update *29:09/2017*

Families of Thue equations associated with a rank one subgroup of the unit group of a number field

by

Claude Levesque and *Michel Waldschmidt*

In memory of Klaus Roth

ABSTRACT. Let K be an algebraic number field of degree $d \geq 3$, $\sigma_1, \sigma_2, \ldots, \sigma_d$ the embeddings of K into $\mathbb{C}$, α a nonzero element in K, $a_0 \in \mathbb{Z}$, $a_0 > 0$ and

$$F_0(X,Y) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha)Y).$$

Let υ be a unit in K. For $a \in \mathbb{Z}$, we twist the binary form $F_0(X,Y) \in \mathbb{Z}[X,Y]$ by the powers υ^a ($a \in \mathbb{Z}$) of υ by setting

$$F_a(X,Y) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha\upsilon^a)Y).$$

Given $m \geq 0$, our main result is an effective upper bound for the size of solutions $(x,y,a) \in \mathbb{Z}^3$ of the Diophantine inequalities

$$0 < |F_a(x,y)| \leq m$$

for which $xy \neq 0$ and $\mathbb{Q}(\alpha\upsilon^a) = K$. Our estimate is explicit in terms of its dependence on m, the regulator of K and the heights of F_0 and of υ; it also involves an effectively computable constant depending only on d.

KEYWORDS: Families of Thue equations, Diophantine equations
MSC: 11D61, 11D41, 11D59

1 Introduction and the main results

Let $d \geq 3$ be a given integer. We denote by $\kappa_1, \kappa_2, \ldots, \kappa_{38}$ positive effectively computable constants which depend only on d.

Let K be a number field of degree d. Denote by $\sigma_1, \sigma_2, \ldots, \sigma_d$ the embeddings of K into $\mathbb{C}$ and by R the regulator of K. Let $\alpha \in K$, $\alpha \neq 0$, and let $a_0 \in \mathbb{Z}$, $a_0 > 0$, be such that the coefficients of the polynomial

$$f_0(X) = a_0 \prod_{i=1}^{d} \bigl(X - \sigma_i(\alpha)\bigr)$$

are in $\mathbb{Z}$. Let υ be a unit in K, not a root of unity. For $a \in \mathbb{Z}$, define the polynomial $f_a(X)$ in $\mathbb{Z}[X]$ and the binary form $F_a(X, Y)$ in $\mathbb{Z}[X, Y]$ by

$$f_a(X) = a_0 \prod_{i=1}^{d} \bigl(X - \sigma_i(\alpha\upsilon^a)\bigr)$$

and

$$F_a(X, Y) = Y^d f_a(X/Y) = a_0 \prod_{i=1}^{d} \bigl(X - \sigma_i(\alpha\upsilon^a)Y\bigr).$$

Define

$$\lambda_0 = a_0 \prod_{i=1}^{d} \max\{1, |\sigma_i(\alpha)|\} \quad \text{and} \quad \lambda = \prod_{i=1}^{d} \max\{1, |\sigma_i(\upsilon)|\}.$$

Let $m \in \mathbb{Z}$, $m > 0$. We consider the family of Diophantine inequalities

$$0 < |F_a(x, y)| \le m, \tag{1}$$

where the unknowns (x, y, a) take their values in the set of elements in $\mathbb{Z}^3$ such that $xy \ne 0$ and $\mathbb{Q}(\alpha\upsilon^a) = K$. It follows from the results in [4] that the set of solutions is finite. However, the proof in [4] relies on Schmidt's subspace theorem, which is not effective. Here by using lower bounds for linear forms in logarithms, we give an upper bound for $\max\{\log|x|, \log|y|, |a|\}$, which is explicit in terms of m, R, λ_0 and λ and which involves an effectively computable constant depending only on d.

For $x \in \mathbb{R}$, $x > 0$, we use the notation $\log^* x$ to denote $\max\{1, \log x\}$. Here is our main result.

