Building Cathedrals and Breaking down Reinforced Concrete Walls

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Institut Henri Poincaré

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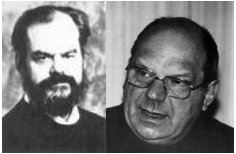

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GG says that when he took his children to see *"Terminator"*, the character reminded him of John, in his ability to rise again and again from apparent destruction to keep attacking "the problem"...

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Tits is a builder, a unifier of thought. He has developed the theory of buildings as a central organizing principle and powerful tool for an astonishingly wide range of problems in group theory and geometry

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Etienne Ghys says that he *"algebraized geometry"* and *"geometrized algebra"*

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First of all, let us examine the role they played in

the classification of finite simple groups,

this fantastic achievement of twentieth century mathematics.

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Sylow, 1872

The maximal p–subgroups of G are all conjugate under G.

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For example, he showed how to proceed from the hypothesis that G is a simple group of even order (and 2–rank at least 3) all of whose local subgroups are solvable (an N-group) to the conclusion that G is a split BN–pair of rank at most 2, defined over a finite field of characteristic 2...

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- Tits' thesis was

  "Sur certaines classes d'espaces homogènes de groupes de Lie",

  giving the final word on Helmholz-Lie problem which had been also considered by Kolmogorov.

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- Was the first to define the Braid Groups attached to Coxeter Groups other than $\mathfrak{S}_n$, called now the Artin–Tits Braid Groups.
- Tits ideas are now an essential ingredient in the arsenal of every geometer. The famous Tits alternative and its "ping–pong lemma" (J. Alg. **20** (1972)), 250–270) is still stimulating Riemannian geometers and polynomial growth type questions...

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1. **Definition** : A family $(C_1, \ldots, C_n)$ of rational conjugacy classes of G is said to be rigid if the set $\{(g_1, \ldots, g_n) \mid (g_i \in C_i)(g_1 \cdots g_n = 1)(G = \langle g_1, \ldots, g_n \rangle\}$ is nonempty and acted on transitively by G.

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2. **Theorem** : If G has a rigid family of rational conjugacy classes, then G is a Galois group over $\mathbb{Q}$.

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Both have maintained a degree of productivity over 50 years which is unusual even among exceptional mathematicians.

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1. For τ in Poincaré upper halfplane and $q := \exp(2\pi i\tau)$,

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2. 196883 is the degree of the smallest nontrivial irreducible complex representation of the Monster group M, the largest sporadic simple group, a group of order
$$|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

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Moreover, as noticed by Andrew Ogg, let $\mathcal{H}/\Gamma_0(p)^+$ be the Riemann surface resulting from taking the quotient of the upper halfplane by $\Gamma_0(p)^+$. Then

$$(\mathcal{H}/\Gamma_0(p)^+ \text{ has genus zero }) \Leftrightarrow (p \text{ divides } |M|) .$$

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There exists a graded $\mathbb{C}M$–module $V = \bigoplus_{n\in\mathbb{N}} V_n$ defining a graded character of M

$$\mathrm{grchar}_V : M \longrightarrow \mathbb{C}[q] \quad , \quad g \mapsto \mathrm{grchar}_V(g) := \sum_{n\in\mathbb{N}} \mathrm{tr}(g, V_n) q^n$$

with the following properties :

For all $g \in M$, there is a genus zero subgroup Γ_g of $\mathrm{PSL}(2,\mathbb{R})$ commensurable with $\mathrm{PSL}(2,\mathbb{Z})$ such that $\mathrm{grchar}_V(q)$ is the normalized main modular function for Γ_g.

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Ultimately proved in 1992 by Richard Borcherds using vertex algebras, generalized Kac–Moody algebras ... after key work on the subject by Thompson and Tits.

