Kempner Colloquium of the Department of Mathematics
University of Colorado at Boulder. February 28, 2005

Multiple Zeta Values

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/

Periods

M. Kontevich and D. Zagier (2000) – *Periods.*

A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains of $\mathbf{R}^n$ given by polynomials (in)equalities with rational coefficients.

Examples:

$$\sqrt{2} = \int_{2x^2 \le 1} dx,$$

$$\pi = \int_{x^2+y^2 \le 1} dxdy,$$

$$\log 2 = \int_{1<x<2} \frac{dx}{x},$$

$$\zeta(2) = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} = \frac{\pi^2}{6}.$$

Relations between periods

1 Additivity

$$\int_a^b \bigl(f(x)+g(x)\bigr)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2 Change of variables

$$\int_{\varphi(a)}^{\varphi(b)} f(t)dt = \int_a^b f\bigl(\varphi(u)\bigr)\varphi'(u)du.$$

3 Newton–Leibniz–Stokes

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Conjecture *(Kontsevich–Zagier). If a period has two representations, then one can pass from one formula to another using only rules* [1], [2] *and* [3] *in which all functions and domains of integrations are algebraic with algebraic coefficients.*

Example:

$$
\begin{aligned}
\pi &= \int_{x^2+y^2\leq 1} dxdy \\
&= 2\int_{-1}^{1} \sqrt{1-x^2}\, dx \\
&= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \\
&= \int_{-\infty}^{\infty} \frac{dx}{1+x^2}\cdot
\end{aligned}
$$

Far reaching consequences:

No "new" algebraic dependence relation among classical constants from analysis.

Zeta Values – Euler Numbers

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For $s \in \mathbf{Z}$ with $s \geq 2$, $\zeta(s)$ is a period:

$$\zeta(s) = \int_{1>t_1>\cdots>t_s>0} \frac{dt_1}{t_1} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}.$$

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Diophantine Question: *Describe all the algebraic relations among the numbers*

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7), \ldots$$

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- Rivoal (2000) + Ball, Zudilin. . . *Infinitely many* $\zeta(2k+1)$ *are irrational + lower bound for the dimension of the* $\mathbf{Q}$*-space they span.*

T. Rivoal: *Let* $\epsilon > 0$. *For any sufficiently large odd integer* a, *the dimension of the* $\mathbf{Q}$*-space spanned by* $1,\ \zeta(3),\ \zeta(5),\ \cdots, \zeta(a)$ *is at least*

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

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W. Zudilin:

- *One at least of the four numbers*

$$\zeta(5), \quad \zeta(7), \quad \zeta(9), \quad \zeta(11)$$

is irrational.

- *There is an odd integer* j *in the range* $[5, 69]$ *such that the three numbers* 1, $\zeta(3)$, $\zeta(j)$ *are linearly independent over* $\mathbf{Q}$.

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From

$$\sum_{n_1\ge 1} n_1^{-s_1} \sum_{n_2\ge 1} n_2^{-s_2} = \sum_{n_1>n_2\ge 1} n_1^{-s_1} n_2^{-s_2} + \sum_{n_2>n_1\ge 1} n_2^{-s_2} n_1^{-s_1} + \sum_{n\ge 1} n^{-s_1-s_2}$$

one deduces, for $s_1 \ge 2$ and $s_2 \ge 2$,

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

with

$$\zeta(s_1, s_2) = \sum_{n_1>n_2\ge 1} n_1^{-s_1} n_2^{-s_2}.$$

For instance

$$\begin{aligned}
\zeta(2)^2 &= \sum_{n_1\geq 1} n_1^{-2} \sum_{n_2\geq 1} n_2{}^{-2} \\
&= \sum_{n_1>n_2\geq 1} n_1^{-2} n_2{}^{-2} + \sum_{n_2>n_1\geq 1} n_2^{-2} n_1{}^{-2} + \sum_{n\geq 1} n^{-4} \\
&= 2\zeta(2,2) + \zeta(4).
\end{aligned}$$

For k, $s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$, define $\underline{s} = (s_1, \ldots, s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$

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For $k = 1$ one recovers Euler's numbers $\zeta(s)$.

