Some Consequences of Schanuel's Conjecture

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The Problems

Set up Related consequences of Schanuel's Conjecture

Main Problem

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Conjecture and Corollaries

Conjecture and Corollaries

Conjecture (Schanuel): Let x_1, \ldots, x_n be \mathbb{Q} -linearly independent complex numbers. Then the transcendence degree over \mathbb{Q} of the field $\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ is at least n.

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Corollaries:

Algebraic independence of π and e over \mathbb{Q} . π , log π , log log π ,... are algebraically independent over $\overline{\mathbb{Q}}$.

Set up Related consequences of Schanuel's Conjecture

Definitions

• Define E_0 to be the set of algebraic numbers $E_0 = \overline{\mathbb{Q}}$.

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Set up Related consequences of Schanuel's Conjecture

Definitions

- Define E_0 to be the set of algebraic numbers $E_0 = \overline{\mathbb{Q}}$.
- ▶ Inductively, for $n \ge 1$, define $E_n = \overline{E_{n-1}(e^x : x \in E_{n-1})}$.

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$$\blacktriangleright E = \bigcup_{n \ge 0} E_n.$$

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• Similarly define
$$L_0 = \overline{\mathbb{Q}}$$
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Set up Related consequences of Schanuel's Conjecture

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- Similarly define $L_0 = \overline{\mathbb{Q}}$.
- ▶ Inductively, for $n \ge 1$, define $L_n = \overline{L_{n-1}(\log(x) : x \in L_{n-1})}$.

Set up Related consequences of Schanuel's Conjecture

Definitions

- Define E_0 to be the set of algebraic numbers $E_0 = \overline{\mathbb{Q}}$.
- ▶ Inductively, for $n \ge 1$, define $E_n = \overline{E_{n-1}(e^x : x \in E_{n-1})}$.

$$\blacktriangleright E = \bigcup_{n\geq 0} E_n.$$

- Similarly define $L_0 = \overline{\mathbb{Q}}$.
- ▶ Inductively, for $n \ge 1$, define $L_n = \overline{L_{n-1}(\log(x) : x \in L_{n-1})}$.

$$\blacktriangleright L = \bigcup_{n \ge 0} L_n.$$

Set up Related consequences of Schanuel's Conjecture

Some Consequences



Some Consequences of Schanuel's Conjecture

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Set up Related consequences of Schanuel's Conjecture

Some Consequences

- *π* ∉ *E*.
- π , log π , log log π , ... are algebraically independent over E.

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Set up Related consequences of Schanuel's Conjecture

Some Consequences

- *π* ∉ *E*.
- π , log π , log log π , ... are algebraically independent over E.
- ► e ∉ L.

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Set up Related consequences of Schanuel's Conjecture

Some Consequences

- *π* ∉ *E*.
- π , log π , log log π , ... are algebraically independent over E.
- ► e ∉ L.
- e, e^e, e^{e^e}, \ldots are algebraically independent over *L*.

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Set up Related consequences of Schanuel's Conjecture

Some Consequences

π ∉ *E*.

• π , log π , log log π , ... are algebraically independent over E.

► e ∉ L.

• e, e^e, e^{e^e}, \ldots are algebraically independent over *L*.

More generally:

E and *L* are linearly disjoint over $\overline{\mathbb{Q}}$.

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Set up Related consequences of Schanuel's Conjecture

An enlightening example

Proposition: Schanuel's Conjecture implies $\pi \notin E$.

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Set up Related consequences of Schanuel's Conjecture

An enlightening example

Proposition: Schanuel's Conjecture implies $\pi \notin E$.

Proof: by induction on $n, \pi \notin E_n$.

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Set up Related consequences of Schanuel's Conjecture

An enlightening example

Proposition: Schanuel's Conjecture implies $\pi \notin E$.

Proof: by induction on $n, \pi \notin E_n$.

Base case: $\pi \notin E_0 = \overline{\mathbb{Q}}$ is clear.

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Set up Related consequences of Schanuel's Conjecture

Key Construction

Induction step:

• π is algebraic over $E_{n-1}(e^x : x \in E_{n-1})$.

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Set up Related consequences of Schanuel's Conjecture

Key Construction

Induction step:

- π is algebraic over $E_{n-1}(e^x : x \in E_{n-1})$.
- π is algebraic over $\overline{E_{n-2}(e^x : x \in E_{n-2})}(e^x : x \in E_{n-1}).$

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Set up Related consequences of Schanuel's Conjecture

Key Construction

Induction step:

- π is algebraic over $E_{n-1}(e^x : x \in E_{n-1})$.
- π is algebraic over $\overline{E_{n-2}(e^x : x \in E_{n-2})}(e^x : x \in E_{n-1}).$
- π is algebraic over $E_{n-2}(e^x : x \in E_{n-2})(e^x : x \in E_{n-1})$.

