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Integer Valued Entire Functions

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Abstract

An integer valued entire function is an entire function which is analytic in the complex plane and takes integer values at the nonnegative integers ; an example is 2^z.

A Hurwitz function is an entire function with derivatives of any order taking integer values at 0 ; an example is e^z.

Lower bounds for the growth of such functions and similar ones when they are not a polynomial have been investigated.

We survey this topic and we present some new results involving Lidstone polynomials.

Introduction : Hilbert's 7th problem (1900)

Hilbert

(1862 – 1943)

Prove that the numbers

$$e^{\pi} = 23.140\,692\,632\ldots$$

and

$$2^{\sqrt{2}} = 2.665\,144\,142\ldots$$

are transcendental.

A *transcendental number* is a number which is not algebraic. The *algebraic numbers* are the roots of the polynomials with rational coefficients.

http://www-history.mcs.st-and.ac.uk/Biographies/Hilbert.html

Values of the exponential function $\mathrm{e}^z = \exp(z)$

$$\mathrm{e}^{\pi} = 1 + \frac{\pi}{1} + \frac{\pi^2}{2} + \frac{\pi^3}{6} + \cdots + \frac{\pi^n}{n!} + \cdots$$

The number

$$\mathrm{e} = \mathrm{e}^1 = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n!} + \cdots$$

is transcendental (Hermite, 1873), while

$$\mathrm{e}^{\log 2} = 1 + \frac{\log 2}{1} + \frac{(\log 2)^2}{2} + \cdots + \frac{(\log 2)^n}{n!} + \cdots = 2$$

$$\mathrm{e}^{i\pi} = 1 + \frac{i\pi}{1} + \frac{(i\pi)^2}{2} + \cdots + \frac{(i\pi)^n}{n!} + \cdots = -1$$

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Constance Reid : Hilbert

The second problem became known as Hilbert's α^β conjecture. As Hilbert notes, corollaries of this conjecture include the transcendence of $2^{\sqrt{2}}$ and of $e^\pi = (e^{\pi i})^{-i} = (-1)^{-i}$.

An amusing incident concerning this conjecture is related in C. Reid's biography of Hilbert [Rei, C]. Carl Ludwig Siegel came to Gottingen as a student in 1919. He always remembered a lecture by Hilbert who, wanting to give his audience examples of problems in the theory of numbers which seem simple at first glance but which are, in fact, incredibly difficult, mentioned the Riemann Hypothesis, Fermat's Last Theorem and the transcendence of $2^{\sqrt{2}}$. Hilbert said that given recent progress he hoped to see the proof of the Riemann Hypothesis in his lifetime. Fermat's problem required totally new methods and possibly the youngest members of the audience would live to see it solved. As for $2^{\sqrt{2}}$, Hilbert said that no one at the lecture would live to see its proof. Hilbert was wrong! Siegel proved the transcendence of $2^{\sqrt{2}}$ about 10 years later (unpublished) and the solution of the α^β conjecture came shortly afterwards. He was right about Fermat's theorem and the Riemann Hypothesis is still unproved.

- Constance Reid. Hilbert. Springer Verlag 1970.
- Jay Goldman. The Queen of Mathematics : A Historically Motivated Guide to Number Theory. Taylor & Francis, 1998.

George Pólya   Aleksandr Osipovich Gel'fond

Growth of integer valued entire functions.

Pólya : $\mathbb{N}$ Gel'fond : $\mathbb{Z}[i]$

G. Pólya

(1887 – 1985)

A.O. Gel'fond

(1906 – 1968)

http://www-history.mcs.st-and.ac.uk/Biographies/Polya.html

http://www-history.mcs.st-and.ac.uk/Biographies/Gelfond.html

Integer valued entire functions on $\mathbb{N}$

G. Pólya (1915) :
An entire function f *which is not a polynomial and satisfies* $f(a) \in \mathbb{Z}$ *for all nonnegative integers* a *grows at least like* 2^z. *It satisfies*

$$\limsup_{R\to\infty} \frac{1}{R} \log |f|_R \geq \log 2.$$

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$$|f|_R := \sup_{|z|\leq R} |f(z)|.$$

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Integer valued entire functions on $\mathbb{N}$

Pólya's proof starts by expanding the function f into a *Newton interpolation series* at the points $0, 1, 2, \ldots$:

$$f(z) = a_0 + a_1 z + a_2 z(z-1) + a_3 z(z-1)(z-2) + \cdots$$

Since $f(n)$ is an integer for all $n \geq 0$, the coefficients a_n are rational and one can bound the denominators. If f does not grow fast, one deduces that these coefficients vanish for sufficiently large n.

