Local group theory : from Frobenius to Derived Categories

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September 2012

LOCAL GROUP THEORY

- Feit–Thompson, 1963

  If G is a non abelian simple finite group, then $2 \mid |G|$.

- Cauchy (1789–1857)

  If $p \mid |G|$, there are non trivial p–subgroups in G.

- Sylow, 1872

  The maximal p–subgroups of G are all conjugate under G.

- Brauer–Fowler, 1956

  There are only a finite number of isomorphism types of finite simple groups with a prescribed type of centralizer of an involution.

Assume $P \subset S$ and $P \subset S'$. There is $g \in G$ such that $S' = S^g$ $(= g^{-1}Sg)$, hence

$$P \subset S \quad \text{and} \quad {}^g P \, (= gPg^{-1}) \subset S \, .$$

This is a *fusion*.

The Frobenius Category

$\mathrm{Frob}_p(G)$:

- Objects : the p–subgroups of G,
- $\mathrm{Hom}(P, Q) := \{g \in G \mid ({}^g P \subset Q)\}/C_G(P)$.

Note that $\mathrm{Aut}(P) = N_G(P)/C_G(P)$.

Alperin's fusion theorem (1967) says essentially that $\mathrm{Frob}_p(G)$ is known as soon as the automorphisms of some of its objects are known.

Main question of local group theory

How much is known about G from the knowledge (up to equivalence of categories) of $\mathrm{Frob}_p(G)$?

Well, certainly not more than $G/O_{p'}(G)$!
(where $O_{p'}(G)$ denotes the largest normal subgroup of G of order not divisible by p)

Indeed, $O_{p'}(G)$ disappears in the Frobenius category, since, for P a p–subgroup,

$$O_{p'}(G) \cap N_G(P) \subseteq C_G(P)\,.$$

But perhaps all of $G/O_{p'}(G)$?

Control subgroup

Let H be a subgroup of G. The following conditions are equivalent :

(i) The inclusion $H \hookrightarrow G$ induces an equivalence of categories

$$\mathrm{Frob}_p(H) \xrightarrow{\sim} \mathrm{Frob}_p(G)\,,$$

(ii) H contains a Sylow p–subgroup of G, and if P is a p–subgroup of H and g is an element of G such that ${}^gP \subseteq H$, then there is $h \in H$ and $z \in C_G(P)$ such that $g = hz$.

If the preceding conditions are satisfied, we say that H controls p–fusion in G, or that H is a control subgroup in G.

The first question may now be reformulated as follows :

If H controls p–fusion in G, does the inclusion $H \hookrightarrow G$ induce an isomorphism

$$H/O_{p'}(H) \xrightarrow{\sim} G/O_{p'}(G)\,?$$

In other words, do we have

$$G = HO_{p'}(G)\,?$$

- Frobenius theorem, 1905

  If a Sylow p–subgroup S of G controls p–fusion in G, then the inclusion induces an isomorphism $S \simeq G/O_{p'}(G)$.

- p–solvable groups, ?

  Assume that G is p–solvable. If H controls p-fusion in G, then the inclusion induces an isomorphism $H/O_{p'}(H) \simeq G/O_{p'}(G)$.

- Z_p^*–theorem (Glauberman, 1966 for $p = 2$, Classification for other primes)

  Assume that $H = C_G(P)$ where P is a p–subgroup of G. If H controls p-fusion in G, then the inclusion induces an isomorphism $H/O_{p'}(H) \simeq G/O_{p'}(G)$.

- But

  Burnside (1852–1927)

  Assume that a Sylow p–subgroup S of G is abelian. Set $H := N_G(S)$. Then H controls p-fusion in G.

Consider the **Monster**, a finite simple group of order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \simeq 8.10^{53} .$$

(predicted in 1973 by Fischer and Griess, constructed in 1980 by Griess, proved to be unique by Thompson)

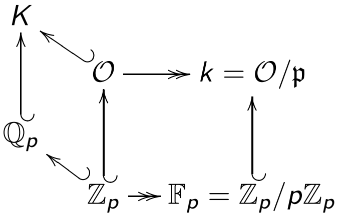
and the normalizer H of one of its Sylow 11–subgroups, a group of order 72600, isomorphic to $(C_{11} \times C_{11}) \rtimes (C_5 \times \mathrm{SL}_2(5))$ (here we denote by C_m the cyclic group of order m).