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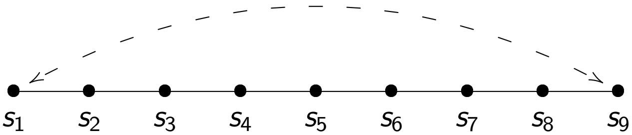

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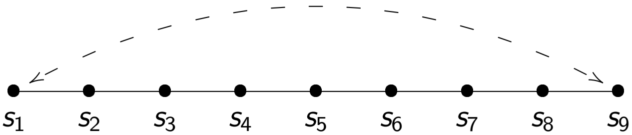

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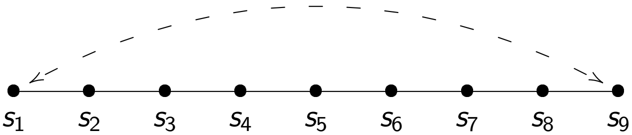

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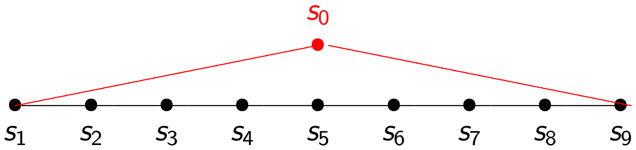

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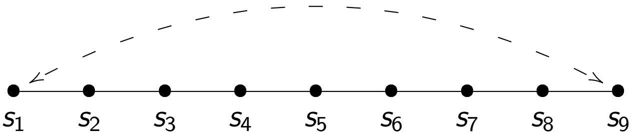

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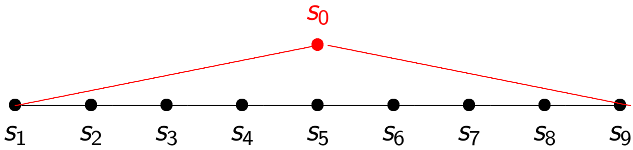
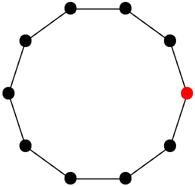
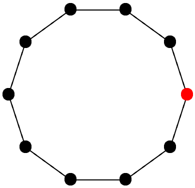
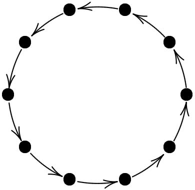
$\tilde{\mathrm{A}}_{n-1} =$ 

viewed as

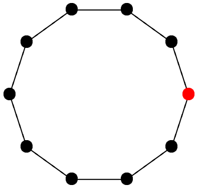




hence C_n-action :



Automorphism group $= C_n \rtimes C_2$



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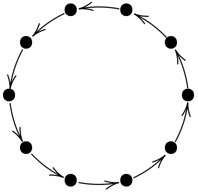

Automorphism group $= C_n \rtimes C_2$

... hence the center of $\mathrm{SL}_n(k)$ is C_n.

The group Spin_{10}

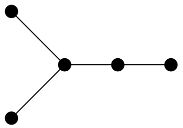

Diagram D_5

Automorphism group : C_2

The group Spin_{10}

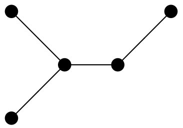

Diagram D_5

Automorphism group : C_2

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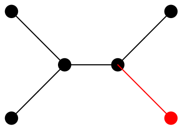
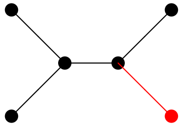


Completed diagram

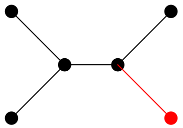
The group Spin_{10}



Completed diagram

Automorphism group : $C_4 \rtimes C_2$

The group Spin_{10}



Completed diagram

Automorphism group : $C_4 \rtimes C_2$

. . . showing that the center of Spin_{10} is cyclic of order 4.