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(1643– 1727)

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$$f(z) = f(\alpha_1) + (z - \alpha_1) f_1(z), \quad f_1(z) = f_1(\alpha_2) + (z - \alpha_2) f_2(z), \ldots$$

we deduce

$$f(z) = a_0 + a_1(z - \alpha_1) + a_2(z - \alpha_1)(z - \alpha_2) + \cdots$$

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An identity due to Hermite

$$\frac{1}{x-z} = \frac{1}{x-\alpha} + \frac{z-\alpha}{x-\alpha} \cdot \frac{1}{x-z}.$$

Ch. Hermite

(1822 – 1901)

Repeat :

$$\frac{1}{x-z} = \frac{1}{x-\alpha_1} + \frac{z-\alpha_1}{x-\alpha_1} \cdot \left(\frac{1}{x-\alpha_2} + \frac{z-\alpha_2}{x-\alpha_2} \cdot \frac{1}{x-z} \right).$$

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Newton interpolation

Integral formula :

$$f(z) = \sum_{j=0}^{n-1} a_j (z - \alpha_1) \cdots (z - \alpha_j) + R_n(z)$$

with

$$a_j = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{f(x)\mathrm{d}x}{(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_{j+1})} \quad (0 \leq j \leq n{-}1)$$

and

$$R_n(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) \cdot \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{f(x)\mathrm{d}x}{(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)(x - z)}.$$

Integer valued entire function on $\mathbb{Z}[i]$

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An entire function f *which is not a polynomial and satisfies* $f(a+ib) \in \mathbb{Z}[i]$ *for all* $a+ib \in \mathbb{Z}[i]$ *grows at least like* e^{cz^2}. *It satisfies*

$$\limsup_{R\to\infty} \frac{1}{R^2} \log |f|_R \geq \gamma.$$

Proof : Expand $f(z)$ into a Newton interpolation series at the Gaussian integers.

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Entire functions vanishing on $\mathbb{Z}[i]$

The canonical product associated with the lattice $\mathbb{Z}[i]$ is the Weierstrass sigma function

$$\sigma(z) = z \prod_{\omega \in \mathbb{Z}[i] \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right),$$

which is an entire function vanishing on $\mathbb{Z}[i]$.

$\sigma(z)$ grows like $e^{\pi z^2/2}$:

$$\limsup_{R \to \infty} \frac{1}{R^2} \log |\sigma|_R = \frac{\pi}{2}.$$

Hence

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Exact value of the constant γ of Gel'fond

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This is best possible : D.W. Masser (1980).

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Irrationality of e^π

The function $e^{\pi z}$ takes the value

$$(e^\pi)^a(-1)^b$$

at the point $a + ib \in \mathbb{Z}[i]$.

If the number

$$e^\pi = 23.140\,692\,632\,779\,269\,005\,729\,086\,367\,\ldots$$

were rational, these values would all be rational numbers.

Gel'fond's proof yields the irrationality of e^π and more generally the fact that e^π is not root of a polynomial $X^N - a$ with $N \geq 1$ and $a \in \mathbb{Q}$.

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Transcendence of e^{π}

A.O. Gel'fond (1929) : e^{π} is transcendental.

More generally, for α nonzero algebraic number with $\log \alpha \neq 0$ and for β imaginary quadratic number,

$$\alpha^{\beta} = \exp(\beta \log \alpha)$$

is transcendental.

Example : $\alpha = -1$, $\log \alpha = i\pi$, $\beta = -i$, $\alpha^{\beta} = (-1)^{-i} = e^{\pi}$.

