Complex reflection groups as Weyl groups

Michel Broué

Institut Henri–Poincaré

July 2007

FINITE COMPLEX REFLECTION GROUPS

FINITE COMPLEX REFLECTION GROUPS

Let K be a characteristic zero field.

FINITE COMPLEX REFLECTION GROUPS

Let K be a characteristic zero field.

A finite reflection group on K is a finite subgroup of $\mathrm{GL}_K(V)$ (V a finite dimensional K–vector space) generated by *pseudo–reflections*, *i.e.*, linear maps represented by

$$\begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

FINITE COMPLEX REFLECTION GROUPS

Let K be a characteristic zero field.

A finite reflection group on K is a finite subgroup of $\mathrm{GL}_K(V)$ (V a finite dimensional K–vector space) generated by *pseudo–reflections*, *i.e.*, linear maps represented by

$$\begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- A finite reflection group on $\mathbb{R}$ is called a Coxeter group.

FINITE COMPLEX REFLECTION GROUPS

Let K be a characteristic zero field.

A finite reflection group on K is a finite subgroup of $\mathrm{GL}_K(V)$ (V a finite dimensional K–vector space) generated by *pseudo–reflections*, *i.e.*, linear maps represented by

$$\begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- A finite reflection group on $\mathbb{R}$ is called a Coxeter group.
- A finite reflection group on $\mathbb{Q}$ is called a Weyl group.

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
    - There is one infinite series $G(de, e, r)$ (d ,e and r integers),

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
   - There is one infinite series $G(de, e, r)$ (d ,e and r integers),
   - ...and 34 exceptional groups G_4 , G_5 , $\ldots$, G_{37}.

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
   - There is one infinite series $G(de, e, r)$ (d ,e and r integers),
   - ...and 34 exceptional groups G_4 , G_5 , . . . , G_{37}.
2. The group $G(de, e, r)$ (d ,e and r integers) consists of all $r \times r$ monomial matrices with entries in μ_{de} such that the product of entries belongs to μ_d.

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
   - There is one infinite series $G(de, e, r)$ (d ,e and r integers),
   - ...and 34 exceptional groups G_4 , G_5 , $\ldots$, G_{37}.
2. The group $G(de, e, r)$ (d ,e and r integers) consists of all $r \times r$ monomial matrices with entries in μ_{de} such that the product of entries belongs to μ_d.
3. We have

$$
\begin{aligned}
&G(d, 1, r) \simeq C_d \wr \mathfrak{S}_r \\
&G(e, e, 2) = D_{2e} \quad \text{(dihedral group of order } 2e\text{)} \\
&G(2, 2, r) = W(\mathrm{D}_r)\,.
\end{aligned}
$$

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

$\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

G is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

Example

$$\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)\ ,\ F : (a_{i,j}) \mapsto (a_{i,j}^q)\ ,\ G = \mathrm{GL}_n(q)$$

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

$\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

Example

$$\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q) \;,\; F : (a_{i,j}) \mapsto (a_{i,j}^q) \;,\; G = \mathrm{GL}_n(q)$$

- Type of $\mathbf{G}$ — The type $\mathbb{G} = (X, Y, R, R^\vee \,;\, W\phi)$ of G consists of the root datum of $\mathbf{G}$ endowed with the outer automorphism $W\phi$ defined by F.

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

$\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

Example

$$\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)\ ,\ F\ :\ (a_{i,j}) \mapsto (a_{i,j}^q)\ ,\ G = \mathrm{GL}_n(q)$$

- **Type of G** — The type $\mathbb{G} = (X, Y, R, R^\vee\ ;\ W\phi)$ of G consists of the root datum of $\mathbf{G}$ endowed with the outer automorphism $W\phi$ defined by F.

Example

$$\mathrm{GL}_n = (X = Y = \mathbb{Z}^n, R = R^\vee = A_n\ ;\ \phi = 1)$$

- Polynomial order — There is a polynomial in $\mathbb{Z}[x]$

$$|\mathbb{G}|(x) = x^N \prod_d \Phi_d(x)^{a(d)}$$

such that $|\mathbb{G}|(q) = |G|$.

- Polynomial order — There is a polynomial in $\mathbb{Z}[x]$

$$|\mathbb{G}|(x) = x^N \prod_d \Phi_d(x)^{a(d)}$$

such that $|\mathbb{G}|(q) = |G|$.

Example

$$|\mathrm{GL}_n|(x) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} (x^d - 1) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} \Phi_d(x)^{[n/d]}$$

- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\bar{\mathbb{F}}^\times \times \cdots \times \bar{\mathbb{F}}^\times$ in $\mathbf{G}$).

