An introduction to
irrationality and transcendence methods.

Michel Waldschmidt

Lecture 5 [11]

Most part of this section if from [27].

5 Conjectures and open problems

We already met a number of open problems in these notes, in particular in § 1.1.1. We collect further conjectures in this field, but this is only a very partial list of questions which deserve to be investigated further.

Part of this section if from [W 2004], especially § 3.

When K is a field and k a subfield, we denote by $\mathrm{trdeg}_k K$ the transcendence degree of the extension K/k. In the case $k = \mathbb{Q}$ we write simply $\mathrm{trdeg} K$ (see [La 1993] Chap. VIII, § 1).

5.1 Schanuel's Conjecture and some consequences

Schanuel's Conjecture is a simple but far-reaching statement – see the historical note to Chap. III of [La 1966].

Conjecture 5.1 (Schanuel)**.** *Let $x_1, \ldots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then the transcendence degree over $\mathbb{Q}$ of the field*

$$\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$$

is at least n.

The special case where $x_1, \ldots, x_n$ are all algebraic is just Theorem 2.41. This is almost the single case where the conjecture is known to be true.

According to S. Lang ([La 1966] p. 31): "From this statement, one would obtain most statements about algebraic independence of values of e^t and $\log t$ which one feels to be true". See also [La 1971] p. 638–639.

A detailed discussion of consequences of Schanuel's Conjecture is given by P. Ribenboim in [Ri 2000], Chap 10 *What kind of Number is $2^{\sqrt{2}}$?*, § 10.7 *Transcendental Numbers*, § 10.7.G *The conjecture of Schanuel.*

[11] http://www.math.jussieu.fr/∼miw/articles/pdf/AWSLecture5.pdf

Exercise 5.2. *1) Deduce from Schanuel's Conjecture that the following numbers are algebraically independent*

$$e,\ \pi,\ e^{\pi},\ \log\pi,\ e^{e},\ \pi^{e},\ \pi^{\pi},\ \log 2,\ 2^{\pi},\ 2^{e},\ 2^{i},\ e^{i},\ \pi^{i},\ \log 3,\ (\log 2)^{\log 3},\ 2^{\sqrt{2}}.$$

2) Define K_0 to be the field of algebraic numbers. Inductively, for $n\ge 1$, define K_n as the algebraic closure of the field generated over K_{n-1} by the numbers e^x, where x ranges over K_{n-1}. Let Ω be the union of K_n, $n\ge 0$. Deduce from Schanuel's Conjecture that the numbers

$$\pi,\ \log\pi,\ \log\log\pi,\ \log\log\log\pi,\dots$$

are algebraically independent over Ω.
(See [La 1966]).
3) Get a (conjectural) generalisation of question 2) involving the field Ω_- defined as follows. Let $E_0=\overline{\mathbb{Q}}$. Inductively, for $n\ge 1$, define L_n as the algebraic closure of the field generated over L_{n-1} by the numbers y, where y ranges over the set of complex numbers such that $e^y\in L_{n-1}$. Define Ω_- as the union of L_n, $n\ge 0$.'

The following statements [Ge 1934] are consequences of Conjecture 5.1.

Question 5.3. *Let $\beta_1,\dots,\beta_n$ be $\mathbb{Q}$-linearly independent algebraic numbers and let $\log\alpha_1,\dots,\log\alpha_m$ be $\mathbb{Q}$-linearly independent logarithms of algebraic numbers. Then the numbers*

$$e^{\beta_1},\dots,e^{\beta_n},\ \log\alpha_1,\dots,\log\alpha_m$$

are algebraically independent over $\mathbb{Q}$.

Question 5.4. *Let $\beta_1,\dots,\beta_n$ be algebraic numbers with $\beta_1\neq 0$ and let $\log\alpha_1,\dots,\log\alpha_m$ be logarithms of algebraic numbers with $\log\alpha_1\neq 0$ and $\log\alpha_2\neq 0$. Then the numbers*

$$e^{\beta_1 e^{\beta_2 e^{\cdot^{\cdot^{\cdot^{\beta_{n-1}e^{\beta_n}}}}}}} \quad \textit{and} \quad \alpha_1^{\alpha_2^{\cdot^{\cdot^{\cdot^{\alpha_m}}}}}$$

are transcendental, and there is no nontrivial algebraic relation between such numbers.

A quantitative refinement of Conjecture 5.1 is suggested in [W 1999b] Conjecture 1.4 and reproduced below (Conjecture 5.48).

A quite interesting approach to Schanuel's Conjecture is given in [Ro 2001a] where D. Roy states the next conjecture which he shows to be equivalent to Schanuel's one. Let $\mathcal{D}$ denote the derivation

$$\mathcal{D}=\frac{\partial}{\partial X_0}+X_1\frac{\partial}{\partial X_1}$$

over the ring $\mathbb{C}[X_0,X_1]$. The *height* of a polynomial $P\in\mathbb{C}[X_0,X_1]$ is defined as the maximum of the absolute values of its coefficients.

Conjecture 5.5 (Roy)**.** *Let k be a positive integer, $y_1, \dots, y_k$ complex numbers which are linearly independent over $\mathbb{Q}$, $\alpha_1, \dots, \alpha_k$ non-zero complex numbers and s_0, s_1, t_0, t_1, u positive real numbers satisfying*

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\} \quad \textit{and} \quad \max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Assume that, for any sufficiently large positive integer N, there exists a non-zero polynomial $P_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\le N^{t_0}$ in X_0, partial degree $\le N^{t_1}$ in X_1 and height $\le e^N$ which satisfies

$$\left| \big(\mathcal{D}^k P_N\big)\Big(\sum_{j=1}^{k} m_j y_j, \prod_{j=1}^{k} \alpha_j^{m_j}\Big) \right| \le \exp(-N^u)$$

for any non-negative integers $k, m_1, \dots, m_k$ with $k \le N^{s_0}$ and $\max\{m_1, \dots, m_k\} \le N^{s_1}$. Then

$$\mathrm{trdeg}\mathbb{Q}(y_1, \dots, y_k, \alpha_1, \dots, \alpha_k) \ge k.$$

This work of Roy's also provides an interesting connection with other open problems related to the Schwarz Lemma for complex functions of several variables (see [Ro 2002] Conjectures 6.1 and 6.3).

The most important special case of Schanuel's Conjecture is the *Conjecture of algebraic independence of logarithms of algebraic numbers.*

Conjecture 5.6 (Algebraic Independence of Logarithms of Algebraic Numbers)**.** *Let $\lambda_1, \dots, \lambda_n$ be $\mathbb{Q}$-linearly independent complex numbers. Assume that the numbers $e^{\lambda_1}, \dots, e^{\lambda_n}$ are algebraic. Then the numbers $\lambda_1, \dots, \lambda_n$ are algebraically independent over $\mathbb{Q}$.*

We are very far from this conjecture. Indeed, it is not yet even known that there exist at least two algebraically independent logarithms of algebraic numbers!

An interesting reformulation of Conjecture 5.6 is due to D. Roy [Ro 1995]. Recall that $\mathcal{L}$ denotes the $\mathbb{Q}$-vector subspace of $\lambda \in \mathbb{C}$ for which e^λ is algebraic. Instead of looking, for a fixed tuple $(\lambda_1, \dots, \lambda_n) \in \mathcal{L}^n$, to the condition $P(\lambda_1, \dots, \lambda_n) = 0$ for some $P \in \mathbb{Z}[X_1, \dots, X_n]$, we fix $P \in \mathbb{Z}[X_1, \dots, X_n]$ and we consider the set of zeros of P in $\mathcal{L}^n$.

Roy's statement is:

Conjecture 5.7. *For any algebraic subvariety V of $\mathbb{C}^n$ defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers, the set $V \cap \mathcal{L}^n$ is the union of the sets $E \cap \mathcal{L}^n$, where E ranges over the set of vector subspaces of $\mathbb{C}^n$ which are contained in V.*

Such a statement is reminiscent of several of Lang's conjectures in Diophantine geometry (e.g., [La 1991] Chap. I, § 6, Conjectures 6.1 and 6.3).

Not much is known about the algebraic independence of logarithms of algebraic numbers, apart from the work of D. Roy on the rank of matrices whose entries are either logarithms of algebraic numbers, or more generally linear combinations of logarithms of algebraic numbers. We refer to [W 2000b] for a detailed study of this question as well as related ones.

Conjecture 5.6 has many consequences. The next three ones are suggested by the work of D. Roy ([Ro 1989] and [Ro 1990]) on matrices whose entries are linear combinations of logarithms of algebraic numbers (see also [W 2000b] Conjecture 11.17, § 12.4.3 and Exercise 12.12).