Theorem 1. *There exists an effectively computable constant $\kappa_1 > 0$, depending only on d, such that any solution $(x, y, a) \in \mathbb{Z}^3$ of (1), which verifies $xy \ne 0$ and $\mathbb{Q}(\alpha\upsilon^a) = K$, satisfies*

$$|a| \le \kappa_1 \lambda^{d^2(d+2)/2} (R + \log m + \log \lambda_0) R \log^\star R.$$

Under the assumptions of Theorem 1, with the help of the upper bound

$$\mathrm{H}(F_a) \le 2^d \lambda_0 \, \lambda^{|a|}$$

for the (usual) height of the form F_a (namely the maximum of the absolute values of the coefficients of the form), it follows from the bound (3.2) in [1, Theorem 3] (see also [2, Th. 9.6.2]) that

$$\log\max\{|x|, |y|\} \le \kappa_2 \bigl(R + \log^\star m + |a| \log\lambda + \log\lambda_0\bigr) R (\log^\star R) \tag{2}$$

where κ_2 is an explicit constant depending only on d. Combining this upper bound with our Theorem 1 provides an effective upper bound for $\max\{\log|x|, \log|y|, |a|\}$.

Corollary 2. *Under the assumptions of Theorem* 1*, there exists an effectively computable constant κ_3 depending only on d such that*

$$\max\{\log|x|, \log|y|, |a|\} \leq \kappa_3 \lambda^{d^2(d+2)/2} (\log\lambda)(R + \log m + \log\lambda_0) R^2 (\log^\star R)^2.$$

Our proof of Theorem 1 actually gives a much stronger estimate for $|a|$; see Theorem 3 below. It involves an extra parameter $\mu > 1$ that we now define.

For $i = 1, \ldots, d$, set $\upsilon_i = \sigma_i(\upsilon)$ and assume

$$|\upsilon_1| \leq |\upsilon_2| \leq \cdots \leq |\upsilon_d|.$$

Define

$$\mu = \begin{cases} \lambda & \text{if } |\upsilon_1| = |\upsilon_{d-1}| \text{ or } |\upsilon_2| = |\upsilon_d|, \\ \min\left\{\dfrac{|\upsilon_{d-1}|}{|\upsilon_1|}, \dfrac{|\upsilon_d|}{|\upsilon_2|}\right\} & \text{if } |\upsilon_1| < |\upsilon_2| = |\upsilon_{d-1}| < |\upsilon_d|, \\ \dfrac{|\upsilon_{d-1}|}{|\upsilon_2|} & \text{if } |\upsilon_2| < |\upsilon_{d-1}|. \end{cases}$$

Notice that the condition $|\upsilon_1| = |\upsilon_{d-1}|$ means $|\upsilon_1| = |\upsilon_2| = \cdots = |\upsilon_{d-1}|$ and that the condition $|\upsilon_2| = |\upsilon_d|$ means $|\upsilon_2| = |\upsilon_3| = \cdots = |\upsilon_d|$; using Lemma 12, we deduce that each of these two conditions implies that d is odd, hence that the field K is almost totally imaginary (namely, with a single real embedding) – compare with [9].

Theorem 3. *There exists a positive effectively computable constant κ_4, depending only on d, with the following property. Let $(x, y, a) \in \mathbb{Z}^3$ satisfy*

$$xy \neq 0, \quad [\mathbb{Q}(\alpha\upsilon^a) : \mathbb{Q}] = d \quad \textit{and} \quad 0 < |F_a(x, y)| \leq m.$$

Then

$$|a| \leq \kappa_4 \frac{\log\lambda}{\log\mu}(R + \log m + \log\lambda_0 + \log\lambda) R \log\left(R \frac{(\log\lambda)^2}{\log\mu}\right). \tag{3}$$

On the one hand, using Lemma 13 (§3.6), we will prove in §5 that

$$\log\mu \geq \kappa_5 \lambda^{-d^2(d+2)/2} (\log\lambda)^2,$$

which will enable us to deduce Theorem 1 from Theorem 3. On the other hand, thanks to (7), we have $\mu \leq \lambda^2$. Hence the largest possible value of

μ is λ^{κ_6} with a positive constant κ_6 depending only on d. For the units υ satisfying such an estimate, the conclusion of Theorem 3 becomes

$$(4) \qquad |a| \le \kappa_7(R + \log m + \log \lambda_0 + \log \lambda) R(\log R + \log^\star \log^\star \lambda)$$

with a positive effectively computable constant κ_7 depending only on d. In §2, we give a few examples where this last bound is valid.