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Fact: *These Multiple Zeta Values are periods*

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Example:

$$\zeta(2,1) = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{dt_3}{1-t_3}.$$

Notation: Define

$$\omega_0 = \frac{dt}{t}, \qquad \omega_1 = \frac{dt}{1-t}.$$

Then for $s \geq 2$ write the relation

$$\zeta(s) = \int_{1>t_1>\cdots>t_s>0} \frac{dt_1}{t_1} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}$$

as

$$\zeta(s) = \int_0^1 \omega_0^{s-1} \omega_1.$$

This defines a non-commutative product of differential forms.

Chen Iterated Integrals

For a holomorphic 1-form φ,

$$\int_0^z \varphi$$

is the primitive of φ which vanishes at $z = 0$.
For 1-forms $\varphi_1, \ldots, \varphi_k$, define inductively

$$\int_0^z \varphi_1 \cdots \varphi_k := \int_0^z \varphi_1(t) \int_0^t \varphi_2 \cdots \varphi_k.$$

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$$\int_0^z \varphi_1 \cdots \varphi_k := \int_0^z \varphi_1(t) \int_0^t \varphi_2 \cdots \varphi_k.$$

If $\varphi_1(t) = \psi_1(t)dt$, then

$$\frac{d}{dz} \int_0^z \varphi_1 \cdots \varphi_k = \psi_1(z) \int_0^z \varphi_2 \cdots \varphi_k.$$

For $\underline{s} = (s_1, \ldots, s_k)$, define

$$\omega_{\underline{s}} = \omega_{s_1} \cdots \omega_{s_k} = \omega_0^{s_1 - 1} \omega_1 \cdots \omega_0^{s_k - 1} \omega_1.$$

Then

$$\zeta(\underline{s}) = \int_0^1 \omega_{\underline{s}}.$$

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Remark on $\omega_0^{s_1-1}\omega_1 \cdots \omega_0^{s_k-1}\omega_1$**:**

- Ends with ω_1
- Starts with ω_0 $\quad (s_1 \geq 2)$.

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Hence the Multiple Zeta Values $\zeta(\underline{s})$ are periods.

Main Fact: *The product of two Multiple Zeta Values is a linear combination, with integer coefficients, of Multiple Zeta Values.*

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Moreover there are two kinds of such quadratic equations: one arising from the definition as series

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}},$$

the other from the integrals

$$\zeta(\underline{s}) = \int_0^1 \omega_{\underline{s}}.$$

These two collections of quadratic equations are essentially distinct. Consequently the Multiple Zeta Values satisfy many linear relations with rational coefficients.

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Example:

Product of series: $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$

Product of integrals: $\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1)$

Hence $\zeta(4) = 4\zeta(3,1).$

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A complete description of these relations would in principle settle the problem of the algebraic independence of

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Goal: *Describe all linear relations among Multiple Zeta Values.*

Further example of linear relation.

Euler:

$$\zeta(2,1)=\zeta(3).$$

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Euler's result follows from $(t_1, t_2, t_3) \mapsto (1-t_3, 1-t_2, 1-t_1)$.

Denote by $\mathfrak{Z}_p$ the $\mathbf{Q}$-vector subspace of $\mathbf{R}$ spanned by the real numbers $\zeta(\underline{s})$ with $\underline{s}$ of weight $s_1+\cdots+s_k=p$, with $\mathfrak{Z}_0=\mathbf{Q}$ and $\mathfrak{Z}_1=\{0\}$.

Here is Zagier's conjecture on the dimension d_p of $\mathfrak{Z}_p$.

Conjecture *(Zagier). For $p\geq 3$ we have*

$$d_p=d_{p-2}+d_{p-3}.$$

$$(d_0,\ d_1,\ d_2,\ldots)=(1,0,1,1,1,2,2,\ldots).$$

Exemples

$d_0 = 1 \qquad \zeta(s_1, \ldots, s_k) = 1$ for $k = 0$.

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$d_3 = 1$ $\qquad \zeta(2,1) = \zeta(3) \neq 0$

$d_4 = 1$ $\qquad \zeta(3,1) = (1/4)\zeta(4),$

$$\zeta(2,2) = (3/4)\zeta(4),$$

$$\zeta(2,1,1) = \zeta(4) = (2/5)\zeta(2)^2$$

Question: $d_5 = 2$?