  The Levi subgroups of G are the subgroups of the shape $\mathbf{L}^F$ where $\mathbf{L}$ is a centralizer of an F–stable torus in $\mathbf{G}$.

- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\bar{\mathbb{F}}^\times \times \cdots \times \bar{\mathbb{F}}^\times$ in $\mathbf{G}$).

  The Levi subgroups of G are the subgroups of the shape $\mathbf{L}^F$ where $\mathbf{L}$ is a centralizer of an F–stable torus in $\mathbf{G}$.

  Example

  The split maximal torus $T_1 = (\mathbb{F}_q^\times)^n$ of order $(q-1)^n$

  The Coxeter maximal torus $T_c = \mathrm{GL}_1(\mathbb{F}_{q^n})$ of order $q^n - 1$

  Levi subgroups have shape $\mathrm{GL}_{n_1}(q^{a_1}) \times \cdots \times \mathrm{GL}_{n_s}(q^{a_s})$

- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\bar{\mathbb{F}}^\times \times \cdots \times \bar{\mathbb{F}}^\times$ in $\mathbf{G}$).

  The Levi subgroups of G are the subgroups of the shape $\mathbf{L}^F$ where $\mathbf{L}$ is a centralizer of an F–stable torus in $\mathbf{G}$.

  Example

  The split maximal torus $T_1 = (\mathbb{F}_q^\times)^n$ of order $(q-1)^n$

  The Coxeter maximal torus $T_c = \mathrm{GL}_1(\mathbb{F}_{q^n})$ of order $q^n - 1$

  Levi subgroups have shape $\mathrm{GL}_{n_1}(q^{a_1}) \times \cdots \times \mathrm{GL}_{n_s}(q^{a_s})$

- Cauchy theorem — The (polynomial) order of an admissible subgroup divides the (polynomial) order of the group.

Levi subgroups and type — For

$$\mathbb{G} = (X, Y, R, R^\vee\, ;\, W\phi)$$

a type, a Levi subtype of $\mathbb{G}$ is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\, ;\, W'w\phi)$$

where R' is a parabolic system of R, with Weyl group W', and where $w \in W$ is such that $w\phi$ stablizes R' and $R'^\vee$.

Levi subgroups and type — For

$$\mathbb{G} = (X, Y, R, R^\vee\,;\, W\phi)$$

a type, a Levi subtype of $\mathbb{G}$ is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\,;\, W'w\phi)$$

where R' is a parabolic system of R, with Weyl group W', and where $w \in W$ is such that $w\phi$ stablizes R' and $R'^\vee$.

There is a natural bijection between

Levi subgroups and type — For

$$\mathbb{G} = (X, Y, R, R^\vee\,;\, W\phi)$$

a type, a Levi subtype of $\mathbb{G}$ is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\,;\, W'w\phi)$$

where R' is a parabolic system of R, with Weyl group W', and where $w \in W$ is such that $w\phi$ stablizes R' and $R'^\vee$.

There is a natural bijection between

- the set of G–conjugacy classes of Levi subgroups of G,

Levi subgroups and type — For

$$\mathbb{G} = (X, Y, R, R^\vee \,;\, W\phi)$$

a type, a Levi subtype of $\mathbb{G}$ is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee \,;\, W'w\phi)$$

where R' is a parabolic system of R, with Weyl group W', and where $w \in W$ is such that $w\phi$ stablizes R' and $R'^\vee$.

There is a natural bijection between

- the set of G–conjugacy classes of Levi subgroups of G,

  and

Levi subgroups and type — For

$$\mathbb{G} = (X, Y, R, R^\vee\,;\, W\phi)$$

a type, a Levi subtype of $\mathbb{G}$ is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\,;\, W'w\phi)$$

where R' is a parabolic system of R, with Weyl group W', and where $w \in W$ is such that $w\phi$ stablizes R' and $R'^\vee$.

There is a natural bijection between

- the set of G–conjugacy classes of Levi subgroups of G,

  and
- the set of W–conjugacy classes of Levi subtypes of $\mathbb{G}$.

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

Sylow theorem —

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

Sylow theorem —

(1) Maximal $\Phi(x)$–subgroups ("Sylow $\Phi(x)$–subgroups") of G have as (polynomial) order the contribution of $\Phi(x)$ to the (polynomial) order of G.

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

Sylow theorem —

(1) Maximal $\Phi(x)$–subgroups ("Sylow $\Phi(x)$–subgroups") of G have as (polynomial) order the contribution of $\Phi(x)$ to the (polynomial) order of G.

(2) Sylow $\Phi(x)$–subgroups are all conjugate by G (*i.e.*, their types are transitively permuted by the Weyl group W).

