuchihashi in [Tsu87], as well as the known toric constructions for Kato surfaces [Oda78].

In all that follows, we fix a smooth toric modification $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ at 0 with $\pi \neq \mathsf{id}$ and denote by $\hat{\Sigma}$ the fan of $\hat{\mathbb{C}}^n$. We fix a toric chart $\varphi_A : \mathbb{C}^n \to X_{\tau_A} \subset \hat{\mathbb{C}}^n$ for some toric Kato matrix $A \in \mathrm{GL}(n,\mathbb{Z})$ and $\tau_A \in \hat{\Sigma}^{(n)}$. Finally, we fix $\underline{\lambda}_0 \in T_N$ so that $\sigma := \underline{\lambda}_0 \circ \varphi_A$ satisfies $\overline{\sigma(\mathbb{B})} \subset \pi^{-1}(\mathbb{B}) =: \hat{\mathbb{B}}$. In particular, $(\pi|_{\hat{\mathbb{B}}}, \sigma|_{\mathbb{B}})$ is a toric Kato data.

Let $\hat{\Sigma}_0 := \hat{\Sigma} \setminus \{\tau_A\}$, and define the infinite fan

$$\Sigma_A := \{A^m \tau \mid \tau \in \hat{\Sigma}_0,\ m \in \mathbb{Z}\} \tag{4.1}$$

on which the group $\Gamma_A := \{A^m \mid m \in \mathbb{Z}\}$ acts naturally. Let $X(\Sigma_A, N)$ be the toric variety associated to the fan Σ_A, let $\widetilde{D}_T := \sum_{\nu \in \Sigma_A^{(1)}} \overline{\mathrm{orb}\,\nu}$ be the associated toric divisor and let

$$D_+ := \sum_{\substack{\nu \in \Sigma_A^{(1)} \\ \nu \not\prec \tau_{P(A)}}} \overline{\mathrm{orb}(\nu)}$$

where we recall that $\tau_{P(A)} = \langle e_k \mid k \in P(A) \rangle$. Define the open set

$$\widetilde{X}_{\underline{\lambda}_0, A} := \mathrm{Int}(\overline{W_T(F_{\underline{\lambda}_0, A})}^{X(\Sigma_A, N)}) \subset X(\Sigma_A, N).$$

Recall that $W_T(F_{\underline{\lambda}_0,A}) \subset T_N \subset X(\Sigma_A, N)$ is $F_{\underline{\lambda}_0,A}$-invariant. Also note that the map F_A extends to an automorphism of the toric variety $X(\Sigma_A, N)$ as the map induced by $N \to N$, $v \mapsto Av$. Similarly, $\underline{\lambda}_0 \in T_N$ extends to an automorphism of $X(\Sigma_A, N)$. Therefore, we have a natural action of $U := \langle F_{\underline{\lambda}_0,A} \rangle$ on $\widetilde{X}_{\lambda_0,A}$.

Before giving the main theorem of the section, we show the following technical lemma needed in the proof.

Lemma 4.1: *Let $V \subset N_{\mathbb{R}}$ be an open subset which satisfies $\forall a \in \mathbb{R}$ with $a \geq 1$, $\forall v \in V$, $av \in V$, and let $\Omega := \mathrm{ord}^{-1}(V) \subset T_N$. Let Σ be a fan in N and let $X(\Sigma, N)$ be the associated toric variety. Denote by $\overline{\Omega}^{\Sigma}$ the closure of Ω in $X(\Sigma, N)$, and by $\overline{V}^N$ the closure of V in $N_{\mathbb{R}}$. Then for any cone $\tau \in \Sigma^{(k)}$ with $1 \leq k \leq n$, one has:*

(i) if $\tau \cap V \neq \emptyset$, then $\mathrm{orb}\,\tau \subset \overline{\Omega}^{\Sigma}$;
(ii) if furthermore V is a cone and $(\tau \setminus \{0\}) \cap \overline{V}^N = \emptyset$, then $\mathrm{orb}\,\tau \cap \overline{\Omega}^{\Sigma} = \emptyset$.

In particular, if $|\Sigma| \subseteq V \cup \{0\}$, then $\Omega \cup D_T$ is an open subset of $X(\Sigma, N)$, where $D_T = \bigcup_{\tau \in \Sigma^{(1)}} \overline{\mathrm{orb}\,\tau}$ is the support of the maximal toric divisor of $X(\Sigma, N)$.

Proof. Let us fix such a cone $\tau \in \Sigma^{(k)}$. In what follows, we let $\mathrm{Int}(S_\tau) := S_\tau \setminus (\tau^\perp \cap M) = \{l \in S_\tau \mid \exists v \in \tau\ \langle l, v \rangle > 0\}$.

Recall that one has a natural map given by extension by 0:

$$j : \operatorname{orb}\tau = \operatorname{Hom}_{gr.}(\tau^\perp \cap M, \mathbb{C}^*) \to X_\tau = \operatorname{Hom}_{u.s.g.}(S_\tau, \mathbb{C})$$
$$j(u)(l) = \begin{cases} u(l), & l \in \tau^\perp \cap S_\tau \\ 0, & \text{otherwise.} \end{cases}$$

Applying the map ord to the above, j induces a natural injection:

$$j_{\mathbb{R}} : \operatorname{orb}\tau_{\mathbb{R}} := \operatorname{Hom}_{gr.}(\tau^\perp \cap M, \mathbb{R}) \to X_{\tau,\mathbb{R}} := \operatorname{Hom}_{u.s.g.}(S_\tau, \mathbb{R} \cup \{\infty\})$$

given this time by extension by ∞. Moreover, its image is given by

$$j_{\mathbb{R}}(\operatorname{orb}\tau_{\mathbb{R}}) = \{u \in \operatorname{Hom}_{u.s.g.}(S_\tau, \mathbb{R} \cup \{\infty\}) \mid \ u^{-1}(\infty) = \operatorname{Int}(S_\tau)\}. \tag{4.2}$$

On the other hand, the natural inclusion $\Omega \subset T_N \subset X_\tau$ reads, after applying the map ord, as follows:

$$k : V \to X_{\tau,\mathbb{R}}, \quad k(v)(l) = \langle l, v\rangle, \quad l \in S_\tau.$$

Since $\operatorname{orb}\tau \subset X_\tau$, one has $\operatorname{orb}\tau \cap \overline{\Omega}^\Sigma = \operatorname{orb}\tau \cap \overline{\Omega}^\tau$, where the latter closure is taken in X_τ. Furthermore, since ord is surjective, this intersection is nonempty precisely when $j_{\mathbb{R}}(\operatorname{orb}\tau_{\mathbb{R}}) \cap \overline{k(V)} \neq \emptyset$.

Suppose first that $\tau \cap V \neq \emptyset$ and let $u \in j_{\mathbb{R}}(\operatorname{orb}\tau_{\mathbb{R}})$. In order to show (i), it suffices to prove that $u \in \overline{k(V)}$. There exists $q \in N_{\mathbb{R}}$ so that $u|_{\tau^\perp \cap S_\tau} = \langle \cdot, q\rangle|_{\tau^\perp \cap S_\tau}$. Since V is open, there exists $v \in V$ which is in the relative interior of τ, which is equivalent to $u_0 := \langle \cdot, v\rangle|_{\operatorname{Int}S_\tau} > 0$. Furthermore, for $m \geq 1$, since $mv \in V$ and using the properties of V, there exists $c_m > m$ big enough so that $v_m := q + c_m mv = c_m(\frac{1}{c_m}q + mv) \in V$. We find:

$$k(v_m)|_{\tau^\perp \cap S_\tau} = \langle \cdot, q\rangle|_{\tau^\perp \cap S_\tau} = u|_{\tau^\perp \cap S_\tau}$$
$$k(v_m)|_{\operatorname{Int}S_\tau} > \langle \cdot, q\rangle|_{\operatorname{Int}S_\tau} + m^2 u_0 \to \infty$$

hence $\lim_{m\to\infty} k(v_m) = u \in \overline{k(V)}$.

For the second part, suppose that V is a cone and that there exists $u = \lim_{m\to\infty} k(v_m) \in j_{\mathbb{R}}(\operatorname{orb}\tau_{\mathbb{R}})$ with $v_m \in \overline{V}^N$ for each $m \geq 1$. Let us fix the standard scalar product on $N_{\mathbb{R}}$ and the orthogonal splitting $N_{\mathbb{R}} = N_0 \oplus \mathbb{R}\tau$, so that the map $d : N_0 \to \operatorname{Hom}(\tau^\perp, \mathbb{R})$, $q \mapsto \langle \cdot, q\rangle$ becomes a continuous isomorphism. In particular, it follows that $u|_{\tau^\perp \cap S_\tau} = d(q)|_{S_\tau}$ for a unique $q \in N_0$. With respect to the splitting, we write $v_m = v_m^0 + v_m^\tau$ with $v_m^0 \in N_0$ and $v_m^\tau \in \mathbb{R}\tau$. Since $d(v_m^0)|_{\tau^\perp \cap S_\tau} = k(v_m)|_{\tau^\perp \cap S_\tau} \to d(q)|_{S_\tau}$ as $m \to \infty$, it follows that $\exists \lim_{m\to\infty} v_m^0 = q$. On the other hand, we have:

$$\infty = \lim_{m\to\infty} k(v_m)|_{\operatorname{Int}S_\tau} - \langle \cdot, q\rangle|_{\operatorname{Int}S_\tau} = \lim_{m\to\infty} k(v_m - v_m^0)|_{\operatorname{Int}S_\tau} = \lim_{m\to\infty} \langle \cdot, v_m^\tau\rangle|_{\operatorname{Int}S_\tau}.$$

Hence there exists $m_0 \geq 1$ so that $\langle \cdot, v_m^\tau\rangle|_{\operatorname{Int}S_\tau} > 0$ for all $m \geq m_0$. In particular, $v_m^\tau \in \tau \setminus \{0\}$ for all $m \geq m_0$ and $a_m := ||v_m^\tau|| \to \infty$. Since the sequence $(\frac{1}{a_m} v_m^\tau)_{m \geq m_0} \subset \tau$ is bounded, up to passing to a subsequence, it converges to an element $v_\infty^\tau \in \tau \setminus \{0\}$ with $||v_\infty^\tau|| = 1$. Now, since $\overline{V}^N$ is a cone, we infer that $w_m := \frac{1}{a_m} v_m \in \overline{V}^N$ and $\lim_{m\to\infty} w_m = v_\infty^\tau \in \overline{V}^N \cap (\tau \setminus \{0\})$, which concludes (ii).

Finally, if $|\Sigma| \subseteq V \cup \{0\}$, it follows by (i) that $D_T = \bigcup_{0 \neq \tau \in \Sigma} \operatorname{orb}\tau \subset \overline{\Omega}^\Sigma$, hence $X(\Sigma, N) \setminus (\Omega \cup D_T) = T_N \setminus \Omega$ is closed in $X(\Sigma, N)$, which concludes the proof. ∎

We are now ready to prove:

Theorem 4.2: *Let (π, σ) be a toric Kato data with corresponding germ $F_{\underline{\lambda}_0,A}$. Then the group U acts freely and properly discontinuously on $\widetilde{X}_{\underline{\lambda}_0,A}$ and we have a biholomorphism*

$$X(\pi,\sigma) \cong \widetilde{X}_{\underline{\lambda}_0,A}/_U.$$

Furthermore, if D denotes the divisor of $X(\pi,\sigma)$ induced by the exceptional divisor of π and $\widetilde{D}$ is the preimage of D in the universal cover, then $\widetilde{D} = D_+$. Finally, we have the following:

i) If $P(A) = \emptyset$, then $\widetilde{X}_{\underline{\lambda}_0,A} = W_T(F_{\underline{\lambda}_0,A}) \cup \widetilde{D}_T$. If furthermore $\underline{\lambda}_0 = \underline{1}$, then $X(\pi,\sigma)$ is one of the manifolds constructed in [Tsu87]*;*
ii) If $|P(A)| = n-1$, then $\widetilde{X}_{\underline{\lambda}_0,A} = X(\Sigma_A, N)$.

Proof. Let us put $\Omega := W_T(F_{\underline{\lambda}_0,A}) \subseteq T_N$ and $F := F_{\underline{\lambda}_0,A}$. Then $H_\infty = \bigcup_{j\in P(A)^c} \mathbb{C}_{\{j\}^c} \subset \overline{W_T(F)}^{\mathbb{C}^n} = \overline{W^s(F)}$. Define the fan:

$$\Sigma_A^+ := \{A^m\tau \mid \tau \in \hat{\Sigma}_0, m \in \mathbb{N}\}$$

so that $|\Sigma_A^+| \subset C_0 = |\Sigma|$ and $X(\Sigma_A^+, N)$ is an open subset of $X(\Sigma_A, N)$. The identity map of N induces a (non-proper) map of toric varieties $\varphi : X(\Sigma_A^+, N) \to \mathbb{C}^n = X(\Sigma, N)$ and we have

$$D_+^+ := \bigcup_{\substack{\nu\in\Sigma_A^{+(1)}\\ \nu\not\prec\tau_{P(A)}}} \overline{\operatorname{orb}\nu} = \varphi^{-1}(H_\infty) \subset \varphi^{-1}(\overline{\Omega}^{\mathbb{C}^n}) = \overline{\Omega}^{X(\Sigma_A^+,N)}$$

from which it follows that $D_+^+ \subset \widetilde{X}_{\underline{\lambda}_0,A}$. Since for each irreducible component Q of D_+ there exists $m \in \mathbb{Z}$ so that $F^m(Q)$ is an irreducible component of D_+^+, it follows then that also $D_+ \subset \widetilde{X}_{\underline{\lambda}_0,A}$. Therefore, if $P(A) = \emptyset$ then $D_+ = \widetilde{D}_T$ and we have $\widetilde{X}_{\underline{\lambda}_0,A} = W_T(F_{\underline{\lambda}_0,A}) \cup \widetilde{D}_T$. If $|P(A)| = n-1$ then $W_T(F) = T_N$ by Lemma 3.10 so $\widetilde{X}_{\underline{\lambda}_0,A} = X(\Sigma_A, A)$.

Let $\mathbb{B}^* := \mathbb{B} \cap T_N$, $\Omega_0 := \mathbb{B}^* \setminus F(\mathbb{B}^*) \subset T_N$ and $W_0 := \overline{\Omega_0}^{X(\Sigma_A,N)}$. Since by the proof of Proposition 2.3, Ω_0 is a fundamental domain for the action of U on Ω, it follows that W_0 is a fundamental domain for the action of U on $\widetilde{X}_{\underline{\lambda}_0,A}$.

Let $V := \operatorname{ord}(\mathbb{B}^*) \subset N_\mathbb{R}$ and note that it is open and satisfies the condition $v \in V, a \geq 1 \Rightarrow av \in V$. Let $C_0 \subset N_\mathbb{R}$ be the standard cone, so that $V \subset C_0$. By Lemma 4.1 (ii), we have that $\operatorname{orb}\tau \cap \overline{\operatorname{ord}^{-1}(C_0)}^{X(\Sigma_A,N)} = \emptyset$ for any $\tau \in \Sigma_A$ with $\tau \not\subset C_0$, which implies that

$$\widetilde{D}_T \cap W_0 \subset \bigcup_{\substack{\nu\in\Sigma_A^{(1)}\\ \nu\subset C_0}} \overline{\operatorname{orb}\nu}.$$

Furthermore, since any ray $\nu \in \Sigma_A^{(1)}$ with $\nu \subset \operatorname{Int}C_0$ satisfies $\nu \cap V \neq \emptyset$, Lemma 4.1 (i) implies that $\overline{\operatorname{orb}\nu} \subset \overline{\mathbb{B}^*}^{X(\Sigma_A,N)}$. In particular, applying A, we infer that

$$\bigcup_{\substack{\nu\in\Sigma_A^{(1)}\\ \nu\subset \operatorname{Int}AC_0}} \overline{\operatorname{orb}\nu} \subset \overline{F_A(\mathbb{B}^*)}^{X(\Sigma_A,N)}$$

and hence:

$$\underline{\lambda}_0 \cdot \bigcup_{\substack{\nu\in\Sigma_A^{(1)}\\ \nu\subset \mathrm{Int}AC_0}} \overline{\mathrm{orb}\,\nu} = \bigcup_{\substack{\nu\in\Sigma_A^{(1)}\\ \nu\subset \mathrm{Int}AC_0}} \overline{\mathrm{orb}\,\nu} \subset \underline{\lambda}_0 \cdot \overline{F_A(\mathbb{B}^*)}^{X(\Sigma_A,N)} = \overline{F(\mathbb{B}^*)}^{X(\Sigma_A,N)}.$$

Therefore $\widetilde{D}_T \cap W_0 = D^{\hat{\Sigma}_0} \cap W_0$, where $D^{\hat{\Sigma}_0} = \bigcup_{\nu\in\hat{\Sigma}_0^{(1)}} \overline{\mathrm{orb}\,\nu}$, and so $W_0 = \overline{\Omega_0}^{X(\hat{\Sigma}_0,N)}$. But $X(\hat{\Sigma}_0, N) = \hat{\mathbb{C}}^n \setminus \{\sigma(0)\}$, hence $W_0 = \pi^{-1}(\overline{\mathbb{B}}) \setminus \sigma(\mathbb{B})$, which is precisely a fundamental domain for the deck group action on $\widetilde{X(\pi,\sigma)}$. Clearly this identification of $\widetilde{X}_{\underline{\lambda}_0,A}$ and of $\widetilde{X(\pi,\sigma)}$ is equivariant, which shows thus that the action of U is free and proper, and that we have the desired isomorphism.

Finally, it is clear from the isomorphism that $\widetilde{D} = D_+$. ∎

For later use, we also show:

Lemma 4.3: *The support of Σ_A is given by*

$$|\Sigma_A| = \begin{cases} H(A) \cup \tau_{P(A)} \setminus \left(\mathbb{R}_{>0} f_A + \tau_{P(A)}\right), & |P(A)| < n-1 \\ H(A) \cup \tau_{P(A)}, & |P(A)| = n-1 \end{cases}$$

where for the case $|P(A)| < n-1$, f_A is a Perron vector for A with positive components.