Since

$$\begin{aligned}\zeta(2,1,1,1) &= \zeta(5),\\ \zeta(3,1,1) &= \zeta(4,1) = 2\zeta(5) - \zeta(2)\zeta(3),\\ \zeta(2,1,2) &= \zeta(2,3) = \frac{9}{2}\zeta(5) - 2\zeta(2)\zeta(3),\\ \zeta(2,2,1) &= \zeta(3,2) = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5),\end{aligned}$$

we have $d_5 \in \{1, 2\}$.

Further, $d_5 = 2$ if and only if the number

$$\zeta(2)\zeta(3)/\zeta(5)$$

is irrational.

Zagier's conjecture can be written

$$\sum_{p\geq 0} d_p X^p = \frac{1}{1 - X^2 - X^3}.$$

M. Hoffman conjectures: *a basis of* $\mathfrak{Z}_p$ *over* $\mathbf{Q}$ *is given by the numbers* $\zeta(s_1, \dots, s_k)$, $s_1 + \cdots + s_k = p$, *where each* s_i *is either* 2 *or* 3.

True for $p \leq 16$ (Hoang Ngoc Minh)

A.G. Goncharov (2000) – *Multiple* ζ*-values, Galois groups and Geometry of Modular Varieties.*

T. Terasoma (2002) – *Mixed Tate motives and Multiple Zeta Values.*

The numbers defined by the recurrence relation of Zagier's Conjecture

$$d_p = d_{p-2} + d_{p-3}.$$

with initial values $d_0 = 1$, $d_1 = 0$ *are actual upper bounds for the actual dimension of* $\mathfrak{Z}_p$.

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To prove a lower bound is the main Diophantine conjecture!

Nothing is known, even $d_p \geq 2$ for a single p!

Algebraic description of the quadratic relations among MZV

1 Integrals:

Shuffle product of differential forms

$$\varphi_1 \cdots \varphi_n \,ш\, \psi_1 \cdots \psi_k \;=\; \varphi_1(\varphi_2 \cdots \varphi_n ш \psi_1 \cdots \psi_k) + \psi_1(\varphi_1 \cdots \varphi_n ш \psi_2 \cdots \psi_k).$$

$$\varphi_1 ш \psi_1 = \varphi_1\psi_1 + \psi_1\varphi_1.$$

Product of iterated integrals:

Let $\varphi_1, \ldots, \varphi_n$, $\psi_1, \ldots, \psi_k$ be differential forms with $n \geq 0$ and $k \geq 0$. Then

$$\int_0^z \varphi_1 \cdots \varphi_n \int_0^z \psi_1 \cdots \psi_k = \int_0^z \varphi_1 \cdots \varphi_n ш \psi_1 \cdots \psi_k.$$

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Proof. Assume $z > 0$. Decompose the Cartesian product

$$\{\underline{t} \in \mathbf{R}^n \,;\; z \geq t_1 \geq \cdots \geq t_n \geq 0\} \times \{\underline{u} \in \mathbf{R}^k \,;\; z \geq u_1 \geq \cdots \geq u_k \geq 0\}$$

into a disjoint union of simplices (up to sets of zero measure)

$$\{\underline{v} \in \mathbf{R}^{n+k} \,;\; z \geq v_1 \geq \cdots \geq v_{n+k} \geq 0\}.$$

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$$ab ш cd = abcd + acbd + acdb + cabd + cadb + cdab$$

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$$\omega_0\omega_1 \sqcup\!\sqcup \omega_0\omega_1 = 4\omega_0^2\omega_1^2 + 2\omega_0\omega_1\omega_0\omega_1$$

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$$\int_0^1 \omega_0\omega_1 \cdot \int_0^1 \omega_0\omega_1 = 4\int_0^1 \omega_0^2\omega_1^2 + 2\int_0^1 \omega_0\omega_1\omega_0\omega_1$$

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$$\zeta(2)^2 = 4\zeta(3,1) + 2\zeta(2,2).$$

Next goal: Extend the definition of Multiple Zeta Values to linear combinations of $\omega_{\underline{s}}$, so that the product of two Multiple Zeta Values is a Multiple Zeta Value.

Write $\widehat{\zeta}(\omega_{\underline{s}})$ in place of $\zeta(\underline{s})$ and define more generally

$$\widehat{\zeta}\left(\sum c_{\underline{s}}\omega_{\underline{s}}\right) = \sum c_{\underline{s}}\zeta(\underline{s})$$

so that

$$\zeta(\underline{s})\zeta(\underline{s}') = \widehat{\zeta}\left(\omega_{\underline{s}} ш \omega_{\underline{s}'}\right).$$

Tool: Free algebra on $\{\omega_0, \omega_1\}$.