R.O. Kuzmin (1930) : $2^{\sqrt{2}}$ is transcendental.

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Solution of Hilbert's seventh problem

A.O. Gel'fond and Th. Schneider (1934).

Transcendence of α^β
and of $(\log\alpha_1)/(\log\alpha_2)$
for algebraic α, β, α_2 *and* α_2.

Entire functions

An *entire function* is a function $\mathbb{C} \to \mathbb{C}$ which is analytic ($=$ holomorphic) in $\mathbb{C}$.

Examples are : polynomials, the exponential function

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trigonometric functions $\sin z$, $\cos z$, $\sinh z$, $\cosh z \ldots$

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The zeroes of an entire function are isolated.

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Entire functions vanishing on $\mathbb{Z}$

The function $\sin(\pi z)$ vanishes on $\mathbb{Z}$.

A finite sum

$$a_1(z)\sin(\pi z) + a_2(z)\sin(2\pi z) + \cdots + a_n(z)\sin(n\pi z)$$

with $a_1(z), a_2(z), \dots, a_n(z)$ in $\mathbb{C}[z]$ vanishes on $\mathbb{Z}$.

The same is true for an infinite sum which is uniformly convergent.

Question : which is the *smallest* nonzero entire function vanishing at each point of $\mathbb{Z}$?

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Order and type of entire functions

For $\varrho \in \mathbb{Z}$, $\varrho \geq 0$, the function e^{z^ϱ} is an entire function of order ϱ.

For $\tau \in \mathbb{C}$, $\tau \neq 0$, the function $e^{\tau z}$ is an entire function of order 1 and exponential type $|\tau|$.

For $\tau \in \mathbb{C}$, $\tau \neq 0$, the function

$$\sin(\tau z) = \frac{e^{i\tau\pi z} - e^{-i\tau\pi z}}{2i}$$

has order 1 and exponential type $|\tau|\pi$.

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$$\sin(\tau z) = \frac{e^{i\tau\pi z} - e^{-i\tau\pi z}}{2i}$$

has order 1 and exponential type $|\tau|\pi$.

Order and type of entire functions

For $\varrho \in \mathbb{Z}$, $\varrho \geq 0$, the function e^{z^ϱ} is an entire function of order ϱ.
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Order and type of entire functions

Maximum modulus principle :

$$|f|_r := \sup_{|z|=r} |f(z)| = \sup_{|z|\leq r} |f(z)|.$$

The order of an entire function f is

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log\log|f|_r}{\log r},$$

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Entire functions of finite exponential type

The exponential type of an entire function is also given by

$$\tau(f) = \limsup_{n\to\infty} |f^{(n)}(z_0)|^{1/n} \quad (z_0 \in \mathbb{C}).$$

Notation :

$$f^{(n)}(z) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n f(z).$$

The proof rests on Cauchy's estimate for the coefficients of the Taylor series and on Stirling's formula for $n!$.

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If the exponential type is finite, then f has order ≤ 1.
If f has order < 1, then the exponential type is 0.

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A polynomial has order 0, hence exponential type 0.
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Jensens's Formula :
A nonzero entire function vanishing on $\mathbb{Z}$ has exponential type ≥ 1.

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F. Carlson (1914) : the *smallest* entire functions vanishing at each point in $\mathbb{N}$ is

$$\sin(\pi z) = \pi z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{n}\right) \mathrm{e}^{z/n}$$

(Hadamard canonical product for $\mathbb{Z}$).

Another example of a function vanishing at each point in $\{0, 1, 2, \dots\}$ is Hadamard canonical product for $\mathbb{N}$ (Weierstrass form of the Gamma function)

$$\frac{1}{\Gamma(-z)} = -z\mathrm{e}^{-\gamma z} \prod_{n=1}^{\infty} (1 - z/n)\mathrm{e}^{z/n}.$$

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with $a_1, \ldots, a_n$ in $\mathbb{C}[z]$ and $n \le \tau(f)/\pi$.