Consider the $\overline{\mathbb{Q}}$-vector space $\widetilde{\mathcal{L}}$ spanned by 1 and $\mathcal{L}$. In other words $\widetilde{\mathcal{L}}$ is the set of complex numbers which can be written

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

where $\beta_0, \beta_1, \ldots, \beta_n$ are algebraic numbers, $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers, and finally $\log \alpha_1, \ldots, \log \alpha_n$ are logarithms of $\alpha_1, \ldots, \alpha_n$ respectively.

Conjecture 5.8 (Strong Four Exponentials Conjecture)**.** *Let x_1, x_2 be two $\overline{\mathbb{Q}}$-linearly independent complex numbers and y_1, y_2 be also two $\overline{\mathbb{Q}}$-linearly independent complex numbers. Then at least one of the four numbers x_1y_1, x_1y_2, x_2y_1, x_2y_2 does not belong to $\widetilde{\mathcal{L}}$.*

The following special case is also open.

Conjecture 5.9 (Strong Five Exponentials Conjecture)**.** *Let x_1, x_2 be two $\mathbb{Q}$-linearly independent complex numbers, and y_1, y_2 be also two $\mathbb{Q}$-linearly independent complex numbers. Further, let β_{ij} $(i = 1, 2,\ j = 1, 2)$, γ_1 and γ_2 be six algebraic numbers with $\gamma_1 \neq 0$. Assume that the five numbers*

$$e^{x_1y_1-\beta_{11}},\ e^{x_1y_2-\beta_{12}},\ e^{x_2y_1-\beta_{21}},\ e^{x_2y_2-\beta_{22}},\ e^{(\gamma_1x_1/x_2)-\gamma_2}$$

are algebraic. Then all five exponents vanish,

$$x_iy_j = \beta_{ij} \quad (i = 1, 2, \quad j = 1, 2) \quad \text{and} \quad \gamma_1x_1 = \gamma_2x_2.$$

.

A consequence of Conjecture 5.9 is the solution of the open problem of the transcendence of the number e^{π^2}, and more generally of $\alpha^{\log \alpha} = e^{\lambda^2}$ when α is a non-zero algebraic number and $\lambda = \log \alpha$ a non-zero logarithm of α.

The next conjecture is proposed in [Ro 1995].

Conjecture 5.10 (Roy)**.** *For any 4×4 skew-symmetric matrix $\mathtt{M}$ with entries in $\mathcal{L}$ and rank ≤ 2, either the rows of $\mathtt{M}$ are linearly dependent over $\mathbb{Q}$, or the column space of $\mathtt{M}$ contains a non-zero element of $\mathbb{Q}^4$.*

Finally a special case of Conjecture 5.10 is the well known Four Exponentials Conjecture (see 2.48) due to Schneider ([Schn 1957] Chap. V, end of § 4, Problem 1), S. Lang ([La 1966] Chap. II, § 1; [La 1971] p. 638) and K. Ramachandra ([R 1968 II], § 4).

Conjecture 5.11 (Four Exponentials Conjecture)**.** *Let x_1, x_2 be two $\mathbb{Q}$-linearly independent complex numbers and y_1, y_2 also be two $\mathbb{Q}$-linearly independent complex numbers. Then at least one of the four numbers*

$$\exp(x_i y_j) \qquad (i = 1, 2, \ j = 1, 2)$$

is transcendental.

We refer to [W 2000b] for a detailed discussion of this topic, including the notion of *structural rank of a matrix* and the result, due to D. Roy, that Conjecture 5.6 is equivalent to a conjecture on the rank of matrices whose entries are logarithms of algebraic numbers.

A classical problem on algebraic independence of algebraic powers of algebraic numbers has been raised by A. O. Gel'fond [Ge 1949] and Th. Schneider [Schn 1957] Chap. V, end of § 4, Problem 7. The data are an irrational algebraic number β of degree d, and a non-zero algebraic number α with a non-zero logarithm $\log \alpha$. We write α^z in place of $\exp\{z \log \alpha\}$. Gel'fond's problem is

Conjecture 5.12 (Gel'fond)**.** *The two numbers*

$$\log \alpha \quad \textit{and} \quad \alpha^\beta$$

are algebraically independent over $\mathbb{Q}$.

Schneider's question is

Conjecture 5.13 (Schneider)**.** *The $d-1$ numbers*

$$\alpha^\beta, \ \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$$

are algebraically independent over $\mathbb{Q}$.

Exercise 5.14. *Let α be a non-zero algebraic number and let ℓ be any non–zero number such that $e^\ell = \alpha$. For $z \in \mathbb{C}$ define α^z as $\exp\{z\ell\}$ (which is the same as $e^{z\ell}$). Show that the following statements are equivalent.*
(i) For any irrational algebraic complex number β, the transcendence degree over $\mathbb{Q}$ of the field

$$\mathbb{Q}\{\alpha^{\beta^i} \ ; \ i \geq 1\}$$

is $d-1$ where d is the degree of β.
(ii) For any algebraic numbers $\beta_1, \ldots, \beta_m$ such that the numbers $1, \beta_1, \ldots, \beta_m$ are $\mathbb{Q}$-linearly independent, the numbers $\alpha^{\beta_1}, \ldots, \alpha^{\beta_m}$ are algebraically independent.

The first partial results towards a proof of Conjecture 5.13 are due to A. O. Gel'fond [Ge 1952]. For the more recent ones, see [NeP 2001], Chap. 13 and 14.

Combining both questions 5.12 and 5.13 yields a stronger conjecture.

Conjecture 5.15 (Gel'fond-Schneider)**.** *The d numbers*

$$\log\alpha,\ \alpha^\beta,\ \alpha^{\beta^2},\ldots,\alpha^{\beta^{d-1}}$$

are algebraically independent over $\mathbb{Q}$.

Partial results are known (see § 2.3.5 and 3.3.3). Large transcendence degree results deal, more generally, with the values of the usual exponential function at products x_iy_j, when $x_1,\ldots,x_d$ and $y_1,\ldots,y_\ell$ are $\mathbb{Q}$-linearly independent complex (or p-adic) numbers. The six exponentials Theorem states that, in these circumstances, the $d\ell$ numbers $e^{x_iy_j}$ $(1\le i\le d,\ 1\le j\le\ell)$ cannot all be algebraic if $d\ell > d+\ell$. Assuming stronger conditions on d and ℓ, namely $d\ell\ge 2(d+\ell)$, one deduces that two at least of these $d\ell$ numbers $e^{x_iy_j}$ are algebraically independent over $\mathbb{Q}$. Other results are available involving, in addition to $e^{x_iy_j}$, either the numbers $x_1,\ldots,x_d$ themselves, or $y_1,\ldots,y_\ell$, or both. But an interesting point is that, if we wish to obtain a higher transcendence degree, say to obtain that three at least of the numbers $e^{x_iy_j}$ are algebraically independent over $\mathbb{Q}$, one needs a further assumption, which is a measure of linear independence over $\mathbb{Q}$ for the tuple $x_1,\ldots,x_d$ as well as for the tuple $y_1,\ldots,y_\ell$. To remove this so-called *technical hypothesis* does not seem to be an easy challenge (see [NeP 2001] Chap. 14, § 2.2 and § 2.3).

The need for such a technical hypothesis seems to be connected with the fact that the actual transcendence methods produce not only a qualititative statement (lower bound for the transcendence degree), but also quantitative statements (transcendence measures and measures of algebraic independence).

Several complex results have not yet been established in the ultrametric situation. Two noticeable instances are

Conjecture 5.16 (p-adic analogue of Lindemann-Weierstrass's Theorem)**.** *Let $\beta_1,\ldots,\beta_n$ be p-adic algebraic numbers in the domain of convergence of the p-adic exponential function $\exp_p$. Then the n numbers $\exp_p\beta_1,\ldots,\exp_p\beta_n$ are algebraically independent over $\mathbb{Q}$.*

Conjecture 5.17 (p-adic analogue of an algebraic independence result of Gel'fond)**.** *Let α be a non-zero algebraic number in the domain of convergence of the p-adic logarithm $\log_p$, and let β be a p-adic cubic algebraic number, such that $\beta\log_p\alpha$ is in the domain of convergence of the p-adic exponential function $\exp_p$. Then*

$$\alpha^\beta=\exp_p(\beta\log_p\alpha)\quad\textit{and}\quad\alpha^{\beta^2}=\exp_p(\beta^2\log_p\alpha)$$

are algebraically independent over $\mathbb{Q}$.