Here we have $G \neq HO_{11'}(G)$ since G is simple.

Remark : one may think of more elementary examples...

LOCAL REPRESENTATION THEORY

Let K be a finite extension of the field of p–adic numbers $\mathbb{Q}_p$ which contains the $|G|$-th roots of unity. Let $\mathcal{O}$ be the ring of integers of K over $\mathbb{Z}_p$, with maximal ideal $\mathfrak{p}$ and residue field $k := \mathcal{O}/\mathfrak{p}$.



Block decomposition

$$\begin{array}{cccll} \mathcal{O}G & = & \bigoplus & B & \text{(indecomposable algebra)} \\ \downarrow & & & & \qquad\qquad\downarrow \\ kG & = & \bigoplus & kB & \text{(indecomposable algebra)} \end{array}$$

The augmentation map $\mathcal{O}G \to \mathcal{O}$ factorizes through a unique block of $\mathcal{O}G$ called *the principal block* and denoted by $B_p(G)$.

$$\begin{array}{ccc} \mathcal{O}G & \longrightarrow & B_p(G) \\ & \searrow & \downarrow \\ & & \mathcal{O} \end{array}$$

Set $e_{p'}(G) := \frac{1}{|O_{p'}(G)|} \sum_{s \in O_{p'}(G)} s$. Then $e_{p'}(G)$ is a central idempotent of $\mathcal{O}G$ and $\mathcal{O}Ge_{p'}(G)$ is a product of blocks containing the principal block $B_p(G)$.

Factorisation and principal block

If H is a subgroup of G, the following assertions are equivalent

(i) $G = HO_{p'}(G)$.

(ii) The map Res_H^G induces an isomorphism from $\mathcal{O}Ge_{p'}(G)$ onto $\mathcal{O}He_{p'}(H)$.

In particular, in that case, the map Res_H^G induces an isomorphism from $B_p(G)$ onto $B_p(H)$.

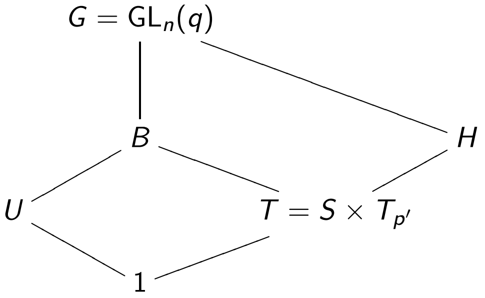
$==$ Let us re-examine the counterexamples to factorization coming from Burnside's theorem.

Assume that a Sylow p–subgroup S of G is abelian, let $H := N_G(S)$ be its normalizer.

Even if $G \neq H\, O_{p'}(G)$, it appears that there is some connection between the (representation theory of) $B_p(G)$ and $B_p(H)$.

First of all, there are many examples where there is no factorization, but where the algebras are *Morita equivalent* — but then *not* through the Res_H^G functor.

A kind of generic example :



$p \nmid q\,, p > n, \quad S$ p-Sylow

$$H := N_G(T) = N_G(S)$$

$$\longleftarrow \qquad H/T = \mathfrak{S}_n$$

We certainly have

$$G \neq HO_{p'}(G)\,.$$

But the principal block algebras of G and H respectively are Morita equivalent.

There exist M and N, respectively a $\mathcal{O}G$–module–$\mathcal{O}H$ and a $\mathcal{O}H$–module–$\mathcal{O}G$ with the following properties :

- (With appropriate cutting by the principal block idempotents)

$$M \otimes_{\mathcal{O}H} N \simeq B_p(G) \text{ as } \mathcal{O}G\text{–modules–}\mathcal{O}G$$

$$N \otimes_{\mathcal{O}G} M \simeq B_p(H) \text{ as } \mathcal{O}H\text{–modules–}\mathcal{O}H$$

- Viewed as an $\mathcal{O}G$–module–$\mathcal{O}T$, we have

$$M \simeq \mathcal{O}(G/U)\,,$$

i.e., the functor $M\otimes_{\mathcal{O}T}?$ is the Harish–Chandra induction.