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Integer valued entire functions

G. Pólya (1915). An *integer valued entire function* is an entire function f (analytic in $\mathbb{C}$) which satisfies $f(n) \in \mathbb{Z}$ for $n = 0, 1, 2, \ldots$.

Example : the polynomials

$$1, z, \frac{z(z-1)}{2}, \ldots, \frac{z(z-1)\cdots(z-n+1)}{n!}, \ldots$$

Any polynomial with complex coefficients which is an integer valued entire function is a linear combination with coefficients in $\mathbb{Z}$ of these polynomials :

$$a_0 + a_1 z + a_2 \frac{z(z-1)}{2} + \cdots + a_n \frac{z(z-1)\cdots(z-n+1)}{n!} + \cdots$$

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The function 2^z is a *transcendental* (= not a polynomial) integer valued entire function.

$$2^{p/q} = \sqrt[q]{2}^p \qquad 2^{\lim p_n/q_n} = \lim 2^{p_n/q_n},$$

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Growth of an integral valued entire function

G. Pólya (1915) : an integral valued entire of exponential type $< \log 2$ is a polynomial.

More precisely, if f is a transcendental integer valued entire function, then

$$\lim_{r\to\infty} \sqrt{r} 2^{-r} |f|_r > 0.$$

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G.H. Hardy (1917)

A refinement of Pólya's result was achieved by G.H. Hardy who proved that if f is an integer valued entire function such that

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G.H. Hardy

(1877 – 1947)

Compare with Pólya's assumption :

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A. Selberg (1941)

A. Selberg proved that if an integer–valued entire function f satisfies

$$\tau(f) \leq \log 2 + \frac{1}{1500},$$

then f is of the form $P_0(z) + P_1(z)2^z$, where P_0 and P_1 are polynomials.

A. Selberg

(1917 – 2007)

There are only countably many such functions.

https://www-history.mcs.st-andrews.ac.uk/Biographies/Selberg.html

Ch. Pisot (1942)

Ch. Pisot proved that if an integer–valued entire function f has exponential type ≤ 0.8, then f is of the form

$$P_0(z) + 2^z P_1(z) + \gamma^z P_2(z) + \overline{\gamma}^z P_3(z),$$

where P_0, P_1, P_2, P_3 are polynomials and $\gamma, \overline{\gamma}$ are the non real roots of the polynomial $z^3 - 3z + 3$.

This contains the result of Selberg, since

$$|\log \gamma| = 0.758\,98 \cdots > \log 2 + \frac{1}{1500} = 0.693.$$

Pisot obtained more general result for functions of exponential type $< 0.9934\ldots$

Ch. Pisot

(1910 – 1984)

https://www-history.mcs.st-andrews.ac.uk/Biographies/Pisot.html

Completely integer–valued entire function

A *completely integer–valued entire function* is an entire function which takes values in $\mathbb{Z}$ at all points in $\mathbb{Z}$.

Let $u > 1$ be a quadratic unit, root of a polynomial $X^2 + aX + 1$ for some $a \in \mathbb{Z}$. Then the functions

$$u^z + u^{-z} \quad \text{and} \quad \frac{u^z - u^{-z}}{u - u^{-1}}$$

are completely integer–valued entire function of exponential type $\log u$.

Examples of such quadratic units are the roots u and u^{-1} of the polynomial $X^2 - 3X + 1$:

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Quizz

Let ϕ be the Golden ratio and let $\tilde{\phi} = -\phi^{-1}$, so that

$$X^2 - X - 1 = (X - \phi)(X - \tilde{\phi}).$$

For any $n \in \mathbb{Z}$ we have

$$\phi^n + \tilde{\phi}^n \in \mathbb{Z}$$

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$$\log \phi = -\log |\tilde{\phi}| < \log 2.$$

Why is $\phi^z + \tilde{\phi}^z$ not a counter example to Pólya's result on the growth of transcendental integer valued entire functions ?

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is a completely integer–valued transcendental entire function.