The p-adic analogue of Conjecture 5.6 would solve Leopoldt's Conjecture on the p-adic rank of the units of an algebraic number field [Le 1962] (see also [N 1990] and [Gra 2002]), by proving the nonvanishing of the p-adic regulator.

Algebraic independence results for the values of the exponential function (or more generally for analytic subgroups of algebraic groups) in several variables have already been established, but they are not yet satisfactory. The conjectures stated p. 292–293 of [W 1986] as well as those of [NeP 2001] Chap. 14, § 2 are not yet proved. One of the main obstacles is the above-mentioned open problem with the technical hypothesis.

The problem of extending the Lindemann-Weierstrass Theorem to commutative algebraic groups is not yet completely solved (see conjectures by P. Philippon in [P 1987]).

Algebraic independence proofs use elimination theory. Several methods are available; one of them, developed by Masser, Wüstholz and Brownawell, relies on the Hilbert Nulstellensatz. In this context we quote the following conjecture of Blum, Cucker, Shub and Smale (see [Sm 1998] and [NeP 2001] Chap. 16, § 6.2), related to the open problem "$P = NP$?" [J 2000].

Conjecture 5.18 (Blum, Cucker, Shub and Smale)**.** *Given an absolute constant c and polynomials $P_1, \dots, P_m$ with a total of N coefficients and no common complex zeros, there is no program to find, in at most N^c step, the coefficients of polynomials A_i satisfying Bézout's relation,*

$$A_1 P_1 + \cdots + A_m P_m = 1.$$

In connection with complexity in theoretical computer science, W. D. Brownawell suggests investigating Diophantine approximation from a new point of view in [NeP 2001] Chap. 16, § 6.3.

Complexity theory may be related to a question raised by M. Kontsevich and D. Zagier in [KZ 2000]. They defined a *period* as a complex number whose real and imaginary part are values of absolutely convergent integrals of rational functions with rational coefficients over domains of $\mathbb{R}^n$ given by polynomial (in)equalities with rational coefficients. Problem 3 in [KZ 2000] is to *produce at least one number which is* not *a period.* This is the analogue for *periods* of Liouville's Theorem for *algebraic numbers.* A more difficult question is to prove that specific numbers like

$$e, \quad 1/\pi, \quad \gamma$$

(where γ is Euler's constant) are not periods. Since every algebraic number is a period, a number which is not a period is transcendental.

Another important tool missing for transcendence proofs in higher dimension is a Schwarz Lemma in several variables. The following conjecture is suggested in [W 1976], § 5. For a finite subset Σ of $\mathbb{C}^n$ and a positive integer t, denote by $\omega_t(\Sigma)$ the least total degree of a non-zero polynomial P in $\mathbb{C}[z_1, \dots, z_n]$ which vanishes on Σ with multiplicity at least t,

$$\left(\frac{\partial}{\partial z_1}\right)^{\tau_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\tau_n} P(z) = 0,$$

$$\text{for any} \quad z \in \Sigma \quad \text{and} \quad \tau = (\tau_1, \dots, \tau_n) \in \mathbb{N}^n \quad \text{with} \quad \tau_1 + \cdots + \tau_n < t.$$

Further, when f is an analytic function in an open neighborhood of a closed polydisc $|z_i| \le r \ \ (1 \le i \le n)$ in $\mathbb{C}^n$, denote by $\Theta_f(r)$ the average mass of the set of zeroes of f in that polydisc (see [BoL 1970]).

Conjecture 5.19. *Let Σ be a finite subset of $\mathbb{C}^n$, and ε be a positive number. There exists a positive number $r_0(\Sigma, \varepsilon)$ such that, for any positive integer t and any entire function f in $\mathbb{C}^n$ which vanishes on Σ with multiplicity $\geq t$,*

$$\Theta_f(r) \geq \omega_t(\Sigma) - t\varepsilon \quad \textit{for} \quad r \geq r_0(\Sigma, \varepsilon).$$

The next question is to compute $r_0(\Sigma, \varepsilon)$. One may expect that for Σ a chunk of a finitely generated subgroup of $\mathbb{C}^n$, say

$$\Sigma = \left\{ s_1 y_1 + \cdots + s_\ell y_\ell \ ; \ (s_1, \ldots, s_\ell) \in \mathbb{Z}^\ell, \ |s_j| \leq S \ (1 \leq j \leq \ell) \right\} \subset \mathbb{C}^n,$$

an admissible value for the number $r_0(\Sigma, \varepsilon)$ will depend only on ε, $y_1, \ldots, y_\ell$, but not on S. This would have interesting applications, especially in the special case $\ell = n + 1$.

Finally we refer to [Chu 1980] for a connection between the numbers $\omega_t(S)$ and Nagata's work on Hilbert's 14th Problem.

5.2 Multiple Zeta Values

Many recent papers (see for instance [C 2001, T 2002, W 2000c, Zu 2003]) are devoted to the study of algebraic relations among "multiple zeta values",

$$\sum_{n_1 > \cdots > n_k \geq 1} n_1^{-s_1} \cdots n_k^{-s_k},$$

where $(s_1, \ldots, s_k)$ is a k-tuple of positive integers with $s_1 \geq 2$. The main Diophantine conjecture, suggested by the work of D. Zagier, A. B. Goncharov, M. Kontsevich, M. Petitot, Minh Hoang Ngoc, K. Ihara, M. Kaneko and others (see [Z 1994], [C 2001] and [Zu 2003]), is that all such relations can be deduced from the linear and quadratic ones arising in the *shuffle* and *stuffle* products (including the relations arising from the study of divergent series – see [W 2000c] for instance). For $p \geq 2$, let $\mathfrak{Z}_p$ denote the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by the real numbers $\zeta(\underline{s})$ satisfying $\underline{s} = (s_1, \ldots, s_k)$ and $s_1 + \cdots + s_k = p$. Set $\mathfrak{Z}_0 = \mathbb{Q}$ and $\mathfrak{Z}_1 = \{0\}$. Then the $\mathbb{Q}$-subspace $\mathfrak{Z}$ spanned by all $\mathfrak{Z}_p$, $p \geq 0$ is a subalgebra of $\mathbb{R}$, and part of the Diophantine conjecture states

Conjecture 5.20 (Goncharov)**.** *As a $\mathbb{Q}$-algebra, $\mathfrak{Z}$ is the direct sum of $\mathfrak{Z}_p$ for $p \geq 0$.*

In other terms, all algebraic relations should be consequences of homogeneous ones, involving values $\zeta(\underline{s})$ with different $\underline{s}$ but with the same weight $s_1 + \cdots + s_k$.

Assuming Conjecture 5.20, the question of *algebraic independence* of the numbers $\zeta(\underline{s})$ is reduced to the question of *linear independence* of the same numbers. The conjectural situation is described by the next conjecture of Zagier [Z 1994] on the dimension d_p of the $\mathbb{Q}$-vector space $\mathfrak{Z}_p$.

Conjecture 5.21 (Zagier)**.** *For $p \geq 3$,*

$$d_p = d_{p-2} + d_{p-3},$$

with $d_0 = 1$, $d_1 = 0$, $d_2 = 1$.

That the actual dimensions of the spaces $\mathfrak{Z}_p$ are bounded above by the integers which are defined inductively in Conjecture 5.21 has been proved by T. Terasoma in [T 2002], who expresses multiple zeta values as periods of relative cohomologies and uses mixed Tate Hodge structures (see also the work of A.G. Goncharov referred to in [T 2002]). The first values of d_p are $d_2 = d_3 = d_4 = 1$. There is no single d for which the lower bound $d \geq 2$ is known. The irrationality of $\zeta(2)\zeta(3)/\zeta(5)$ is equivalent to $d_5 = 2$, the irrationality of $(\zeta(3)/\pi^3)^2$ is equivalent to $d_6 = 2$.

Further work on Conjectures 5.20 and 5.21 is due to J. Écalle. In case $k = 1$ (values of the Riemann zeta function) the conjecture is

Conjecture 5.22. *The numbers π, $\zeta(3), \zeta(5), \ldots, \zeta(2n+1), \ldots$ are algebraically independent over $\mathbb{Q}$.*

So far the only known results on this topic [Fis 2002] are:

• $\zeta(2n)$ *is transcendental for* $n \geq 1$ (because π is transcendental and $\zeta(2n)\pi^{-2n} \in \mathbb{Q}$),

• $\zeta(3)$ *is irrational* (Apéry, 1978),

and

• *For any $\varepsilon > 0$ the $\mathbb{Q}$-vector space spanned by the $n+1$ numbers*

$$1, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)$$

has dimension

$$\geq \frac{1-\varepsilon}{1+\log 2} \log n$$

for $n \geq n_0(\varepsilon)$ (see [Riv 2000] and [BalR 2001]). For instance infinitely many of these numbers $\zeta(2n+1)$ $(n \geq 1)$ are irrational. W. Zudilin proved that at least one of the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.