In Theorem 1, the hypothesis that υ is not a root of unity cannot be omitted. Here is an example with $\alpha = a_0 = m = 1$. Let $\Phi_n(X)$ be the cyclotomic polynomial of index n and degree $\varphi(n)$ (Euler totient function). Let ζ_n be a primitive n–th root of unity. Set $f_0 = \Phi_n$ and $u = \zeta_n$. For $a \in \mathbb{Z}$ with $\gcd(a, n) = 1$, the irreducible polynomial f_a of ζ_n^a is nothing else than f_0. Hence, if the equation

$$F_0(x, y) = \pm 1$$

has a solution $(x, y) \in \mathbb{Z}^2$ with $xy \neq 0$, then for infinitely many $a \in \mathbb{Z}$ the twisted Thue equation $F_a(x, y) = \pm 1$ has also the solution (x, y), since $F_a = F_0$. For instance, when $n = 12$, we have $\Phi_{12}(X) = X^4 - X^2 + 1$ and the equation

$$x^4 - x^2y^2 + y^4 = 1$$

has the solutions $(1, 1)$, $(-1, 1)$, $(1, -1)$, $(-1, -1)$.

Let us compare the results of the present paper with our previous work.

The main result of [5], which deals only with non totally real cubic equations, is a special case of Theorem 3; the "constants" in [5] depend on α and υ, while here they depend only on d. The main result of [6] deals with Thue equations twisted by a set of units which is not supposed to be a group of rank 1, but it involves an assumption (namely that at least two of the conjugates of υ have a modulus as large as a positive power of $\lceil\overline{\upsilon}\rceil$) which we do not need here. Our Theorem 3 also improves the main result of [7]: we remove the assumption that the unit is totally real (besides, the result of [7] is not explicit in terms of the heights and regulator). We also notice that part (iii) of Theorem 1.1 of [8] follows from our Theorem 3. The main result of [9] does not assume that the twists are done by a group of units of rank 1, but it needs a strong assumption which does not occur here, namely that the field K has at most one real embedding.

2 Examples

The lower bound $\mu \ge \lambda^{\kappa_6}$ quoted in Section 1 is true
- when $d = 3$ and the cubic field K is not totally real;
- for the Salem numbers;
- for the roots of the polynomials in the families giving the simplest fields of degree 3 (see [8]), and also the simplest fields of degrees 4 and 6;

• when $|\upsilon_1| = |\upsilon_2|$ and $|\upsilon_{d-1}| = |\upsilon_d|$ with $d \geq 4$. In particular when $-\upsilon$ is a Galois conjugate of υ (which means that the irreducible polynomial of υ is in $\mathbb{Z}[X^2]$).

Here is an example of this last situation. Let ϵ be an algebraic unit, not a root of unity, of degree $\ell \geq 2$ and conjugates $\epsilon_1, \epsilon_2, \ldots, \epsilon_\ell$. Let $h \geq 2$ and let $d = \ell h$. For $a \in \mathbb{Z}$, define

$$F_a(X,Y) = \prod_{i=1}^{\ell} (X^h - \epsilon_i^a Y^h). \tag{5}$$

Let R be the regulator of the field $\mathbb{Q}(\epsilon^{1/h})$.

From Theorem 3 we deduce the following corollary.

Corollary 4. *Let $m \geq 1$. If the form F_a in (5) is irreducible and if there exists $(x,y) \in \mathbb{Z}^2$ with $xy \neq 0$ and $|F_a(x,y)| \leq m$, then*

$$|a| \leq \kappa_8 (R + \log m + \log \overline{\lceil \epsilon \rceil}) R \log^\star (R \log \overline{\lceil \epsilon \rceil}).$$

Proof. Without loss of generality, assume $|\epsilon_1| \leq |\epsilon_2| \leq \cdots \leq |\epsilon_\ell|$, so that $|\epsilon_\ell| = \overline{\lceil \epsilon \rceil}$. Let ζ be a primitive h-th root of unity. Let $\upsilon = \epsilon^{1/h}$. We apply Theorem 3 with $\alpha = \zeta$, $a_0 = 1$, $\lambda_0 = 1$, $\lambda \leq \overline{\lceil \epsilon \rceil}^\ell$, $F_0(X,Y) = (X^h - Y^h)^\ell$ and

$$\upsilon_{ih+j} = \zeta^{j-1} \epsilon_{i+1}^{1/h} \quad (0 \leq i \leq \ell - 1,\ 1 \leq j \leq h).$$

From $|\upsilon_1| = |\upsilon_2| = |\epsilon_1|^{1/h} < 1$ and $|\upsilon_{d-1}| = |\upsilon_d| = |\epsilon_\ell|^{1/h}$ we deduce

$$\mu = \left| \frac{\epsilon_\ell}{\epsilon_1} \right|^{1/h} = \left| \frac{\upsilon_d}{\upsilon_1} \right|$$

and using (7) we conclude

$$\log \mu \geq \frac{2}{d-1} \log \lambda. \qquad \square$$

A variant of this proof is to take $\alpha = 1$, $\lambda_0 = 1$, $F_0(X,Y) = (X - Y)^d$, and to use the fact that ζ^a is also a primitive h-th root of unity since F_a is irreducible.