In 1921, F. Carlson proved that if the type $\tau(f)$ of a completely integer–valued entire function f satisfies

$$\tau(f) < \log\left(\frac{3+\sqrt{5}}{2}\right) = 0.962\ldots,$$

then f is a polynomial.

Completely integer–valued entire function

The function

$$\frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^{z}-\frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^{-z}.$$

is a completely integer–valued transcendental entire function.

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A. Selberg (1941)

A. Selberg : if the type $\tau(f)$ of a completely integer–valued entire function f satisfies

$$\tau(f) \leq \log\left(\frac{3+\sqrt{5}}{2}\right) + 2 \cdot 10^{-6},$$

then f is of the form

$$P_0(z) + P_1(z)\left(\frac{3+\sqrt{5}}{2}\right)^z + P_2(z)\left(\frac{3+\sqrt{5}}{2}\right)^{-z}$$

where P_0, P_1, P_2 are polynomials.

Hurwitz functions

A *Hurwitz function* is an entire function f such that $f^{(n)}(0) \in \mathbb{Z}$ for all $n \geq 0$.

A. Hurwitz

(1859 – 1919)

The polynomials which are Hurwitz functions are the polynomials of the form

$$a_0 + a_1 z + a_2 \frac{z^2}{2} + a_3 \frac{z^3}{6} + \cdots + a_n \frac{z^n}{n!}$$

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https://www-history.mcs.st-andrews.ac.uk/Biographies/Hurwitz.html

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Hurwitz functions

The exponential function

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots$$

is a transcendental Hurwitz function of exponential type 1. For $a \in \mathbb{Z}$, the function e^{az} is also a Hurwitz function of exponential type $|a|$.

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Kakeya (1916)

S. Kakeya (1916) : a Hurwitz function of exponential type < 1 is a polynomial.

More precisely, he proved that a Hurwitz function satisfying

$$\limsup_{r\to\infty} \sqrt{r}\mathrm{e}^{-r}|f|_r = 0$$

is a polynomial.

Question : is $\sqrt{r}$ superfluous ? Is e^z the *smallest* Hurwitz function ?

Recall Pólya vs Hardy : an integer valued entire functions of low growth is a polynomial.

Pólya's assumption : $\lim_{r\to\infty} \sqrt{r}2^{-r}|f|_r = 0.$

Hardy's assumption : $\lim_{r\to\infty} 2^{-r}|f|_r = 0.$

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is a polynomial.
(Kakeya's assumption : $\limsup = 0$).

This is best possible for uncountably many functions, as shown by the functions

$$f(z) = \sum_{n\geq 0} \frac{e_n}{2^n!} z^{2^n}$$

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Sato and Straus (1964)

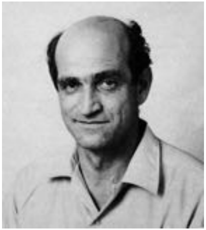
D. Sato and E.G. Straus proved that for every $\epsilon > 0$, there exists a transcendental Hurwitz function with

$$\limsup_{r\to\infty} \sqrt{2\pi r}\, \mathrm{e}^{-r} \left(1 + \frac{1+\epsilon}{24r}\right)^{-1} |f|_r < 1,$$

while every Hurwitz function for which

$$\limsup_{r\to\infty} \sqrt{2\pi r}\, \mathrm{e}^{-r} \left(1 + \frac{1-\epsilon}{24r}\right)^{-1} |f|_r \le 1$$

is a polynomial.



E.G. Straus

(1922 – 1983)

https://www-history.mcs.st-andrews.ac.uk/Biographies/Straus.html

Integer–valued functions vs Hurwitz functions :

Let us display horizontally the rational integers and vertically the derivatives.

integer–valued functions : horizontal

$$\begin{array}{cccccccc} f & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ & 0 & 1 & 2 & \cdots & n & \cdots \end{array}$$

Hurwitz functions : vertical

$$\begin{array}{cc} & \vdots \\ f^{(n)} & \bullet \\ & \vdots \\ f' & \bullet \\ f & \bullet \\ & 0 \end{array}$$

Several points and / or several derivatives

There are several natural ways to mix integer–valued functions and Hurwitz functions :

- *horizontally*, one may include finitely may derivatives in the study of integer–valued functions.