Proof. Since $|\hat{\Sigma}_0| = C_0 \setminus \mathrm{Int}(AC_0)$, we find

$$|\Sigma_A| = \bigcup_{m\in\mathbb{Z}} A^m |\hat{\Sigma}_0| = \bigcup_{m\in\mathbb{Z}} A^m C_0 \setminus \bigcap_{m\in\mathbb{Z}} \mathrm{Int}(A^m C_0).$$

As in the proof of Lemma 3.7, we can suppose without loss of generality that $A_{P(A)^c} = \mathsf{id}$ and that $B := A_{P(A)^c}$ is a positive matrix.

We first show that

$$\bigcup_{m\in\mathbb{Z}} A^m C_0 = H(A) \cup \tau_{P(A)}. \tag{4.3}$$

Indeed, the inclusions $\bigcup_{m\in\mathbb{Z}} A^m C_0 \subseteq H(A) \cup \tau_{P(A)}$ and $\tau_{P(A)} \subseteq \bigcup_{m\in\mathbb{Z}} A^m C_0$ are straightforward. Let now $v \in H(A)$. We shall prove there exists $l > 0$ such that $A^l v \in C_0$. For the case $|P(A)| < n-1$, $f_B = (f_A)_{P(A)^c}$ is the Perron vector of B. Let f_B^* be a Perron vector for B^T with $\langle f_B^*, f_B\rangle = 1$. For the case $P(A)^c = \{j\}$, we put $f_B = e_j$ and $f_B^* = e_j^*$. If $\alpha > 0$ is the dominant eigenvalue of A, or also of B, then we have:

$$\lim_{m\to\infty} \frac{1}{\alpha^m} B^m = f_B \cdot (f_B^*)^T \tag{4.4}$$

which is obvious in the case $|P(A)| = n-1$ and implied by the Perron-Frobenius theorem for the case $|P(A)| < n-1$. In particular, we have:

$$\lim_{m\to\infty} \left(\frac{1}{\alpha^m} A^m v\right)_{P(A)^c} = \langle f_B^*, v\rangle f_B.$$

Since $\langle f_B^*, v\rangle > 0$, then there is some $l_0 \in \mathbb{N}$ so that for any $l > l_0$, $(A^l v)_{P(A)^c}$ has positive components. For any $m \in \mathbb{N}$, put $Q_m := (a_{jk}^{(m)})_{\substack{j\in P(A)\\ k\in P(A)^c}}$, where $a_{\cdot\cdot}^{(m)}$ denote the components of A^m. Then we have:

$$(A^{l+m} v)_{P(A)} = (A^l v)_{P(A)} + Q_m (A^l v)_{P(A)^c}$$

and since Q_m is a positive matrix, we obtain that for big enough m, $A^{l+m}v \in C_0$.

In order to compute the intersection, we distinguish between the two cases. Suppose first that $|P(A)| < n-1$. Then we wish to show that

$$\bigcap_{m\in\mathbb{Z}} \mathrm{Int}(A^m C_0) = \mathbb{R}_{>0} f_A + \tau_{P(A)}. \tag{4.5}$$

It is clear that $\mathbb{R}_{>0}f_A + \tau_{P(A)}$ is invariant by A and is contained in $\mathrm{Int}C_0$. Moreover, for any vector $v \in C_0 \setminus \tau_{P(A)} \subset H(A)$, we have that

$$\lim_{m\to+\infty} \frac{A^m v}{\alpha^m} = f_A \langle f_A^*, v\rangle = k f_A$$

for a suitable $k > 0$. It follows that the cones $A^m C_0 = \langle A^m e_1, \dots, A^m e_n\rangle$ tend to $\mathbb{R}_{>0}f_A + \tau_{P(A)}$ when $m \to +\infty$, and so we have:

$$\mathbb{R}_{>0}f_A + \tau_{P(A)} \subseteq \bigcap_{m\in\mathbb{Z}} A^m \mathrm{Int}C_0 \subseteq \bigcap_{m\geq 0} A^m C_0 = \mathbb{R}_{>0}f_A + \tau_{P(A)}$$

which proves (4.5).

Suppose now that $P(A)^c = \{j\}$. Then we have:

$$\bigcap_{m\in\mathbb{Z}} A^m C_0 = \tau_{P(A)}.$$

Indeed, clearly $\tau_{P(A)} \subset A^m C_0$ for any $m \in \mathbb{Z}$. Conversely, let $v \in \bigcap_{m\in\mathbb{Z}} A^m C_0$, so that for each $m \in \mathbb{Z}$ there exist $v_m \in \mathbb{R}_{P(A)}$ and $a_m \in \mathbb{R}_{\geq 0}$ with

$$v = A^m(v_m + a_m e_j) = v_m + a_m Q_m e_j + a_m e_j.$$

In particular, we find that $v_j = a_m e_j$ so that $a_m = a_0$ for any $m \in \mathbb{Z}$. If $a_0 > 0$, then $||a_m Q_m e_j||$ tends to ∞ with m, so $||v||$ is unbounded, which is absurd. Thus $a_0 = 0$ and so $v \in \tau_{P(A)}$.

Finally, we have

$$\bigcap_{m\in\mathbb{Z}} \mathrm{Int}(A^m C_0) \subseteq \mathrm{Int}C_0 \cap \bigcap_{m\in\mathbb{Z}} A^m C_0 = \emptyset.$$

which concludes the proof. ■

Example 4.4: Consider the toric Kato data (π, σ), where π is associated to the fan depicted in the left side of Figure 2, and $\sigma = \lambda_0 \varphi_A$ with $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. In this case, $P(A) = \{1\}$, and a direct computation shows that the eigenvalues of A are $1, \xi^2, \xi^{-2}$, where $\xi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The right side of Figure 2 shows the fan $\hat{\Sigma}_2$ associated to the toric modification π^2 given by Lemma 3.3.

A Perron vector f_A is given by $(\xi, \xi, 1)$, while the dual vector f_A^* is a positive multiple of $(0, \xi, 1)$. In this case $H(A) = \{x, y, z \in \mathbb{R}^3 |\ \xi y + z > 0\}$, and $|\Sigma_A| = H(A) \cup \mathbb{R}_{\geq 0}(1,0,0) \setminus \{(s + t\xi, t\xi, t) \mid s \in \mathbb{R}_{\geq 0}, t \in \mathbb{R}_{>0}\}$.

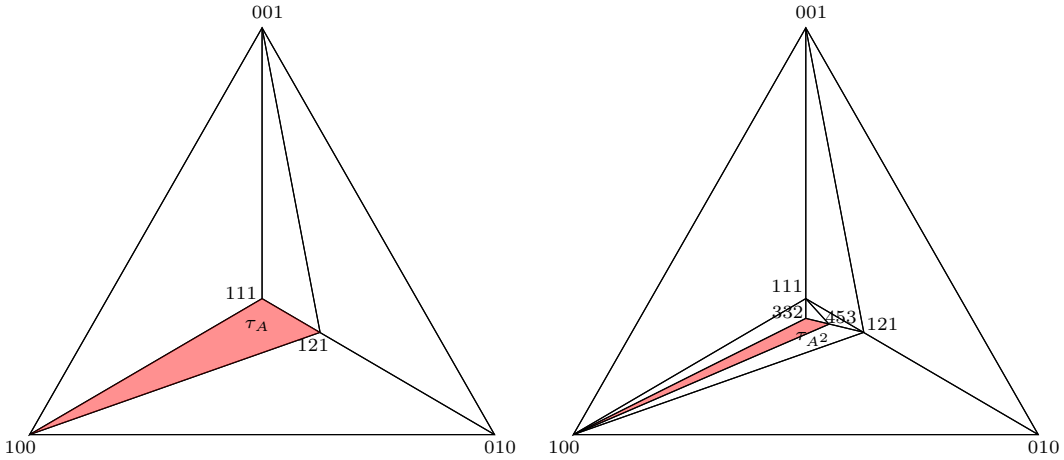

FIGURE 2. Toric Kato data with $P(A) = \{1\}$.

5. Geometric properties of toric Kato manifolds

In complex dimension $n = 2$, the germ $F_{\underline{\lambda}_0,A}$ determines the toric Kato surface. In the case $P(A) = \emptyset$, the corresponding toric Kato surfaces are precisely the properly blow-up Inoue-Hirzebruch surfaces. As we will see later, $X(F_{\underline{\lambda}_0,A}) \cong X(F_A)$ for any compatible $\underline{\lambda}_0 \in T_N$. In the case $|P(A)| = 1$, the corresponding toric Kato surfaces are precisely the properly blown-up parabolic Inoue surfaces, and for fixed A, they are parametrized by $\mathbb{D}^*$. Finally, the case $|P(A)| = 2$ gives finite quotients of diagonal Hopf surfaces. In higher dimension n, it is still true that the case $|P(A)| = n$ gives finite quotients of diagonal Hopf manifolds, while the dichotomy persists between the cases $|P(A)| = n - 1$ and $|P(A)| < n - 1$. For this reason, we propose the following definition:

Definition 5.1: We call a toric Kato manifold X of matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ of *parabolic type* if $|P(A)| = n - 1$. If $|P(A)| < n - 1$, then we call X of *hyperbolic type*.

In analogy to Nakamura's result in dimension $n = 2$ [Nak84], we can detect the type of a toric Kato manifold by looking at its curves. This shows in particular that the above definition makes sense, and does not depend on the chosen Kato data. In order to make this precise, we first introduce one more definition.

We will call a compact complex subspace Y of a toric Kato manifold X T-invariant if its preimage in the universal cover is invariant under the $\mathbb{T}^n$- action, or equivalently, under the (local) T_N-action. All such subspaces correspond to cones in Σ_A, and are of two types: either they are immersed toric Kato manifolds, if the corresponding cone is A-invariant, or they are complete toric algebraic varieties, if the corresponding cone is not A-invariant. In particular, we have the following description of T-invariant curves:

Theorem 5.2: *Let X be a toric Kato manifold. Then we have the following characterization in terms of T-invariant curves:*

(1) X is a finite quotient of a diagonal Hopf manifold if and only if any of its T-invariant curves is elliptic;
(2) X is of hyperbolic type if and only if any T-invariant curve is rational;

(3) *X is of parabolic type if and only if X contains a unique T-invariant elliptic curve E, and at least one rational T-invariant curve. In this case, E is smooth and any other of the T-invariant curves are rational. Moreover, we have $[E]\cdot[D]=0$ in $H_\bullet(X,\mathbb{Z})$.*

Proof. Let $A\in \mathrm{GL}(n,\mathbb{Z})$ denote the Kato matrix of X and suppose, without loss of generality, that $A_{P(A)}=\mathsf{id}$. Any T-invariant curve C of X is given by a cone $\tau\in\Sigma_A^{(n-1)}$, either as $\operatorname{orb}\tau/\Gamma$, in case τ is Γ-invariant, or as the image of $\overline{\operatorname{orb}\tau}\cap\widetilde{X}$. If Moreover, C is an elliptic curve in the first case, and toric, thus rational, in the second case.

If $|P(A)|=n$, then $\Sigma_A=\Sigma\setminus\{C_0\}$, so it is clear that X contains n T-invariant elliptic curves and no T-invariant rational curve. If $|P(A)|<n-1$, then there is no A-invariant cone $\tau\in\Sigma_A^{(n-1)}$, so all T-invariant curves of X are rational.

Suppose now that $P(A)^c=\{j\}$. Then the only A-invariant cone $\tau\in\Sigma_A^{(n-1)}$ is $\tau_{P(A)}$, so $E:=\operatorname{orb}\tau_{P(A)}/\Gamma\cong\mathbb{C}^*/\lambda_j$ is the only elliptic curve in this last case, and is clearly smooth. Moreover, it is clear that there exits at least one $\tau_{P(A)}\neq\tau\in\Sigma_A^{(n-1)}$, giving rise to a T-invariant rational curve.

Recall that by Theorem 4.2, D is the image in X of

$$D_+=\sum_{\substack{\nu\in\Sigma_A^{(1)}\\ \nu\not\subset\tau_{P(A)}}}\overline{\operatorname{orb}\nu}\cap\widetilde{X}.$$

Thus, in order to see that $[E]\cdot[D]=0$, it is enough to see that $\operatorname{orb}\tau_{P(A)}$ does not intersect $\overline{\operatorname{orb}\nu}$ for any $\nu\in\Sigma_A^{(1)}$ with $\nu\not\subset\tau_{P(A)}$. But this is equivalent to say that any such ν is not a face of $\tau_{P(A)}$, which is obvious. ∎

Next we want to give an explicit description of the divisors defined by

$$\begin{aligned}\widetilde{D}_T^X &:= \widetilde{D}_T\cap\widetilde{X}=\widetilde{X}_{\underline{\lambda}_0,A}\setminus W_T(F_{\underline{\lambda}_0,A}),\\ D_T &= \widetilde{D}_T^X/_\Gamma.\end{aligned}$$

It is clear that $\widetilde{D}_T^X=D_+ +\sum_{k\in P(A)}X_k$, where $X_k:=\overline{\operatorname{orb}\langle e_k\rangle}^{X(\Sigma_A,N)}\cap\widetilde{X}_{\underline{\lambda}_0,A}$. We will call D_T the *toric divisor* of X. It satisfies $q^*D_T=\widetilde{D}_T^X$, where $q:\widetilde{X}\to X$ is the universal cover map.

Suppose first that the permutation s given by Lemma 3.7 corresponding to A is the identity. Fix $k\in P(A)$, let $\hat{\mathbb{C}}_{\{k\}^c}=\overline{\operatorname{orb}\langle e_k\rangle}\subset\hat{\mathbb{C}}^n$ and let $\pi_k:=\pi|_{\hat{\mathbb{C}}_{\{k\}^c}}:\hat{\mathbb{C}}_{\{k\}^c}\to\mathbb{C}_{\{k\}^c}$. Then the proof of Lemma 3.8 shows that the toric chart φ_A of $\hat{\mathbb{C}}^n$ induces a toric chart $\varphi_{A_{\{k\}^c}}$ of $\hat{\mathbb{C}}_{\{k\}^c}$. Let $\hat{\Sigma}_k$ be the fan of $\hat{\mathbb{C}}_{\{k\}^c}$ and let $\Sigma_{A_{\{k\}^c}}:=\{A^m_{\{k\}^c}\tau\mid\tau\in\hat{\Sigma}_k\setminus\{\tau_{A_{\{k\}^c}}\},m\in\mathbb{Z}\}$. Then we find that

$$\overline{\operatorname{orb}\langle e_k\rangle}^{X(\Sigma_A,N)}\cong X(\Sigma_{A_{\{k\}^c}},N_{\{k\}^c})$$

and it is easy to check that $F_{\underline{\lambda}_0,A}|_{\overline{\operatorname{orb}\langle e_k\rangle}}=F_{(\underline{\lambda}_0)_{\{k\}^c},A_{\{k\}^c}}$. In particular, we have:

$$X_k\cong\overline{p_{\{k\}^c}(W_T(F_{\underline{\lambda}_0,A}))}^{\overline{\operatorname{orb}\langle e_k\rangle}}=\overline{W_T(F_{(\underline{\lambda}_0)_{\{k\}^c},A_{\{k\}^c}})}^{X(\Sigma_{A_{\{k\}^c}},N)}.$$

We have thus found:

Lemma 5.3: *If A is a toric Kato matrix with $s = \mathsf{id}$, and $k \in P(A)$, then X_k is Γ-equivariantly biholomorphic to $\widetilde{X}_{(\underline{\lambda}_0)_{\{k\}^c}, A_{\{k\}^c}}$, where Γ acts by $F_{\underline{\lambda}_0, A}$ on the first manifold and by $F_{(\underline{\lambda}_0)_{\{k\}^c}, A_{\{k\}^c}}$ on the second one. In particular, $X_k/_\Gamma \cong X(\pi_k, \sigma_k)$, where $\sigma_k = (\underline{\lambda}_0)_{\{k\}^c} \varphi_{A_{\{k\}^c}}$.*

In the general case, i.e. when $s \neq \mathsf{id}$, X_k need not be invariant by $F := F_{\underline{\lambda}_0, A}$, however F still acts on $\sum_{k \in P(A)} X_k$. In this case, there exists $d \in \mathbb{N}$ with $s^d = \mathsf{id}$. Then it is easy to see that $W_T(F) = W_T(F^d)$ and that $F^d = F_{\underline{\lambda}', A^d}$ where $\underline{\lambda}' = \underline{\lambda}_0^{\mathsf{id} + \ldots + A^{d-1}}$. Consider the toric Kato data (π^d, σ^d) given by Lemma 3.3, with corresponding germ F^d. Then clearly $\Sigma_{A^d} = \Sigma_A$. The above argument thus implies that $X_k \cong \widetilde{X}_{(\underline{\lambda}')_{\{k\}^c}, A^d_{\{k\}^c}}$. To each cycle $J \subset P(A)$ of s then corresponds a compact hypersurface which autointersects transversely

$$Y_J = \bigcup_{k \in J} X_k/_\Gamma \subset X(\pi, \sigma)$$

so that the divisor $D_J := |J| \cdot Y_J$ on $X(\pi, \sigma)$ satisfies $q^* D_J = \sum_{k \in J} X_k$. Finally, we have found:

Proposition 5.4: *Let (π, σ) be a toric Kato data with germ $F = F_{\underline{\lambda}_0, A}$ and let s be the permutation corresponding to A. Let $C := \{J \subset P(A) \mid J \text{ is a cycle of } s\}$. Then we have:*

$$D_T = D + \sum_{J \in C} |J| \cdot Y_J.$$

Moreover, for each $J \in C$, Y_J is a compact (possibly singular) hypersurface whose normalization is a toric Kato manifold.