SOME NUMERICAL MIRACLES

Let us consider the case $G = \mathfrak{A}_5$ and $p = 2$. Then we have $H \simeq \mathfrak{A}_4$.

Remark : on a larger screen, we might as well consider the above case of the Monster and of the prime $p = 11$.

Table : Character table of $\mathfrak{A}_5$

	(1)	(2)	(3)	(5)	(5')
1	1	1	1	1	1
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ_3	3	-1	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$
χ_3'	3	-1	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$

Table : Character table of $B_2(\mathfrak{A}_5)$

	(1)	(2)	(5)	(5')	(3)
1	1	1	1	1	1
χ_5	5	1	0	0	-1
χ_3	3	-1	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$	0
χ_3'	3	-1	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$	0

Table : Character table of $\mathfrak{A}_4$

	(1)	(2)	(3)	(3')
1	1	1	1	1
$-\alpha_3$	-3	1	0	0
$-\alpha_1$	-1	-1	$(1+\sqrt{-3})/2$	$(1-\sqrt{-3})/2$
$-\alpha_1'$	-1	-1	$(1-\sqrt{-3})/2$	$(1+\sqrt{-3})/2$

ABELIAN SYLOW CONJECTURE

Assume that a Sylow p–subgroup S of G is abelian, let $H := N_G(S)$ be its normalizer.

- (ASC) :

  The algebras $B_p(G)$ and $B_p(H)$ are derived equivalent.

Which means :
There exist M and N, respectively a bounded complex of $B_p(G)$–modules–$B_p(H)$ and a bounded complex of $B_p(H)$–modules–$B_p(G)$ with the following properties :

$$M \otimes_{B_p(H)} N \simeq B_p(G) \text{ as complexes of } B_p(G)\text{–modules–}B_p(G)$$

$$N \otimes_{B_p(G)} M \simeq B_p(H) \text{ as complexes of } B_p(H)\text{–module–}B_p(H)$$

- (Strong ASC) :

  They are Rickard equivalent, that is, derived equivalent in a way which is compatible with the equivalence of Frobenius categories

  Which means : There is a G–equivariant collection of derived equivalences

$$\{\, \mathcal{E}(P) \, : \, \mathcal{D}^b(B_p(C_G(P)) \xrightarrow{\sim} \mathcal{D}^b(B_p(C_H(P))\,\}_{P \subseteq S}$$

  compatible with Brauer morphisms.

RICKARD'S EXPLANATION FOR $\mathfrak{A}_5$

- $G := \mathfrak{A}_5$
- $H := N_G(S_2)$, (S_2 a Sylow 2–subgroup of G)
- View $B_2(G)$ as acted on as follows

$$ {}_{B_2(G)} \circlearrowright B_2(G) \circlearrowleft {}_{B_2(H)} $$

- ${}_{B_2(G)} \circlearrowright IB_2(G) \circlearrowleft {}_{B_2(H)}$:= kernel of augmentation map $B_2(G) \twoheadrightarrow \mathcal{O}$.
- C := a projective cover of the bimodule $IB_2(G)$.

Thus we have

$$\begin{array}{ccccccc} & & C & & & & \\ & & \downarrow & \searrow & & & \\ \{0\} & \longrightarrow & IB_2(G) & \longrightarrow & B_2(G) & \longrightarrow & \{0\} \\ & & \downarrow & & & & \\ & & \{0\} & & & & \end{array}$$

set

$$\Gamma_2 := \{0\} \to C \to B_2(G) \to \{0\}$$

where $B_2(G)$ is in degree 0 (and C in degree -1).

