Complex reflection groups and associated braid groups

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September 2008

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- Let G be a finite subgroup of $\mathrm{GL}(V)$. The group G acts on the algebra S, and we let $R := S^G$ denote the subalgebra of G–fixed polynomials.

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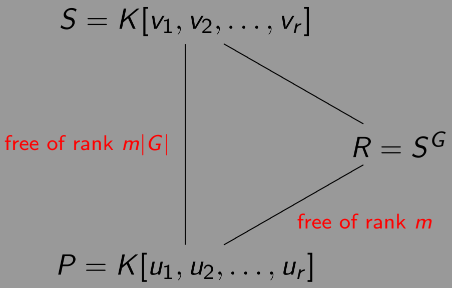

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such that

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free of rank $m|G|$

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$$S = K[x, y]$$

not free

free of rank 4

$$R = S^G = K[x^2, y^2] \oplus K[x^2, y^2]xy$$

free of rank 2

$$P = K[x^2, y^2]$$

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- A finite reflection group on $\mathbb{Q}$ is called a Weyl group.

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Theorem (Shephard–Todd, Chevalley–Serre)

Let G be a finite subgroup of $\mathrm{GL}(V)$ (V an r–dimensional vector space over a characteristic zero field K). Let S denote the symmetric algebra of V, isomorphic to the polynomial ring $K[v_1, v_2, \ldots, v_r]$.
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In other words, unless... $m = 1$, *i.e.*, $R = P$.

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free of rank $m|G|$

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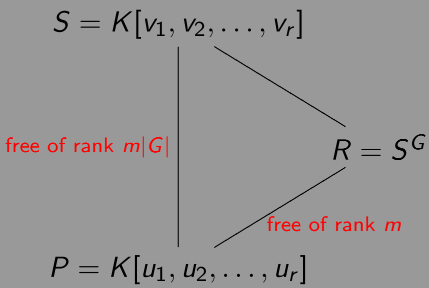
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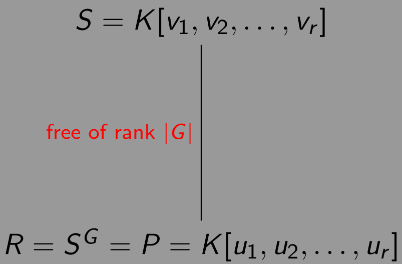
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free of rank $|G|$
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- For $G = \mathfrak{S}_r$, one may choose

$$\begin{cases} u_1 = v_1 + \cdots + v_r \\ u_2 = v_1 v_2 + v_1 v_3 + \cdots + v_{r-1} v_r \\ \vdots \qquad \vdots \\ u_r = v_1 v_2 \cdots v_r \end{cases}$$

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- For $G = \langle e^{2\pi i/d} \rangle$, cyclic group of order d acting by multiplication on $V = \mathbb{C}$, we have

$$S = K[x] \quad \text{and} \quad R = K[x^d] \,.$$

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&G(2, 2, r) = W(\mathrm{D}_r) \\
&G_{23} = H_3 \ , \ \ G_{28} = F_4 \ , \ \ G_{30} = H_4 \\
&G_{35,36,37} = E_{6,7,8} \, .
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- The fixator G_H (pointwise stabilizer) of H is a cyclic group consisting of reflections with reflecting hyperplane H and reflecting line L.

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Now the extension $\begin{array}{c} S \\ \uparrow \\ R = S^G \end{array}$ (corresponding to the covering $\begin{array}{c} V \\ \downarrow \\ V/G \end{array}$) is ramified at $\mathfrak{q} = SL$ if and only if L is a reflecting line.

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3. The set $\mathrm{Par}(G)$ of fixators ("parabolic subgroups" of G) is in (reverse–order) bijection with the set $\mathrm{I}(\mathcal{A})$ of intersections of elements of $\mathcal{A}$:

$$\mathrm{I}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Par}(G) \quad , \quad X \mapsto G_X \, .$$

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Notation around H

- Let $H \in \mathcal{A}$, with associated line L. For $x \in V$, we set

$$x = x_L + x_H \quad (\text{with } x_L \in L \text{ and } x_H \in H)\,.$$

$=$ Thus, we have $\qquad s_H(x) = \mathrm{e}^{2i\pi/e_H} x_L + x_H\,.$

- If $t \in \mathbb{R}$, we set :

$$s_H^t(x) = \mathrm{e}^{2i\pi t/e_H} x_L + x_H \quad \text{defining a path } \mathbf{s}_{H,x} \text{ from } x \text{ to } s_H(x)$$

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In other words,

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Braid reflections

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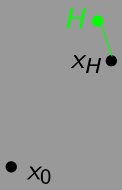