We also want to establish when the divisors $D, \widetilde{D}, D_T, \widetilde{D}_T^X$ are connected. In complex dimension 2, the situation is well known by the works of Nakamura [Nak84] and Dloussky [Dlo84], and can be easily read off from the fan Σ_0. Namely, in the case of parabolic Inoue surfaces, i.e. when $|P(A)| = 1$, if $P(A) = \{j\}$, then $\overline{\mathrm{orb}\langle e_j \rangle}$ is fixed by A and disconnected from $\widetilde{D}$, which itself is connected. It follows that both $\widetilde{D}_T^X$ and D_T have two connected components, respectively $\widetilde{D}$ and $\overline{\mathrm{orb}\langle e_j \rangle}$ for $\widetilde{D}_T$, and D and the elliptic curve $\overline{\mathrm{orb}\langle e_j \rangle}/\Gamma$ for D_T. In the case of Inoue-Hirzebruch surfaces, $\widetilde{D}_T^X = \widetilde{D}$ has two connected components, given by the two connected components of $|\Sigma_0| \setminus \{0\}$. When $\det A = 1$, A fixes the components, so that D has also two connected components. When $\det A = -1$, A interchanges the components, and in this case D is connected. In higher dimension, things are much simpler:

Lemma 5.5: *Let $X = X(\pi, \sigma)$ be a toric Kato manifold of complex dimension $n \geq 3$. Then the divisors $\widetilde{D}_T^X, \widetilde{D}, D_T$ and D are connected.*

Proof. Given a divisor F on a toric variety $X(\Sigma, N)$, $F = \sum_{j \in I} \overline{\mathrm{orb}\, \nu_j}$, let $G_{F,\Sigma} = (V, E)$ be the graph whose vertexes are given by $V = \{\nu_j \mid j \in I\}$ and edges given by $E = \{(\nu_i, \nu_j) \mid i, j \in I, \nu_i \oplus \nu_j \in \Sigma^{(2)}\}$. It is clear that F is connected if and only if the graph $G_{F,\Sigma}$ is connected.

Let A be the matrix of X. Consider first $\hat{\mathbb{C}}^n = X(\hat{\Sigma}, N)$, the exceptional divisor E of π, and the divisors $D_1 = E + \sum_{j=1}^n \overline{\mathrm{orb}\langle e_j \rangle}$ and $D_2 = E + \sum_{j \notin P(A)} \overline{\mathrm{orb}\langle e_j \rangle}$. Since E is connected and intersects each component $\overline{\mathrm{orb}\langle e_j \rangle}$, $1 \leq j \leq n$, also D_1 and D_2 are connected, so their graphs $G_{D_1, \hat{\Sigma}}$, $G_{D_2, \hat{\Sigma}}$ are connected. Next, consider $\hat{\mathbb{C}}^n \setminus \{\sigma(0)\} = X(\hat{\Sigma}_0, N) \subset \hat{\mathbb{C}}^n$ and

$D'_j = D_j \cap X(\hat{\Sigma}_0, N)$, $j = 1, 2$. As $n \geq 3$, we have $G_{D_j,\hat{\Sigma}} = G_{D'_j,\hat{\Sigma}_0}$, so D'_j are also connected, $j = 1, 2$. From this we infer that $\widetilde{D}_T = \sum_{m\in\mathbb{Z}} \gamma^m D'_1$ and $\widetilde{D} = \sum_{m\in\mathbb{Z}} \gamma^m D'_2$ are connected on $X(\hat{\Sigma}, N)$, hence also their respective restrictions to $\widetilde{X}$ are connected and their images in X are connected. This concludes the proof. ∎

We also have the following geometric description of Kato manifolds with $P(A) \neq \emptyset$:

Proposition 5.6: *Let $X_A := X(\pi, \sigma)$ be a toric Kato manifold with $\sigma = \underline{\lambda}_0 \sigma_A$. Let $B = A_{P(A)^c}$ and let $(\pi', \sigma' = \underline{\lambda}' \sigma_B)$ be the naturally induced Kato data from (π, σ) and B. Let $j : X_B := X(\pi', \sigma') \to X_A$ be the natural embedding. We denote by D_A the divisor of X_A induced by the exceptional divisor of π, and by $D_B = j^* D_A$ the corresponding divisor of X_B. Then $X_A \setminus D_A$ has a natural structure of a holomorphic vector bundle V of rank $|P(A)|$ over $X_B \setminus D_B$. Moreover, up to passing to a finite unramified cover of X_A, we have $V = \oplus_{k\in P(A)} L_k$, where each $L_k \in \mathrm{Pic}(X_B \setminus D_B)$. In particular, if $P(A)^c = \{j\}$, then $X_B = X_B \setminus D_B$ is the unique elliptic curve $E = \mathbb{C}^*/\lambda_j$ of X_A, and for each $k \in P(A)$, L_k has positive Chern class determined by A.*

Proof. Putting

$$\Sigma' := \Sigma_A \setminus \bigcup_{\substack{\nu\in\Sigma_A^{(1)} \\ \nu\not\subset\tau_{P(A)}}} St_\nu = \{\tau \in \Sigma_A \mid \tau \prec \tau_{P(A)}\}$$

we have that

$$X(\Sigma_A, N) \setminus D_{A,+} = X(\Sigma', N) \cong X(\Sigma', N_{P(A)}) \times X(\{0\}, N_{P(A)^c}).$$

Moreover, $X(\{0\}, N_{P(A)^c}) = X(\Sigma_B, N_{P(A)^c}) \setminus D_{B,+}$, and the natural projection $u_{\mathbb{Z}} : N \to N_{P(A)^c}$ induces a toric morphism $u : X(\Sigma', N) \to X(\{0\}, N_{P(A)^c})$ satisfying $u(\gamma z) = \gamma'(u(z))$ for any $z \in X(\Sigma', N)$, where γ is the positive generator of Γ acting on $X(\Sigma_A, N)$ and γ' is the positive generator of Γ acting on $X(\Sigma_B, N_{P(A)^c})$. Since moreover $W_T(F_A) = T_{P(A)} \times W_T(F_B)$ by Lemma 3.10, it follows that we have an induced vector bundle:

$$u|_V : V := X_A \setminus D_A \to X_B \setminus D_B.$$

With respect to the standard basis $e_1, \dots e_n$ of $N = \mathbb{Z}^n$, we have a natural splitting

$$X(\Sigma', N_{P(A)}) = \oplus_{k\in P(A)} \mathbb{C} e_k.$$

Up to taking a finite cover of X_A, which corresponds to taking a positive power of A, suppose that $A_{P(A)} = \mathsf{id}$. Then the action of Γ preserves each $\mathbb{C}e_k \times W_T(F_B)$ and γ acts on it by

$$\gamma(z_k, w) = (\lambda_k w^{l_k} z_k, \gamma'(w)), \quad (z_k, w) \in \mathbb{C}e_k \times W_T(F_B),$$

where $l_k \in \mathbb{N}^{|P(A)^c|}$ is the line $(A_{ks})_{s\in P(A)^c}$. It follows that $\mathbb{C}e_k$ descends to a line bundle L_k over $X_B \setminus D_B$ and that $V = \oplus_{k\in P(A)} L_k$.

In particular, when $P(A)^c = \{j\}$, each line bundle L_k is given by the multiplier $\epsilon_{k,\gamma}(t) = \lambda_k t^{A_{kj}}$, $t \in \mathbb{C}^* = W_T(F_B)$. From this we infer that

$$c_1(L_k) = A_{kj} \in H^2(E, \mathbb{Z}) = \mathbb{Z}.$$

∎

Finally, we also have:

Proposition 5.7: *Let X be a toric Kato manifold with toric divisor D_T and matrix A, and let $L_{\det A}$ be the flat line bundle on X determined by $\rho : \Gamma \to \mathbb{C}^*$, $\rho(\gamma) = (-1)^{\det A}$. Then we have*

$$K_X \otimes L_{\det A} = \mathcal{O}_X(-D_T).$$

In particular, X has negative Kodaira dimension.

Proof. Let $e_1^* \dots e_n^*$ be the standard basis of M and consider the meromorphic n-form on $\widetilde{X}$:

$$\omega := \frac{d\widetilde{e}_1^*}{\widetilde{e}_1^*} \wedge \ldots \wedge \frac{d\widetilde{e}_n^*}{\widetilde{e}_n^*}.$$

Then [Ful93, Proposition at page 85] shows that $\operatorname{div}(\omega) = -\widetilde{D}_T$. Since moreover $\gamma^*\omega = (-1)^{\det A}\omega$, the conclusion follows. ∎

6. Betti numbers

In this section, we fix a toric Kato manifold $X = X(\pi, \sigma)$ of matrix A, with $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$. Let D be the divisor of X induced by the exceptional divisor of π. We also let $\hat{\Sigma}$ denote the fan of $\hat{\mathbb{C}}^n$, we put $\mathbb{C}\hat{\mathbb{P}}^n = \hat{\mathbb{C}}^n \cup H$, where H is the divisor at infinity, and we let $\Sigma_{\mathbb{P}}$ be the corresponding fan of $\mathbb{C}\hat{\mathbb{P}}^n$. Let us put $a_j := |\hat{\Sigma}^{(j)}|$, for each $0 \leq j \leq n$. Then we can express the Betti numbers of X purely in terms of the combinatorial data $a_0, \dots, a_n$.

Theorem 6.1: *The n-dimensional toric Kato manifold X has the following Betti numbers:*

$$b_0(X) = b_1(X) = b_{2n-1}(X) = b_{2n}(X) = 1$$

$$b_{2j+1}(X) = 0, \quad 1 \leq j \leq n-2$$

$$b_{2j}(X) = -1 + \sum_{s=j}^{n} (-1)^{s-j} \binom{s}{j} \left(a_{n-s} + \binom{n}{s+1} \right), \quad 1 \leq j \leq n-1.$$

In particular, we have

$$b_2(X) = \sharp D, \quad \chi(X) = a_n - 1$$

where $\sharp$ denotes the number of irreducible components, and $\chi(X)$ denotes the Euler characteristic of X.

Proof. Let us suppose that $n > 2$, since the statement for $n = 2$ is well known. In this proof, we use the notation $H^\bullet(M)$ to denote the cohomology of M with coefficients in $\mathbb{C}$.

We will first show that $b_j(X) = b_j(\hat{\mathbb{C}}^n)$ for $2 \leq j \leq 2n-2$. By [Kat77], $X = X(\pi, \sigma)$ is diffeomorphic to a Kato manifold $X' = X(\pi, \sigma')$, so that $\sigma'(0)$ no longer meets the exceptional divisor of π. In particular, there exists a Hopf manifold $Y \cong \mathbb{S}^{2n-1} \times \mathbb{S}^1$, $p \in Y$ and a modification $\mu : X' \to Y$ at p induced by π. Let $B \subset Y$ be a ball centered at p, so that $V := \mu^{-1}(B) \subset X'$ is a neighborhood of the exceptional divisor F of μ which is homotopy equivalent to $\hat{\mathbb{C}}^n$, and let $U := X' \setminus F \cong Y \setminus \{p\}$. Using the fact that $U \cap V \cong \mathbb{S}^{2n-1}$ and that $b_j(U) = b_j(Y)$ for $0 \leq j \leq 2n-1$, the Mayer-Vietoris sequence in cohomology for (U, V) gives $H^j(X) = H^j(X') = H^j(\hat{\mathbb{C}}^n)$ for $2 \leq j \leq 2n-2$.

Next, we wish to compare the cohomology of $\hat{\mathbb{C}}^n$ with that of $\mathbb{C}\hat{\mathbb{P}}^n$. Let U be a neighborhood of H in $\mathbb{C}\hat{\mathbb{P}}^n$ biholomorphic to a $\mathbb{D}$-bundle over H inside $\mathcal{N}_{\mathbb{C}\hat{\mathbb{P}}^n} H$, let $V := \hat{\mathbb{C}}^n \subset \mathbb{C}\hat{\mathbb{P}}^n$

and let $U^* := U \cap V$, which is a $\mathbb{D}^*$-bundle over H. We have $H^\bullet(U) = H^\bullet(H)$, while $H^j(U^*) = H^j(H) \oplus H^{j-1}(H) = \mathbb{C}$ for $j \leq 2n-2$. Using this, the Mayer-Vietoris sequence in cohomology then gives:

$$\longrightarrow H^j(\mathbb{C}\hat{\mathbb{P}}^n) \longrightarrow H^j(H) \oplus H^j(\hat{\mathbb{C}}^n) \xrightarrow{r_U^* - r_V^*} \mathbb{C} \longrightarrow H^{j+1}(\mathbb{C}\hat{\mathbb{P}}^n) \longrightarrow$$

where r_U, r_V are the corresponding restriction maps, so that $r_V^* = 0$ and r_U^* is injective. From this we infer

$$b_j(\hat{\mathbb{C}}^n) = b_j(\mathbb{C}\hat{\mathbb{P}}^n), \quad j \text{ odd}\ , \quad b_j(\hat{\mathbb{C}}^n) = b_j(\mathbb{C}\hat{\mathbb{P}}^n) - 1, \quad j \text{ even}. \tag{6.1}$$

Finally, since $\mathbb{C}\hat{\mathbb{P}}^n = X(\Sigma_{\mathbb{P}}, N)$ is a complete toric algebraic variety, it satisfies [Dan78, Theorem 10.8]

$$b_{2j+1}(\mathbb{C}\hat{\mathbb{P}}^n) = 0, \quad b_{2j}(\mathbb{C}\hat{\mathbb{P}}^n) = \sum_{s=j}^{n} (-1)^{s-j} \binom{s}{j} \hat{a}_{n-s}, \quad j \in \mathbb{N}$$

where $\hat{a}_j := |\Sigma_{\mathbb{P}}^{(j)}| = a_j + \binom{n}{j-1}$, $j \in \{1, \dots, n\}$, $\hat{a}_0 = a_0$. Together with (6.1), this gives the desired formula for $b_\bullet(X)$.

In particular, since $a_1 = \sharp D + n$, we find

$$b_2(X) = b_{2n-2}(X) = -1 + (a_1 + 1) - (n \cdot a_0) = \sharp D.$$

Furthermore, we have:

$$\chi(\mathbb{C}\hat{\mathbb{P}}^n) = \sum_{j=0}^{n} \sum_{s=j}^{n} \hat{a}_{n-s} (-1)^{s-j} \binom{s}{j} = \sum_{s=0}^{n} \hat{a}_{n-s} \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} = \hat{a}_n = a_n + n$$

$$\chi(X) = \sum_{j=1}^{n-1} b_{2j}(X) = \sum_{j=1}^{n-1} (b_{2j}(\mathbb{C}\hat{\mathbb{P}}^n) - 1) = a_n + n - 2 - (n-1) = a_n - 1$$

which concludes the proof. ∎

7. Toric degenerations

7.1. **Nakamura degenerations.** In this section, we describe how the Nakamura degenerations [Nak83] of toric Kato surfaces, generalizing constructions of Miyake-Oda [Oda78], can be generalized to any dimension. Let us fix a toric Kato manifold $X = X(\pi, \sigma)$ with germ $F = \underline{\lambda} F_A$. Recall that $\widetilde{X}$ can be seen as a partial compactification of $\Omega_F = W_T(F)$ inside the toric manifold $X(\Sigma_A, N)$.

Also recall that we denoted by $\hat{\Sigma}_0$ the fan of the toric manifold $\hat{\mathbb{C}}^n \setminus \{\sigma(0)\}$, which is then a fundamental domain of the action of Γ on Σ_A. Here Γ is the deck group of $\widetilde{X} \to X$, generated by γ. Let $\mathbb{C}\hat{\mathbb{P}}^n$ be $\hat{\mathbb{C}}^n$ with the hyperplane at infinity H, and let $\widetilde{\mathbb{C}\mathbb{P}^n}$ be the blowup of $\mathbb{C}\hat{\mathbb{P}}^n$ along $\sigma(0)$, with exceptional divisor E. Then $\widetilde{\mathbb{C}\mathbb{P}^n} = X(\Sigma_c, N)$ is a toric manifold, with fan

Σ_c given by the completion of Σ_0 with the n-dimensional cones:

$$\tau_j = \langle Ac, Ae_1, \dots, \widehat{Ae_j}, \dots Ae_n\rangle, \quad 1 \le j \le n$$
$$\tau'_j = \langle -c, e_1, \dots, \widehat{e_j}, \dots, e_n\rangle, \quad 1 \le j \le n$$
$$c := \sum_{j=1}^{n} e_j.$$

We now wish to construct a manifold $\widetilde{\mathcal{X}}_N$ as a subset in a toric variety, together with an action of Γ, giving rise to the Nakamura degeneration. Consider then $\widetilde{N} = N \oplus \mathbb{Z}e$, and define an action $\widetilde{A}$ on $\widetilde{N}$ by putting $\widetilde{A}|_N = A$ and $\widetilde{A}e = e + Ac$. Put $\nu_k := \widetilde{A}^k e$, $k \in \mathbb{Z}$, and define regular fans $\widetilde{\Sigma}_0, \widetilde{\Sigma}$ in $\widetilde{N}$ as follows:

$$\widetilde{\Sigma}_0 := \{\widetilde{\tau} := \tau \oplus \langle e\rangle \mid \tau \in \Sigma_0\} \cup \{\widetilde{\tau}' := \tau \oplus \langle e, \nu_{-1}\rangle \mid \tau \in \Sigma_0, \tau \subset \partial C_0\}$$
$$\widetilde{\Sigma} := \bigcup_{k\in\mathbb{Z}} \widetilde{A}^k \widetilde{\Sigma}_0.$$

Let us also put $\widetilde{\underline{\lambda}} = (\underline{\lambda}, 1) \in T_{\widetilde{N}}$ and let $\widetilde{\gamma}$ be the automorphism of $X(\widetilde{\Sigma}, \widetilde{N})$ given by $\widetilde{\underline{\lambda}}\widetilde{A}$, where we also denote by $\widetilde{A}$ the natural action induced by $\widetilde{A}$ on the toric variety. Note that for $z \in T_{\widetilde{N}}$, $\widetilde{\gamma}(z) = \widetilde{F}(z) := \widetilde{\underline{\lambda}} z^{\widetilde{A}}$.