Remark. There are cases where μ is very small when compared to λ. Let D be an integer ≥ 2. Consider the algebraic number field $K = \mathbf{Q}(\omega)$ where $\omega = \sqrt[d]{D^d - 1}$. The number $\upsilon = D - \omega$ is a *Bernstein-Hasse* unit of K. When d is fixed, λ is larger than $\kappa_9 D^{d-1}$, while μ is bounded above by κ_{10}. In this example, when d is odd, the field K is almost totally imaginary in the sense of [9] and our proof yields the estimate (4). However, when d is even, we are not able to prove the estimate (4) : the estimate (3) has one extra factor $\log \lambda$.

.

3 Auxiliary results

3.1 Mahler measure, house and height

When f is a polynomial in one variable of degree d with coefficients in $\mathbb{Z}$ and leading coefficient $c_0 > 0$, the Mahler measure of f is

$$\mathrm{M}(f) = c_0 \prod_{i=1}^{d} \max\{1, |\gamma_i|\},$$

where $\gamma_1, \gamma_2, \ldots, \gamma_d$ are the roots of f in $\mathbb{C}$.

Let us recall[1] that the logarithmic height $\mathrm{h}(\gamma)$ of an algebraic number γ of degree d is $\frac{1}{d}\log \mathrm{M}(\gamma)$ where $\mathrm{M}(\gamma)$ is the Mahler measure of the irreducible polynomial of γ. We have

$$\mathrm{M(f)} \le \sqrt{\mathrm{d}+1}\,\mathrm{H(f)} \quad \text{and} \quad \mathrm{H(f)} \le 2^{\mathrm{d}}\mathrm{M(f)} \tag{6}$$

(see [12], Annex to Chapter 3, *Inequalities Between Different Heights of a Polynomial,* pp. 113–114; see also [2, §1.9]). The second upper bound in (6) could be replaced by the sharper one

$$\mathrm{H}(f) \le \binom{d}{\lfloor d/2 \rfloor}\mathrm{M}(f),$$

but we shall not need it.

Let υ be a unit of degree d and conjugates $\upsilon_1, \ldots, \upsilon_d$ with

$$|\upsilon_1| \le |\upsilon_2| \le \cdots \le |\upsilon_d|,$$

so that $\lceil\overline{\upsilon}\rceil = |\upsilon_d|$. Let $\lambda = \mathrm{M}(\upsilon)$ and let s be an index in $\{1, \ldots, d-1\}$ such that

$$|\upsilon_1| \le |\upsilon_2| \le \cdots \le |\upsilon_s| \le 1 \le |\upsilon_{s+1}| \le \cdots \le |\upsilon_d|.$$

We have

$$\lambda = \mathrm{M}(\upsilon) = |\upsilon_{s+1} \cdots \upsilon_d| \le |\upsilon_d|^{d-s} \le |\upsilon_d|^{d-1}$$

and

$$\mathrm{M}(\upsilon^{-1}) = |\upsilon_1 \cdots \upsilon_s|^{-1} = \mathrm{M}(\upsilon) = \lambda$$

with

$$\lambda \le |\upsilon_1|^{-s} \le |\upsilon_1|^{-(d-1)}.$$

Therefore we have

$$\lambda^{1/(d-1)} \le |\upsilon_d| \le \lambda \quad \text{and} \quad \lambda^{-1} \le |\upsilon_1| \le \lambda^{-1/(d-1)}. \tag{7}$$

[1] Our h is the same as in [2], it corresponds to the logarithm of the h in [1].

3.2 An elementary result

For the convenience of the reader, we include the following elementary result – similar arguments are often used without explicit mention in the literature.

Lemma 5. *Let U and V be positive numbers satisfying $U \leq V \log^\star U$. Then $U < 2V \log^\star V$.*

Proof. If $\log U \leq 1$, the assumption is $U \leq V$ and the conclusion follows. Assume $\log U > 1$. Then $\log U \leq \sqrt{U}$, hence the hypothesis of the lemma implies $U \leq V\sqrt{U}$ and therefore we have $U \leq V^2$. We deduce

$$\log U \leq 2 \log V,$$

hence

$$U \leq V \log U \leq 2V \log V. \qquad \square$$

3.3 Diophantine tool

In this section only, the positive integer d is not restricted to $d \geq 3$.