  A k*–times integer–valued function* is an entire function f such that $f^{(j)}(n) \in \mathbb{Z}$ for all $n \geq 0$ and $j = 0, 1, \ldots, k-1$.

- *Vertically*, one may consider entire functions with all derivatives at finitely many points taking integer values.

  A k*–point Hurwitz function* is an entire function having all its derivatives at $0, 1, \ldots, k-1$ taking integer values.

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k–times integer–valued functions (horizontal)

$k=2$: $f(n)\in\mathbb{Z}$, $f'(n)\in\mathbb{Z}$ $(n\geq 0)$.

$$\begin{array}{ccccccc} f' & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ f & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\ & 0 & 1 & 2 & \cdots & n & \cdots \end{array}$$

According to Gel'fond (1929), a k–times integer–valued function of exponential type $< k\log\left(1+\mathrm{e}^{-\frac{k-1}{k}}\right)$ is a polynomial.

The function $(\sin(\pi z))^k$ has exponential type $k\pi$ and vanishes with multiplicity k on $\mathbb{Z}$.

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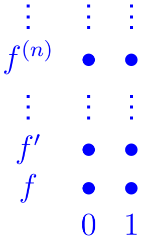
Two–point Hurwitz functions (vertical)

$k = 2$: $f^{(n)}(0) \in \mathbb{Z}$, $f^{(n)}(1) \in \mathbb{Z}$ ($n \geq 0$).



D. Sato (1971) : every two point Hurwitz entire functions for which there exists a positive constant C such that

$$|f|_r \leq C \exp\left(r^2 - r - \log r\right)$$

is a polynomial.

Also, there exist transcendental two point Hurwitz entire functions with

$$|f|_r \leq \exp\left(r^2 + r - \log r + O(1)\right).$$

k–point Hurwitz functions

For $k \geq 3$ our knowledge is more limited.
D. Sato (1971) proved that the order of k–point Hurwitz functions is $\geq k$.
This is best possible, as shown by the function $\mathrm{e}^{z(z-1)\cdots(z-k+1)}$.

For an entire function f of order $\leq \varrho$, define

$$\tau_\varrho(f) = \limsup_{r\to\infty} \frac{\log |f|_r}{r^\varrho}.$$

f grows like $e^{\tau_\varrho(f) z^\varrho}$.

Example : for $k \geq 1$, the function $f(z) = \mathrm{e}^{z(z-1)\cdots(z-k+1)}$ has order k and $\tau_k(f) = 1$.

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k–point Hurwitz functions

L. Bieberbach (1953) stated that if a transcendental entire function f of order ϱ is a k–point Hurwitz entire function, then either $\varrho > k$, or $\varrho = k$ and the type $\tau_k(f)$ of f satisfies $\tau_k(f) \geq 1$.

L. Bieberbach

(1886 – 1982)

https://www-history.mcs.st-andrews.ac.uk/Biographies/Bieberbach.html

k–point Hurwitz functions

However, as noted by D. Sato, since the polynomial

$$a(z) = \frac{1}{2}z(z-1)(z-2)(z-3)$$

can be written

$$a(z) = \frac{1}{2}z^4 - 3z^3 - \frac{11}{2}z^2 - 3z,$$

it satisfies $a'(z) \in \mathbb{Z}[z]$.

It follows that the function $e^{a(z)}$ is a 4-point Hurwitz transcendental entire function of order $\varrho = 4$ and $\tau_4(f) = 1/2$.

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Utterly integer–valued entire functions

Another way of mixing the horizontal and the vertical generalizations is to introduce *utterly integer–valued entire function*, namely entire functions f which satisfy $f^{(n)}(m) \in \mathbb{Z}$ for all $n \geq 0$ and $m \in \mathbb{Z}$.