We have homotopy equivalences :

$$\Gamma_2 \underset{\mathcal{O}H}{\otimes} \Gamma_2^* \simeq B_2(G) \quad \text{as complexes of } (B_2(G), B_2(G))\text{–bimodules,}$$

$$\Gamma_2^* \underset{\mathcal{O}G}{\otimes} \Gamma_2 \simeq B_2(H) \quad \text{as complexes of } (B_2(H), B_2(H))\text{–bimodules.}$$

Case of finite reductive groups

Usual notation

- $\mathbf{G}$ is a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius endomorphism F defining a rational structure over $\mathbf{F}_q$.
  Here we assume that $(\mathbf{G}, F)$ is split.
- $G := \mathbf{G}^F$ is the corresponding finite reductive group, with order

  $$|G| = q^N \prod_{d>0} \Phi_d(q)^{a(d)}$$

  a polynomial which depends only on the reflection representation of W on $\mathbb{Q} \otimes Y(\mathbf{T})$.

  Indeed, that polynomial is

  $$q^{\sum_i d_i - 1} \prod_i (q^{d_i} - 1)\,.$$

Sylow Φ_d–subgroups, d–cyclotomic Weyl group

- There exists a rational torus $\mathbf{S}_d$ of $\mathbf{G}$, unique up to G–conjugation, such that

$$|S_d| = |\mathbf{S}_d^F| = \Phi_d(q)^{a(d)}\,.$$

- Set $\quad \mathbf{L}_d := C_{\mathbf{G}}(\mathbf{S}_d) \quad$ and $\quad \mathbf{N}_d := N_{\mathbf{G}}(\mathbf{S}_d) = N_{\mathbf{G}}(\mathbf{L}_d)$

- $W_d := N_d/L_d$ is a true finite group, a complex reflection group in its action on $\mathbb{C} \otimes Y(\mathbf{S}_d)$.

  $=$ This is the *d–cyclotomic Weyl group* of the finite reductive group G.

Example : For $G = \mathrm{GL}_n(q)$ and $n = dm + r\,(r < d)$, then

$$L_d = \mathrm{GL}_1(q^d)^m \times \mathrm{GL}_r(q) \quad , \quad W_d = \boldsymbol{\mu}_d \wr \mathfrak{S}_m$$

The Sylow ℓ–subgroups and their normalizers

- ℓ a prime number, prime to q, $\ell \mid |G|$, $\ell \nmid |W|$

  $\Longrightarrow$ There exists one d ($a(d) > 0$) such that $\ell \mid \Phi_d(q)$, and the Sylow ℓ–subgroup S_ℓ of S_d is a Sylow of G.

- $L_d = C_G(S_\ell)$ and $N_d = N_\ell = N_G(S_\ell)$:

$$\begin{array}{c} N_\ell \\ \Big| \big\} W_d \\ L_d \\ \Big| \\ 1 \end{array}$$

Since the "local" block is

$$B_\ell(\mathbb{Z}_\ell N_\ell) \simeq \mathbb{Z}_\ell[S_\ell \rtimes W_d]$$

our conjecture reduces to

Conjecture

$$\mathcal{D}^b(B_\ell(\mathbb{Z}_\ell G)) \simeq \mathcal{D}^b(\mathbb{Z}_\ell[S_\ell \rtimes W_d])$$

Role of Deligne–Lusztig varieties

- Let $\mathbf{P}$ be a parabolic subgroup with Levi subgroup $\mathbf{L}_d$, and with unipotent radical $\mathbf{U}$.

Note that $\mathbf{P}$ is never rational if $d \neq 1$.

- The Deligne–Lusztig variety is

$$\mathcal{V}_{\mathbf{P}} := {}_{G} \circlearrowleft \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g\mathbf{U} \cap F(g\mathbf{U}) \neq \emptyset\} \circlearrowleft {}_{L_d}$$

hence defines an object

$$\mathrm{R}\Gamma_c(\mathcal{V}_{\mathbf{P}}, \mathbb{Z}_\ell) \in \mathcal{D}^b({}_{\mathbb{Z}_\ell G}\,\mathbf{mod}\,{}_{\mathbb{Z}_\ell L_d})$$

Conjecture

There is a choice of $\mathbf{U}$ such that

1. $\mathrm{R}\Gamma_c(\mathcal{V}_{\mathbf{P}}, \mathbb{Z}_\ell)_0$ is a Rickard complex between $B_\ell(\mathbb{Z}_\ell G)$ and its commuting algebra $\mathcal{C}(\mathbf{U})$.
2. $\mathcal{C}(\mathbf{U}) \simeq B_\ell(\mathbb{Z}_\ell N_\ell)$.