The main tool is the following Diophantine estimate ([6, Proposition 2], [12, Theorem 9.1] or [2, Th. 3.2.4]), the proof of which uses transcendental number theory.

Proposition 6. *Let s and D be two positive integers. There exists an effectively computable positive constant $\kappa(s, D)$, depending only upon s and D, with the following property. Let $\eta_1, \dots, \eta_s$ be nonzero algebraic numbers generating a number field of degree $\leq D$. Let $c_1, \dots, c_s$ be rational integers and let $H_1, \dots, H_s$ be real numbers ≥ 1 satisfying*

$$H_i \geq \mathrm{h}(\eta_i) \quad (1 \leq i \leq s).$$

Let C be a real number with $C \geq 2$. Suppose that one of the following two statements is true:

(i) $C \geq \max_{1 \leq j \leq s} |c_j|$

or

(ii) $H_j \leq H_s$ *for* $1 \leq j \leq s$ *and*

$$C \geq \max_{1 \leq j \leq s} \left\{ \frac{H_j}{H_s} |c_j| \right\}.$$

Suppose also $\eta_1^{c_1} \cdots \eta_s^{c_s} \neq 1$. Then

$$|\eta_1^{c_1} \cdots \eta_s^{c_s} - 1| > \exp\{-\kappa(s, D) H_1 \cdots H_s \log C\}.$$

The statement (ii) of Proposition 6 implies the statement (i) by permuting the indices so that $H_j \le H_s$ for $1 \le j \le s$; however, we find it more convenient to use part (i) so that we can use the estimate without permuting the indices.

We will use Proposition 6 several times. Here is a first consequence.

Corollary 7. *Let $d \ge 1$. There exists an effectively computable constant κ_{11}, which depends only on d, with the following property. Let K be a number field of degree d. Let α_1, α_2, υ_1, υ_2 be nonzero elements in K and let a be a nonzero integer. Set $\gamma_1 = \alpha_1\upsilon_1^a$ and $\gamma_2 = \alpha_2\upsilon_2^a$. Let λ_0 and λ satisfy*

$$\max\{\mathrm{h}(\alpha_1), \mathrm{h}(\alpha_2)\} \le \log\lambda_0, \quad \max\{\mathrm{h}(\upsilon_1), \mathrm{h}(\upsilon_2)\} \le \log\lambda$$

and assume $\gamma_1 \ne \gamma_2$. Define

$$\chi = (\log^\star \lambda_0)(\log^\star \lambda) \log^\star\left(|a| \min\left\{1, \frac{\log^\star\lambda}{\log^\star\lambda_0}\right\}\right).$$

Then

$$|\gamma_1 - \gamma_2| \ge \max\{|\gamma_1|, |\gamma_2|\}\, e^{-\kappa_{11}\chi}.$$

Proof. By symmetry, without loss of generality, we may assume $|\gamma_2| \ge |\gamma_1|$. Set

$$s = 2, \quad \eta_1 = \frac{\upsilon_1}{\upsilon_2}, \quad \eta_2 = \frac{\alpha_1}{\alpha_2}, \quad c_1 = a, \quad c_2 = 1,$$

$$H_1 = 2\log^\star\lambda, \quad H_2 = 2\log^\star\lambda_0, \quad C = \max\left\{2, |a| \min\left\{1, \frac{H_1}{H_2}\right\}\right\}.$$

The conclusion of Corollary 7 follows from Proposition 6 (via part (i) if $H_1 \ge H_2$, via part (ii) otherwise), thanks to the relation

$$|\eta_1^{c_1}\eta_2^{c_2} - 1| = |\gamma_2|^{-1}|\gamma_1 - \gamma_2|.$$

□

3.4 Lower bound for the height and the regulator

For the record, we quote Kronecker's Theorem and its effective improvement.