The free monoid X^*

Let $X = \{x_0, x_1\}$ denote the *alphabet* with two letters x_0, x_1 and X^* the free monoid on X. The elements of X^* are *words*. A word can be written

$$x_{\epsilon_1} \cdots x_{\epsilon_k}$$

with $k \geq 0$ and where each ϵ_j is 0 or 1.

This law is called *concatenation.* It is not commutative:

$$x_0 x_1 \neq x_1 x_0.$$

Its unit is the *empty word* $e \in X^*$: the word for which $k = 0$.

The Algebra $\mathfrak{H} = \mathbf{Q}\langle x_0, x_1\rangle$

The free $\mathbf{Q}$-vector space with basis X^* is the free algebra on X, denoted by $\mathfrak{H} = \mathbf{Q}\langle X\rangle$. Its elements are non commutative polynomials in the two variables x_0, x_1.

Its unit is the $empty$ word e.

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Let $w \in X^*x_1$. Write $w = x_{\epsilon_1} \cdots x_{\epsilon_p}$ where each ϵ_i is 0 or 1 and $\epsilon_p = 1$.
If k is the number of x_1, we define positive integers $s_1, \ldots, s_k$ by

$$w = x_0^{s_1 - 1} x_1 \cdots x_0^{s_k - 1} x_1.$$

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$$w = x_0^{s_1-1}x_1 \cdots x_0^{s_k-1}x_1.$$

For $s \geq 1$ define $y_s = x_0^{s-1}x_1$. For $\underline{s} = (s_1, \ldots, s_k)$ with $s_i \geq 1$, set

$$y_{\underline{s}} = y_{s_1} \cdots y_{s_k} = x_0^{s_1-1}x_1 \cdots x_0^{s_k-1}x_1.$$

$y_{\underline{s}}$ *is a word on the alphabet*

$$Y = \{y_1, y_2, \ldots, y_s, \ldots\}.$$

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$$Y = \{y_1, y_2, \ldots, y_s, \ldots\}.$$

The free monoid Y^* on Y

$$Y^* = \{y_{\underline{s}}\ ;\ \underline{s} = (s_1, \ldots, s_k),\ k \geq 0,\ s_j \geq 1\ (1 \leq j \leq k)\}$$

is the set $\{e\} \cup X^* x_1$ of words which do not end with x_0, hence Y^* is a submonoid of X^*.

Any message can be coded with only two letters.

The Subalgebra $\mathfrak{H}^1 = \mathbf{Q}e + \mathfrak{H}x_1$ of $\mathfrak{H}$

The free $\mathbf{Q}$-vector space with basis Y^* is the free algebra

$$\mathfrak{H}^1 = \mathbf{Q}\langle Y \rangle$$

on Y. Its elements are non commutative polynomials in the variables $\{y_1, \ldots, y_s, \ldots\}$. It is a subalgebra of $\mathfrak{H}$.

The Subalgebra $\mathfrak{H}^0 = \mathbf{Q}e + x_0\mathfrak{H}x_1$ of $\mathfrak{H}$

The set of words in X^* which start with x_0 and end with x_1 is $x_0X^*x_1$.

The set of words in X^* which do not start with x_1 and do not end with x_0 is $\{e\} \cup x_0X^*x_1$.

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The set of words in X^* which do not start with x_1 and do not end with x_0 is $\{e\} \cup x_0X^*x_1$.

This is NOT the same as the free monoid on the infinite alphabet $\{y_2, y_3, \ldots\}$.
Example: $y_2y_1 \in x_0X^*x_1$.

The Subalgebra $\mathfrak{H}^0 = \mathbf{Q}e + x_0\mathfrak{H}x_1$ of $\mathfrak{H}$

The set of words in X^* which start with x_0 and end with x_1 is $x_0X^*x_1$.

The set of words in X^* which do not start with x_1 and do not end with x_0 is $\{e\} \cup x_0X^*x_1$.