The case where $d = 1$

If $d = 1$,

- $\mathbf{S}_d = \mathbf{T} = \mathbf{L}_d$ and $W_d = W$
- $\mathcal{V}_{\mathbf{B}} = G/U$ and $\mathrm{R}\Gamma_c(\mathcal{V}_{\mathbf{P}}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(G/U)$
- $$\mathbb{Z}_\ell G \circlearrowleft \mathbb{Z}_\ell(G/U) \circlearrowleft \mathcal{C}(U)$$
- where
  1. $\mathcal{C}(U) \simeq \mathbb{Z}_\ell T.\mathbb{Z}_\ell \mathcal{H}(W, q)$
  2. $\overline{\mathbb{Q}}_\ell \mathcal{H}(W, q) \simeq \overline{\mathbb{Q}}_\ell W$

The unipotent part

- Extend the scalar to $\overline{\mathbb{Q}}_\ell =: K \quad \Rightarrow$ Get into a semisimple situation
  - $\mathrm{R}\Gamma_c(\mathcal{V}(\mathbf{U}), \mathbb{Z}_\ell)$ becomes

$$H_c^\bullet(\mathcal{V}(\mathbf{U}), K) := \bigoplus_i H_c^i(\mathcal{V}(\mathbf{U}), K)$$

- Replace $\mathcal{V}(\mathbf{U})$ by $\quad \mathcal{V}(\mathbf{U})^{\mathrm{un}} := \mathcal{V}(\mathbf{U})/L_d \quad \Rightarrow$

  Only unipotent characters of G are involved

Semisimplified unipotent conjecture

1. The different $H_c^i(\mathcal{V}(\mathbf{U})^{\mathrm{un}}, K)$ are disjoint as KG–modules,
2. $\mathcal{H}(\mathbf{U}) := \mathrm{End}_{KG}\, H_c^\bullet(\mathcal{V}(\mathbf{U})^{\mathrm{un}}, K) \simeq KW_d$

Case where d is regular

- $\mathbf{L}_d =: \mathbf{T}_d$ is a torus $\iff$ d is a regular number for W
- The set of tori $\mathbf{L}_d$ is a single orbit of rational maximal tori under G, hence corresponds to a conjugacy class of W.
- For w in that class, we have $W_d \simeq C_W(w)$.
- The choice of $\mathbf{U}$ corresponds to the choice of an element w in that class.
- We then have

$$\mathcal{V}(\mathbf{U}_w)^{\mathrm{un}} = \mathbf{X}_w := \{\mathbf{B} \in \mathcal{B} \mid \mathbf{B} \stackrel{w}{\to} F(\mathbf{B})\}$$

  - $\mathcal{B}$ is the variety of all Borel subgroups of $\mathbf{G}$
  - The orbits of $\mathbf{G}$ on $\mathcal{B} \times \mathcal{B}$ are canonically in bijection with W and we write $\mathbf{B} \stackrel{w}{\to} \mathbf{B}'$ if the orbit of $(\mathbf{B}, \mathbf{B}')$ corresponds to w.

Relevance of the braid groups

Notation

- $V := \mathbb{C} \otimes Y(\mathbf{T})$ acted on by W,
  $\mathcal{A} :=$ set of reflecting hyperplanes of W
- $V^{\mathrm{reg}} := V - \bigcup_{H \in \mathcal{A}} H$
- $B_W := \Pi_1(V^{\mathrm{reg}}/W, x_0)$
- "Section" $W \to B_W$, $w \mapsto \mathbf{w}$, since

$$\text{If} \quad W = < S \mid \underbrace{ststs\dots}_{m_{s,t} \text{ factors}} = \underbrace{tstst\dots}_{m_{s,t} \text{ factors}} , s^2 = t^2 = 1 >$$

$$\text{then} \quad B_W = < \mathbf{S} \mid \underbrace{\mathbf{ststs}\dots}_{m_{s,t} \text{ factors}} = \underbrace{\mathbf{tstst}\dots}_{m_{s,t} \text{ factors}} >$$