Lemma 8. (a) *If a nonzero algebraic integer α has all its conjugates in the closed unit disc $\{z \in \mathbb{C} \mid |z| \le 1\}$, then α is a root of unity.*

(b) *More precisely, given $d \ge 1$, there exists an effectively computable positive constant κ_{12}, depending only on d, such that, if α is a nonzero algebraic integer of degree d satisfying $\mathrm{h}(\alpha) < \kappa_{12}$, then α is a root of unity.*

Proof. Voutier (1996) refined an earlier estimate due to Dobrowolski (1979) by proving that the conclusion of part (b) in Lemma 8 holds with

$$\kappa_{12} = \begin{cases} \log 2 & \text{if } d = 1, \\ \dfrac{2}{d(\log(3d))^3} & \text{if } d \geq 2. \end{cases}$$

See for instance [2, Prop. 3.2.9] and [12, §3.6]. □

Lemma 9. *There exists an explicit absolute constant $\kappa_{13} > 0$ such that the regulator R of any number field of degree ≥ 2 satisfies $R > \kappa_{13}$.*

Proof. According to a result of Friedman (1989 – see [2, (1.5.3)]) the conclusion of Lemma 9 holds with $\kappa_{13} = 0.2052$. □

3.5 A basis of units of an algebraic number field

Here is Lemma 1 of [1]. See also [2, Proposition 4.3.9]. The result is essentially due to C.L. Siegel [11].

Proposition 10. *Let d be a positive integer with $d \geq 3$. There exist effectively computable constants $\kappa_{14}, \kappa_{15}, \kappa_{16}$ depending only on d, with the following property. Let K be a number field of degree d, with unit group of rank r. Let R be the regulator of this field. Denote by $\varphi_1, \varphi_2, \dots, \varphi_r$ a set of r embeddings of K into $\mathbb{C}$ containing the real embeddings and no pair of conjugate embeddings. Then there exists a fundamental system of units $\{\epsilon_1, \epsilon_2, \dots, \epsilon_r\}$ of K which satisfies the following:*

(i) $\displaystyle\prod_{1 \leq i \leq r} \mathrm{h}(\epsilon_i) \leq \kappa_{14} R$;

(ii) $\displaystyle\max_{1 \leq i \leq r} \mathrm{h}(\epsilon_i) \leq \kappa_{15} R$;

(iii) *The absolute values of the entries of the inverse matrix of*

$$(\log |\varphi_j(\epsilon_i)|)_{1 \leq i,j \leq r}$$

do not exceed κ_{16}.

The next result is [10, Lemma A.15].

Lemma 11. *Let ϵ_1, ϵ_2, ..., ϵ_r be an independent system of units for K satisfying the condition (ii) of Proposition 10. Let $\beta \in \mathbb{Z}_K$ with $\mathrm{N}_{K/\mathbb{Q}}(\beta) = m \neq 0$. Then there exist $b_1, b_2, \dots, b_r$ in $\mathbb{Z}$ and $\tilde{\beta} \in \mathbb{Z}_K$ with conjugates $\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_d$, satisfying*

$$\beta = \tilde{\beta} \epsilon_1^{b_1} \epsilon_2^{b_2} \cdots \epsilon_r^{b_r}$$

and

$$|m|^{1/d} e^{-\kappa_{17} R} \leq |\tilde{\beta}_j| \leq |m|^{1/d} e^{\kappa_{17} R} \quad \textit{for} \quad j = 1, \dots, d.$$

The conclusion of Lemma 11 can be written

$$\left|\log\left(|m|^{-1/d}|\tilde{\beta}_j|\right)\right| \le \kappa_{17}R \quad \text{for} \quad j=1,\dots,d.$$

3.6 Estimates for the conjugates

Lemma 12. *Let γ be an algebraic number of degree $d \ge 3$. Let γ_1, γ_2, ..., γ_d be the conjugates of γ with $|\gamma_1| \le |\gamma_2| \le \cdots \le |\gamma_d|$.*
(a) *If $|\gamma_1| < |\gamma_2|$ and $\gamma_2 \in \mathbb{R}$, then $|\gamma_2| < |\gamma_3|$.*
(b) *If $|\gamma_{d-1}| < |\gamma_d|$ and $\gamma_{d-1} \in \mathbb{R}$, then $|\gamma_{d-2}| < |\gamma_{d-1}|$.*

Proof. (a) The conditions $|\gamma_1| < |\gamma_2| \le |\gamma_i|$ for $3 \le i \le d$ imply that γ_1 is real and that $-\gamma_1$ is not a conjugate of γ_1. Hence the minimal polynomial of γ is not a polynomial in X^2. Assume $|\gamma_2| = |\gamma_3|$. Since $-\gamma_2$ is not a conjugate of γ_2, we deduce $\gamma_3 \notin \mathbb{R}$, hence $d \ge 4$. We may assume $\gamma_4 = \overline{\gamma_3}$. Let σ be an automorphism of $\overline{\mathbb{Q}}$ which maps γ_2 to γ_1; via σ, let γ_j be the image of γ_3 and γ_k the image of γ_4. From

$$\gamma_2^2 = \gamma_3\gamma_4$$

we deduce $\gamma_1^2 = \gamma_j\gamma_k$ and $|\gamma_1|^2 = |\gamma_j\gamma_k|$. This is not possible since $|\gamma_j| > |\gamma_1|$ and $|\gamma_k| > |\gamma_1|$.