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

Sylow theorem —

(1) Maximal $\Phi(x)$–subgroups ("Sylow $\Phi(x)$–subgroups") of G have as (polynomial) order the contribution of $\Phi(x)$ to the (polynomial) order of G.

(2) Sylow $\Phi(x)$–subgroups are all conjugate by G (*i.e.*, their types are transitively permuted by the Weyl group W).

(3) The (polynomial) index of the normalizer in G of a Sylow $\Phi(x)$–subgroup is congruent to 1 modulo $\Phi(x)$.

The centralizers of $\Phi_d(x)$–subgroups are called the *d–split Levi subgroups*.

The centralizers of $\Phi_d(x)$–subgroups are called the *d–split Levi subgroups*.

The minimal d–split Levi subgroups are the centralizers of Sylow $\Phi_d(x)$–subgroups. They are all conjugate under G.

The centralizers of $\Phi_d(x)$–subgroups are called the d–split Levi subgroups.

The minimal d–split Levi subgroups are the centralizers of Sylow $\Phi_d(x)$–subgroups. They are all conjugate under G.

Example

The centralizers of $\Phi_d(x)$–subgroups are called the *d–split Levi subgroups*.

The minimal d–split Levi subgroups are the centralizers of Sylow $\Phi_d(x)$–subgroups. They are all conjugate under G.

Example

For each d ($1 \leq d \leq n$), $\mathrm{GL}_n(q)$ contains a subtorus of order $\Phi_d(x)^{[\frac{n}{d}]}$

The centralizers of $\Phi_d(x)$–subgroups are called the d–split Levi subgroups.

The minimal d–split Levi subgroups are the centralizers of Sylow $\Phi_d(x)$–subgroups. They are all conjugate under G.

Example

For each d $(1 \leq d \leq n)$, $\mathrm{GL}_n(q)$ contains a subtorus of order $\Phi_d(x)^{[\frac{n}{d}]}$

Assume $n = md + r$ with $r < d$. Then a minimal d–split Levi subgroup has shape $\mathrm{GL}_1(q^d)^m \times \mathrm{GL}_r(q)$.

GENERIC AND ORDINARY SYLOW SUBGROUPS

GENERIC AND ORDINARY SYLOW SUBGROUPS

Let ℓ be a prime number which does not divide $|W|$.

GENERIC AND ORDINARY SYLOW SUBGROUPS

Let ℓ be a prime number which does not divide $|W|$.

- If ℓ divides $|G| = \mathbb{G}(q)$, there is a unique integer d such that ℓ divides $\Phi_d(q)$.

GENERIC AND ORDINARY SYLOW SUBGROUPS

Let ℓ be a prime number which does not divide $|W|$.

- If ℓ divides $|G| = \mathbb{G}(q)$, there is a unique integer d such that ℓ divides $\Phi_d(q)$.
- Then the Sylow ℓ–subgroups of G are nothing but the Sylow ℓ–subgroups S_ℓ of $S = \mathbf{S}^F$ ($\mathbf{S}$ a Sylow $\Phi_d(x)$–subgroup of $\mathbf{G}$).

GENERIC AND ORDINARY SYLOW SUBGROUPS

Let ℓ be a prime number which does not divide $|W|$.

- If ℓ divides $|G| = \mathbb{G}(q)$, there is a unique integer d such that ℓ divides $\Phi_d(q)$.
- Then the Sylow ℓ–subgroups of G are nothing but the Sylow ℓ–subgroups S_ℓ of $S = \mathbf{S}^F$ ($\mathbf{S}$ a Sylow $\Phi_d(x)$–subgroup of $\mathbf{G}$).
- We have
$$N_G(S_\ell) = N_G(\mathbf{S}) \text{ and } C_G(S_\ell) = C_G(\mathbf{S})\,.$$

CYCLOTOMIC WEYL GROUPS AND SPRINGER THEOREM

CYCLOTOMIC WEYL GROUPS AND SPRINGER THEOREM

Let L (or $\mathbf{L}$, or $\mathbb{L}$) be a minimal d–split Levi subgroup, the centralizer of a Sylow $\Phi_d(x)$–subgroup $\mathbf{S}$.

CYCLOTOMIC WEYL GROUPS AND SPRINGER THEOREM

Let L (or $\mathbf{L}$, or $\mathbb{L}$) be a minimal d–split Levi subgroup, the centralizer of a Sylow $\Phi_d(x)$–subgroup $\mathbf{S}$.

(1) We have $N_G(\mathbf{L})/L \simeq N_G(\mathbf{S})/C_G(\mathbf{S}) \simeq N_W(\mathbb{L})/W'$ (where W' is the Weyl group of $\mathbf{L}$).
Denote that group by $W_{\mathbb{G}}(\mathbb{L})$.