The $\mathbf{Q}$-vector subspace of $\mathfrak{H}^1$ spanned by $\{e\} \cup x_0X^*x_1$ is the sub-algebra

$$\mathfrak{H}^0 = \mathbf{Q}e + x_0\mathfrak{H}x_1 \subset \mathfrak{H}^1 \subset \mathfrak{H}.$$

Multizeta values associated to words

For $w \in x_0X^*x_1$, write $w = y_{\underline{s}}$ with $\underline{s} = (s_1, \ldots, s_k)$ and $s_1 \geq 1$, and define

$$\widehat{\zeta}(w) = \zeta(\underline{s}).$$

Define also $\widehat{\zeta}(e) = 1$ and extend by $\mathbf{Q}$-linearity the definition of $\widehat{\zeta}$ to $\mathfrak{H}^0$. Hence we get a mapping

$$\widehat{\zeta} : \mathfrak{H}^0 \longrightarrow \mathbf{R}.$$

Shuffle relations among MZV

For w and w' in $\mathfrak{H}^0$, the shuffle product $w ш w'$ belongs to $\mathfrak{H}^0$. Furthermore,

$$\widehat{\zeta}(w)\widehat{\zeta}(w') = \widehat{\zeta}(w ш w')$$

for any w and w' in $\mathfrak{H}^0$.

Proposition. *The map $\widehat{\zeta} : \mathfrak{H}^0 \rightarrow \mathbf{R}$ is a morphism of algebras of $\mathfrak{H}^0_ш$ into $\mathbf{R}$.*

2 Series:

The Harmonic Algebra

The product $\zeta(\underline{s})\cdot\zeta(\underline{s}')$*:*

$$\sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}\cdot\sum_{n'_1>n'_2>\cdots>n'_{k'}\geq 1}\frac{1}{{n'}_1^{s'_1}\cdots {n'}_{k'}^{{s'}_{k'}}}$$

is a linear combination of MZV.
Shuffle like product (*stuffle*) on the alphabet Y.

The map $\star : Y^* \times Y^* \to \mathfrak{H}$ is defined by induction

$$y_s u \star y_t v = y_s(u \star y_t v) + y_t(y_s u \star v) + y_{s+t}(u \star v)$$

for u and v in Y^*, s and t positive integers.

This defines *Hoffman's harmonic algebra* denoted by $\mathfrak{H}_\star$.

Examples.

$$y_2^{\star 2} = y_2 \star y_2 = 2y_2^2 + y_4.$$

$$y_2^{\star 3} = y_2 \star y_2 \star y_2 = 6y_2^3 + 3y_2y_4 + 3y_4y_2 + y_6.$$

Quadratic relations arising from the product of series

The map $\widehat{\zeta}:\mathfrak{H}^0\rightarrow\mathbf{R}$ is a morphism of algebras of $\mathfrak{H}^0_\star$ into $\mathbf{R}$:

$$\widehat{\zeta}(u\star v)=\widehat{\zeta}(u)\widehat{\zeta}(v).$$

for u and v in $\mathfrak{H}^0$.

Quadratic relations arising from the product of series

The map $\widehat{\zeta} : \mathfrak{H}^0 \to \mathbf{R}$ is a morphism of algebras of $\mathfrak{H}^0_\star$ into $\mathbf{R}$:

$$\widehat{\zeta}(u \star v) = \widehat{\zeta}(u)\widehat{\zeta}(v).$$

for u and v in $\mathfrak{H}^0$.

Consequence of the two sets of quadratic relations:

$$\widehat{\zeta}(u ш v - u \star v) = 0$$

for u and v in $\mathfrak{H}^0$.

Hoffman Third Standard Relations

For any $w \in \mathfrak{H}^0$, we have $x_1 \sqcup\!\sqcup w - x_1 \star w \in \mathfrak{H}^0$ and

$$\widehat{\zeta}(x_1 \sqcup\!\sqcup w - x_1 \star w) = 0.$$

Hoffman Third Standard Relations

For any $w \in \mathfrak{H}^0$, we have $x_1 ш w - x_1 \star w \in \mathfrak{H}^0$ and

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Example. For $w = x_0 x_1$,

$$x_1 ш x_0 x_1 = x_1 x_0 x_1 + 2 x_0 x_1^2 = y_1 y_2 + 2 y_2 y_1,$$

$$x_1 \star x_0 x_1 = y_1 \star y_2 = y_1 y_2 + y_2 y_1 + y_3,$$

hence

$$y_2 y_1 - y_3 \in \ker \widehat{\zeta}$$

and (Euler) $$\zeta(2,1) = \zeta(3).$$

Euler's proof with divergent series:

Product of series: $\zeta(1)\zeta(2) = \zeta(1,2) + \zeta(2,1) + \zeta(3)$

Product of integrals: $\zeta(1)\zeta(2) = \zeta(1,2) + 2\zeta(2,1)$

Hence $\zeta(3) = \zeta(2,1).$

Diophantine Conjecture *(simple form)*

Conjecture (Petitot, Hoang Ngoc Minh. . .). *The kernel of* $\widehat{\zeta}$ *is spanned by the standard relations*

$$\widehat{\zeta}(u ш v - u \star v) = 0 \quad \textit{and} \quad \widehat{\zeta}(x_1 ш w - x_1 \star w) = 0$$

for u, v *and* w *in* $x_0 X^* x_1$.

Minh, H.N, Jacob, G., Oussous, N. E., Petitot, M. –
Aspects combinatoires des polylogarithmes et des sommes d'Euler-Zagier.
J. Électr. Sém. Lothar. Combin. **43** (2000), Art. B43e, 29 pp.

Regularized Double Shuffle Relations

The map $\widehat{\zeta} : \mathfrak{H}^0 \to \mathbf{R}$ is a morphism of algebras for $ш$ and for $\star$:

$$\widehat{\zeta}(u ш v) = \widehat{\zeta}(u)\widehat{\zeta}(v) \quad \text{and} \qquad \widehat{\zeta}(u \star v) = \widehat{\zeta}(u)\widehat{\zeta}(v).$$

Question: Is-it possible to extend $\widehat{\zeta}$ to $\mathfrak{H}^1$ into a morphism of algebras both for $ш$ and $\star$?

Regularized Double Shuffle Relations

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Question: Is-it possible to extend $\widehat{\zeta}$ to $\mathfrak{H}^1$ into a morphism of algebras both for ш and $\star$?

Answer: NO!

$$x_1 ш x_1 = 2x_1^2, \qquad x_1 \star x_1 = y_1 \star y_1 = 2x_1^2 + y_2$$

$$\widehat{\zeta}(y_2) = \zeta(2) \neq 0.$$

Radford's Theorem:

$$\mathfrak{H}_{ш} = \mathfrak{H}^1_{ш}[x_0]_{ш} = \mathfrak{H}^0_{ш}[x_0, x_1]_{ш} \quad \textit{and} \quad \mathfrak{H}^1_{ш} = \mathfrak{H}^0_{ш}[x_1]_{ш}.$$

Radford's Theorem:

$$\mathfrak{H}_{ш} = \mathfrak{H}^1_{ш}[x_0]_{ш} = \mathfrak{H}^0_{ш}[x_0, x_1]_{ш} \quad \textit{and} \qquad \mathfrak{H}^1_{ш} = \mathfrak{H}^0_{ш}[x_1]_{ш}.$$

Hoffman's Theorem:

$$\mathfrak{H}_{\star} = \mathfrak{H}^1_{\star}[x_0]_{\star} = \mathfrak{H}^0_{\star}[x_0, x_1]_{\star} \quad \textit{and} \quad \mathfrak{H}^1_{\star} = \mathfrak{H}^0_{\star}[x_1]_{\star}.$$

Radford's Theorem:

$$\mathfrak{H}_{ш} = \mathfrak{H}^1_{ш}[x_0]_{ш} = \mathfrak{H}^0_{ш}[x_0, x_1]_{ш} \quad \textit{and} \qquad \mathfrak{H}^1_{ш} = \mathfrak{H}^0_{ш}[x_1]_{ш}.$$

Hoffman's Theorem:

$$\mathfrak{H}_\star = \mathfrak{H}^1_\star[x_0]_\star = \mathfrak{H}^0_\star[x_0, x_1]_\star \quad \textit{and} \quad \mathfrak{H}^1_\star = \mathfrak{H}^0_\star[x_1]_\star.$$

From $\mathfrak{H}^1_{ш} = \mathfrak{H}^0_{ш}[x_1]_{ш}$ and $\mathfrak{H}^1_\star = \mathfrak{H}^0_\star[x_1]_\star$ we deduce that there are two uniquely determined algebra morphisms

$$\widehat{Z}_{ш} : \mathfrak{H}^1_{ш} \longrightarrow \mathbf{R}[T] \quad \text{and} \qquad \widehat{Z}_\star : \mathfrak{H}^1_\star \longrightarrow \mathbf{R}[T]$$

which extend $\widehat{\zeta}$ and map x_1 to T.