Further, more recent results are due to T. Rivoal and W. Zudilin. For instance, in a joint paper they have proved that infinitely many numbers among

$$\sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^{2s}} \qquad (s \in \mathbb{Z}, \quad s \geq 1)$$

are irrational, but, as pointed out in § 1.1.1, the irrationality of Catalan's constant G (1.5) is still an open problem.

It may turn out to be more efficient to work with a larger set of numbers, including special values of multiple polylogarithms,

$$\sum_{n_1 > \cdots > n_k \geq 1} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}} \cdot$$

An interesting set of points $\underline{z} = (z_1, \ldots, z_k)$ to consider is the set of k-tuples consisting of roots of unity. The function of a single variable,

$$\mathrm{Li}_{\underline{s}}(z) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}},$$

is worth of study from a Diophantine point of view. For instance, Catalan's constant (1.5) is the imaginary part of $\mathrm{Li}_2(i)$,

$$\mathrm{Li}_2(i) = \sum_{n \geq 1} \frac{i^n}{n^2} = -\frac{1}{8}\zeta(2) + iG.$$

Also no proof for the irrationality of the numbers

$$\zeta(4,2) = \sum_{n > k \geq 1} \frac{1}{n^4 k^2} = \zeta(3)^2 - \frac{4\pi^6}{2835},$$

$$\mathrm{Li}_2(1/2) = \sum_{n \geq 1} \frac{1}{n^2 2^n} = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2$$

and

(Ramanujan) $$\mathrm{Li}_{2,1}(1/2) = \sum_{n \geq k \geq 1} \frac{1}{2^n n^2 k} = \zeta(3) - \frac{1}{12}\pi^2 \log 2,$$

is known so far.

According to P. Bundschuh [Bun 1979], the transcendence of the numbers

$$\sum_{n=2}^{\infty} \frac{1}{n^s - 1}$$

for even $s \geq 4$ is a consequence of Schanuel's Conjecture 5.1. For $s = 2$ the sum is $3/4$, and for $s = 4$ the value is $(7/8) - (\pi/4)\coth\pi$, which is a transcendental number since π and e^π are algebraically independent over $\mathbb{Q}$ (Yu. V. Nesterenko [NeP 2001]).

Nothing is known about the arithmetic nature of the values of the Riemann zeta function at rational or algebraic points which are not integers.

5.3 Gamma, Elliptic, Modular, G and E-Functions

The transcendence problem of the values of the Euler Beta function at rational points was solved as early as 1940, by Th. Schneider. *For any rational numbers a and b which are not integers and such that $a+b$ is not an integer, the number*

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental. Transcendence results for the values of the gamma function itself are not so precise: apart from G. V. Chudnovsky's results, which imply the transcendence of $\Gamma(1/3)$ and $\Gamma(1/4)$ (and Lindemann's result on the transcendence of π which implies that $\Gamma(1/2) = \sqrt{\pi}$ is also transcendental), not much is known. For instance, as we said earlier, there is no proof so far that $\Gamma(1/5)$ is transcendental. This is because the Fermat curve of exponent 5, viz. $x^5 + y^5 = 1$, has genus 2. Its Jacobian is an Abelian surface, and the algebraic independence results known for elliptic curves like $x^3 + y^3 = 1$ and $x^4 + y^4 = 1$ which were sufficient for dealing with $\Gamma(1/3)$ and $\Gamma(1/4)$, are not yet known for Abelian varieties (see [Grin 2002]).

Among many open problems (we already quoted Schneider's second problem 3.18 on the transcendence of the values of the modular function and we introduced a number of conjectures at the end of § 3.3.5; see also for instance 3.35), we mention

Conjecture 5.23. *The three numbers π, $\Gamma(1/3)$, $\Gamma(1/4)$ are algebraically independent.*
The four numbers e, π, e^π and $\Gamma(1/4)$ are algebraically independent.

One might expect that Nesterenko's results (see [NeP 2001], Chap. 3) on the algebraic independence of π, $\Gamma(1/4)$, e^π and of π, $\Gamma(1/3)$, $e^{\pi\sqrt{3}}$ should be extended as follows.

Conjecture 5.24. *At least three of the four numbers*

$$\pi,\ \Gamma(1/5),\ \Gamma(2/5),\ e^{\pi\sqrt{5}}$$

are algebraically independent over $\mathbb{Q}$.

So the challenge is to extend Nesterenko's results on modular functions in one variable (and elliptic curves) to several variables (and Abelian varieties).

This may be one of the easiest questions to answer on this topic (but it is still open). But one may ask for a general statement which would produce all algebraic relations between gamma values at rational points. Here is a conjecture of Rohrlich [La 1978a] (see also Conjecture 3.8 in § 3). Define

$$G(z) = \frac{1}{\sqrt{2\pi}}\Gamma(z).$$

According to the multiplication theorem of Gauss and Legendre [WhW 1927], § 12.15, for each positive integer N and for each complex number x such that $Nx \not\equiv 0 \pmod{\mathbb{Z}}$,

$$\prod_{i=0}^{N-1} G\left(x + \frac{i}{N}\right) = N^{(1/2)-Nx} G(Nx).$$

The gamma function has no zero and defines a map from $\mathbb{C}\backslash\mathbb{Z}$ to $\mathbb{C}^\times$. We restrict that function to $\mathbb{Q}\backslash\mathbb{Z}$ and we compose it with the canonical map $\mathbb{C}^\times \to \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$

- which amounts to considering its values modulo the algebraic numbers. The composite map has period 1, and the resulting mapping,

$$\overline{G} : \frac{\mathbb{Q}}{\mathbb{Z}} \setminus \{0\} \to \frac{\mathbb{C}^\times}{\overline{\mathbb{Q}}^\times},$$

is an odd *distribution* on $(\mathbb{Q}/\mathbb{Z}) \setminus \{0\}$,

$$\prod_{i=0}^{N-1} \overline{G}\left(x + \frac{i}{N}\right) = \overline{G}(Nx) \quad \text{for} \quad x \in \frac{\mathbb{Q}}{\mathbb{Z}} \setminus \{0\} \quad \text{and} \quad \overline{G}(-x) = \overline{G}(x)^{-1}.$$

Rohrlich's Conjecture ([La 1978a], [La 1978c] Chap. II, Appendix, p. 66) asserts that

Conjecture 5.25 (Rohrlich)**.** *$\overline{G}$ is a universal odd distribution with values in groups where multiplication by 2 is invertible.*

In other terms, any multiplicative relation between gamma values at rational points

$$\pi^{b/2} \prod_{a \in \mathbb{Q}} \Gamma(a)^{m_a} \in \overline{\mathbb{Q}}$$

with b and m_a in $\mathbb{Z}$ can be derived for the standard relations satisfied by the gamma function. This leads to the question whether the distribution relations, the oddness relation and the functional equations of the gamma function generate the ideal over $\overline{\mathbb{Q}}$ of all algebraic relations among the values of $G(x)$ for $x \in \mathbb{Q}$.

In [NeP 2001] (Chap. 3, § 1, Conjecture 1.11) Yu. V. Nesterenko proposed another conjectural extension of his algebraic independence result on Eisenstein series of weight 2, 4 and 6:

$$\begin{aligned} P(q) &= 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} &&= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \\ Q(q) &= 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} &&= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \\ R(q) &= 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} &&= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n. \end{aligned}$$

Conjecture 5.26 (Nesterenko)**.** *Let $\tau \in \mathbb{C}$ have positive imaginary part. Assume that τ is not quadratic. Set $q = e^{2i\pi\tau}$. Then at least 4 of the 5 numbers*

$$\tau,\ q,\ P(q),\ Q(q),\ R(q)$$

are algebraically independent.

Finally we remark that essentially nothing is known about the arithmetic nature of the values of either the beta or the gamma function at algebraic irrational points.

A wide range of open problems in transcendental number theory, including not only Schanuel's Conjecture 5.1 and Rohrlich's Conjecture 5.25 on the values of the gamma function, but also a conjecture of Grothendieck on the periods of an algebraic variety (see [La 1966] Chap. IV, Historical Note; [La 1971] p. 650; [And 1989] p. 6 and [Ch 2001], § 3), are special cases of very general conjectures due to Y. André [And 1997], which deal with periods of mixed motives. A discussion of André's conjectures for certain 1-motives related to the products of elliptic curves and their connections with elliptic and modular functions is given in [Ber 2002]. Here is a special case of the *elliptico-toric Conjecture* in [Ber 2002].