CYCLOTOMIC WEYL GROUPS AND SPRINGER THEOREM

Let L (or $\mathbf{L}$, or $\mathbb{L}$) be a minimal d–split Levi subgroup, the centralizer of a Sylow $\Phi_d(x)$–subgroup $\mathbf{S}$.

(1) We have $N_G(\mathbf{L})/L \simeq N_G(\mathbf{S})/C_G(\mathbf{S}) \simeq N_W(\mathbb{L})/W'$ (where W' is the Weyl group of $\mathbf{L}$).
Denote that group by $W_{\mathbb{G}}(\mathbb{L})$.

(2) For ζ a primitive d-th root of the unity, we have

$$|W_{\mathbb{G}}(\mathbb{L})| = \mathbb{G}(\zeta)/\mathbb{L}(\zeta)\,.$$

CYCLOTOMIC WEYL GROUPS AND SPRINGER THEOREM

Let L (or $\mathbf{L}$, or $\mathbb{L}$) be a minimal d–split Levi subgroup, the centralizer of a Sylow $\Phi_d(x)$–subgroup $\mathbf{S}$.

(1) We have $N_G(\mathbf{L})/L \simeq N_G(\mathbf{S})/C_G(\mathbf{S}) \simeq N_W(\mathbb{L})/W'$ (where W' is the Weyl group of $\mathbf{L}$).
Denote that group by $W_{\mathbb{G}}(\mathbb{L})$.

(2) For ζ a primitive d-th root of the unity, we have

$$|W_{\mathbb{G}}(\mathbb{L})| = \mathbb{G}(\zeta)/\mathbb{L}(\zeta)\,.$$

Example

For $n = mr + d$ $(d < r)$, we have $W_{\mathbb{G}}(\mathbb{L}) \simeq C_d \wr \mathfrak{S}_r$

The case $d = 1$ — The Sylow $\Phi_1(x)$–subgroups, as well as the minimal d–split subgroups, coincide with the split maximal tori.

The case $d = 1$ — The Sylow $\Phi_1(x)$–subgroups, as well as the minimal d–split subgroups, coincide with the split maximal tori.

In case $\mathbf{G}$ is split (*i.e.*, the automorphism ϕ induced by F is the identity), then the group $W_{\mathbb{G}}(\mathbb{L})$ coincides with W.

The case $d = 1$ — The Sylow $\Phi_1(x)$–subgroups, as well as the minimal d–split subgroups, coincide with the split maximal tori.

In case $\mathbf{G}$ is split (*i.e.*, the automorphism ϕ induced by F is the identity), then the group $W_{\mathbb{G}}(\mathbb{L})$ coincides with W.

Springer and Springer–Lehrer theorem — The group $W_{\mathbb{G}}(\mathbb{L})$ is a complex reflection group (in its representation over the complex vector space $\mathbb{C} \otimes X((Z\mathbf{L})_{\Phi_d})$).

The case $d = 1$ — The Sylow $\Phi_1(x)$–subgroups, as well as the minimal d–split subgroups, coincide with the split maximal tori.

In case $\mathbf{G}$ is split (*i.e.*, the automorphism ϕ induced by F is the identity), then the group $W_{\mathbb{G}}(\mathbb{L})$ coincides with W.

Springer and Springer–Lehrer theorem — The group $W_{\mathbb{G}}(\mathbb{L})$ is a complex reflection group (in its representation over the complex vector space $\mathbb{C} \otimes X((Z\mathbf{L})_{\Phi_d}))$.

Example

For $n = mr + d$ $(d < r)$, we have $W_{\mathbb{G}}(\mathbb{L}) \simeq C_d \wr \mathfrak{S}_r$

The case $d = 1$ — The Sylow $\Phi_1(x)$–subgroups, as well as the minimal d–split subgroups, coincide with the split maximal tori.

In case $\mathbf{G}$ is split (*i.e.*, the automorphism ϕ induced by F is the identity), then the group $W_{\mathbb{G}}(\mathbb{L})$ coincides with W.

Springer and Springer–Lehrer theorem — The group $W_{\mathbb{G}}(\mathbb{L})$ is a complex reflection group (in its representation over the complex vector space $\mathbb{C} \otimes X((Z\mathbf{L})_{\Phi_d})$).

Example

For $n = mr + d$ $(d < r)$, we have $W_{\mathbb{G}}(\mathbb{L}) \simeq C_d \wr \mathfrak{S}_r$

The group $W_{\mathbb{G}}(\mathbb{L})$ is called the d–cyclotomic Weyl group.

If G is split, the 1–cyclotomic Weyl group is nothing but the ordinary Weyl group W.