Theorem (Boutet de Monvel, Zagier). *There is a* $\mathbf{R}$*-linear isomorphism* $\varrho : \mathbf{R}[T] \to \mathbf{R}[X]$ *which makes commutative the following diagram:*

$$
\begin{array}{ccc}
 & & \mathbf{R}[X] \\
 & \widehat{Z}_{Ш} \nearrow & \\
\mathfrak{H}^1 & & \uparrow \varrho \\
 & \widehat{Z}_{\star} \searrow & \\
 & & \mathbf{R}[T]
\end{array}
$$

An explicit formula for ϱ is given by means of the generating series

$$\sum_{\ell\geq 0}\varrho(T^\ell)\frac{t^\ell}{\ell!}=\exp\left(Xt+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}t^n\right).$$

Compare with the formula giving the expansion of the logarithm of Euler Gamma function:

$$\Gamma(1+t)=\exp\left(-\gamma t+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}t^n\right).$$

One may see ϱ as the differential operator of infinite order

$$\exp\left(\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}\left(\frac{\partial}{\partial T}\right)^n\right)$$

(just consider the image of e^{tT}).

Denote by $\text{reg}_ш$ the $\mathbf{Q}$-linear map $\mathfrak{H} \to \mathfrak{H}^0$ which maps $w \in \mathfrak{H}$ onto its constant term when w is written as a polynomial in x_0, x_1 in the shuffle algebra $\mathfrak{H}^0[x_0, x_1]_ш$. Then $\text{reg}_ш$ is a morphism of algebras $\mathfrak{H}_ш \to \mathfrak{H}^0_ш$.

Theorem. *(Regularized Double Shuffle Relations – Ihara and Kaneko). For $w \in \mathfrak{H}^1$ and $w_0 \in \mathfrak{H}^0$,*

$$\text{reg}_ш(w ш w_0 - w \star w_0) \in \ker \widehat{\zeta}.$$

Example. Take $w = x_1$. Since $x_1 ш w_0 - x_1 \star w_0 \in \mathfrak{H}^0$ for any $w_0 \in \mathfrak{H}^0$, one recovers the third standard relations of Hoffman.

Diophantine Conjectures

Conjecture (Zagier, Cartier, Ihara-Kaneko,. . .). *All existing algebraic relations between the real numbers* $\zeta(\underline{s})$ *are in the ideal generated by the ones described above.*

Petitot and Hoang Ngoc Minh: up to weight $s_1 + \cdots s_k \le 16$, the three standard relations for u, v and w in $x_0 X^* x_1$

$$\widehat{\zeta}(u)\widehat{\zeta}(v) = \widehat{\zeta}(u \sqcup\!\sqcup v), \quad \widehat{\zeta}(u)\widehat{\zeta}(v) = \widehat{\zeta}(u \star v),$$

$$\widehat{\zeta}(x_1 \sqcup\!\sqcup w - x_1 \star w) = 0$$

suffice.

Goncharov's Conjecture

Let $\mathfrak{Z}$ denote the $\mathbf{Q}$-vector space spanned in $\mathbf{C}$ by the numbers

$$(2i\pi)^{-|s|}\zeta(\underline{s})$$

$\underline{s} = (s_1, \ldots, s_k) \in \mathbf{N}^k$ with $k \geq 1$, $s_1 \geq 2$, $s_i \geq 1$ $(2 \leq i \leq k)$.

Hence $\mathfrak{Z}$ is a $\mathbf{Q}$-subalgebra of $\mathbf{C}$ bifiltered by the weight and by the depth.

For a graded Lie algebra $C_\bullet$ denote by $\mathfrak{U}C_\bullet$ its universal envelopping algebra and by

$$\mathfrak{U}C_\bullet^\vee = \bigoplus_{n\geq 0} (\mathfrak{U}C)_n^\vee$$

its graded dual, which is a commutative Hopf algebra.

Conjecture (Goncharov). *There exists a free graded Lie algebra $C_\bullet$ and an isomorphism of algebras*

$$\mathfrak{Z} \simeq \mathfrak{U}C_\bullet^\vee$$

filtered by the weight on the left and by the degree on the right.

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