- $\boldsymbol{\pi} := t \mapsto e^{2i\pi t} x_0 \quad \Longrightarrow \quad \boldsymbol{\pi} \in ZB_W$
  Moreover $\boldsymbol{\pi} = \mathbf{w}_0^2 = \mathbf{c}^h$ ($\mathbf{c}$ Coxeter element, h Coxeter number).

A theorem of Deligne

Theorem (Deligne)

Whenever $b \in B_W^+$ there is a well defined variety $\mathbf{X}_b^{(F)}$) such that

- $\mathbf{X}_{\mathbf{w}}^{(F)} = \mathbf{X}_w^{(F)}$.
- For $b = \mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_n$ we have

$$\mathbf{X}_b^{(F)} = \{ (\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_n) \mid \mathbf{B}_0 \stackrel{w_1}{\to} \mathbf{B}_1 \stackrel{w_2}{\to} \cdots \stackrel{w_n}{\to} \mathbf{B}_n \text{ and } \mathbf{B}_n = F(\mathbf{B}_0) \}$$

The variety $\mathbf{X}_\pi$

$$\mathbf{X}_\pi = \{ (\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2) \mid \mathbf{B}_0 \xrightarrow{w_0} \mathbf{B}_1 \xrightarrow{w_0} \mathbf{B}_2 \text{ and } \mathbf{B}_2 = F(\mathbf{B}_0 \}$$
$$= \{ (\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_h) \mid \mathbf{B}_0 \xrightarrow{c} \mathbf{B}_1 \xrightarrow{c} \cdots \xrightarrow{c} \mathbf{B}_h \text{ and } \mathbf{B}_h = F(\mathbf{B}_0) \}$$

The (opposite) monoid B_W^+ acts on $\mathbf{X}_\pi$: For $\mathbf{w} \in B_W^{\mathrm{red}}$, and $\pi = \mathbf{w}b = b\mathbf{w}$,

$$\text{if} \quad B \xrightarrow{\mathbf{w}} B_0 \xrightarrow{b} F(B)$$
$$D_\mathbf{w} : (\mathbf{B}, \mathbf{B}_0, B_1 = F(\mathbf{B})) \mapsto (\mathbf{B}_0, F(\mathbf{B}), F(\mathbf{B}_0))$$

Hence B_W acts on $H_c^\bullet(\mathbf{X}_\pi)$

- Proposition : The action of B_W on $H_c^\bullet(\mathbf{X}_\pi)$ factorizes through the (ordinary) Hecke algebra $\mathcal{H}(W)$.
- Conjecture :

$$\mathrm{End}_{KG}\, H_c^\bullet(\mathbf{X}_\pi) = \mathcal{H}(W)$$

Relevance of roots of $\boldsymbol{\pi}$

Proposition

d regular for $W \iff$ there exists $\mathbf{w} \in B_W^+$ such that $\mathbf{w}^d = \boldsymbol{\pi}$.

Application

1. $\mathbf{X}_{\mathbf{w}}^{(F)}$ embeds into $\mathbf{X}_{\boldsymbol{\pi}}^{(F^d)}$:

$$\mathbf{X}_{\mathbf{w}}^{(F)} \hookrightarrow \mathbf{X}_{\boldsymbol{\pi}}^{(F^d)}$$
$$\mathbf{B} \mapsto (\mathbf{B}, F(\mathbf{B}), \ldots, F^d(\mathbf{B}))$$

2. Its image is

$$\{ \mathbf{x} \in \mathbf{X}_{\boldsymbol{\pi}}^{(F^d)} \mid D_{\mathbf{w}}(\mathbf{x}) = F(\mathbf{x}) \}$$

3. $C_{B_W^+}(\mathbf{w})$ acts on $\mathbf{X}_{\mathbf{w}}^{(F)}$.