Conjecture 5.27 (Bertolin)**.** *Let $\mathcal{E}_1,\dots,\mathcal{E}_n$ be pairwise non isogeneous elliptic curves with modular invariants $j(\mathcal{E}_h)$. For $h=1,\dots,n$, let ω_{1h},ω_{2h} be a pair of fundamental periods of $\wp_h$ where η_{1h},η_{2h} are the associated quasi-periods, P_{ih} points on $\mathcal{E}_h(\mathbb{C})$ and p_{ih} (resp. d_{ih}) elliptic integrals of the first (resp. second) kind associated to P_{ih}. Define $\kappa_h=[k_h:\mathbb{Q}]$ and let d_h be the dimension of the k_h-subspace of $\mathbb{C}/(k_h\omega_{1h}+k_h\omega_{2h})$ spanned by $p_{1h},\dots,p_{r_hh}$. Then the transcendence degree of the field*

$$\mathbb{Q}\Big(\big\{j(\mathcal{E}_h),\omega_{1h},\omega_{2h},\eta_{1h},\eta_{2h},P_{ih},p_{ih},d_{ih}\big\}_{1\le i\le r_h\ \frac{1}{\le}h\le n}\Big)$$

is at least

$$2\sum_{h=1}^{n}d_h+4\sum_{h=1}^{n}\kappa_h^{-1}-n+1.$$

A new approach to Grothendieck's Conjecture via Siegel's G-functions was introduced in [And 1989] Chap. IX. A development of this method led Y. André to his conjecture on the special points on Shimura varieties [And 1989] Chap. X, § 4, which gave rise to the André–Oort Conjecture [Oo 1997] (for a discussion of this topic, including a precise definition of "Hodge type", together with relevant references, see [Co 2003]).

Conjecture 5.28 (André–Oort)**.** *Let $\mathcal{A}_g(\mathbb{C})$ denote the moduli space of principally polarized complex Abelian varieties of dimension g. Let Z be an irreducible algebraic subvariety of $\mathcal{A}_g(\mathbb{C})$ such that the complex multiplication points on Z are dense for the Zariski topology. Then Z is a subvariety of $\mathcal{A}_g(\mathbb{C})$ of Hodge type.*

Conjecture 5.28 is a far-reaching generalization of Schneider's Theorem on the transcendence of $j(\tau)$, where j is the modular invariant and τ an algebraic point in the Poincaré upper half plane $\mathfrak{H}$, which is not imaginary quadratic ([Schn 1957] Chap. II, § 4, Th. 17). We also mention a related conjecture of D. Bertrand (see [NeP 2001] Chap. 1, § 4 Conjecture 4.3) which may be viewed as a nonholomorphic analogue of Schneider's result and which would answer the following question raised by N. Katz.

Question 5.29. *Assume that a lattice $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ in $\mathbb{C}$ has algebraic invariants $g_2(L)$ and $g_3(L)$ and no complex multiplication. Does this implies* *that the number*

$$G_2^*(L) = \lim_{s\to 0} \sum_{\omega\in L\setminus\{0\}} \omega^{-2}|\omega|^{-s}$$

is transcendental?

Many open transcendence problems dealing with elliptic functions are consequences of André's conjectures (see [Ber 2002]), most of which are likely to be very hard. Some of them are Conjectures 3.35, 3.42, 3.43, 3.44, 3.45. The next one, which is still open, may be easier, since a number of partial results are already known, as a result of the work of G. V. Chudnovsky and others (see [Grin 2002]).

Conjecture 5.30. *Given an elliptic curve with Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$, a non-zero period ω, the associated quasi-period η of the zeta function and a complex number u which is not a pole of $\wp$,*

$$\mathrm{trdeg}\mathbb{Q}\bigl(g_2, g_3, \pi/\omega, \wp(u), \zeta(u) - (\eta/\omega)u\bigr) \geq 2.$$

Given a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$ with invariants $g_2(L)$ and $g_3(L)$, denote by $\eta_i = \zeta_L(z+\omega_i) - \zeta_L(z)$ $(i = 1, 2)$ the corresponding fundamental quasi-periods of the Weierstrass zeta function. Conjecture 5.30 implies that the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}\bigl(g_2(L), g_3(L), \omega_1, \omega_2, \eta_1, \eta_2\bigr)$ is at least 2.This would be optimal in the CM case, while in the non CM case, we expect it to be ≥ 4. These lower bounds are given by the period conjecture of Grothendieck applied to an elliptic curve.

According to [Di 2000] conjectures 1 and 2, p. 187, the following special case of Conjecture 5.30 can be stated in two equivalent ways: either in terms of values of elliptic functions, or in terms of values of Eisenstein series E_2, E_4 and E_6 (which are P, Q and R in Ramanujan's notation).

Conjecture 5.31. *For any lattice L in $\mathbb{C}$ without complex multiplication and for any non-zero period ω of L,*

$$\mathrm{trdeg}\mathbb{Q}\bigl(g_2(L), g_3(L), \omega/\sqrt{\pi}, \eta/\sqrt{\pi}\bigr) \geq 2.$$

Conjecture 5.32. *For any $\tau \in \mathfrak{H}$ which is not imaginary quadratic,*

$$\mathrm{trdeg}\mathbb{Q}\bigl(\pi E_2(\tau), \pi^2 E_4(\tau), \pi^3 E_6(\tau)\bigr) \geq 2.$$

Moreover, each of these two statements implies the following one, which is stronger than one of Lang's conjectures ([La 1971] p. 652).

Conjecture 5.33. *For any $\tau \in \mathfrak{H}$ which is not imaginary quadratic,*

$$\mathrm{trdeg}\mathbb{Q}\bigl(j(\tau), j'(\tau), j''(\tau)\bigr) \geq 2.$$

Further related open problems are proposed by G. Diaz in [Di 1997] and [Di 2000], in connection with conjectures due to D. Bertrand on the values of the modular function $J(q)$, where $j(\tau) = J(e^{2i\pi\tau})$ (see [Bert 1997b] as well as [NeP 2001] Chap. 1, § 4 and Chap. 2, § 4).

Conjecture 5.34 (Bertrand)**.** *Let $q_1, \ldots, q_n$ be non-zero algebraic numbers in the unit open disc such that the $3n$ numbers*

$$J(q_i),\ DJ(q_i),\ D^2J(q_i) \qquad (i = 1, \ldots, n)$$

are algebraically dependent over $\mathbb{Q}$. Then there exist two indices $i \neq j$ $(1 \leq i \leq n,\ 1 \leq j \leq n)$ such that q_i and q_j are multiplicatively dependent.

Conjecture 5.35 (Bertrand)**.** *Let q_1 and q_2 be two non-zero algebraic numbers in the unit open disc. Suppose that there is an irreducible element $P \in \mathbb{Q}[X, Y]$ such that*

$$P\bigl(J(q_1), J(q_2)\bigr) = 0.$$

Then there exist a constant c and a positive integer s such that $P = c\Phi_s$, where Φ_s is the modular polynomial of level s. Moreover q_1 and q_2 are multiplicatively dependent.

Among Siegel's G-functions are the algebraic functions. Transcendence methods produce some information, in particular in connection with Hilbert's Irreducibility Theorem. Let $f \in \mathbb{Z}[X, Y]$ be a polynomial which is irreducible in $\mathbb{Q}(X)[Y]$. According to Hilbert's Irreducibility Theorem, the set of positive integers n such that $P(n, Y)$ is irreducible in $\mathbb{Q}[Y]$ is infinite. Effective upper bounds for an admissible value for n have been studied (especially by M. Fried, P. Dèbes and U. Zannier), but do not yet answer the next question.

Question 5.36. *Is there such a bound depending polynomially on the degree and height of P?*

Such questions are also related to the *Galois inverse Problem* [Se 1989].

Also the polylogarithms

$$\mathrm{Li}_s(z) = \sum_{n \geq 1} \frac{z^m}{n^s},$$

where s is a positive integer, are G-functions; unfortunately no way has yet been found to use the Siegel-Shidlovskii method to prove the irrationality of the values of the Riemann zeta function ([FeN 1998] Chap. 5, § 7, p. 247).

With G-functions, the other class of analytic functions introduced by C. L. Siegel in 1929 is the class of E-functions, which includes the hypergeometric ones. One main open question is the arithmetic nature of the values at algebraic points of hypergeometric functions with *algebraic* parameters,

$${}_2F_1\left(\begin{matrix}\alpha\ ,\ \beta \\ \gamma\end{matrix}\middle| z\right) = \sum_{n \geq 0} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \cdot \frac{z^n}{n!},$$

defined for $|z| < 1$ and $\gamma \notin \{0, -1, -2, \ldots\}$.