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & & \vdots & \\
f^{(n)} & \bullet & \bullet & \cdots & \bullet & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
f' & \bullet & \bullet & \cdots & \bullet & \cdots \\
f & \bullet & \bullet & \cdots & \bullet & \cdots \\
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\end{array}
$$

G.A. Fridman (1968), M. Welter (2005)

E.G. Straus (1951) suggested that transcendental utterly integer–valued entire function may not exist.

G.A. Fridman (1968) showed that there exists transcendental utterly integer–valued function f with

$$\limsup_{r\to\infty} \frac{\log\log|f|_r}{r} \le \pi$$

and proved that a transcendental utterly integer–valued function f satisfies

$$\limsup_{r\to\infty} \frac{\log\log|f|_r}{r} \ge \log(1+1/\mathrm{e}).$$

The bound $\log(1+1/\mathrm{e})$ was improved by M. Welter (2005) to $\log 2$: hence f grows like e^{2^z} (double exponential).

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Sato's examples

An utterly integer–valued transcendental entire functions has infinite order : it grows like a double exponential $e^{e^{\alpha z}}$.

D. Sato (1985) constructed a nondenumerable set of utterly integer–valued transcendental entire functions.

He selected inductively the coefficients a_n with

$$\frac{1}{n!(2\pi)^n} \le |a_n| \le \frac{3}{n!(2\pi)^n}$$

and defined

$$f(z) = \sum_{n\ge 0} a_n \sin^n(2\pi z).$$

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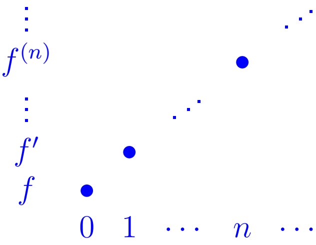
Abel series

There is also a *diagonal* way of mixing the questions of integer–valued functions and Hurwitz functions by considering entire functions f such that $f^{(n)}(n) \in \mathbb{Z}$. The source of this question goes back to N. Abel.



Niels Abel

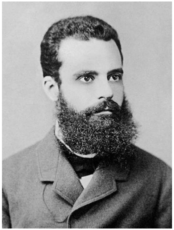
(1802 – 1829)

https://www-history.mcs.st-andrews.ac.uk/Biographies/Abel.html

Abel series

G.H. Halphén

(1844 – 1889)



V. Pareto

(1848 – 1923)

Abel's interpolation problem is to find an entire function f for which the values $f^{(n)}(n)$ are prescribed. It was studied by G. Halphén (1882), V. Pareto (1892), W. Gontcharoff (1930), R.C. Buck (1946).

https://www-history.mcs.st-andrews.ac.uk/Biographies/Halphen.html

https://fr.wikipedia.org/wiki/Vilfredo_Pareto

Abel's interpolation problem

The lack of unicity arises from nonzero entire functions f, like $\sin(\pi z/2)$, satisfying $f^{(n)}(n) = 0$ for $n \geq 0$.

Let us start with polynomials. Given a polynomial f, we are looking for a finite expansion

$$f(z) = \sum_{n\geq 0} f^{(n)}(n) P_n(z).$$

We need a sequence of polynomials $(P_n)_{n\geq 0}$ satisfying

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amount to $P_0 = 1$,

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The solution

$$P_n(z) = \frac{1}{n!} z(z-n)^{n-1} \qquad (n \geq 1)$$

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It follows that any polynomial f has a finite expansion

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Such an expansion (with a series in the right hand side which is absolutely and uniformly convergent on any compact of $\mathbb{C}$) holds also for any entire function f of finite exponential type $< \omega$, where $\omega = 0.278\,464\,542\ldots$ is the positive real number defined by $\omega \mathrm{e}^{\omega+1} = 1$.

Example (Legendre, Abel). For $|\tau| < \omega$, we have

$$e^{\tau z} = 1 + \tau e^{\tau} z + \frac{1}{2}(\tau e^{\tau})^2 z(z-2) + \frac{1}{6}(\tau e^{\tau})^3 z(z-3)^2 + \cdots$$

If an entire function f of exponential type $< \omega$ satisfies $f^{(n)}(n) = 0$ for all sufficiently large n, then f is a polynomial.