Let $p_{\mathbb{Z}} : \widetilde{N} \to \mathbb{Z}e$ be the natural projection, and let $\Sigma_1 = \{0, \langle e\rangle\}$ be the fan of $\mathbb{C}$ in $\mathbb{Z}e$. Then $p_{\mathbb{Z}}$ is a map of fans, and so defines naturally a map of toric varieties $p : X(\widetilde{\Sigma}, \widetilde{N}) \to X(\Sigma_1, \mathbb{Z}e)$. We have

$$p^{-1}(\mathbb{C}^*) = p^{-1}(\operatorname{orb} 0) = \bigcup_{\tau\in\widetilde{\Sigma}, p_{\mathbb{Z}}(\tau)=0} \operatorname{orb}\tau = X(\widetilde{\Sigma} \cap N, \widetilde{N}) = X(\Sigma_A, N) \times \mathbb{C}^*$$

and for $(x, t) \in X(\Sigma_A, N) \times \mathbb{C}^*$ we have $\widetilde{\gamma}(x, t) = (\underline{t}^A \gamma(x), t)$, where $\underline{t} := (t, \dots, t) \in T_N$. Moreover

$$p^{-1}(0) = p^{-1}(\overline{\operatorname{orb} e}) = \bigcup_{\tau\in\widetilde{\Sigma}, p_{\mathbb{Z}}(\tau)=\langle e\rangle} \overline{\operatorname{orb}\tau} = \bigcup_{m\in\mathbb{Z}} \overline{\operatorname{orb}\nu_m}$$

and $\widetilde{\gamma}(\overline{\operatorname{orb}\nu_m}) = \overline{\operatorname{orb}\nu_{m+1}}$. Let us note that $\overline{\operatorname{orb} e}$ is the toric variety $X(\widehat{St}_e, N_e)$, where $N_e := \widetilde{N}/\mathbb{Z}e$ and

$$St_e = \{\tau \in \widetilde{\Sigma} \mid \langle e\rangle \prec \tau\}, \quad \widehat{St}_e = \{\tau \mod \mathbb{Z}e \mid \tau \in St_e\}.$$

Under the natural isomorphism $N_e \cong N$ given by the projection, we have $\widehat{St}_e \cong \Sigma_c$. Thus $\overline{\operatorname{orb}\nu_0}$ is equivariantly isomorphic to $\widetilde{\mathbb{CP}^n}$, so that $\overline{\operatorname{orb}\langle\nu_{-1}, \nu_0\rangle} = E$, $\overline{\operatorname{orb}\langle\nu_0, \nu_1\rangle} = H$ and $\widetilde{\gamma}$ sends E biholomorphically to H.

Let $\Delta := \{t \in \mathbb{C} | F(t\mathbb{B}) \subset \mathbb{B}\} \subset \mathbb{C}$, which is an open disk containing 1. Put

$$\Omega := \{(z, t) \in T_N \times \Delta^* \mid z \in W_T(F(t\cdot))\} \subset T_{\widetilde{N}}$$

and define the smooth manifold

$$\widetilde{\mathcal{X}}_N := \operatorname{Int}(\overline{\Omega}^{X(\widetilde{\Sigma}, \widetilde{N})}) \subset X(\widetilde{\Sigma}, \widetilde{N})$$

together with a free and proper action of $\Gamma = \langle\widetilde{\gamma}\rangle$ and a Γ-invariant map $p : \widetilde{\mathcal{X}}_N \to \Delta$ so that $p^{-1}(t) = \widetilde{X(\pi, \sigma(t\cdot))}$ for $t \neq 0$ and $p^{-1}(0)$ is given by $\mathbb{Z}$ copies of $\widetilde{\mathbb{CP}^n}$ glued along H and E

via $\widetilde{\gamma}$. Thus we obtain a flat proper holomorphic map

$$p_N : \mathcal{X}_N := \widetilde{\mathcal{X}}_N/\Gamma \to \Delta$$

with $p_N^{-1}(t) = X(\pi, \sigma(t\cdot))$ for $t \in \Delta \setminus \{0\}$ and $p_N^{-1}(0) = \widetilde{\mathbb{CP}^n}/_{E \underset{\gamma}{\sim} H}$. This is precisely the Nakamura degeneration.

Define the toric divisor on $\widetilde{\mathcal{X}}_N$:

$$\widetilde{\mathcal{D}} = \sum_{\nu \in \widetilde{\Sigma}^{(1)}} \overline{\operatorname{orb} \nu} \cap \widetilde{\mathcal{X}}_N.$$

It is clear that $\widetilde{\mathcal{D}}|_{\Delta^*} = \widetilde{D}_T^X \times \Delta^*$, so that $\widetilde{\mathcal{D}} \cap p^{-1}(t) = \widetilde{D}_T^X$ is the maximal toric divisor of $\widetilde{X(\pi, \sigma(t\cdot))}$ for each $t \in \Delta^*$. Since moreover $\widetilde{\mathcal{D}}$ is Γ-invariant, we obtain a divisor $\mathcal{D} := \widetilde{\mathcal{D}}/\Gamma$ on $\mathcal{X}_N$ so that for each $t \in \Delta^*$, $\mathcal{D} \cap p_N^{-1}(t) = D_T$ and in particular $K_{\mathcal{X}_t} \otimes L_{\det A} = \mathcal{O}(-\mathcal{D})_t$. We have thus obtained:

Theorem 7.1: *Let $X(\pi, \sigma)$ be an n-dimensional toric Kato manifold, let $\widetilde{\mathbb{CP}^n} = X(\Sigma_c, N)$ be defined as above and let $S = \widetilde{\mathbb{CP}^n}/_{E \underset{\gamma}{\sim} H}$, where $\widetilde{\gamma}$ was defined above. There exists a flat holomorphic proper family $p_N : (\mathcal{X}_N, \mathcal{D}) \to \Delta$ over some disk $\Delta \subset \mathbb{C}$ of radius $R > 1$, so that:*

- *$\mathcal{X}_N$ is smooth*
- *$\mathcal{D}$ is an effective divisors on $\mathcal{X}_N$*
- *for each $t \in \Delta^*$, $p_N^{-1}(t) \cong X(\pi, \sigma(t\cdot))$ and $\mathcal{D} \cap p_N^{-1}(t)$ is the toric divisor of $p_N^{-1}(t)$*
- *$p_N^{-1}(0) = S$.*

Note that by the discussion of Section 9, the smooth fibers of the family $\mathcal{X}_N$ are all isomorphic if $c \in \operatorname{im}(\mathsf{id} - A)$.

7.2. **Isotrivial degenerations.** Nakamura degenerations are defined for all Kato manifolds, however generally the smooth fibers are not all isomorphic. On the other hand, for most toric Kato manifolds we can define other degenerations, whose smooth fibers will all be isomorphic, but whose central fiber is less concrete and generally more singular. Moreover, the total space of these degenerations needs not be smooth. These follow the same ideas as [Tsu87, Section 2].

In order to do so, we fist need the following:

Lemma 7.2: *Let A be a toric Kato matrix. Then $\operatorname{im}(\mathsf{id} - A) \cap \operatorname{Int} C_0 \cap N \neq \emptyset$ if and only if $|P(A)| < n - 1$.*

Proof. If $|P(A)| = n - 1$, so that $j \notin P(A)$, then it is clear that $\operatorname{im}(\mathsf{id} - A) \subset \ker e_j^*$, thus $\operatorname{im}(\mathsf{id} - A) \cap \operatorname{Int} C_0 = \emptyset$.

Suppose now that $|P(A)| < n - 1$, and let $j \neq k \notin P(A)$. Let C_j and C_k be the j-th and k-th columns of $A - \mathsf{id}$. Then $C_j + C_k - e_j - e_k \in \operatorname{Int} C_0 \cap \operatorname{im}(\mathsf{id} - A) \cap N \neq \emptyset$. ∎

Let us thus suppose that $X = X(\pi, \sigma)$ is an n-dimensional toric Kato manifold with $\sigma = \underline{\lambda}\sigma_A$ and $|P(A)| < n - 1$. Let $F = \underline{\lambda} F_A$ be its corresponding germ. We will again define a toric manifold $X(\widetilde{\Sigma}, \widetilde{N})$ together with a toric map to $\mathbb{C}$.

In order to do so, choose, via Lemma 7.2, a primitive element $u \in \operatorname{im}(\mathsf{id} - A) \cap \operatorname{Int} C_0 \cap N$, and let $v \in N$ with $Au = Av - v \in \operatorname{Int} AC_0$. Consider again $\widetilde{N} = N \oplus \mathbb{Z}e$, and define an action of $\widetilde{A}'$ on $\widetilde{N}$ by putting $\widetilde{A}'|_N = A$ and $\widetilde{A}'e = e$. Put $\nu_k := (\widetilde{A}')^k(e + v)$, $k \in \mathbb{Z}$, and define fans

$\widetilde{\Sigma}_0, \widetilde{\Sigma}_u$ in $\widetilde{N}$ as before. Putting again $\widetilde{\underline{\lambda}} = (\underline{\lambda}, 1) \in T_{\widetilde{N}}$, we have the natural action of $\widetilde{\gamma}'$ given by $\widetilde{\underline{\lambda}}\widetilde{A}'$, which restricted to $T_{\widetilde{N}}$ reads $\widetilde{\gamma}'(z) = \widetilde{F}'(z) := \widetilde{\underline{\lambda}}.z^{\widetilde{A}'}$.

Let again $p_{\mathbb{Z}} : \widetilde{N} \to \mathbb{Z}e$ be the natural projection, and let $\Sigma_1 = \{0, \langle e\rangle\}$ be the fan of $\mathbb{C}$ in $\mathbb{Z}e$. Then $p_{\mathbb{Z}}$ induces a map of toric varieties $p : X(\widetilde{\Sigma}_u, \widetilde{N}) \to X(\Sigma_1, \mathbb{Z}e)$, with:

$$p^{-1}(\mathbb{C}^*) = X(\Sigma_A, N) \times \mathbb{C}^*$$
$$p^{-1}(0) = \bigcup_{m\in\mathbb{Z}} \overline{\mathrm{orb}\,\nu_m}.$$

For $(x,t) \in X(\Sigma_A, N) \times \mathbb{C}^*$ we have $\widetilde{\gamma}'(x,t) = (\gamma(x), t)$.

The toric variety $\overline{\mathrm{orb}\,\nu_0} = X(\widehat{St}_{\nu_0}, N_{\nu_0})$ is isomorphic to $X(\Sigma_u, N)$, via the projection $q : \widetilde{N} \to N$, $q|_N = \mathsf{id}$, $qe = -v$ which maps the fan $\widehat{St}_{\nu_0}$ isomorphically to Σ_u given by:

$$\Sigma_u := \Sigma_0 \cup \{\tau' = \langle Au\rangle \oplus A\tau,\ \tau'' = \langle -u\rangle \oplus \tau \mid \tau \in \Sigma_0, \tau \subset \partial C_0\}.$$

Under this isomorphism, $\widetilde{\gamma}'$ sends $E' := \overline{\mathrm{orb}\langle -u\rangle}$ biholomorphically to $H' := \overline{\mathrm{orb}\langle Au\rangle}$.

Taking now $\Omega_{\widetilde{F}'} := W_T(F_{\underline{\lambda},A}) \times \mathbb{C}^*$ and

$$\widetilde{\mathcal{X}}_u := \mathrm{Int}(\overline{\Omega_{\widetilde{F}'}}^{X(\widetilde{\Sigma}_u, \widetilde{N})}) \subset X(\widetilde{\Sigma}_u, \widetilde{N})$$

with the $\Gamma = \langle\widetilde{\gamma}'\rangle$-invariant projection $p : \widetilde{\mathcal{X}}_u \to \mathbb{C}$, we obtain a proper holomorphic family

$$p_u : \mathcal{X}_u := \widetilde{\mathcal{X}}_u/\Gamma \to \mathbb{C}$$

with $p_u^{-1}(t) = X$ for $t \in \mathbb{C}^*$ and $p_u^{-1}(0) = X(\Sigma_u, N)/_{E' \sim_{\gamma'} H'}$. Moreover, p_u is a flat map by [Gro65, 6.1.5] since it is equidimensional and $\mathcal{X}_u$ is Cohen-Macaulay.

As before, define the divisors

$$\widetilde{\mathcal{D}} = \sum_{\nu\in\widetilde{\Sigma}^{(1)}} \overline{\mathrm{orb}\,\nu} \cap \widetilde{\mathcal{X}}_N \in \mathrm{Div}\widetilde{\mathcal{X}},\quad \mathcal{D} = \widetilde{\mathcal{D}}/\Gamma \in \mathrm{Div}\mathcal{X}$$

Then again one obtains that $\mathcal{D} \cap p_u^{-1}(t)$ is the toric divisor of X, for each $t \in \mathbb{C}^*$.

Note that the central fiber of the family $\mathcal{X}_u$ depends on u, and that any primitive element $u \in \mathrm{im}(\mathsf{id} - A) \cap \mathrm{Int}C_0 \cap N$ gives rise to such a family. However, we cannot in general ensure that $\mathcal{X}_u$ is a smooth manifold, or equivalently that the normalization of the central fiber is smooth, as the following result shows:

Proposition 7.3: *The toric variety $X(\Sigma_u, N)$ is smooth if and only if $u = c = \sum_{j=1}^n e_j$. In this case, the family $(\mathcal{X}_u, \mathcal{D})$ is equivalent to the Nakamura family $(\mathcal{X}_N, \mathcal{D}_N)$, in the sense that there exists a biholomorphism $\varphi_X : \mathcal{X}_N \to \mathcal{X}_c|_\Delta = p_c^{-1}(\Delta)$ with $p_c \circ \varphi_X = p_N$, $\varphi_X^*\mathcal{D}|_\Delta = \mathcal{D}_N$.*

Proof. It is clear that if $u = c$, then $X(\Sigma_u, N) \cong \widetilde{\mathbb{CP}^n}$, so it is smooth. Conversely, suppose that $X(\Sigma_u, N)$ is smooth. Since for any $j \in \{1, \dots, n\}$, $-u, e_1, \dots, \widehat{e_j}, \dots, e_n$ is a basis of N, and since $u \in \mathrm{Int}C_0$, this implies that $u = c$.

Suppose now that $u = c$. Define $\varphi_{\mathbb{Z}} : \widetilde{N} \to \widetilde{N}$ by $\varphi_{\mathbb{Z}}|_N = \mathsf{id}$, $\varphi_{\mathbb{Z}}e = e + v$, where v is chosen to satisfy $Ac = Av - v$. We have then $\varphi_{\mathbb{Z}}(\widetilde{\Sigma}) = \widetilde{\Sigma}_c$ and $\varphi_{\mathbb{Z}}\widetilde{A}x = \widetilde{A}'\varphi_{\mathbb{Z}}x$ for any $x \in \widetilde{N}$. This implies that $\varphi_{\mathbb{Z}}$ induces a Γ-equivariant biholomorphism $\varphi : X(\widetilde{\Sigma}, \widetilde{N}) \to X(\widetilde{\Sigma}_c, \widetilde{N})$ commuting with the projections to $\mathbb{C}$ and satisfying $\varphi \circ \widetilde{F} = \widetilde{F}' \circ \varphi$. In particular, it is clear that $\varphi(\Omega) = \Omega_{\widetilde{F}'} \cap p^{-1}(\Delta)$ and that with $\varphi^*\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}_N}$, so φ also induces a biholomorphism

between $\widetilde{\mathcal{X}}_N$ and $\widetilde{\mathcal{X}}_c|_\Delta$, and so a biholomorhism $\varphi_X : \mathcal{X}_N \to \mathcal{X}_c|_\Delta$ with $p_c \circ \varphi_X = p_N$ and $\varphi_X^* \mathcal{D}|_\Delta = \mathcal{D}_N$.■

The conclusion of the discussion is the following:

Theorem 7.4: *Let X be a toric Kato manifold of hyperbolic type of matrix A, and let $u \in \mathrm{im}(\mathsf{id} - A) \cap \mathrm{Int} C_0 \cap N$ be a primitive element. Then there exists a flat holomorphic proper family $p_u : (\mathcal{X}_u, \mathcal{D}) \to \mathbb{C}$ so that:*

- *$\mathcal{D}$ is an effective divisor on $\mathcal{X}_u$*
- *for each $t \in \mathbb{C}^*$, $p_u^{-1}(t) \cong X$ and $\mathcal{D} \cap p_u^{-1}(t)$ is the toric divisor of X*
- *$p_u^{-1}(0) = X(\Sigma_u, N)/_{E' \sim_{\widetilde{\gamma'}} H'}$.*

Moreover, $\mathcal{X}_u$ is smooth if and only if $u = c$, in which case the family $(\mathcal{X}_u, \mathcal{D})$ is equivalent to the Nakamura family $(\mathcal{X}_N, \mathcal{D})$ of Theorem 7.1.

8. Analytic invariants

Proposition 8.1: *Let $\widetilde{X}$ be the universal cover of a toric Kato manifold X. If X is of hyperbolic type, then $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \mathbb{C}$. If X is of parabolic type, then $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \mathcal{O}_{\mathbb{C}}(\mathbb{C})$.*

Proof. Let A be the matrix of X. Let $f \in H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Since Ω is a $\mathbb{T}$-invariant domain of T_N, f expresses as a series on Ω:

$$f(z) = \sum_{m \in M} c_m z^m, \quad z \in \Omega.$$

Since the right hand side extends naturally to a meromorphic function on $\widetilde{X}$, f writes on $\widetilde{X}$ as

$$f = \sum_{m \in M} c_m \widetilde{m} \tag{8.1}$$

where for each $m \in M$, $\widetilde{m}$ is the natural meromorphic function induced by m on $X(\Sigma_A, N)$.

Now since for any $\tau \in \Sigma_A^{(1)}$, f has no poles on $\overline{\mathrm{orb}\,\tau}$, it follows that for any $m \in M$ with $c_m \neq 0$ we have

$$\langle m, v \rangle \geq 0, \ \forall v \in \tau.$$

Since $C := \overline{|\Sigma_A|} = \overline{H(A)}$ is a convex cone by Lemma 4.3, we readily infer:

$$f = \sum_{m \in \check{C} \cap M} c_m \widetilde{m}.$$

Now from Lemma 4.3 we have $\check{C} \cap M = \mathbb{R}_{\geq 0} f_A^* \cap M = \{0\}$ if $|P(A)| < n-1$ and $\check{C} \cap M = \mathbb{N} e_j^*$ if $P(A) = \{j\}$. From this it follows that in the parabolic case, f is an entire function in the z_j-variable. ■

Remark 8.2: For a parabolic toric Kato manifold X with $P(A)^c = \{j\}$, define

$$\rho : \Gamma \cong \mathbb{Z}\langle F_{\underline{\lambda}_0, A} \rangle \to \mathbb{C}^*, \quad \rho(F_{\underline{\lambda}_0, A}) = \lambda_j$$

$$\mathcal{L} = \widetilde{X(\pi, \sigma)} \times_\rho \mathbb{C} \tag{8.2}$$

where $\lambda_j \in \mathbb{D}$ is the j-th component of $\underline{\lambda}_0$. Then it follows that the holomorphic function on the universal cover defined by $f(z) = z_j$ on T_N gives rise to a holomorphic section of $\mathcal{L}$.