Belief

A good choice for $\mathbf{U}_w$ is : $\mathbf{w}$ a d–th root of π.

Theorem (David Bessis)

There is a natural isomorphism

$$B_{C_W(w)} \xrightarrow{\sim} C_{B_W}(\mathbf{w})$$

From which follows :

Theorem

The braid group $B_{C_W(w)}$ of the complex reflections group $C_W(w)$ acts on $H_c^\bullet(\mathbf{X}_\mathbf{w})$.

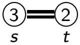
Conjecture

The braid group $B_{C_W(w)}$ acts on $H_c^\bullet(\mathbf{X}_\mathbf{w})$ through a d–cyclotomic Hecke algebra $\mathcal{H}_W(w)$.

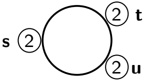
d–cyclotomic Hecke algebras

- A d–cyclotomic Hecke algebra for $C_W(w)$ is in particular
  - an image of the group algebra of the braid group $B_{C_W(w)}$,
  - a deformation in one parameter q of the group algebra of $C_W(w)$,
  - which specializes to that group algebra when q becomes $e^{2\pi i/d}$
- Examples :
  - The ordinary Hecke algebra $\mathcal{H}(W)$ is 1–cyclotomic,
  - Case where $W = \mathfrak{S}_6$, $d = 3$:

$$C_W(w) = B_2(3) = \boldsymbol{\mu}_3 \wr \mathfrak{S}_2 \quad \longleftrightarrow$$



$$\mathcal{H}_W(w) = \left\langle S, T \, ; \begin{Bmatrix} STST = TSTS \\ (S-1)(S-q)(S-q^2) = 0 \\ (T-q^3)(T+1) = 0 \end{Bmatrix} \right\rangle$$

- $W = D_4$, $d = 4$, $C_W(w) = G(4,2,2)$ $\longleftrightarrow$ 

$$\mathcal{H}_W(w) = \left\langle S, T, U \,;\, \left\{ \begin{array}{l} STU = TUS = UST \\ (S - q^2)(S-1) = 0 \end{array} \right\} \right\rangle$$

Let us summarize

1. $\ell \rightsquigarrow d$, d regular, *i.e.*, $L_d = T_w$, $\mathbf{w}^d = \boldsymbol{\pi}$, $\mathcal{V}(\mathbf{U}_w)/L_d = \mathbf{X_w}$
2. $\mathrm{End}_{KG}\, H_c^\bullet(\mathbf{X}_w) \simeq \mathcal{H}_W(w)$
3. $\mathbb{Z}_\ell \mathcal{H}_W(w) \xrightarrow{\sim} \mathbb{Z}_\ell C_W(w)$
4. $\mathrm{End}_{\mathbb{Z}_\ell G}\, \mathrm{R}\Gamma_c(\mathcal{V}(\mathbf{U}_w), \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell(T_w)_\ell \cdot \mathrm{End}_{\mathbb{Z}_\ell G}\, \mathrm{R}\Gamma_c(\mathbf{X}_w, \mathbb{Z}_\ell) \simeq B_\ell(\mathbb{Z}_\ell N_\ell)$

What is really proven today

- Everything
  - if $d = 1$ (Puig),
  - for $G = \mathrm{GL}_2(q)$ (Rouquier), $\mathrm{SL}_2(q)$ (cf. a book by Bonnafé)
  - for $G = \mathrm{GL}_n(q)$ and $d = n$ (Bonnafé–Rouquier)
- About : $\mathrm{End}_{KG}\, H_c^\bullet(\mathbf{X}_w) \simeq \mathcal{H}_W(w)$?
  - All $\mathcal{H}_W(w)$ are known, all cases (Malle)
  - Assertion $\mathrm{End}_{KG} H_c^\bullet(\mathbf{X}_w) \simeq \mathcal{H}_W(w)$ known for
    - $d = h$ (Lusztig),
    - $d = 2$ (Lusztig, Digne–Michel),
    - small rank GL,
    - $d = 4$ for $D_4(q)$ (Digne–Michel).