(b) We deduce (b) from (a), by using $\gamma \mapsto 1/\gamma$ (or by repeating the proof, *mutatis mutandis*). $\square$

Remark. Here is an example showing that the assumptions of Lemma 12 are sharp. The polynomial $X^4 - 4X^2 + 1$ is irreducible, its roots are

$$\upsilon_1 = \sqrt{2-\sqrt{3}}, \quad \upsilon_2 = -\upsilon_1, \quad \upsilon_3 = 1/\upsilon_1 = \sqrt{2+\sqrt{3}}, \quad \upsilon_4 = -\upsilon_3$$

with

$$\upsilon_1 = |\upsilon_2| < \upsilon_3 = |\upsilon_4|.$$

More generally, if $h \ge 2$ is a positive integer and ϵ is a quadratic unit with Galois conjugate ϵ' and if $\epsilon^{1/h}$ has degree $2h$, then it has h conjugates of absolute value $|\epsilon|^{1/h}$ and h conjugates of absolute value $|\epsilon'|^{1/h}$. See also §2.

Lemma 13. *Let υ be an algebraic unit of degree $d \ge 3$. Set $\lambda = \mathrm{M}(\upsilon)$. Let υ' and υ'' be two conjugates of υ with $|\upsilon'| < |\upsilon''|$. Then*

$$\log\frac{|\upsilon''|}{|\upsilon'|} \ge \kappa_{18}\lambda^{-(d^3+2d^2-d+2)/2}.$$

We will deduce Lemma 13 from Theorem 1 of [3] which[2] states the following.

[2]This reference was kindly suggested to us by Yann Bugeaud.

Lemma 14 (X. Gourdon and B. Salvy [3])**.** *Let P be a polynomial of degree $d \geq 2$ with integer coefficients and with Mahler measure* $\mathrm{M}(P)$*. If α' and α'' are two roots of P with $|\alpha'| < |\alpha''|$, then*

$$|\alpha''| - |\alpha'| \geq \kappa_{19} M(P)^{-d(d^2+2d-1)/2}$$

with

$$\kappa_{19} = \frac{\sqrt{3}}{2}\bigl(d(d+1)/2\bigr)^{-d(d+1)/4-1}.$$

Proof of Lemma 13. We apply Lemma 14 to the minimal polynomial of υ. To conclude the proof of Lemma 13, we use the bounds $|\upsilon'| \leq \lambda$ and

$$\log(1+x) \geq \frac{x}{2} \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{with} \quad x = \frac{|\upsilon''|}{|\upsilon'|} - 1. \qquad \square$$

4 Proof of Theorem 3

In order to prove Theorem 3 with the assumption $|F_a(x,y)| \leq m$, it suffices to consider the equation $F_a(x,y) = m$ with $m \neq 0$.

Let $(a,x,y,m) \in \mathbb{Z}^4$ satisfy $m \neq 0$, $xy \neq 0$, $[\mathbb{Q}(\alpha\upsilon^a) : \mathbb{Q}] = d$ and

$$F_a(x,y) = m.$$

Without loss of generality, we may restrict (a,y) to $a \geq 0$ (otherwise, replace υ by υ^{-1}) and to $y > 0$ (otherwise replace $F_a(X,Y)$ by $F_a(X,-Y)$).

The form $\tilde{F}_a(X,Y) = a_0^{d-1}F_a(X,Y)$ has coefficients in $\mathbb{Z}$, and if we set $\tilde{x} = a_0x$, $\tilde{y} = y$, $\tilde{m} = a_0^{d-1}m$ we have $\tilde{F}_a(\tilde{x},\tilde{y}) = \tilde{m}$ with $(\tilde{x},\tilde{y}) \in \mathbb{Z}^2$. Therefore, there is no loss of generality to assume $a_0 = 1$.