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Set up Related consequences of Schanuel's Conjecture

Key Construction

Induction step:

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- π is algebraic over $E_{n-1}(e^x : x \in E_{n-1})$.
- π is algebraic over $\overline{E_{n-2}(e^x : x \in E_{n-2})}(e^x : x \in E_{n-1}).$
- π is algebraic over $E_{n-2}(e^x : x \in E_{n-2})(e^x : x \in E_{n-1})$.
- π is algebraic over $E_{n-2}(e^x : x \in E_{n-1})$.

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Set up Related consequences of Schanuel's Conjecture

Key Construction

Induction step:

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- π is algebraic over $E_{n-1}(e^x : x \in E_{n-1})$.
- π is algebraic over $\overline{E_{n-2}(e^x : x \in E_{n-2})}(e^x : x \in E_{n-1}).$
- π is algebraic over $E_{n-2}(e^x : x \in E_{n-2})(e^x : x \in E_{n-1})$.
- π is algebraic over $E_{n-2}(e^x : x \in E_{n-1})$.
- π is algebraic over $\mathbb{Q}(e^x : x \in E_{n-1})$.

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Set up Related consequences of Schanuel's Conjecture

Key Construction

Induction step:

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- π is algebraic over $E_{n-1}(e^x : x \in E_{n-1})$.
- π is algebraic over $\overline{E_{n-2}(e^x : x \in E_{n-2})}(e^x : x \in E_{n-1}).$
- π is algebraic over $E_{n-2}(e^x : x \in E_{n-2})(e^x : x \in E_{n-1})$.
- π is algebraic over $E_{n-2}(e^x : x \in E_{n-1})$.
- π is algebraic over $\mathbb{Q}(e^x : x \in E_{n-1})$.

Therefore π is algebraic over $\mathbb{Q}(\exp(A_{n-1}))$ for some finite $A_{n-1} \subseteq E_{n-1}$.

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Set up Related consequences of Schanuel's Conjecture

Key Construction

Following similarly:

 A_{n-1} is algebraic over Q(exp(A_{n-2})) for some finite A_{n-2} ⊆ E_{n-2}.

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Set up Related consequences of Schanuel's Conjecture

Key Construction

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Following similarly:

- A_{n-1} is algebraic over $\mathbb{Q}(\exp(A_{n-2}))$ for some finite $A_{n-2} \subseteq E_{n-2}$.
- A_{n-2} is algebraic over $\mathbb{Q}(\exp(A_{n-3}))$ for some finite $A_{n-3} \subseteq E_{n-3}$.

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Set up Related consequences of Schanuel's Conjecture

Key Construction

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Following similarly:

- A_{n-1} is algebraic over $\mathbb{Q}(\exp(A_{n-2}))$ for some finite $A_{n-2} \subseteq E_{n-2}$.
- ► A_{n-2} is algebraic over $\mathbb{Q}(\exp(A_{n-3}))$ for some finite $A_{n-3} \subseteq E_{n-3}$.

• A_1 is algebraic over $\mathbb{Q}(\exp(A_0))$ for some finite $A_0 \subseteq E_0 = \overline{\mathbb{Q}}$.

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Set up Related consequences of Schanuel's Conjecture

End of proof

• Set
$$A = \bigcup_{m < n-1} A_m$$
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Some Consequences of Schanuel's Conjecture

Set up Related consequences of Schanuel's Conjecture

End of proof

- Set $A = \bigcup_{m \le n-1} A_m$.
- Take B ⊆ A such that exp(B) is a transcendence basis of Q(exp(A)).

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Set up Related consequences of Schanuel's Conjecture

End of proof

- Set $A = \bigcup_{m \le n-1} A_m$.
- Take B ⊆ A such that exp(B) is a transcendence basis of Q(exp(A)).
- ▶ $\{i\pi\} \cup B$ are \mathbb{Q} -linearly independent.

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Set up Related consequences of Schanuel's Conjecture

End of proof

- Set $A = \bigcup_{m \le n-1} A_m$.
- Take B ⊆ A such that exp(B) is a transcendence basis of Q(exp(A)).
- $\{i\pi\} \cup B$ are \mathbb{Q} -linearly independent.
- By Schanuel's Conjecture trdeg_QQ(iπ, B, exp(B)) ≥ |B| + 1.