Group of type E_6

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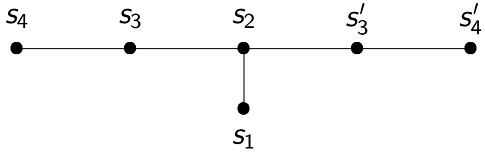

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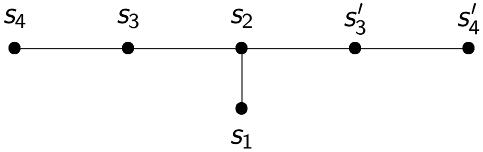

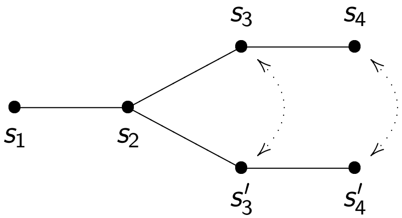

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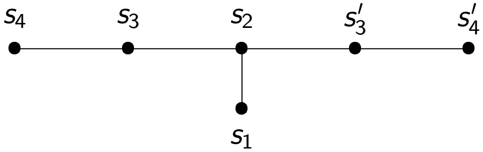
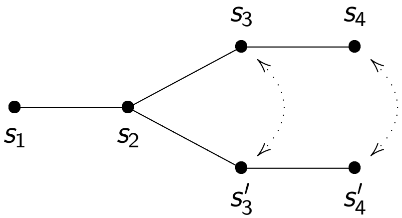
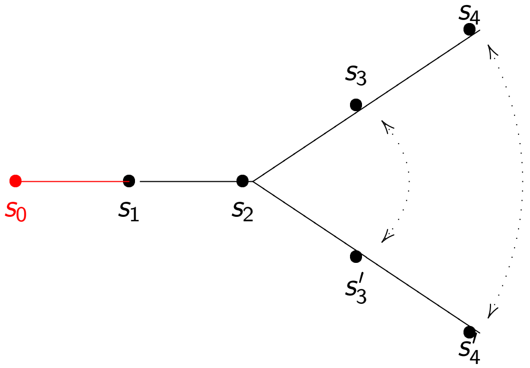


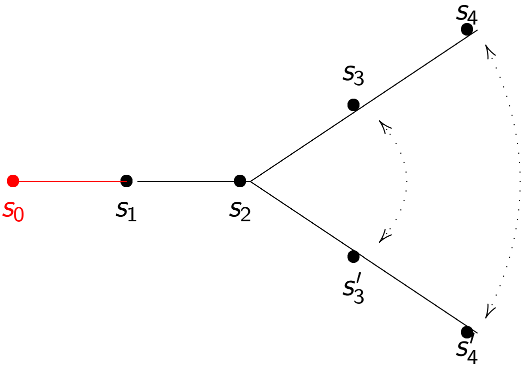


Automorphism group $= C_2$

Completed Dynkin diagram of type $\tilde{E}_6$



Completed Dynkin diagram of type $\tilde{E}_6$



Automorphism group $= \mathfrak{S}_3 = C_3 \rtimes C_2$ hence $Z(G) = C_3$.

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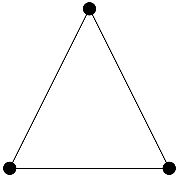
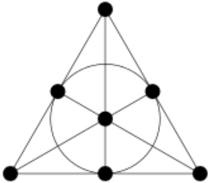
There are q^2+q+1 points and q^2+q+1 lines.

Examples

Projective Planes of order 1 and 2 :





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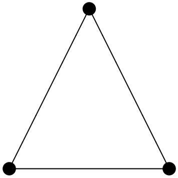

Whenever q is a prime power, there is a projective plane of order q, namely $\mathbb{P}^2(\mathbb{F}_q)$.

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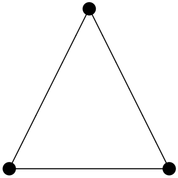

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$=$ So there exist projective planes of order 2,3,4,5, ,7,8,9, ,11.

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3	1	2

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22	31	13
33	12	21

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$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array} \qquad \begin{array}{ccc} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{array} \qquad \begin{array}{ccc} 11 & 23 & 32 \\ 22 & 31 & 13 \\ 33 & 12 & 21 \end{array}$$

This is related to "36 officers problem" considered by Euler :

Is it possible to arrange in a square 36 officers from 6 different regiments and with 6 different ranks in such a way that in each row and each column regiments and ranks are different ?

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Answer : No ! (Gaston Tarry) There is no Projective Plane of order 6.

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> *"Thompson forced Group Theory into a problem where it had nothing to do. "*

TRUTH AND BEAUTY.