Corollary 8.3: *If X is a toric Kato manifold and $L \in \mathrm{Pic}_0(X)$ is non-trivial, then $H^0(X, L) \neq 0$ if and only if X is of parabolic type and L is a positive power of the line bundle defined in Remark 8.2. In this case, $H^0(X, L) \cong \mathbb{C}$.*

Next, let $\Theta_X(-\log D_T)$ denote the locally free subsheaf of the tangent sheaf given by germs of holomorphic vector fields tangent to D_T, let $\Omega^1_X(\log D_T)$ be the dual sheaf and let $\Omega^p(\log D_T) := \bigwedge^p(\Omega^1(\log D_T))$ for $p \geq 2$.

Proposition 8.4: *If X is any toric Kato manifold, then $H^0(X, \Omega^p_X) = 0$ for any $p \geq 1$.*

Proof. For Hopf manifolds the result is well known, so we only treat the other cases. We start by noting that the map

$$M \otimes \mathcal{O}_{\widetilde{X}} \to \Omega^1_{\widetilde{X}}(\log \widetilde{D}_T), \quad m \otimes f \to f \cdot \frac{d\widetilde{m}}{\widetilde{m}}$$

where $\widetilde{m}$ is the natural meromorphic function on $\widetilde{X}$ associated to m, is an isomorphism of $\mathcal{O}_{\widetilde{X}}$-modules [Ful93, Propostion at page 87]. In particular, for any $p \geq 1$, $\Omega^p_{\widetilde{X}}(\log \widetilde{D}_T) \cong \bigwedge^p M \otimes \mathcal{O}_{\widetilde{X}}$.

Let $f_1, \ldots, f_n$ be a $\mathbb{Z}$-basis of N so that $\nu_j := \langle f_j \rangle \in \Sigma^{(1)}_A$ for each $1 \leq j \leq n$ and let $m_1, \ldots, m_n$ be the dual basis of M. For each $J = (1 \leq j_1 < \ldots < j_p \leq n)$, $1 \leq p \leq n$, denote by

$$\omega_J := \frac{d\widetilde{m}_{j_1}}{\widetilde{m}_{j_1}} \wedge \ldots \wedge \frac{d\widetilde{m}_{j_p}}{\widetilde{m}_{j_p}} \in H^0(\widetilde{X}, \Omega^p_{\widetilde{X}}(\log D_T)).$$

Using Proposition 8.1, we find that

$$H^0(\widetilde{X}, \Omega^p_{\widetilde{X}}(\log \widetilde{D}_T)) = \begin{cases} \mathrm{span}_{\mathbb{C}}\langle \omega_J \mid |J| = p \rangle, & |P(A)| < n-1 \\ \mathrm{span}_{\mathcal{O}_{\mathbb{C}}(\mathbb{C})}\langle \omega_J \mid |J| = p \rangle, & P(A)^c = \{j\} \end{cases}$$

where $\mathcal{O}_{\mathbb{C}}(\mathbb{C})$ denotes the ring of entire functions in $z_j = \widetilde{e}^*_j$.

Since $\Omega^p_{\widetilde{X}}$ is a subsheaf of $\Omega^p_{\widetilde{X}}(\log \widetilde{D}_T)$ and since each ω_J has poles along $\sum_{j \in J} \overline{\mathrm{orb}\, \nu_j}$, it is clear that in the case $|P(A)| < n-1$ we have $H^0(X, \Omega^p_X) = H^0(\widetilde{X}, \Omega^p_{\widetilde{X}}) = 0$ for any $p \geq 1$. In the case $P(A)^c = \{j\}$, the same argument shows that $H^0(X, \Omega^p_X) = H^0(\widetilde{X}, \Omega^p_{\widetilde{X}}) = 0$ for $p \geq 2$. For $p = 1$, we find that

$$H^0(\widetilde{X}, \Omega^1_{\widetilde{X}}) = \{\omega = f(z_j)dz_j \mid f \in \mathcal{O}_{\mathbb{C}}(\mathbb{C})\}.$$

Since $\gamma = F_{\underline{\lambda}, A}$ acts on such an element $\omega = \sum_{k=0}^{\infty} c_k z_j^k dz_j$ by $\gamma^* \omega = \sum_{k=0}^{\infty} \lambda_j^{k+1} c_k z_j^k dz_j$ and as $|\lambda_j| < 1$, we find

$$H^0(X, \Omega^1_X) = H^0(\widetilde{X}, \Omega^1_{\widetilde{X}})^{\Gamma} = 0$$

which concludes the proof. ∎

Theorem 8.5: *Let X be a toric Kato manifold of hyperbolic type. Then we have:*

$$H^0(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X) = \mathbb{C}, \quad H^p(X, \mathcal{O}_X) = 0, \ p \geq 2.$$

Proof. Let A be a toric Kato matrix for X. Since $|P(A)| < n-1$, there exists $u \in \mathrm{im}(\mathsf{id} - A) \cap \mathrm{Int}C_0 \cap N$ a primitive element. Consider the family $p_u : \mathcal{X}_u \to \mathbb{C}$ of Theorem 7.4, of central fiber S and generic fiber X. Then by upper-semicontinuity [BS76, Theorem 4.12] we have:

$$\dim H^p(X, \mathcal{O}_X) \leq \dim H^p(S, \mathcal{O}_S), \quad p \in \mathbb{N}. \tag{8.3}$$

Let us first compute $H^\bullet(S, \mathcal{O}_S)$. Let $n : \hat{S} = X(\Sigma_u, N) \to S$ be the normalization map, and let $Y = n(E') = n(H')$ be the double locus of S, with inclusion map $j : Y \to S$. Then we have an exact sequence of sheaves on S:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow n_*\mathcal{O}_{\hat{S}} \longrightarrow j_*\mathcal{O}_Y \longrightarrow 0 \tag{8.4}$$

and the corresponding long exact sequence in cohomology. Since Y and $\hat{S}$ are compact toric varieties, we have $H^p(Y, \mathcal{O}_Y) = H^p(\hat{S}, \mathcal{O}_{\hat{S}}) = 0$ for any $p \geq 1$ cf. [Dan78, Corollary 7.4]. This implies that $H^p(S, \mathcal{O}_S) = 0$ for $p \geq 2$ and gives rise to the exact sequence:

$$0 \longrightarrow H^0(S, \mathcal{O}_S) \longrightarrow \mathbb{C} \xrightarrow{0} \mathbb{C} \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow 0.$$

We infer that $H^0(S, \mathcal{O}_S) = H^1(S, \mathcal{O}_S) = \mathbb{C}$.

The cohomology of $\mathcal{O}_S$ together with (8.3) immediately gives the desired result for $H^p(X, \mathcal{O}_X)$ for $p \neq 1$. On the other hand, since $h^{1,0}(X) = 0$ by Proposition 8.4, using again (8.3) we find:

$$1 = b_1(X) \leq h^{1,0}(X) + h^{0,1}(X) = h^{0,1}(X) \leq h^{0,1}(S) = 1$$

hence we also have $H^1(X, \mathcal{O}_X) = \mathbb{C}$. This concludes the proof. ∎

Corollary 8.6: *Let X be a toric Kato manifold of hyperbolic type. Then we have an isomorphism $\mathrm{Pic}_0(X) \cong H^1(X, \mathbb{C}^*) \cong \mathbb{C}^*$ given by*

$$\lambda \in \mathbb{C}^* \mapsto L_\lambda := \widetilde{X} \times_{\rho_\lambda} \mathbb{C}, \quad \rho_\lambda \in \mathrm{Hom}(\Gamma, \mathbb{C}^*),\ \rho_\lambda(n) = \lambda^n.$$

Moreover, we have a short exact sequence:

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow 0.$$

Proof. Since by Theorem 8.5 we have $H^1(X, \mathcal{O}_X) = \mathbb{C} = H^1(X, \mathbb{C})$, we find that $\mathrm{Pic}_0(X) := H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) \cong H^1(X, \mathbb{C}^*) = \mathbb{C}^*$. Since moreover $H^2(X, \mathcal{O}_X) = 0$, the exponential sequence gives rise to the exact sequence:

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) = 0$$

from which we conclude. ∎

Theorem 8.7: *Let X be a toric Kato manifold with toric divisor D_T and matrix A. Then we have:*

$$H^0(X, \Theta_X(-\log D_T)) \cong \ker(A - \mathsf{id})$$
$$H^0(X, \Omega^1_X(\log D_T)) \cong \ker(A^T - \mathsf{id}).$$

If moreover X is of hyperbolic type, then we also have:

$$H^1(X, \Theta_X(-\log D_T)) \cong \mathrm{coker}(A - \mathsf{id}), \quad H^p(X, \Theta_X(-\log D_T)) = 0,\ p \geq 2$$
$$H^1(X, \Omega^1_X(\log D_T)) \cong \mathrm{coker}(A^T - \mathsf{id}), \quad H^p(X, \Omega^1_X(\log D_T)) = 0,\ p \geq 2.$$

Proof. We only show the statements for the sheaf $\Theta_X(-\log D_T)$, since for $\Omega^1_X(\log D_T)$ the proof is dual. We have

$$H^0(X, \Theta_X(-\log D_T)) = H^0(\widetilde{X}, \Theta_{\widetilde{X}}(-\log \widetilde{D}_T))^\Gamma.$$

Since $\widetilde{X}$ is an open set of the toric manifold $X(\Sigma_A, N)$, we find $\Theta_{\widetilde{X}}(-\log \widetilde{D}_T) \cong \mathcal{O}_{\widetilde{X}} \otimes N$ and so, by Proposition 8.1, the conclusion follows for the case $|P(A)| < n-1$.

If $P(A)^c = \{j\}$, then by Proposition 8.1, any element $s \in H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}} \otimes N)$ expresses as:

$$s = \sum_{m \in \mathbb{N}} a_m z_j^m, \quad a_m \in N \otimes \mathbb{C}.$$

The Γ-invariance of s reads:

$$A a_m = \lambda_j^{-m} a_m, \quad \forall m \in \mathbb{N}$$

Using that $|\lambda_j| < 1$ and that for any $\mu \in \mathrm{Spec}(A)$, $|\mu| = 1$, we find:

$$H^0(X, \Theta_X(-\log D_T)) \cong \ker(A - \mathsf{id}).$$

Suppose next that $|P(A)| < n-1$, so that there exists a primitive element $u \in \mathrm{im}(\mathsf{id} - A) \cap \mathrm{Int} C_0 \cap N$, and consider the family $p_u : (\mathcal{X}_u, \mathcal{D}) \to \mathbb{C}$ of Theorem 7.4, of central fiber S and generic fiber X. Consider the sheaf $\mathcal{F} := \Theta_{\mathcal{X}}(-\log \mathcal{D})$, and denote by $\mathcal{F}_t$ its restriction to any fiber $p_u^{-1}(t)$, $t \in \mathbb{C}$. Then for any $t \in \mathbb{C}^*$, we have $\mathcal{F}_t \cong \Theta_X(-\log D_T) \oplus \mathcal{O}_X$, so by upper-semicontinuity [BS76, Theorem 4.12] we find:

$$\dim H^p(X, \Theta_X(-\log D_T)) + \dim H^p(X, \mathcal{O}_X) \leq \dim H^p(S, \mathcal{F}_0), \quad p \in \mathbb{N}. \tag{8.5}$$

We must thus compute $H^\bullet(S, \mathcal{F}_0)$. Consider again the exact sequence (8.4) and tensorize it with the flat sheaf $\mathcal{F}_0$, obtaining:

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow n_* \mathcal{O}_{\hat{S}} \otimes \widetilde{N} \longrightarrow j_* \mathcal{O}_Y \otimes \widetilde{N} \longrightarrow 0.$$

As before, passing to the long exact sequence in cohomology we find that $H^p(S, \mathcal{F}_0) = 0$ for $p \geq 2$ and so $H^p(X, \Theta_X(-\log D_T)) = 0$ for $p \geq 2$. Moreover, we have the short exact sequence:

$$0 \longrightarrow H^0(S, \mathcal{F}_0) \longrightarrow \widetilde{N} \otimes \mathbb{C} \xrightarrow{\widetilde{A}' - \mathsf{id}} \widetilde{N} \otimes \mathbb{C} \longrightarrow H^1(S, \mathcal{F}_0) \longrightarrow 0$$

which implies $H^1(S, \mathcal{F}_0) = \mathrm{coker}(\widetilde{A}' - \mathsf{id}) = \mathrm{coker}(A - \mathsf{id}) \oplus \mathbb{C}$. Thus, from (8.5) and Theorem 8.5 we infer the inequality:

$$\dim H^1(X, \Theta_X(-\log D_T)) \leq \dim \mathrm{coker}(A - \mathsf{id}). \tag{8.6}$$

At the same time, $H^1(X, \Theta_X(-\log D_T))$ can be computed, as Γ-equivariant cohomology, by a spectral sequence [Gro57, Subsection 5.2]. Namely, if $q : \widetilde{X} \to X$ denotes the covering map, then $\Theta_{\widetilde{X}}(-\log \widetilde{D}_T)$ is a Γ-sheaf on $\widetilde{X}$ and $\Theta_X(-\log D_T) = q_*^\Gamma \Theta_{\widetilde{X}}(-\log \widetilde{D}_T)$. Therefore, we have a spectral sequence

$$E_2^{p,q} = H^p(\Gamma, H^q(\widetilde{X}, \Theta_{\widetilde{X}}(-\log \widetilde{D}_T))) \Rightarrow H^{p+q}(X, \Theta_X(-\log D_T)).$$

Moreover, since $\Gamma \cong \mathbb{Z}$, we have $E_2^{p,\bullet} = 0$ for any $p \geq 2$. Hence this spectral sequence degenerates at the second page, implying that $H^1(X, \Theta_X(-\log D_T)) = E_2^{1,0} \oplus E_2^{0,1}$. In particular, we have

$$\dim H^1(X, \Theta_X(-\log D_T)) \geq \dim H^1(\Gamma, H^0(\widetilde{X}, \Theta_{\widetilde{X}}(-\log \widetilde{D}_T))). \tag{8.7}$$

But using Proposition 8.1, the fact that $\mathbb{S}^1$ is the Eilenberg–MacLane space of Γ and Poincaré duality for local systems, we find:

$$\begin{aligned} H^1(\Gamma, H^0(\widetilde{X}, \Theta_{\widetilde{X}}(-\log \widetilde{D}_T))) &= H^1(\Gamma, N \otimes \mathbb{C}) \cong H^0(\Gamma, N^* \otimes \mathbb{C})^* \\ &= (\ker(A^T - \mathsf{id}))^* = \operatorname{coker}(A - \mathsf{id}). \end{aligned}$$

This, together with (8.7) and (8.6) gives us the desired conclusion. ∎

Theorem 8.8: *Let X be a toric Kato manifold of hyperbolic type, with divisor D induced by the exceptional divisor of π. Then we have:*

$$\begin{aligned} h^{1,p}(X) &= 0, \ p \neq 1 \\ h^{1,1}(X) &= \sharp D > 0. \end{aligned}$$

Proof. Let us write $D_T = \sum_{j=1}^k D_j$, where each D_j is an immersed irreducible hypersurface, possibly equal to D_s for some $s \neq j$. Let $\hat{D}_j$ be the normalization of D_j for each $j = 1, \ldots, k$ and let $\alpha : \hat{D}_T = \bigsqcup_{j=1}^k \hat{D}_j \to X$ be the natural map given by inclusion and normalization. Then we have an exact sequence:

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log D_T) \xrightarrow{P} \alpha_* \mathcal{O}_{\hat{D}_T} \longrightarrow 0 \tag{8.8}$$

where P is the Poncaré residue map.

Each component of $\hat{D}_T$ is either a compact toric variety, or a toric Kato manifold of hyperbolic type, and we have precisely $|P(A)|$ components of the last type. Using Theorem 8.5 and Theorem 8.7, we infer that $H^p(X, \Omega^1_X) = 0$ for $p \geq 3$ and we find the following exact sequence:

$$\begin{aligned} 0 \longrightarrow H^0(X, \Omega^1_X(\log D_T)) \longrightarrow \mathbb{C}^k \longrightarrow H^1(X, \Omega^1_X) \longrightarrow \\ \longrightarrow H^1(X, \Omega^1_X(\log D_T)) \xrightarrow{P_1} \mathbb{C}^{|P(A)|} \longrightarrow H^2(X, \Omega^1_X) \longrightarrow 0. \end{aligned} \tag{8.9}$$

Now the point is to show that the map P_1 induced by P is surjective. This will readily imply that $h^{1,2}(X) = 0$. Furthermore, as the Euler characteristic of the exact sequence is 0, and as $\dim H^0(X, \Omega^1_X(\log D_T)) = \dim H^1(X, \Omega^1_X(\log D_T))$ by Theorem 8.7, we also infer that $h^{1,1} = k - |P(A)| = \sharp D$.