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Set up Related consequences of Schanuel's Conjecture

End of proof

- Set $A = \bigcup_{m \le n-1} A_m$.
- Take B ⊆ A such that exp(B) is a transcendence basis of Q(exp(A)).
- $\{i\pi\} \cup B$ are \mathbb{Q} -linearly independent.
- ▶ By Schanuel's Conjecture $trdeg_{\mathbb{Q}}\mathbb{Q}(i\pi, B, \exp(B)) \ge |B| + 1$.
- ► But $trdeg_{\mathbb{Q}}\mathbb{Q}(i\pi, B, \exp(B)) = trdeg_{\mathbb{Q}}\mathbb{Q}(i\pi, B, \exp(A)) = trdeg_{\mathbb{Q}}\mathbb{Q}(\exp(A)) = trdeg_{\mathbb{Q}}\mathbb{Q}(\exp(B)) \le |B|.$

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Main Result Corollaries The Proof

Main result

We say K_1 and K_2 are linearly disjoint over k iff:

 $\{x_1, \ldots, x_n\} \subseteq K_1$ linearly independent over $k \Rightarrow$ linearly independent over K_2 .

Theorem: Schanuel's Conjecture implies E and L are linearly disjoint over $\overline{\mathbb{Q}}$.

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Main Result Corollaries The Proof

Corollaries:

$\blacktriangleright E \cap L = \overline{\mathbb{Q}}.$

Some Consequences of Schanuel's Conjecture

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Main Result Corollaries The Proof

Corollaries:

$\blacktriangleright E \cap L = \overline{\mathbb{Q}}.$

• π , log π , log log π , ... are algebraically independent over E.

Some Consequences of Schanuel's Conjecture

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Main Result Corollaries The Proof

Corollaries:

$\blacktriangleright \ E \cap L = \overline{\mathbb{Q}}.$

• π , log π , log log π , ... are algebraically independent over E.

• e, e^e, e^{e^e}, \ldots are algebraically independent over L.

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Main Result Corollaries The Proof

The Proof

Let's prove E_m and L_n are linearly disjoint. Take $\{l_i\} \subseteq L_n$ linearly independent over $\overline{\mathbb{Q}}$ and $\{e_i\} \subseteq E_m$ such that $\sum l_i e_i = 0$.

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Main Result Corollaries The Proof

The Proof

Let's prove E_m and L_n are linearly disjoint. Take $\{I_i\} \subseteq L_n$ linearly independent over $\overline{\mathbb{Q}}$ and $\{e_i\} \subseteq E_m$ such that $\sum I_i e_i = 0$.

Proceeding as before:

- \exists finite $A \subseteq E_{m-1}$ such that $A \cup \{e_i\}$ algebraic over $\mathbb{Q}(\exp(A))$.
- \exists finite $C \subseteq L_n$ finite such that $\exp(C) \cup \{l_i\}$ algebraic over $\mathbb{Q}(C)$.

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Main Result Corollaries The Proof

The Proof

Let's prove E_m and L_n are linearly disjoint. Take $\{I_i\} \subseteq L_n$ linearly independent over $\overline{\mathbb{Q}}$ and $\{e_i\} \subseteq E_m$ such that $\sum I_i e_i = 0$.

Proceeding as before:

∃ finite $A \subseteq E_{m-1}$ such that $A \cup \{e_i\}$ algebraic over $\mathbb{Q}(\exp(A))$. ∃ finite $C \subseteq L_n$ finite such that $\exp(C) \cup \{I_i\}$ algebraic over $\mathbb{Q}(C)$.

Take $B \subseteq A$ such that $\exp(B)$ is a transcendence basis of $\mathbb{Q}(\exp(A))$. Take $D \subseteq C$ such that D is a transcendence basis of $\mathbb{Q}(C)$.

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Main Result Corollaries The Proof

The Proof

We have $B \cup D$ linearly independent over \mathbb{Q} .

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Main Result Corollaries The Proof

The Proof

We have $B \cup D$ linearly independent over \mathbb{Q} .

By Schanuel's Conjecture $trdeg_{\mathbb{Q}}\mathbb{Q}(B, D, \exp(B), \exp(D)) \ge |B| + |D|.$

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Main Result Corollaries The Proof

The Proof

We have $B \cup D$ linearly independent over \mathbb{Q} .

By Schanuel's Conjecture $trdeg_{\mathbb{Q}}\mathbb{Q}(B, D, \exp(B), \exp(D)) \ge |B| + |D|.$

However

 $trdeg_{\mathbb{Q}}\mathbb{Q}(B, D, \exp(B), \exp(D)) = trdeg_{\mathbb{Q}}\mathbb{Q}(\exp(B), D) \leq |B| + |D|.$

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Main Result Corollaries The Proof

The Proof

Therefore $\mathbb{Q}(\exp(B))$ and $\mathbb{Q}(D)$ are free over $\overline{\mathbb{Q}}$, and the same is true for $\overline{\mathbb{Q}}(\exp(B))$ and $\overline{\mathbb{Q}(D)}$.

Since $\overline{\mathbb{Q}}$ is algebraically closed, $\overline{\mathbb{Q}(\exp(B))}$ and $\overline{\mathbb{Q}(D)}$ are linearly independent over $\overline{\mathbb{Q}}$ (see Lang's Algebra).

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Main Result Corollaries The Proof

References

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