In 1949, C. L. Siegel ([Si 1949] Chap. 2, § 9, p. 54 and 58; see also [FeS 1967] p. 62 and [FeN 1998] Chap. 5, § 1.2) asked whether *any E-function satisfying* *a linear differential equation with coefficients in $\mathbb{C}(z)$ can be expressed as a polynomial in z and a finite number of hypergeometric E-functions or functions obtained from them by a change of variables of the form $z \mapsto \gamma z$ with algebraic γ's?*

Finally, we quote from [W 1999b]: a folklore conjecture is that the zeroes of the Riemann zeta function (say their imaginary parts, assuming it > 0) are algebraically independent. As suggested by J-P. Serre, one might also be tempted to consider

•The eigenvalues of the zeroes of the hyperbolic Laplacian in the upper half plane modulo $\mathrm{SL}_2(\mathbb{Z})$ (i.e., to study the algebraic independence of the zeroes of the Selberg zeta function).

•The eigenvalues of the Hecke operators acting on the corresponding eigenfunctions (Maass forms).

5.4 Fibonacci and Miscellanea

Many further open problems arise in transcendental number theory. An intriguing question is to study the arithmetic nature of real numbers given in terms of power series involving the Fibonacci sequence

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \quad F_1 = 1.$$

Several results are due to P. Erdős, R. André-Jeannin, C. Badea, J. Sándor, P. Bundschuh, A. Pethő, P.G. Becker, T. Töpfer, D. Duverney, Ku. et Ke. Nishioka, I. Shiokawa and T. Tanaka. It is known that the number

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1$$

is rational, while

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1-\sqrt{5}}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}+1} = \frac{\sqrt{5}}{2}$$

are irrational algebraic numbers. Each of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n + F_{n+2}} \quad \text{and} \quad \sum_{n \geq 1} \frac{1}{F_1 F_2 \cdots F_n}$$

is irrational, but it is not known whether they are algebraic or transcendental. The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2n}},$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n-1}+F_{2^n+1}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{F_{2^n+1}}$$

are all transcendental (further results of algebraic independence are known). The first challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.

There is a similar situation for infinite sums $\sum_n f(n)$ where f is a rational function [Ti 2000]. While

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

and

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

are rational numbers, the sums

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2, \qquad \sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{2\pi}{e^{\pi}-e^{-\pi}}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)} = \frac{1}{4320}(192 \log 2 - 81 \log 3 - 7\pi\sqrt{3})$$

are transcendental. The simplest example of the Euler sums $\sum_n n^{-s}$ (see § 3.2) illustrates the difficulty of the question. Here again, even a sufficiently general conjecture is missing. One may remark that there is no known algebraic irrational number of the form

$$\sum_{\substack{n\geq 0 \\ Q(n)\neq 0}} \frac{P(n)}{Q(n)},$$

where P and Q are non-zero polynomials having rational coefficients and $\deg Q \geq 2 + \deg P$.

The arithmetic study of the values of power series suggests many open problems. We shall only mention a few of them.

The next question is due to K. Mahler [M 1984].

Question 5.37 (Mahler)**.** *Are there entire transcendental functions $f(z)$ such that if x is a Liouville number then so is $f(x)$?*

The study of integral valued entire functions gives rise to several open problems; we quote only one of them which arose in the work of D. W. Masser and F. Gramain on entire functions f of one complex variable which map the ring of Gaussian integers $\mathbb{Z}[i]$ into itself. The initial question (namely to derive an analogue of Pólya's Theorem in this setting) has been solved by F. Gramain in [Gr 1981] (following previous work of Fukasawa, Gel'fond, Gruman and Masser). *If f is not a polynomial, then*

$$\limsup_{r\to\infty} \frac{1}{r^2} \log |f|_r \geq \frac{\pi}{2e}\cdot$$

Here,

$$|f|_r = \max_{|z|=r} |f(z)|.$$

Preliminary works on this estimate gave rise to the following problem, which is still unsolved. For each integer $k \geq 2$, let A_k be the minimal area of a closed disk in $\mathbb{R}^2$ containing at least k points of $\mathbb{Z}^2$, and for $n \geq 2$ define

$$\delta_n = -\log n + \sum_{k=2}^{n} \frac{1}{A_k}\cdot$$

The limit $\delta = \lim_{n\to\infty} \delta_n$ exists (it is an analogue in dimension 2 of the Euler constant), and the best known estimates for it are [GrW 1985]

$$1.811\cdots < \delta < 1.897\ldots$$

(see also [Fi 2003]). F. Gramain conjectures that

$$\delta = 1 + \frac{4}{\pi}\bigl(\gamma L(1) + L'(1)\bigr),$$

where γ is Euler's constant and

$$L(s) = \sum_{n\geq 0} (-1)^n (2n+1)^{-s}$$

is the L function of the quadratic field $\mathbb{Q}(i)$ (Dirichlet beta function). Since $L(1) = \pi/4$ and

$$L'(1) = \sum_{n\geq 0} (-1)^{n+1} \cdot \frac{\log(2n+1)}{2n+1} = \frac{\pi}{4}\bigl(3\log\pi + 2\log 2 + \gamma - 4\log\Gamma(1/4)\bigr),$$

Gramain's conjecture is equivalent to

$$\delta = 1 + 3\log\pi + 2\log 2 + 2\gamma - 4\log\Gamma(1/4) = 1.822825\ldots$$

Other problems related to the lattice $\mathbb{Z}[i]$ are described in the section "*On the borders of geometry and arithmetic*" of [Sie 1964].

5.5 Diophantine Approximation

One of the main open problems in Diophantine approximation is to produce an effective version of the Thue-Siegel-Roth Theorem 2.34. In connexion with the negative answer to Hilbert's 10th Problem by Yu. Matiyasevich, it has been suggested by M. Mignotte that an effective version of Schmidt's Subspace Theorem 2.39.

5.6 The *abc* Conjecture

For a positive integer n, we denote by

$$R(n) = \prod_{p|n} p$$

the *radical* or *square free part* of n.

The *abc* Conjecture resulted from a discussion between D. W. Masser and J. Œsterlé ([Œ 1988] p. 169; see also [Mas 1990], as well as [La 1990], [La 1991] Chap. II § 1; [La 1993] Ch. IV § 7; [Guy 1994] B19; [Bro 1999]; [Ri 2000], § 9.4.E; [V 2000], [Maz 2000] and [Ni].

Conjecture 5.38. *[abc Conjecture] For each $\varepsilon > 0$ there exists a positive number $\kappa(\varepsilon)$ which has the following property: if a, b and c are three positive rational integers which are relatively prime and satisfy $a + b = c$, then*

$$c < \kappa(\varepsilon) R(abc)^{1+\varepsilon}.$$

M. Langevin noticed that the abc Conjecture yields a stronger inequality than Roth's,

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\varepsilon)}{R(pq)q^{\varepsilon}}\cdot$$

Connexions between the *abc* Conjecture and measures of linear independence of logarithms of algebraic numbers have been pointed out by A. Baker [B 1998] and P. Philippon [P 1999a] (see also [W 2000b] exercise 1.11). We reproduce here the main conjecture of the addendum of [P 1999a]. For a rational number a/b with relatively prime integers a, b, we denote by $\mathrm{h}(a/b)$ the number $\log \max\{|a|, |b|\}$.

Conjecture 5.39. *[Philippon] There exist real numbers ε, α and β with $0 < \varepsilon < 1/2$, $\alpha \geq 1$ and $\beta \geq 0$, and a positive integer B, such that for any non-zero rational numbers x, y satisfying $xy^B \neq 1$, if S denotes the set of prime numbers for which $|xy^B + 1|_p < 1$, then*

$$-\sum_{p\in S} \log |xy^B + 1|_p \leq B\Big(\alpha \mathrm{h}(x) + \varepsilon \mathrm{h}(y) + \big(\alpha B + \varepsilon\big)\big(\beta + \sum_{p\in S} \log p\big)\Big).$$

The conclusion is a lower bound for the p-adic distance between $-xy^B$ and 1; the main point is that several p's are involved. Conjecture 5.39 is telling us something about the prime decomposition of all numbers xy^B+1 for some fixed but unspecified value of B – and it implies the *abc* Conjecture.

Examples of optimistic Archimedean estimates related to measures of linear independence of logarithms of algebraic numbers are the Lang-Waldschmidt Conjectures in [La 1978b] (introduction to Chap. X and XI, p. 212–217). Here is a simple example.