Let us thus show that P_1 is surjective. We recall that for each $j \in P(A)$, we denoted by $X_j := \overline{\operatorname{orb}\langle e_j \rangle} \cap \widetilde{X}$. We also denote by Y_j the quotient of X_j by a finite index subgroup of Γ, so that Y_j is a smooth Kato manifold of hyperbolic type which is the normalization of one of the components of D_T. In particular, we have $\hat{D}_T = \hat{D} \sqcup \bigsqcup_{j \in P(A)} Y_j$, where $\hat{D}$ is the

normalization of D. The map P_1 is induced by the Γ-equivariant map of Γ-sheaves on $\widetilde{X}$:

$$\begin{gathered}\widetilde{P}_1 : \Omega^1_{\widetilde{X}}(\log \widetilde{D}_T) \cong M \otimes \mathcal{O}_{\widetilde{X}} \to \oplus_{j\in P(A)}\mathcal{O}_{X_j} \\ m \otimes f \mapsto (\langle m, e_j\rangle \cdot f|_{X_j})_{j\in P(A)}\end{gathered}$$

so that

$$P_1 = \widetilde{P}_1^* : H^1_\Gamma(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log \widetilde{D}_T)) = H^1(X, \Omega^1_X(\log D_T) \to \oplus_{j\in P(A)} H^1(Y_j, \mathcal{O}_{Y_j}).$$

On the other hand, using again the spectral sequence of [Gro57] which degenerates at the second page, together with Theorem 8.5 and Theorem 8.7, we find

$$\begin{gathered}\mathbb{C} = H^1(Y_j, \mathcal{O}_{Y_j}) = H^1(\Gamma, H^0(X_j, \mathcal{O}_{X_j})) = H^1(\Gamma, \mathbb{C}) \\ \operatorname{coker}(A^T - \mathsf{id}) = H^1(X, \Omega^1_X(\log D_T)) = H^1(\Gamma, H^0(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log \widetilde{D}_T))) = H^1(\Gamma, M \otimes \mathbb{C})\end{gathered}$$

and so P_1 is the map induced in group cohomology by $Q : M \otimes \mathbb{C} \to \mathbb{C}^{|P(A)|}$, $m \mapsto (\langle m, e_j\rangle)_{j\in P(A)}$. Clearly Q is surjective, and let $K = \ker Q$, so that we have an exact sequence

$$0 \longrightarrow K \longrightarrow M \otimes \mathbb{C} \xrightarrow{Q} \mathbb{C}^{|P(A)|} \longrightarrow 0.$$

This further induces an exact sequence

$$H^1(\Gamma, M \otimes \mathbb{C}) \xrightarrow{P_1} H^1(\Gamma, \mathbb{C}^{|P(A)|}) \longrightarrow H^2(\Gamma, K).$$

But $H^2(\Gamma, K) = 0$ since $\Gamma = \mathbb{Z}$, so indeed P_1 is a surjective map. This concludes the proof of the theorem. ∎

Remark 8.9: We note here that for toric Kato manifolds $X(\pi, \sigma_A(t\cdot))$ of parabolic type, the conclusions of Theorem 8.5, Theorem 8.7 and Theorem 8.8 also hold if $|t|$ is small enough. The arguments are exactly the same, except that one has to replace the families p_u by the Nakamura family p_N in Theorem 8.5 and Theorem 8.7. Since in the Nakamura family the smooth fibers need not be isomorphic, the semi-continuity argument then works only for parabolic Kato manifolds close to the singular fiber.

9. Isomorphism classes of toric Kato manifolds

The aim of this section is to describe isomorphism classes of toric Kato manifolds, according to the combinatorial data provided by the toric modification $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ (specifically the fan $\hat{\Sigma}$ of $\hat{\mathbb{C}}^n$), and the Kato germ $F = F_{\underline{\lambda},A}$.

9.1. **Collapsing models.** To this aim, we need first to introduce some more constructions related to toric Kato manifolds, inspired by the collapsing models of [Dlo84], and generalizing constructions introduced above.

Denote by Σ the fan of $\mathbb{C}^n$, by $\hat{\Sigma}$ the fan of $\hat{\mathbb{C}}^n$ and set $\hat{\Sigma}^0 = \hat{\Sigma}\backslash A\Sigma$, where $A\Sigma = \{A\tau \mid \tau \in \Sigma\}$. For any $\ell \in \mathbb{Z} \cup \{-\infty\}$ and $m \in \mathbb{Z} \cup \{+\infty\}$ with $\ell \leq m$ we set

$$\Sigma_A^{\ell\cdot m} := \bigcup_{\ell\leq k<m} A^k\hat{\Sigma}^0 \cup A^m\Sigma,$$

where we set $A^{\pm\infty}\tau = \emptyset$ for any cone τ.

Lemma 9.1: *The set $\Sigma_A^{\ell\cdot m}$ defines a regular fan, which is finite if and only if $\ell, m \in \mathbb{Z}$. When $\ell = -\infty$, the support is given by $|\Sigma_A^{-\infty\cdot m}| = H(A) \cup \tau_{P(A)}$ for any $m \in \mathbb{Z}$. In general, we have $|\Sigma_A^{\ell\cdot m}| \supseteq |\Sigma_A^{\ell\cdot m'}|$ for all $\ell \in \mathbb{Z} \cup \{-\infty\}$, $m \leq m' \in \mathbb{Z} \cup \{+\infty\}$, $\ell \leq m$, with strict inclusion if and only if $m \in \mathbb{Z}$ and $m' = +\infty$.*

Proof. The first part of the statement is trivial, the rest follows directly from (4.3). ∎

To simplify notations, we will omit ℓ and m when they are infinite. Notice that $\Sigma_A^{0\cdot d} = \hat{\Sigma}^d$ is the fan associated to π^d as defined in the proof of Lemma 3.3; $\Sigma_A^{\cdot}$ corresponds to the infinite fan Σ_A defined by (4.1), while $\Sigma_A^{0\cdot}$ corresponds to the infinite fan Σ_A^+ defined in the proof of Theorem 4.2.

To the fan $\Sigma_A^{\ell\cdot m}$ is associated the toric variety (possibly of non-finite type) $X(\Sigma_A^{\ell\cdot m}, N)$. When $m \in \mathbb{Z}$, we denote by 0_m the point associated to the cone $A^m\tau_0$, where $\tau_0 \in \Sigma$ is the cone associated to the origin of $\mathbb{C}^n$. Notice that when $\ell = 0$ and $m \in \mathbb{Z}_{>0}$, the point 0_m is exactly the point $\sigma^m(0)$, where σ^m is the second entry of the Kato data given by Lemma 3.3.

All these toric varieties are related one to the other via some natural morphisms.

Definition 9.2: Let $\ell \in \mathbb{Z} \cup \{-\infty\}$, and $m \leq m' \in \mathbb{Z} \cup \{+\infty\}$, with $\ell \leq m$. The toric morphism $p_\ell^{m\cdot m'} : X(\Sigma_A^{\ell\cdot m'}, N) \to X(\Sigma_A^{\ell\cdot m}, N)$, induced by the identity of N, is called the *collapsing map* from $X(\Sigma_A^{\ell\cdot m'}, N)$ to $X(\Sigma_A^{\ell\cdot m}, N)$.

The F-invariant open set $W_T(F)$ sits inside the torus T_N, which is an open dense subset of $X(\Sigma_A^{\ell\cdot m}, N)$ for any $\ell \leq m$. Similarly to Section 4, we set

$$\widetilde{X}_F^{\ell\cdot m} := \mathrm{Int}(\overline{W_T(F)}^{X(\Sigma_A^{\ell\cdot m}, N)}) \subset X(\Sigma_A^{\ell\cdot m}, N).$$

The variety $\widetilde{X}_F^{\ell\cdot m}$ is called the ($\ell\cdot m$-)*collapsed model* associated to the Kato data (π, σ). When $\ell = -\infty$ and $m \in \mathbb{Z}$, this corresponds to the collapsed model defined in [Dlo84, Chapter I.3]. We note that since for any $\ell \in \mathbb{Z}$, $\Sigma_A^{\ell\cdot} \subset \Sigma_A$, $\widetilde{X}_F^{\ell\cdot}$ naturally sits inside $\widetilde{X}_F^{\cdot}$ as an open subset.

Notice that $p_\ell^{m\cdot m'}$ leaves $W_T(F)$ invariant, and hence defines a regular map $p_\ell^{m\cdot m'} : \widetilde{X}_F^{\ell\cdot m'} \to \widetilde{X}_F^{\ell\cdot m}$, which is a proper modification of 0_m.

Notice also that for any $m \in \mathbb{Z}_{\geq 1}$, the collapsing maps $p_0^{0\cdot m} : \widetilde{X}_F^{0\cdot m} \to \widetilde{X}_F^{0\cdot 0} = W^s(F)$ correspond to the toric modifications π^m given by Lemma 3.3.

Remark 9.3: The natural diagrams involving collapsing maps commute, and in particular, for any $\ell \leq m \leq m' \leq m''$, one has $p_\ell^{m\cdot m''} = p_\ell^{m\cdot m'} \circ p_\ell^{m'\cdot m''}$. This allows to construct projective limits of the families $\left(X(\Sigma_A^{\ell\cdot m}, N)\right)_{m\in\mathbb{Z}_{\geq\ell}}$ and $\left(\widetilde{X}_F^{\ell\cdot m}\right)_{m\in\mathbb{Z}_{\geq\ell}}$. The spaces $X(\Sigma_A^{\ell\cdot}, N)$ and $\widetilde{X}_F^{\ell\cdot}$ sit naturally inside such projective limits as open dense subsets. Their complement can be interpreted in terms of valuations (see, e.g., [Tei18]), and the whole projective limits in terms of hybrid spaces (see, e.g., [Ber09, BJ17, Fav20]).

9.2. **Isomorphism classes.** We will say that two toric Kato manifolds X and X' are *equivariantly isomorphic* if there exists a biholomorphism $\Phi : X \to X'$ and a group isomorphism $\nu \in \mathrm{Aut}_{gr.}(T_N)$ so that the lift $\tilde{\Phi} : \tilde{X} \to \tilde{X}'$ satisfies $\tilde{\Phi}(\underline{\lambda}x) = \nu(\underline{\lambda})\tilde{\Phi}(x)$ for any $x \in X$ and $\underline{\lambda} \in T_N$ for which this is defined.

Also, recall that a holomorphic germ Φ satisfying $\Phi \circ F = G \circ \Phi$ is called a *semi-conjugacy* between F and G. When Φ is moreover an invertible germ, we call Φ a *conjugacy*.

We start by describing the compact hypersurfaces in the universal covering of a toric Kato variety.

Lemma 9.4: *Let X be a toric Kato manifold, and $\widetilde{X}$ its universal covering. Then the compact hypersurfaces of $\widetilde{X}$ are exactly the toric hypersurfaces whose sum gives D_+: they correspond to the rays of the fan Σ_A associated to X not contained in $\tau_{P(A)}$.*

Proof. Let H be any compact hypersurface of $\widetilde{X}$. Being compact, it must be contained in the open set $(p_\ell^{m\bullet})^{-1}\left(\widetilde{X}_F^{\ell\cdot m} \setminus \{0_m\}\right) \subset \widetilde{X}_F^{\cdot}$, for some $\ell \leq m \in \mathbb{Z}$. Notice that $p_\ell^{\ell\cdot m} : \widetilde{X}_F^{\ell\cdot m} \to \widetilde{X}_F^{\ell\cdot \ell} \subseteq X(\Sigma_A^{\ell\cdot\ell}, N) \cong \mathbb{C}^n$ is a toric modification above the origin of $\mathbb{C}^n$, and we infer that H must correspond to one of the exceptional primes of the toric modification. This means that H corresponds to one of the rays of $\Sigma_A^{\ell\cdot m}$, and so to one of the rays of Σ_A. Finally, it is clear that a ray ν of Σ_A gives rise to a compact hypersurface in $\tilde{X}$ if and only if ν is not contained in $\tau_{P(A)}$. ■

Theorem 9.5: *Let X and X' be two isomorphic toric Kato manifolds, with associated Kato germs $F = F_{\underline{\lambda},A}$ and $G = F_{\underline{\mu},B}$. Set $H_\infty = \bigcup_{m>0} F^{-m}(0)$ and $H'_\infty = \bigcup_{m>0} G^{-m}(0)$. Then there exists a holomorphic germ $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ which is an isomorphism on $\mathbb{C}^n \setminus H_\infty$, satisfying $\Phi^{-1}(\mathbb{C}^n \setminus H'_\infty) = \mathbb{C}^n \setminus H_\infty$ and $\Phi \circ F = G \circ \Phi$.*

Proof. Let $\Psi : X \to X'$ be an isomorphism, and let $\widetilde{\Psi} : \widetilde{X} \to \widetilde{X}'$ be a lift of Ψ to the universal coverings.

To distinguish collapsing maps for X and X', we denote collapsing maps for X with the letter p, and collapsing maps for X' with the letter q. To simplify notations, we also set $p_m := p_m^{m\bullet}$ and $q_m := q_m^{m\bullet}$. Finally, we denote by D_+ (resp., D'_+) the union of the compact hypersurfaces of $\widetilde{X}$ (resp., $\widetilde{X}'$), and set $\widetilde{D}'^{+}_{+} = q_m^{-1}(0_m) \subset D'_+$.

Let us take $m \ll 0$ small enough so that the images through $\widetilde{\Psi}$ of the compact hypersurfaces in $p_0^{-1}(0)$ belong to $(q_m)^{-1}(0_m)$. This can be done since all such (infinitely many) compact hypersurfaces project to finitely many hypersurfaces of X.

For any $\ell \in \mathbb{Z}$, let $\widetilde{B}^{-\ell} : \widetilde{X}_G^{\ell\cdot\ell} \to \mathbb{C}^n$ be the isomorphism induced by $B^{-\ell}$, and let $\tilde{q}_\ell := \widetilde{B}^{-\ell} \circ q_\ell : \widetilde{X}_G^{\ell\cdot} \to \mathbb{C}^n$. Note that for $\ell < \ell'$, $\tilde{q}_\ell|_{\widetilde{X}_G^{\ell'\cdot}} = \tilde{q}_{\ell'}$. Define

$$\Phi := \tilde{q}_m \circ \widetilde{\Psi} \circ p_0^{-1} : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n.$$

By Hartogs' theorem, the map Φ extends to a holomorphic map on $\mathbb{C}^n$. Moreover, it is a biholomorphism outside $p_0(\widetilde{\Psi}^{-1}(\widetilde{D}'^{+}_{+})) \subseteq p_0(D_+)$. Thanks to the description of compact hypersurfaces given by Lemma 9.4, we have $\widetilde{\Psi}(D_+) = D'_+$, from which we infer that $\Phi(\mathbb{C}^n \setminus H_\infty) = \mathbb{C}^n \setminus H'_\infty$ and that Φ is an isomorphism outside H_∞.

We are left with verifying that $\Phi \circ F = G \circ \Phi$. Let γ_F be the deck group generator of $\tilde{X}$, which is induced by F, and let γ_G be the deck group generator of $\tilde{X}'$. Being $\widetilde{\Psi}$ the lift of Ψ, we have $\widetilde{\Psi} \circ \gamma_F = \gamma_G \circ \widetilde{\Psi}$.

Note that γ_F induce isomorphisms $\widetilde{X}_F^{\ell\cdot} \to \widetilde{X}_F^{\ell+1\cdot}$ for any $\ell \in \mathbb{Z}$. Moreover, it is straightforward to check that $F \circ p_0 = p_0|_{\widetilde{X}_F^{1\cdot}} \circ \gamma_F$, and similarly $G \circ \tilde{q}_m = \tilde{q}_m|_{\widetilde{X}_G^{m+1\cdot}} \circ \gamma_G$. Then we

have

$$\begin{aligned}\Phi \circ F &= \tilde{q}_m \circ \widetilde{\Psi} \circ p_0^{-1} \circ F = \tilde{q}_m \circ \widetilde{\Psi} \circ \gamma_F \circ p_0^{-1} \\ &= \tilde{q}_m \circ \gamma_G \circ \widetilde{\Psi} \circ p_0^{-1} = G \circ \tilde{q}_m \circ \widetilde{\Psi} \circ p_0^{-1} = G \circ \Phi\end{aligned}$$

which concludes the proof. ■

In general, the semi-conjugacy given by Theorem 9.5 is not necessarily equivariant (see Example 9.7). The next proposition shows that this is the case, as soon as the union of the coordinate hyperplanes is Φ-invariant. Notice that this condition is automatically satisfied when $P(A) = \emptyset$.

Proposition 9.6: *Let $F = F_{\underline{\lambda},A}$ and $G = F_{\underline{\mu},B}$ be two Kato germs. Let $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a semi-conjugacy between F and G which is a local isomorphism on T_N and so that $\Phi^{-1}(T_N) = T_N$. Then Φ is ν-equivariant for some $\nu \in \mathrm{Aut}_{gr.}(T_N)$.*

Proof. Being $K := \{z_1 \cdots z_n = 0\}$ totally invariant for the action of Φ, we infer that Φ takes the form

$$\Phi(z) = \underline{\chi} z^Q (\mathbb{1} + u(z)),$$

where $\mathbb{1} = (1, \dots, 1)$, $u : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, and Q is a suitable matrix in $\mathrm{GL}(n, \mathbb{Z})$, with non-negative entries since Φ is regular.

From the conjugacy relation we get

$$\underline{\chi}\underline{\lambda}^Q z^{QA} \Big(\mathbb{1} + u\big(\underline{\lambda} z^A\big) \Big) = \underline{\mu\chi}^B z^{BQ} \big(\mathbb{1} + u(z)\big)^B. \tag{9.1}$$

By checking the lowest degrees of (9.1), we get $QA = BQ$ and $\underline{\chi}\underline{\lambda}^Q = \underline{\mu\chi}^B$. After simplifying the common factor in (9.1), we get

$$\mathbb{1} + u\big(\underline{\lambda} z^A\big) = \big(\mathbb{1} + u(z)\big)^B. \tag{9.2}$$

Let $u_i z^i$, with $u_i \in \mathbb{C}^n$ and $i \in \mathbb{N}^n \setminus \{0\}$, be any ($n$-uple of) monomial(s) appearing in the formal power series expansion of u.

This monomial contributes with $\underline{\lambda}^i u_i z^{iA}$ in the left-hand-side, and with $B u_i z^i + \text{h.o.t.}$ in the right-hand-side of (9.2), where h.o.t. denotes a suitable formal power series with monomials z^j with $|j| > |i|$.

Since $\sum_k a_{hk} \geq 1$ for all $h = 1, \dots, n$, we have $|iA| \geq |i|$ for any $i \in \mathbb{N}^n$.

Assume that this inequality is strict for any $i \in \mathbb{N}^n \setminus \{0\}$. This implies that $u \equiv 0$. In fact, if this is not the case, there exists $i \in \mathbb{N}^n \setminus \{0\}$ so that $u_i \neq 0$. We may assume that $|i|$ is minimal among the multi-indices satisfying the above condition. In this case we get a contradiction, since the left-hand-side of (9.2) contains a non-trivial monomial or order $|i|$, while the right-hand side does not.