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Let $\tau_0 = 0.567\,143\,290\ldots$ be the positive real number defined by $\tau_0 e^{\tau_0} = 1$.

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The example of the function $f(z) = e^{\tau_0 z}$ which has $f^{(n)}(n) = 1$ for all $n \geq 0$ shows that this result is sharp.

Let τ_1 be the complex number defined by $\tau_1 e^{\tau_1} = (1 + i\sqrt{3})/2$. Then an entire function f of exponential type $< |\tau_1| = 0.616\ldots$ such that $f^{(n)}(n) \in \mathbb{Z}$ for all sufficiently large integers $n \geq 0$ is of the form $P(z) + Q(z)e^{\tau_0 z}$, where P and Q are polynomials.

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Variations on this theme

- q analogues and multiplicative versions (geometric progressions) :
  Gel'fond (1933, 1952), J.A. Kazmin (1973), J.P. Bézivin (1984, 1992) F. Gramain (1990), M. Welter (2000, 2005), J-P. Bézivin (2014).
- analogs in finite characteristic :
  D. Adam (2011), D. Adam and M. Welter (2015).
- congruences :
  A. Perelli and U. Zannier (1981), J. Pila (2003, 2005).
- several variables :
  S. Lang (1965), F. Gross (1965), A. Baker (1967), V. Avanissian and R. Gay (1975), F. Gramain (1977, 1986), P. Bundschuh (1980). . . .

Connection with transcendental number theory

In 1950, E. G. Straus introduced a connection between integer–valued functions and transcendence results, including the Hermite–Lindemann Theorem on the transcendence of e^{α} for $\alpha \neq 0$ algebraic.

However, as he pointed out in a footnote, at the same time, Th. Schneider obtained more far reaching results, which ultimately gave rise to the Schneider–Lang Criterion (1962).

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The Masser–Gramain–Weber constant

D.W. Masser (1980) and F. Gramain–M. Weber (1985) studied an analog of Euler's constant for $\mathbb{Z}[i]$, which arises in a 2–dimensional analogue of Stirling's formula :

$$\delta = \lim_{n\to\infty} \left(\sum_{k=2}^{n} (\pi {r_k}^2)^{-1} - \log n \right),$$

where r_k is the radius of the smallest disc in $\mathbb{R}^2$ that contains at least k integer lattice points inside it or on its boundary.

In 2013, G. Melquiond, W. G. Nowak and P. Zimmermann computed the first four digits :

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Lidstone and Whittaker interpolation

George James Lidstone

(1870 – 1952)

⋮	⋮	⋮
$f^{(2n+1)}$	○	○
$f^{(2n)}$	●	●
⋮	⋮	⋮
f''	●	●
f'	○	○
f	●	●
	s_0	s_1

John Macnaghten Whittaker

(1905 – 1984)

⋮	⋮	⋮
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$f^{(2n)}$	○	●
⋮	⋮	⋮
f''	○	●
f'	●	○
f	○	●
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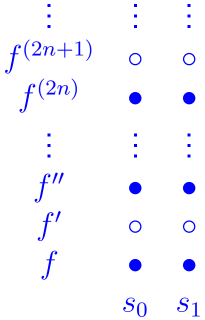
Arithmetic result for Lidstone interpolation

Let s_0 and s_1 be two complex numbers and f an entire function satisfying $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n.



If

$$\tau(f) < \min\left\{1, \frac{\pi}{|s_0 - s_1|}\right\},$$

then f is a polynomial.

This is best possible.

• values in $\mathbb{Z}$ ○ no condition

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The function

$$f(z) = \frac{\sinh(z - s_1)}{\sinh(s_0 - s_1)}$$

has exponential type 1 and satisfies $f^{(2n)}(s_0) = 1$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

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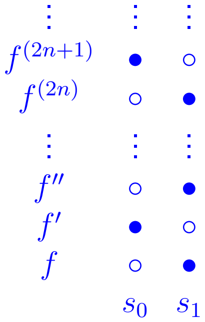

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Integer Valued Entire Functions

Professeur Émérite, Sorbonne Université,
Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/

01/11/2019