Conjecture 5.40. *[Lang-Waldschmidt] For any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that, for any non-zero rational integers $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ with $a_1^{b_1} \cdots a_m^{b_m} \neq 1$,*

$$\left| a_1^{b_1} \cdots a_m^{b_m} - 1 \right| \geq \frac{C(\varepsilon)^m B}{(|b_1| \cdots |b_m| \cdot |a_1| \cdots |a_m|)^{1+\varepsilon}},$$

where $B = \max_{1 \leq i \leq m} |b_i|$.

Similar questions related to Diophantine approximation on tori are discussed in [La 1991] Chap. IX, § 7.

Conjecture 5.40 deals with rational integers; a more general setting involving algebraic numbers is considered in [W 2004].

5.7 Irrationality and linear independence measures

As we mentioned in § 1.1.1, the class of "interesting" real numbers which are known to be irrational is not as large as one would expect [KZ 2000]. For instance no proof of irrationality has been given so far for numbers like Euler's constant, Catalan's constant (1.5), $\Gamma(1/5)$, $e+\pi$, $\zeta(5)$, $\zeta(3)/\pi^3$, $e^\gamma = 1.781072\ldots$ and

$$\sum_{n\geq 1} \frac{\sigma_k(n)}{n!} \quad (k=1,2) \quad \text{where} \quad \sigma_k(n) = \sum_{d|n} d^k$$

(see [Guy 1994] B14).

Here is another irrationality question raised by P. Erdős and E. Straus in 1975 (see [E 1988] and [Guy 1994] E24). Define an *irrationality sequence* as an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that, for any sequence $(t_k)_{k\geq 1}$ of positive integers, the real number

$$\sum_{k\geq 1} \frac{1}{n_k t_k}$$

is irrational. On the one hand, it has been proved by Erdős that $(2^{2^k})_{k\geq 1}$ is an irrationality sequence. On the other hand, the sequence $(k!)_{k\geq 1}$ is not, since

$$\sum_{k\geq 1} \frac{1}{k!(k+2)} = \frac{1}{2}\cdot$$

An open question is whether an irrationality sequence must increase very rapidly. No irrationality sequence $(n_k)_{k\geq 1}$ is known for which $n_k^{1/2^k}$ tends to 1 as k tends to infinity.

Many further open irrationality questions are raised in [E 1988].

Assume now that the first step has been completed and that we know our number θ is irrational. Then there are (at least) two directions for further investigation.

•(1) Considering several real numbers $\theta_1, \ldots, \theta_n$, a fundamental question is to decide whether or not they are linearly independent over $\mathbb{Q}$. One main example is to start with the successive powers of one number, $1, \theta, \theta^2, \ldots, \theta^{n-1}$. The goal is to decide whether θ is algebraic of degree $< n$. If n is not fixed, the question is whether θ is transcendental. This question, which is relevant also for complex numbers, will be considered in the next section. Observe also that the problem of algebraic independence is included here. It amounts to the linear independence of monomials.

•(2) Another direction of research is to consider a quantitative refinement of the irrationality statement, namely an *irrationality measure*. We wish to bound from below the non-zero number $|\theta - (p/q)|$ when p/q is any rational number; this lower bound will depend on θ as well as the denominator q of the rational approximation. In case where a statement weaker than an irrationality result is known, namely if one can prove only that at least one of n numbers $\theta_1, \ldots, \theta_n$ is irrational, then a quantitative refinement will be a lower bound (in terms of q) for

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \ldots, \left|\theta_n - \frac{p_n}{q}\right|\right\},$$

when $p_1/q, \ldots, p_n/q$ are n rational numbers and $q > 0$ a common denominator.

On the one hand, the study of rational approximation of real numbers is achieved in a satisfactory way for numbers whose regular continued fraction expansion is known. This is the case for rational numbers (!), for quadratic numbers, as well as for a small set of transcendental numbers, like (1.6). On the other hand, even for a real number x for which an irregular continued fraction expansion is known, like

$$\log 2 = \frac{1|}{|1} + \frac{1|}{|1} + \frac{4|}{|1} + \frac{9|}{|1} + \cdots \frac{n^2|}{|\,1} + \cdots$$

or

$$\frac{\pi}{4} = \frac{1|}{|1} + \frac{9|}{|2} + \frac{25|}{|\,2} + \frac{49|}{|\,2} + \cdots \frac{(2n+1)^2|}{|\quad 2} + \cdots$$

one does not know how well x can be approximated by rational numbers. No regular pattern has been observed or can be expected from the regular continued fraction of π,

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, \ldots],$$

nor from any number "easily" related to π.

One expects that for any $\varepsilon > 0$ there are constants $C(\varepsilon) > 0$ and $C'(\varepsilon) > 0$ such that

$$\left|\log 2 - \frac{p}{q}\right| > \frac{C(\varepsilon)}{q^{2+\varepsilon}} \quad \text{and} \quad \left|\pi - \frac{p}{q}\right| > \frac{C'(\varepsilon)}{q^{2+\varepsilon}}$$

hold for any $p/q \in \mathbb{Q}$, but this is known only with larger exponents, namely $3.8913\ldots$ and $8,0161\ldots$ respectively (Rukhadze and Hata). The sharpest known exponent for an irrationality measure of

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1.202056\ldots$$

is $5.513891\ldots$, while for π^2 (or for $\zeta(2) = \pi^2/6$) it is $5.441243\ldots$ (both results due to Rhin and Viola). For a number like $\Gamma(1/4)$, the existence of absolute positive constants C and κ for which

$$\left|\Gamma(1/4) - \frac{p}{q}\right| > \frac{C}{q^\kappa}$$

has been proved only recently [P 1999b]. The similar problem for e^π is not yet solved. In other terms there is no proof so far that e^π is not a Liouville number.

Earlier we distinguished two directions for research once we know the irrationality of some given numbers. Either, on the qualitative side, one studies the linear dependence relations, or else, on the quantitative side, one studies the quality of rational approximation. One can combine both. A quantitative version of a result of $\mathbb{Q}$-linear independence of n real numbers $\theta_1, \ldots, \theta_n$, is a lower bound, in terms of $\max\{|p_1|, \ldots, |p_n|\}$, for

$$|p_1\theta_1 + \cdots + p_n\theta_n|$$

when $(p_1, \ldots, p_n)$ is in $\mathbb{Z}^n \setminus \{0\}$.

For some specific classes of transcendental numbers, A. I. Galochkin [G 1983], A. N. Korobov (Th. 1.22 of [FeN 1998] Chap. 1 § 7) and more recently P. Ivankov proved extremely sharp measures of linear independence (see [FeN 1998] Chap. 2 § 6.2 and § 6.3).

A general and important problem is to improve the known measures of linear independence for logarithms of algebraic numbers, as well as elliptic logarithms, Abelian logarithms, and more generally logarithms of algebraic points on commutative algebraic groups. For instance the conjecture that e^π is not a Liouville number should follow from improvements of known linear independence measures for logarithms of algebraic numbers.

The next step, which is to obtain sharp measures of algebraic independence for transcendental numbers, will be considered later (see § 4.2 and § 4.3).

The so-called Mahler Problem (see [W 2001] § 4.1) is related to linear combination of logarithms $|b - \log a|$.

Conjecture 5.41. *[Mahler] There exists an absolute constant $c > 0$ such that*

$$\|\log a\| > a^{-c}$$

for all integers $a \geq 2$.

Equivalently,

$$|a - e^b| > a^{-c}$$

for some absolute constant $c > 0$ for all integers $a,\ b > 1$.

A stronger conjecture is suggested in [W 2001] (4.1)[12]

$$\|\log a\| > a^{-1}(\log a)^{-c}$$

for some absolute constant $c > 0$ for all integers $a \geq 3$, or equivalently

$$|a - e^b| > b^{-c}$$

for some absolute constant $c > 0$ for all integers $a,\ b > 1$. So far the best known estimate is

$$|a - e^b| > e^{-c(\log a)(\log b)},$$

so the problem is to replace the product $(\log a)(\log b)$ in the exponent by the sum $\log a + \log b$.

Such explicit lower bounds have interest in theoretical computer science [MüT 1996]. The sharpest known estimate on Mahler's problem is

$$|e^b - a| > b^{-20b}.$$

In a joint work with Yu.V. Nesterenko [NeW-1996] in 1996, we considered an extension of this question when a and b are rational numbers. A refinement of our estimate has been obtained by S. Khemira in 2005 and is currently being sharpened in a joint work of S. Khemira and P. Voutier.