Assume now that $|iA| = |i|$ for some $i \in \mathbb{N}^n \setminus \{0\}$. Notice that $\sum_k a_{jk} = 1$ for some $j \in \{1, \dots, n\}$ if and only if $P(A)^c = \{j\}$. We infer that in this case $P(A)^c = \{j\}$ and $i = m e_j$ for some $m > 0$. Then the coefficient of the monomial z^i in (9.2) gives $\lambda_j u_i = B u_i$, where

$\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Since $|\lambda_j| < 1$ and B has only eigenvalues of modulus 1, we deduce that $u_i = 0$, and hence $u \equiv 0$. ∎

Example 9.7: Consider a Kato 4-fold $X = X(\pi, \sigma)$ whose associated Kato germ is $F = F_{\underline{\lambda},A}$, with

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

For any $\alpha \in \mathbb{C}$, let H_α be the hyperplane of equation $z_1 - \alpha z_4 = 0$. Then H_α is F-invariant if and only if $\lambda_1 = \lambda_4$. In this case, for any $\alpha \in \mathbb{C}$, the map $\Phi(z) = (z_1 - \alpha z_4, z_2, z_3, z_4)$ commutes with F.

Suppose moreover that Φ lifts to an automorphism $\hat{\Phi} : \hat{\mathbb{C}}^n \to \hat{\mathbb{C}}^n$ on the total space $\hat{\mathbb{C}}^n$ of the toric modification π. Then Φ induces a (non-equivariant) automorphism of X.

This example also shows that the vector space $H^0(X, \Theta_X)$ is generally larger than the vector space $H^0(X, \Theta_X(-\log D_T)) \cong \ker(A - \mathsf{id})$ computed in Theorem 8.7.

Since the semi-conjugacy Φ induced by an equivariant isomorphism Ψ as in Theorem 9.5 automatically satisfies the hypothesis of Proposition 9.6, we deduce the following classification of toric Kato manifolds, associated to the same toric modification, up to equivariant isomorphisms.

Corollary 9.8: *Let $X = X(\pi, \sigma)$ be a toric Kato manifold with Kato germ $F = F_{\underline{\lambda},A}$. Let $X' = X(\pi, \sigma')$ be another toric Kato manifold associated to the same toric modification π, and denote by $G = F_{\underline{\mu},B}$ its associated Kato germ. Set $\underline{\lambda} = e^{2\pi \mathrm{i} \underline{\ell}}$ and $\underline{\mu} = e^{2\pi \mathrm{i} \underline{m}}$, where $\underline{\ell}$ and $\underline{m}$ belong to $\mathbb{C}^n$.*

Then X' is equivariantly isomorphic to X if and only if there exists $Q \in \mathrm{GL}(n, \mathbb{Z})$ such that $QA = BQ$, $Q(\Sigma_A) = \Sigma_B$, and so that

$$Q\underline{\ell} - \underline{m} \in \mathrm{Im}(B - \mathsf{id}) + \mathbb{Z}^n. \tag{9.3}$$

Proof. By Proposition 9.6, if X and X' are equivariantly isomorphic, then there exists a map $\Phi(z) = \underline{\chi} z^Q$ conjugating F and G and so that $Q\Sigma_A = \Sigma_B$. Conversely, any such map Φ induces a natural equivariant biholomorphism $\tilde{\Phi} : X \to \tilde{X}'$ commuting with the deck group action, and so an equivariant biholomorphism between X and X'.

By $\Phi \circ F = G \circ \Phi$, we infer $QA = BQ$, and $\underline{\chi}\underline{\lambda}^Q = \underline{\mu}\underline{\chi}^B$. By applying $\frac{1}{2\pi \mathrm{i}} \log$ to both sides, we get

$$Q\underline{\ell} + \underline{v} = \underline{m} + B\underline{v} \ (\mathrm{mod}\ \mathbb{Z}^n),$$

where $\underline{v}$ is such that $\underline{\chi} = e^{2\pi \mathrm{i} \underline{v}}$. The statement follows. ∎

Notice that, as in Theorem 9.5, we may assume that the matrix Q in the statement of Corollary 9.8 has non-negative entries. As an immediate corollary, we get:

Corollary 9.9: *Let X and X' be two toric Kato manifolds as in Corollary 9.8, both with the same associated Kato matrix A. If $1 \notin \mathrm{Spec}(A)$, then X and X' are equivariantly isomorphic.*

Proof. It suffices to apply Corollary 9.8 with $A = B$ and $Q = \mathsf{id}$. ∎

The non-triviality of the moduli space of toric Kato varieties with prescribed toric modification is related to the existence of (possibly non-equivariant) invariant families of subvarieties

(or more generally foliations) for the associated Kato germs, which in turn corresponds to the existence of families of compact subvarieties (or foliations) on the toric Kato variety.

Example 9.10: (**Parabolic Inoue surfaces**) For $n = 2$, the only toric Kato matrices A with $\ker(\mathsf{id} - A) \neq 0$ are $A_a := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $A'_a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, which is $\mathrm{GL}_2(\mathbb{Z})$-conjugated to A_a, where $a \in \mathbb{N}^*$. Let $\pi_0 : \mathrm{Bl}_0 \mathbb{C}^2 \to \mathbb{C}^2$ be the blow-up of the origin, and $\sigma_0 : \mathbb{C}^2 \to \mathrm{Bl}_0 \mathbb{C}^2$ be given by the chart $\tau := \langle e_1, e_1 + e_2 \rangle$. Let now (π, σ) be the composition of (π_0, σ_0) with itself a times, as given by Lemma 3.3.

Being e_1 invariant, the locus $\{x = 0\}$ is invariant by $F = F_{\underline{\lambda}, A_a}$, and induces an elliptic curve $\mathbb{C}^*/\lambda_2$ on the parabolic Inoue surface $X_{\underline{\lambda}, A_a} = X(\pi, \sigma)$. In particular if $X_{\underline{\lambda}, A_a}$ and $X_{\underline{\mu}, A_a}$ are isomorphic, then we must have $\lambda_2 = \mu_2$.

A direct computation shows that the matrices Q satisfying the conditions of Corollary 9.8 for $A = B = A_a$ are of the form $Q = A_k$ for some $k \in \mathbb{N}$. Then condition (9.3) is satisfied if and only if there exists $t \in \mathbb{R}$ so that

$$\begin{cases} \ell_1 + k\ell_2 - m_1 - km_2 - t \in \mathbb{Z}, \\ \ell_2 - m_2 \in \mathbb{Z}, \end{cases}$$

which holds if and only if $\ell_2 - m_2 \in \mathbb{Z}$, i.e., if and only if $\lambda_2 = \mu_2$.

Example 9.11: Be π be a toric modification whose associated fan contains the cone generated by $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ Let $X = X(\pi, \sigma)$ be a toric Kato data whose associated Kato germ takes the form $F = F_{\underline{\lambda}, A}$. In this example, $1 \in \mathrm{Spec}(A)$, and $\mathrm{Im}(A - \mathsf{id})$ is generated by $e_1 + e_2 + e_3$ and e_1, hence it coincides with $\{v \in \mathbb{R}^3 \mid v_2 - v_3 = 0\}$. In fact, an eigenvector for 1 is given by $(0, 1, -1)$, and can be also interpreted by the fact that $z_2/z_3 = \mathrm{const}$ defines an invariant family of surfaces for F. These families induce a regular foliation on X, whose geometrical properties depend on the value of $\lambda_2/\lambda_3 =: \lambda$.

Let now $X' = X(\pi, \sigma')$ be another toric Kato data with associated germ $G = F_{\underline{\mu}, A}$. The condition (9.3) to have X and X' isomorphic, when $Q = \mathsf{id}$, gives

$$\ell_2 - \ell_3 - (m_2 - m_3) \in \mathbb{Z}.$$

Hence X and X' are isomorphic if $\lambda = \mu := \mu_2/\mu_3$.

Notice that the matrix A commutes with the permutation matrix $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. If the fan associated to π is invariant by the action of Q (see for example Figure 3), then $\Phi = \underline{\chi} z^Q$ induces an isomorphism between X and X', as long as

$$\ell_3 - \ell_2 - (m_2 - m_3) \in \mathbb{Z}.$$

Hence in this case X and X' are isomorphic also whenever $\lambda\mu = 1$.

9.3. **Deformations of Kato manifolds of hyperbolic type.**

Theorem 9.12: *Let $X = X(\pi, \sigma)$ be a toric Kato manifold of non-parabolic type, with toric divisor D_T. There exists a smooth versal family of deformations of the pair $p : (\mathcal{X}, \mathcal{D}) \to B$, where B is a centered ball in $H^1(X, \Theta_X(-\log D_T)) \cong \ker(A - \mathsf{id}) \subset N \otimes \mathbb{C}$. Moreover, for*

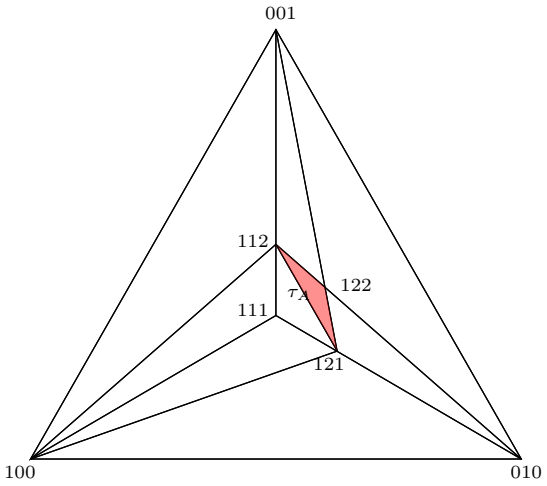

FIGURE 3. Fan associated to the toric modification π, invariant by the permutation Q of Example 9.11.

each $u \in B$, $(X_u, \mathcal{D} \cap X_u)$ is the toric Kato manifold $X(\pi, (\exp u).\sigma)$ together with its toric divisor. In particular, the family p is versal for each of its fibers.

Proof. The existence of a smooth versal family of deformations of the pair follows simply form the fact that $H^2(X, \Theta_X(-\log D_T)) = 0$ and [KNS58], but we can in fact construct such a family explicitly. Let A be the matrix of X. Note first that since

$$H^1(X, \Theta_X(-\log D_T)) = \operatorname{coker}(A - \mathsf{id}) = H^1(\Gamma, H^0(\widetilde{X}, \Theta_{\widetilde{X}}(-\log \widetilde{D}_T)))$$

the family $(\mathcal{X}, \mathcal{D})$ must come from a trivial family

$$\widetilde{p} : (\widetilde{\mathcal{X}} = \widetilde{X} \times B, \widetilde{\mathcal{D}} = \widetilde{D}_T \times B) \to B$$

and a deformation of the action of Γ. Indeed, let γ denote the positive generator of the deck group of $\widetilde{X}$, and consider an action of $\Gamma := \langle \widetilde{\gamma} \rangle$ on $\widetilde{\mathcal{X}}$ by

$$\widetilde{\gamma}(z, u) = (\gamma(\exp v.z), u), \quad (z, u) \in \widetilde{X} \times \ker(A - \mathsf{id})$$

where $\exp : N \otimes \mathbb{C} \to T_N = N \otimes \mathbb{C}^*$ is induced by $\mathbb{C} \ni t \mapsto \mathrm{e}^{2\pi i t} \in \mathbb{C}^*$. Since γ commutes with $\exp v$ for any $v \in \ker(A - \mathsf{id})$, $\widetilde{X} \subset X(\Sigma_A, N)$ is invariant under the action of $\exp(\ker(A - \mathsf{id}))$, so $\widetilde{\gamma}$ is well-defined. Moreover, we take the ball $B \subset \ker(A - \mathsf{id})$ small enough so that $\sigma(\exp u.\overline{\mathbb{B}}) \subset \mathbb{B}$ for each $u \in B$. It follows that for each $u \in B$,

$$\widetilde{\gamma}|_{\widetilde{X} \times \{u\}} = \gamma(\exp u \cdot) = \exp u\gamma \in \operatorname{Aut}(\widetilde{X}).$$

Therefore Γ acts freely and properly on $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})$ and $p : (\mathcal{X} := \widetilde{\mathcal{X}}/\Gamma, \mathcal{D} := \widetilde{\mathcal{D}}/\Gamma) \to B$ gives the desired smooth versal family. ∎

10. Hermitian geometry of Kato manifolds

Kato manifolds cannot admit Kähler metrics since their fundamental group is isomorphic to $\mathbb{Z}$. However, a large class of them carry *locally conformally Kähler metrics (lcK)*, as shown in [IOP20, Theorem 2.2]. In this section, we give a characterization for the existence of

lcK metrics on Kato manifolds. Furthermore, we investigate the possibility of endowing a Kato manifold with other special Hermitian metrics, such as balanced, pluriclosed, Hermitian symplectic or strongly Gauduchon.

Recall the following definition:

Definition 10.1: A *locally conformally Kähler metric (lcK)* on a complex manifold X is a Hermitian metric Ω satisfying $d\Omega = \theta \wedge \Omega$ for a closed non-exact one-form θ.

This definition is equivalent to any of the following formulations:

(1) There exists a covering with open sets $(U_i)_i$ of X and smooth non-constant functions f_i on every U_i such that $e^{-f_i}\Omega$ is Kähler.
(2) The universal cover $\widetilde{X}$ admits a Kähler metric $\widetilde{\Omega}$ such that $\gamma^*\widetilde{\Omega} = c_\gamma \widetilde{\Omega}$, for any deck-transformation γ, where $c_\gamma > 0$, not all equal to 1.

For our characterization purpose, we start by giving a general principle used to modify Kähler metrics on modifications of balls, already used in [Bru11] and [IOP20], which will be needed.

Lemma 10.2: *Let $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ be a proper modification at 0, let $\rho(z) = \rho(||z||) \in \mathcal{C}^\infty(\mathbb{C}^n, \mathbb{R})$ be a strictly plurisubharmonic function depending only on $||z||$, and let ω be a Kähler metric on $\hat{\overline{\mathbb{B}}} := \pi^{-1}(\overline{\mathbb{B}})$. Then for any $0 < s < 1$, there exists $\lambda > 0$ and a Kähler metric $\widetilde{\omega}$ on $\hat{\mathbb{C}}^n$ so that*

$$\widetilde{\omega}|_{\hat{\mathbb{B}}_s} = \omega, \quad \widetilde{\omega}|_{\hat{\mathbb{C}}^n \setminus \hat{\mathbb{B}}} = \lambda \cdot \pi^* dd^c \rho.$$

Proof. There exists a pluriharmonic function ψ on $\mathbb{C}^n$, smooth and strictly plurisubharmonic on $\mathbb{C}^n \setminus \{0\}$, so that $\pi_*\omega = dd^c\psi$. By the maximum principle, the functions

$$u, v : (0,1] \to \mathbb{R}, \quad u(t) = \max_{\partial \mathbb{B}_t} \rho, \quad v(t) = \max_{\partial \mathbb{B}_t} \psi$$

are strictly increasing. It follows that for any $0 < s < 1$,

$$r := u(s) \equiv \rho|_{\partial \mathbb{B}_s} < R := u(1) \equiv \rho|_{\partial \mathbb{B}}, \quad m := \min_{\partial \mathbb{B}_s} \psi \leq v(s) < M := v(1).$$

Thus, for any $\alpha > 1$, we can take

$$\lambda := \alpha \frac{M - m}{R - r} > 0, \quad c \in (M - \lambda R, m - \lambda r) \neq \emptyset.$$

so that $\lambda\rho + c > \psi$ on $\partial\mathbb{B}$ and $\lambda\rho + c < \psi$ on $\partial\mathbb{B}_s$. Therefore, if $\widetilde{\psi}$ is the function on $\mathbb{C}^n$ defined as the regularized maximum between ψ and $\lambda\rho + c$ on $\mathbb{B} \setminus \mathbb{B}_s$, equal to ψ on $\mathbb{B}_s$ and equal to $\lambda\rho + c$ on $\mathbb{C}^n \setminus \mathbb{B}$, then the metric $\pi^*(dd^c\widetilde{\psi})$ on $\hat{\mathbb{C}}^n \setminus \pi^{-1}(0)$ glues to ω to define the desired metric $\widetilde{\omega}$ on $\hat{\mathbb{C}}^n$. ∎

Now we are ready to characterize the existence of lcK metrics. For this, let us note that any proper modification $\pi : \hat{\mathbb{B}} \to \mathbb{B}$ induces naturally proper modifications $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ and $\pi : \hat{\mathbb{CP}}^n \to \mathbb{CP}^n$, where $\hat{\mathbb{CP}}^n$ is the compactification of $\hat{\mathbb{C}}^n$ with a hyperplane at infinity.

Theorem 10.3: *Let $(\pi : \hat{\mathbb{B}} \to \mathbb{B}, \sigma : \overline{\mathbb{B}} \to \hat{\mathbb{B}})$ be a Kato data and let $X = X(\pi, \sigma)$ be the corresponding Kato manifold. The following are equivalent:*

(1) X admits an lcK metric;
(2) $\widetilde{X}$ admits a Kähler metric;

(3) $\hat{\mathbb{CP}}^n$ *is a projective manifold;*
(4) $\hat{\mathbb{C}}^n$ *admits a Kähler metric;*

Proof. The implications (1) ⇒ (2) and (3) ⇒ (4) are clear, while the implication (4) ⇒ (1) is precisely Brunella's theorem [Bru11] adapted to all complex dimensions in [IOP20, Theorem 2.2]. We are thus left with showing (2) ⇒ (3).

If $\widetilde{X}$ is Kähler, then Proposition 2.4 implies that $\hat{\mathbb{B}} \setminus \{\sigma(0)\}$ is also Kähler, so by Miyaoka's extension theorem [Miy74, Proposition A] $\hat{\mathbb{B}}$ admits a Kähler metric ω. Now putting $\rho := \log(1 + ||z||^2)$ in Lemma 10.2, we find that $\hat{\mathbb{C}}^n$ admits a Kähler metric $\widetilde{\omega}$ which glues to $\lambda \cdot \pi^* \omega_{FS}$ on $\mathbb{C}\hat{\mathbb{P}}^n \setminus \hat{\mathbb{B}}$ for some $\lambda > 0$, where ω_{FS} is the standard Fubini-Study metric on $\mathbb{CP}^n$. Thus we have shown that $\mathbb{C}\hat{\mathbb{P}}^n$ is Kähler. Since on the other hand $\mathbb{C}\hat{\mathbb{P}}^n$ is a modification of $\mathbb{CP}^n$, it is Moishezon, and therefore $\mathbb{C}\hat{\mathbb{P}}^n$ is projective by Moishezon's theorem [Moi67]. This concludes the proof. ∎

Remark 10.4: We note here that a non-Hopf toric Kato manifold X admitting an lcK metric can in no way be a toric lcK manifold, because in this case it would admit a Vaisman metric by [Ist19], which is impossible by [IOP20, Propoposition 2.6]. At the same time, the compact torus $\mathbb{T}$ acts effectively and holomorphically on the universal cover $\widetilde{X}$, and any lcK metric on X gives rise to a Kähler metric ω on $\widetilde{X}$, which can be made $\mathbb{T}$-invariant after eventual averaging. Since $\widetilde{X}$ is moreover simply connected, it follows that $(\widetilde{X}, \omega, \mathbb{T})$ is a toric Kähler manifold in the classical sense. In this manner, lcK toric Kato manifolds give interesting generalizations of the class of toric lcK manifolds.