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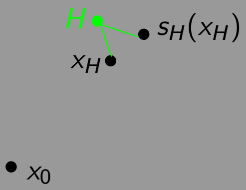

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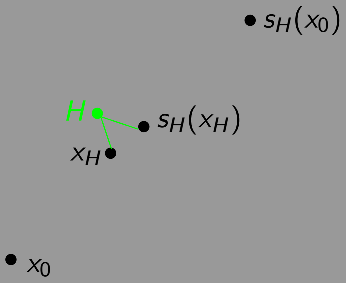

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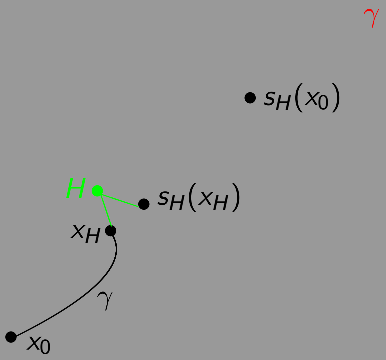

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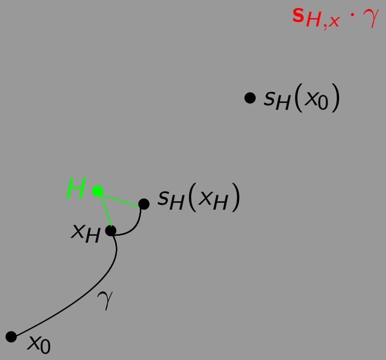

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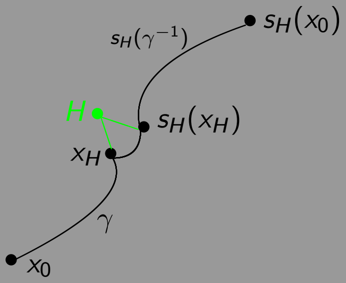

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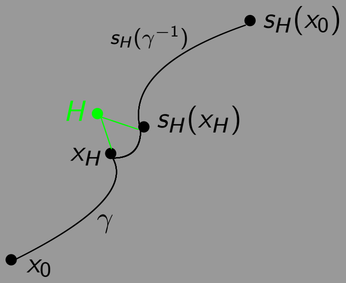

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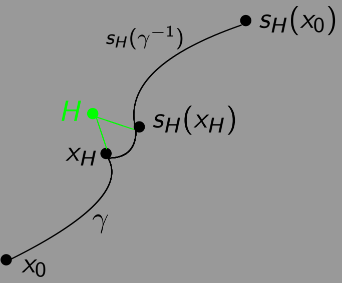

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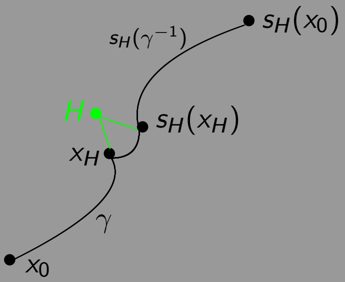

Definition

We call *braid reflections* the elements $\mathbf{s}_{H,\gamma} \in B$ defined by the paths $\sigma_{H,\gamma}$.

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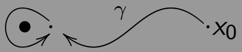

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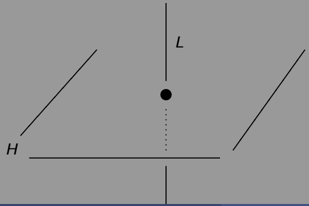

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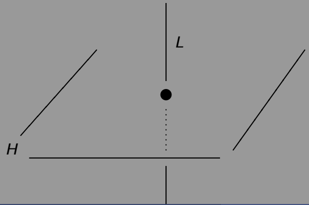

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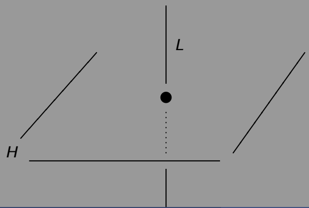

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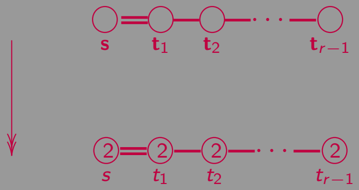

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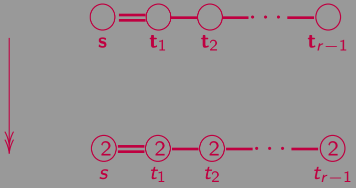

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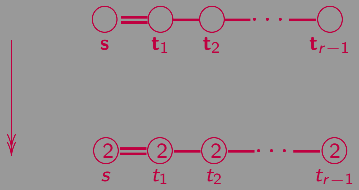