Define $H(p/q) = \max\{|p|,\ q\}$. Then for a and b in $\mathbb{Q}$ with $b \neq 0$, the estimate is

$$|e^b - a| \geq \exp\{-1,3 \cdot 10^5 (\log A)(\log B)\}$$

where $A = \max\{H(a),\ A_0\}$, $B = \max\{H(b),\ 2\}$. The numerical value of the absolute constant A_0 is explicitly computed.

5.8 Expansions of irrational algebraic numbers

A reference for this section is [W 2008].

The digits of the expansion (in any basis ≥ 2) of an irrational, real, algebraic number should be equidistributed – in particular any digit should appear infinitely often. But even the following special case is unknown.

Conjecture 5.42. *[Mahler]. Let $(\varepsilon_n)_{n\geq 0}$ be a sequence of elements in $\{0,1\}$. Assume that the real number*

$$\sum_{n\geq 0} \varepsilon_n 3^{-n}$$

is irrational, then it is transcendental.

[12] As pointed out to me by Iam Ho (Ho Chi Minh), in [W 2004] p. 266 the factor a^{-1} is missing.

In two papers, the first one published in 1909 [Bor 1909] and the second one in 1950 [Bor 1950], É. Borel studied the g–ary expansion of real numbers, where $g \geq 2$ is a positive integer. In his second paper he suggested that this expansion for an algebraic irrational real number should satisfy some of the laws shared by almost all numbers (for Lebesgue's measure). More precisely, for a positive integer $g \geq 2$, a *normal number* in base g is a real number such that the sequence $(xg^n)_{n\geq 1}$ is equidistributed modulo 1. Almost all real numbers for Lebesgue measure are normal (i.e., normal in basis g for any $g > 1$), but it is not known whether any irrational real algebraic number is normal to any integer basis.

Conjecture 5.43 (É. Borel, 1950)**.** *Let x be an irrational algebraic real number and $g \geq 2$ a positive integer. Then x is normal in base g.*

Also it is not known whether there is an integer g for which any number like e, π, $\zeta(3)$, $\Gamma(1/4)$, γ, G, $e+\pi$, e^γ is normal in basis g (see [Ra 1976]).

Few results are known on the expansion in a basis g of irrational algebraic numbers, essentially nothing is known about the continued fraction expansion of a real algebraic number of degree ≥ 3; one does not know the answer to any of the following two questions.

Question 5.44. *Does there exist a real algebraic number of degree ≥ 3 with bounded partial quotients?*

Question 5.45. *Does there exist a real algebraic number of degree ≥ 3 with unbounded partial quotients?*

It is usually expected is that the continued fraction expansion of a real algebraic number of degree at least 3 always has unbounded partial quotients.

5.9 Logarithms of algebraic numbers

We have already suggested several questions related to linear independence measures over the field of rational numbers for logarithms of rational numbers (see Conjectures 5.39, 5.40 and 5.41). Now that we have a notion of height for algebraic numbers at our disposal, we can extend our study to linear independence measures over the field of algebraic numbers for the logarithms of algebraic numbers.

The next statement is Conjecture 14.25 of [W 2000b].

Conjecture 5.46. *There exist two positive absolute constants c_1 and c_2 with the following property. Let $\lambda_1, \ldots, \lambda_m$ be logarithms of algebraic numbers with $\alpha_i = e^{\lambda_i}$ $(1 \leq i \leq m)$, let $\beta_0, \ldots, \beta_m$ be algebraic numbers, D the degree of the number field*

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_m, \beta_0, \ldots, \beta_m),$$

and, finally, let $h \geq 1/D$ satisfy

$$h \geq \max_{1 \leq i \leq m} \mathrm{h}(\alpha_i), \quad h \geq \frac{1}{D} \max_{1 \leq i \leq m} |\lambda_i| \quad \textit{and} \quad h \geq \max_{0 \leq j \leq m} \mathrm{h}(\beta_j).$$

(1) Assume that the number

$$\Lambda = \beta_0 + \beta_1 \lambda_1 + \cdots + \beta_m \lambda_m$$

is non-zero. Then

$$|\Lambda| \geq \exp\bigl\{-c_1 m D^2 h\bigr\}.$$

(2) Assume that $\lambda_1, \ldots, \lambda_m$ are linearly independent over $\mathbb{Q}$. Then

$$\sum_{i=1}^{m} |\lambda_i - \beta_i| \geq \exp\bigl\{-c_2 m D^{1+(1/m)} h\bigr\}.$$

Connection between Conjecture 5.46 and Conjecture 5.1 are described in [W 2000a] and [W 2000b] Chap. 15; see [W 2004]).

Conjecture 5.47. *There exists a positive absolute constant C with the following property. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers and $\log \alpha_1, \ldots, \log \alpha_n$ logarithms of $\alpha_1, \ldots, \alpha_n$ respectively. Assume that the numbers $\log \alpha_1, \ldots, \log \alpha_n$ are $\mathbb{Q}$-linearly independent. Let $\beta_0, \beta_1, \ldots, \beta_n$ be algebraic numbers, not all of which are zero. Denote by D the degree of the number field*

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n)$$

over $\mathbb{Q}$. Further, let $A_1, \ldots, A_n$ and B be positive real numbers, each $\geq e$, such that

$$\log A_j \geq \max\left\{\mathrm{h}(\alpha_j), \ \frac{|\log \alpha_j|}{D}, \ \frac{1}{D}\right\} \quad (1 \leq j \leq n),$$

$$B \geq \max_{1 \leq j \leq n-1} \mathrm{h}(\beta_j).$$

Then the number

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

satisfies

$$|\Lambda| > \exp\{-C^n D^{n+2} (\log A_1) \cdots (\log A_n)(\log B + \log D)(\log D)\}.$$

One is rather close to such an estimate (see [W 2001], § 5 and § 6, as well as [Matv 2000]). The result is proved now in the so-called rational case, where

$$\beta_0 = 0 \quad \text{and} \quad \beta_i \in \mathbb{Q} \quad \text{for} \quad 1 \leq i \leq n.$$

In the general case, one needs a further condition, namely

$$B \geq \max_{1 \leq i \leq n} \log A_i.$$

Removing this extra condition would enable one to prove that numbers like e^π or $2^{\sqrt{2}}$ are not Liouville numbers.

These questions are the first and simplest ones concerning transcendence measures,measures of Diophantine approximation, measures of linear independence and measures of algebraic independence. One may ask many further questions on this topic, including an effective version of Schanuel's conjecture. It is interesting to notice that in this case a "technical condition" cannot be omitted ([W 1999b] Conjecture 1.4).

Recall that the rank of a prime ideal $\mathfrak{P} \subset \mathbb{Q}[T_1, \ldots, T_m]$ is the largest integer $r \geq 0$ such that there exists an increasing chain of prime ideals

$$(0) = \mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_r = \mathfrak{P}.$$

The rank of an ideal $\mathfrak{I} \subset \mathbb{Q}[T_1, \ldots, T_m]$ is the minimum rank of a prime ideal containing $\mathfrak{I}$.

Conjecture 5.48. (Quantitative Refinement of Schanuel's Conjecture). *Let $x_1, \ldots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Assume that for any $\varepsilon > 0$, there exists a positive number H_0 such that, for any $H \geq H_0$ and n-tuple $(h_1, \ldots, h_n)$ of rational integers satisfying $0 < \max\{|h_1|, \ldots, |h_n|\} \leq H$, the inequality*

$$|h_1 x_1 + \cdots + h_n x_n| \geq \exp\{-H^\varepsilon\}$$

is valid. Let d be a positive integer. Then there exists a positive number $C = C(x_1, \ldots, x_n, d)$ with the following property: for any integer $H \geq 2$ and any $n+1$ tuple $P_1, \ldots, P_{n+1}$ of polynomials in $\mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ with degrees $\leq d$ and usual heights $\leq H$, which generate an ideal of $\mathbb{Q}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ of rank $n+1$,

$$\sum_{j=1}^{n+1} \bigl|P_j(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})\bigr| \geq H^{-C}.$$

A consequence of Conjecture 5.48 is a quantitative refinement to Conjecture 5.6 on the algebraic independence of logarithms of algebraic numbers [W 1999b].

Conjecture 5.49. *If $\log\alpha_1, \ldots, \log\alpha_n$ are $\mathbb{Q}$-linearly independent logarithms of algebraic numbers and d a positive integer, there exists a constant $C > 0$ such that, for any non-zero polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$ of degree $\leq d$ and height $\leq H$, with $H \geq 2$,*

$$|P(\log\alpha_1, \ldots, \log\alpha_n)| \geq H^{-C}.$$

Further open questions are related to a question by B. Mazur on density of points on a variety (see [Maz 1992,Maz 1994,Maz 1995] and [W 2004]).

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