Next, we recall the definitions of the other classes of Hermitian metrics whose existence on Kato manifolds we discuss.

Definition 10.5: Let Ω be a Hermitian metric on a complex manifold X. Ω is called:

- *balanced* if $d\Omega^{n-1} = 0$, or equivalently, if Ω is co-closed [Mic82].
- *pluriclosed*, or *strongly Kähler with torsion*, if $\partial\overline{\partial}\Omega = 0$ [Bis89].
- *Hermitian symplectic* if Ω is the $(1,1)$-component of a real d-closed two-form [ST10].
- *strongly Gauduchon* if $\partial\Omega^{n-1}$ is $\overline{\partial}$-exact [Pop13].

There are certain inclusion relations between the classes of manifolds which admit one of the above metrics. For instance, it is easy to see that any balanced metric is strongly Gauduchon. Similarly, it is easy to see that any Hermitian symplectic metric is pluriclosed. Finally, [YZZ19, Lemma 1] shows that any manifold admitting a Hermitian symplectic metric also admits a strongly Gauduchon one.

We have the following non-existence results:

Theorem 10.6: *A Kato manifold X admits no strongly Gauduchon metric, and in particular no balanced or Hermitian symplectic metric.*

Proof. Let $\mathcal{X} \xrightarrow{\pi} \mathbb{D}$ be the complex analytic deformation of X given by [Kat77, Theorem 1], so that $\pi^{-1}(0) = X$ and $X_t := \pi^{-1}(t)$ is a proper modification at a finite number of points of a Hopf manifold for $t \neq 0$. Assume that X admits a strongly Gauduchon metric. Then by [Pop14, Theorem 3.1], stating that the strongly Gauduchon property is open with respect to holomorphic deformations, X_t also admits a strongly Gauduchon metric if t is close to 0. However, according to [Pop10, Theorem 2.2], if $\mu : \hat{Y} \to Y$ is a proper modification of

complex manifolds and $\hat{Y}$ carries a strongly Gauduchon metric, then so does Y. Consequently, this would imply that Hopf manifolds admit strongly Gauduchon metrics, which is false by [Pop14, Section 2]. Therefore, X cannot be endowed with strongly Gauduchon metrics, nor with balanced or Hermitian symplectic metrics [YZZ19, Lemma 1]. ■

Remark 10.7: The above result is related to the open question [ST10, Question 1.7], asking whether there exist any non-Kähler Hermitian symplectic manifolds in dimension ≥ 3. We answer it negatively for the class of Kato manifolds.

Theorem 10.8: *If X is a Kato manifold of dimension $n \geq 3$ satisfying $H^{1,2}_{\overline{\partial}}(X) = 0$, then it cannot be endowed with a pluriclosed metric. In particular, if X is a toric Kato manifold of hyperbolic type, then it does not admit a pluriclosed metric unless it is a surface.*

Proof. Suppose that X carries a pluriclosed metric. Let $\mathcal{X} \xrightarrow{\pi} \mathbb{D} \subset \mathbb{C}$ be the same deformation as above. Then by [Cav12, Theorem 8.5], for any $t \in \mathbb{D}$ close to 0, $\pi^{-1}(t)$ also admits a pluriclosed metric. We recall that for each $t \neq 0$, $\pi^{-1}(t)$ is a modification at a finite number of points of a Hopf manifold M_t. According to the analogue of Miyaoka's theorem for pluriclosed metrics [FT09, Theorem 4.3], a complex manifold M without a point admits a pluriclosed metric if and only if M admits a pluriclosed metric. Therefore, the Hopf manifolds M_t also admit pluriclosed metrics. However, by [Cav12, Example 5.17, Theorem 5.16], this can only happen if $n = 2$. Finally, in the toric case the conclusion follows by Theorem 8.8. ■

Examples of non-lcK toric Kato manifolds. We have shown in [IOP20] that a toric Kato manifold of dimension $n \geq 4$ does not need to be lcK, however the examples we constructed there are not toric. We end this section by constructing examples of toric Kato manifolds in any dimension $n \geq 4$ which admit no lcK metric.

In order to do so, we will start from the simplest example of a complete smooth toric algebraic variety which admits no Kähler metric, cf. [Oda88]

Consider the toric manifold $\mathbb{CP}^3$ with the standard action of $T_N = (\mathbb{C}^*)^3$, $N = \mathbb{Z}^3$. Let e_1, e_2, e_3 denote the standard basis of N and let $e_0 = -e_1 - e_2 - e_3$. Then the fan of $\mathbb{CP}^3$ is generated by the 3-dimensional cones

$$\sigma_0 = \langle e_1, e_2, e_3\rangle, \quad \sigma_1 = \langle e_0, e_2, e_3\rangle, \quad \sigma_2 = \langle e_0, e_1, e_3\rangle, \quad \sigma_3 = \langle e_0, e_1, e_2\rangle.$$

Consider the T_N-fixed points $P_j = \overline{\mathrm{orb}\,\sigma_j}$, $j = 1, 2, 3$, and the one-dimensional orbits $d_{0j} = \overline{\mathrm{orb}\langle e_0, e_j\rangle}$, $j = 1, 2, 3$, so that we have $d_{01} \cap d_{02} = P_3$ and so on.

Let $\mu : \hat{\mathbb{CP}}^3 \to \mathbb{CP}^3$ be the toric modification of $\mathbb{CP}^3$ constructed as follows. Around the point P_1, blow-up d_{02} and then the strict transform of d_{03}, around P_2 blow-up d_{03} and then the strict transform of d_{01}, and around the point P_3 blow up d_{01} and then the strict transform of d_{02}. It is easy to check that this is a well defined operation globally, resulting in a new smooth toric variety $\hat{\mathbb{CP}}^3$. The corresponding fan Σ_3 has 3 new rays generated by $v_j = e_0 + e_j$, $j = 1, 2, 3$, and the following 3-dimensional cones:

$$\begin{array}{lll} \langle e_1, v_1, e_2\rangle, & \langle v_1, e_2, v_2\rangle, & \langle v_1, v_2, e_0\rangle \\ \langle e_2, e_3, v_2\rangle, & \langle v_2, e_3, v_3\rangle, & \langle v_2, v_3, e_0\rangle \\ \langle e_1, e_3, v_3\rangle, & \langle e_1, v_3, v_1\rangle, & \langle v_1, v_3, e_0\rangle \\ & \langle e_1, e_2, e_3\rangle. & \end{array}$$

Proposition 10.9: ([Oda78, Proposition 9.4], [Oda88, Example on page 84]) *The toric variety* $\hat{\mathbb{CP}}^3$ *is not projective and so admits no Kähler metric.*

We now show:

Proposition 10.10: *In any complex dimension* $n \geq 4$*, there exist toric Kato manifolds* X *with holomorphic immersions* $\hat{\mathbb{CP}}^3 \to X$*. In particular,* X *admits no lcK metric.*

Proof. We wish first to construct a toric modification $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ over 0 which contains $\hat{\mathbb{CP}}^3$ as a submanifold. Let $\pi'' : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ be the blow-up at 0 of $\mathbb{C}^n$, let $N = \mathbb{Z}^n$ of standard basis $f_1, \dots, f_n$, let $f_0 = \sum_{j=1}^n f_j$ and denote by $\widetilde{\Sigma}$ the fan of $\widetilde{\mathbb{C}}^n$ with the standard T_N-action. Its n-dimensional cones are:

$$\tau_j := \langle f_0, \dots, \widehat{f_j}, \dots, f_n \rangle, \quad j = 0, 1 \dots n.$$

Let $E = \overline{\mathrm{orb}\langle f_0 \rangle} \cong \mathbb{CP}^{n-1}$ be the exceptional divisor of $\widetilde{\mathbb{C}}^n$ and identify

$$\mathbb{CP}^3 = \overline{\mathrm{orb}\langle f_0, f_5 \dots, f_n \rangle}.$$

For $j = 1, 2, 3$ denote by $L_j = \overline{\mathrm{orb}\langle f_0, f_j, f_4, f_5 \dots, f_n \rangle}$ the T_N-invariant one-dimensional submanifolds. Again, put $P_j = \overline{\mathrm{orb}\, \tau_j}$, $j = 1, 2, 3$. Define the toric modification $\pi' : \hat{\mathbb{C}}^n \to \widetilde{\mathbb{C}}^n$ similarly as before. Namely, around P_1, blow up L_2 and then the strict transform of L_3, around P_2 blow-up L_3 and then the strict transform of L_1 and around the point P_3 blow up L_1 and then the strict transform of L_2. Equivalently, define $\hat{\mathbb{C}}^n = X(\hat{\Sigma}, N)$, where the fan $\hat{\Sigma}$ has the rays of $\widetilde{\Sigma}$ to which we add $\nu_j = f_0 + f_j + \sum_{k=4}^n f_k$, $j = 1, 2, 3$, and is generated by the n-dimensional cones:

$$\begin{aligned}
\tau_0, \quad \tau_j, \quad 4 \leq j \leq n \\
\tau_{1k} = \langle \nu_2, f_s \mid s \in \{0, \dots, n\} \setminus \{1, k\} \rangle, \quad k \in \{0, 4, \dots, n\} \\
\tau_{12l} = \langle \nu_2, \nu_3, f_s \mid s \in \{0, \dots, n\} \setminus \{1, 2, l\} \rangle, \quad l \in \{0, 3, 4, \dots, n\} \\
\tau_{2k} = \langle \nu_3, f_s \mid s \in \{0, \dots, n\} \setminus \{2, k\} \rangle, \quad k \in \{0, 4, \dots, n\} \\
\tau_{23l} = \langle \nu_3, \nu_1, f_s \mid s \in \{0, \dots, n\} \setminus \{2, 3, l\} \rangle, \quad l \in \{0, 1, 4, \dots, n\} \\
\tau_{3k} = \langle \nu_1, f_s \mid s \in \{0, \dots, n\} \setminus \{3, k\} \rangle, \quad k \in \{0, 4, \dots, n\} \\
\tau_{31l} = \langle \nu_1, \nu_2, f_s \mid s \in \{0, \dots, n\} \setminus \{1, 3, l\} \rangle, \quad l \in \{0, 2, 4, \dots, n\}.
\end{aligned}$$

It is clear that the strict transform of $\mathbb{CP}^3$ under π' is isomorphic to $\hat{\mathbb{CP}}^3$, so $\pi = \pi'' \circ \pi' : \hat{\mathbb{C}}^n \to \mathbb{C}^n$ has the desired properties. Now it suffices to take any toric chart σ with $\sigma(\mathbb{B}) \subset (\hat{\mathbb{B}})$ and $\sigma(0) \notin \hat{\mathbb{CP}}^3$, for instance one corresponding to the cone τ_{10}, in order to obtain a toric Kato manifold $X = X(\pi, \sigma)$. Clearly $\hat{\mathbb{CP}}^3 \subset \widetilde{X}$, therefore $\widetilde{X}$ is not Kähler and so X admits no lcK metric. ∎

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References

[BS76] C. Bănică, O. Stănăşilă, *Algebraic methods in the global theory of complex spaces*, Editura Academiei, Bucureşti and John Wiley & Sons, London, New York, Sydney (1976). (Cited on pages 28 and 29.)

[Ber09] V.G. Berkovich, *A non-Archimedean interpretation of the weight zero subspaces of limit mixed Hodge structures*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progress in Math. **202**, Birkhäuser Boston, Boston, MA (2009), 49–67. (Cited on page 32.)

[Bis89] J.-M. Bismut, *A local index theorem for non-Kähler manifolds*, Math. Ann. **284** no. 4 (1989), 681–699. (Cited on page 39.)

[BJ17] Boucksom, Sébastien, Jonsson, Mattias, *Tropical and non-Archimedean limits of degenerating families of volume forms*, J. Éc. polytech. Math. **4** (2017), 87–139. (Cited on page 32.)

[Bru11] M. Brunella, *Locally conformally Kähler metrics on Kato surfaces*, Nagoya Math. J. **202** (2011), 77–81. (Cited on pages 1, 38, and 39.)

[Cav12] G. Cavalcanti, *Hodge theory and deformations of SKT manifolds*, arXiv:1203.0493. (Cited on page 40.)

[Dan78] V. I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33**(2) (1978), 97–154. (Cited on pages 22 and 28.)

[Dlo84] G. Dloussky, *Structure des surfaces de Kato*, Mém. Soc. Math. France (N.S.) **14** (1984), I1–II120. (Cited on pages 1, 4, 19, 31, and 32.)

[DOT03] G. Dloussky, K. Oeljeklaus, M. Toma, *Class VII_0 surfaces with b_2 curves*, Tôhoku Math. J. (2) **55** (2003), 283–309. (Cited on page 1.)

[Fav20] C. Favre *Degeneration of endomorphisms of the complex projective space in the hybrid space*, J. Inst. Math. Jussieu **19** (2020), 1141–1183. (Cited on page 32.)

[FP10] A. Fujiki, M. Pontecorvo *Anti-self-dual bihermitian structures on Inoue surfaces*, J. Differ. Geom. **85** (2010), 15–71. (Cited on page 1.)

[FT09] A. Fino, A. Tomassini, *Blow-ups and resolutions of strong Kähler with torsion metrics*, Adv. Math. **221** (3) (2009), 914–935. (Cited on page 40.)

[Ful93] W. Fulton, *Introduction to toric varieties*, Princeton University Press, 1993. (Cited on pages 6, 9, 21, and 27.)

[Gro57] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. **9**(2) (1957), 119–221. (Cited on pages 29 and 31.)

[Gro65] A. Grothendieck, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Seconde partie*, Publ. Math. I.H.E.S. **24** (1965), 5–231. (Cited on page 25.)

[Ist19] N. Istrati, *A characterisation of toric locally conformally Kähler manifolds*, J. Symplectic Geom. **17** (5) (2019), 1297–1316. (Cited on page 39.)

[IOP20] N. Istrati, A. Otiman, M. Pontecorvo, *On a class of Kato manifolds*, Int. Math. Res. Notices, doi.org/10.1093/imrn/rnz354. (Cited on pages 1, 6, 8, 9, 37, 38, 39, and 40.)

[Kat77] M. Kato, *Compact complex manifolds containing "global" spherical shells*, I, Intl. Symp on Algebraic Geometry Kyoto (1977), 45–84. (Cited on pages 1, 4, 21, and 39.)

[KNS58] K. Kodaira, L. Nirenberg, D. C. Spencer. *On the Existence of Deformations of Complex Analytic Structures*, Annals of Math. **68** (1958), 450–59. (Cited on page 37.)

[Mic82] M.L. Michelson, *On the existence of special metrics in complex geometry*, Acta Math. **149** (1982), 261–295. (Cited on page 39.)

[Miy74] Y. Miyaoka, *Extension Theorems for Kähler metrics*, Proc. Japan Acad. **50** (1974), 407–410. (Cited on page 39.)

[Moi67] B. Moishezon, *On n-dimensional compact varieties with n algebraically independent meromorphic functions*, I. Amer. Math. Soc. Transl. Ser. 2, **63** (1967), 51–93. (Cited on page 39.)

[Nak83] I. Nakamura, *VII_0 surfaces and a duality of cusp singularities*, in *Classification of algebraic and analytic manifolds*, (Ed. K. Ueno), Progress in Math. **39** (1983), 333–378. (Cited on pages 3 and 22.)

[Nak84] I. Nakamura, *On surfaces of class VII_0 with curves*, Invent. Math. **78** (1984), 393–443. (Cited on pages 1, 2, 17, and 19.)

[Oda78] T. Oda, *Lectures on Torus Embeddings and Applications* (Based on joint work with K. Miyake), Tata. Inst. of Fund. Res., Bombay, No.58, Springer-Verlag, Berlin- Heiderberg-New York, 1978. (Cited on pages 2, 6, 12, 22, and 41.)

[Oda88] T. Oda, *Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties*, Springer-Verlag, 1988. (Cited on pages 6, 7, 40, and 41.)

[Pop10] D. Popovici, *Stability of Strongly Gauduchon Manifolds under Modifications*, J. Geom. Anal. **23** (2013), 653–659. (Cited on page 39.)
[Pop13] D. Popovici, *Deformation Limits of Projective Manifolds: Hodge Numbers and Strongly Gauduchon Metrics*, Invent. Math. **194** (2013), 515–534. (Cited on page 39.)
[Pop14] D. Popovici, *Deformation Openness and Closedness of Various Classes of Compact Complex*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), 255–305. (Cited on pages 39 and 40.)
[Sha92] B.V. Shabat, *Introduction to complex analysis Part II. Functions of several variables*, Translations of Mathematical Monographs Volume 110. Providence, Rhode Island 02904- 2294, USA: American Mathematical Society, 1992. (Cited on page 5.)
[ST10] J. Streets, G. Tian, *A Parabolic Flow of Pluriclosed Metrics*, Int. Math. Res. Notices **16** (2010), 3101–3133. (Cited on pages 39 and 40.)
[Tei18] B. Teissier, *Two points of the boundary of toric geometry*, Singularities, algebraic geometry, commutative algebra, and related topics, Springer, Cham (2018), 107–117. (Cited on page 32.)
[Tel05] A. Teleman, *Donaldson theory on non-Kählerian surfaces and class VII surfaces with $b_2 = 1$*, Invent. Math. **162** (2005), 493–521. (Cited on page 1.)
[Tsu87] H. Tsuchihashi, *Certain compact complex manifolds with infinite cyclic fundamental groups*, Tôhoku Math. J. **39** (1987), 519–32. (Cited on pages 2, 3, 12, 14, and 24.)
[YZZ19] S.-T. Yau, Q. Zhao, F. Zheng, *On Strominger Kähler-like Manifolds with Degenerate Torsion*, arXiv: 1908.05322v2. (Cited on pages 39 and 40.)

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