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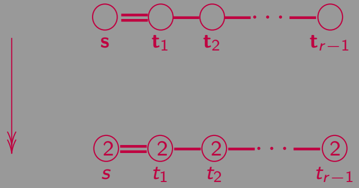

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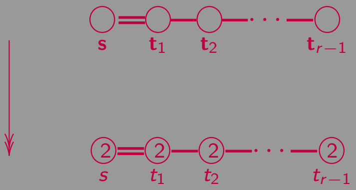

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where

- $\mathbf{S}$ is a finite set of distinguished braid reflections,
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2. Either $\Gamma_G = r$ or $\Gamma_G = r + 1$, and the group B_G has an Artin–like presentation by Γ_G braid reflections.

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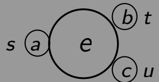
Let $\mathcal{D}$ be a diagram like 

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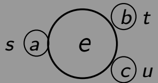

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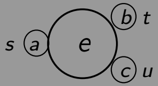

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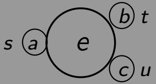

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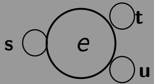

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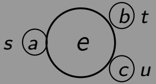

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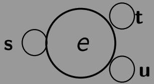

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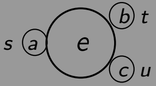

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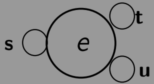

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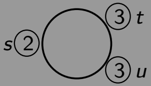

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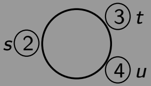

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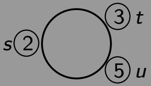

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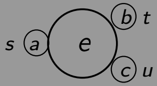
Let $\mathcal{D}$ be a diagram like 

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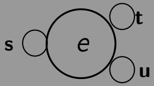
We denote by $\mathcal{D}_{\mathrm{br}}$ and call *braid diagram* the diagram 

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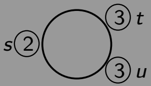
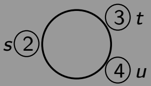
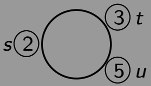
G_7 : 

G_{11} : 

G_{19} : 

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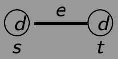
- The groups G_n for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type  ,

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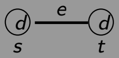
- The groups G_n for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type 

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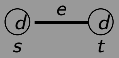
- The groups G_n for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type 

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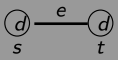
- The groups G_n for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type 

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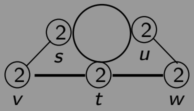
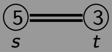
- The group G_{31} has diagram 

- The group G_{18} has diagram 

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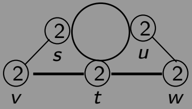
- The group G_{31} has diagram 

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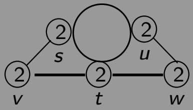
- The group G_{31} has diagram 

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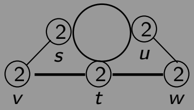
- The group G_{31} has diagram 

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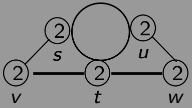
- The group G_{31} has diagram 

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A monodromy representation

(after Knizhnik–Zamolodchikov, Cherednik, Dunkl, Opdam, Kohno, Broué-Malle-Rouquier)

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$$\forall v \in V \,,\, x \in V^{\mathrm{reg}} \,, \quad df(x)(v) = \frac{1}{2i\pi} \sum_{H \in \mathcal{A}} \frac{\alpha_H(v)}{\alpha_H(x)} z_H f(x)$$

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- The associated generic Hecke algebra is defined from such a presentation :

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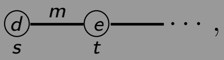
if $G =$ 

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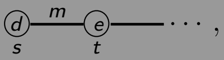
if $G =$ 

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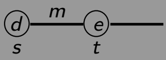
if $G =$ 

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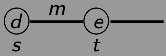
if $G =$ 

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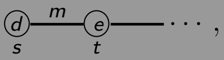
if $G =$ 

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- The above specialisation defines a bijection

$$\mathrm{Irr}(G) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}(G)) \quad , \quad \chi \mapsto \chi_{\mathcal{H}} .$$

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- $t_{\mathbf{q}}$ specializes to the canonical linear form on the group algebra.
- For all $b \in B$, we have

$$t_{\mathbf{q}}(b^{-1})^{\vee} = \frac{t_{\mathbf{q}}(b\boldsymbol{\pi})}{t_{\mathbf{q}}(\boldsymbol{\pi})} .$$

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The canonical forms $t_{\mathbf{q}}$ are hidden behind Lusztig's theory of characters of finite reductive groups, their generic degrees and Fourier transform matrices.