Theorem 3 includes the assumption that υ is not a root of unity, hence $\lambda > 1$. More precisely, it follows from part (b) of Lemma 8 that

$$\log\lambda \geq \kappa_{12}.$$

In particular, we have

$$\log^\star\lambda \leq \max\left\{1, \frac{1}{\kappa_{12}}\right\}\log\lambda,$$

an inequality which can be written

$$\log^\star\lambda \leq \kappa_{20}\log\lambda \tag{8}$$

with an effectively computable constant $\kappa_{20} > 0$.

From Lemma 9, we deduce that $R > \kappa_{13}$. Therefore, there is no loss of generality to assume that, for a sufficiently large constant κ_{21}, we have

$$a \geq \kappa_{21}\bigl(\log|m| + (\log^\star\lambda_0)\log^\star\log^\star\lambda\bigr). \tag{9}$$

This hypothesis will frequently be used, sometimes without explicit mention.

By assumption, $\mathbb{Q}(\alpha\upsilon^a) = K$. If some conjugate $\sigma_j(\alpha\upsilon^a)$ of $\alpha\upsilon^a$ is real, then it follows that $\sigma_j(K) \subset \mathbb{R}$, hence the embedding σ_j is real, and α_j and υ_j are both real. We also notice that if $\sigma_j(\upsilon) = -\sigma_i(\upsilon)$ with $i \neq j$, then it follows that υ and $-\upsilon$ are conjugate, hence the irreducible polynomial of υ belongs to $\mathbb{Z}[X^2]$.

Recall that $\upsilon_i = \sigma_i(\upsilon)$ $(i = 1, \dots, d)$ and that

$$|\upsilon_1| \leq |\upsilon_2| \leq \cdots \leq |\upsilon_d|.$$

Let us write α_i for $\sigma_i(\alpha)$ $(i = 1, \dots, d)$. Let

$$\gamma = \alpha\upsilon^a \quad \text{and} \quad \beta = x - \gamma y.$$

Since $a_0 = 1$, it follows that α, β and γ are algebraic integers in K. For $j = 1, 2, \dots, d$, define γ_j and β_j by

$$\gamma_j = \sigma_j(\gamma) = \alpha_j\upsilon_j^a, \quad \beta_j = \sigma_j(\beta) = x - \alpha_j\upsilon_j^a y = x - \gamma_j y.$$

The assumption $F_a(x, y) = m$ yields $\beta_1\beta_2\cdots\beta_d = m$. Let $i_0 \in \{1, 2, \dots, d\}$ be an index such that

$$|\beta_{i_0}| = \min_{1\leq i\leq d} |\beta_i|.$$

We define $\Psi_1, \Psi_2, \dots, \Psi_d$ by the following conditions:

$$\beta_i = \begin{cases} \gamma_{i_0} y\Psi_i & \text{for } 1 \leq i < i_0, \\ \gamma_i y\Psi_i & \text{for } i_0 < i \leq d \end{cases}$$

and

$$\beta_{i_0} = \frac{m}{y^{d-1}} \cdot \frac{\gamma_1\gamma_2\cdots\gamma_{i_0-1}}{\gamma_{i_0}^{i_0-2}} \Psi_{i_0}.$$

We split the proof into several steps.

`Step 1.` We start by proving that

$$|x| \leq 2\lambda_0\lambda^a y \tag{10}$$

and that there exists an effectively computable positive constant κ_{22} depending only on d such that

$$e^{-\kappa_{22}\chi} \leq |\Psi_i| \leq e^{\kappa_{22}\chi} \qquad (i = 1, 2, \dots, d) \tag{11}$$

with

$$\chi = (\log^\star \lambda_0)(\log \lambda) \log\left(a \min\left\{1, \ \frac{\log \lambda}{\log^\star \lambda_0}\right\}\right).$$

From the estimate (11) we will deduce

$$|\beta_{i_0}| < |\beta_i|$$

for $i \neq i_0$, which implies $\alpha_{i_0} \in \mathbb{R}$ and $\upsilon_{i_0} \in \mathbb{R}$.

`Remark.` The estimate (11) can be written as follows:

$$\left|\log\left(|\beta_i| y^{-1} \max\{|\gamma_i^{-1}|, |\gamma_{i_0}^{-1}|\}\right)\right| \leq \kappa_{22}\chi$$

for $i \neq i_0$ and

$$\left|\log\left(|\beta_{i_0}| \frac{y^{d-1}}{|m|} \left|\gamma_1^{-1} \cdots \gamma_{i_0-1}^{-1} \gamma_{i_0}^{i_0-2}\right|\right)\right| \leq \kappa_{22}\chi.$$

`Proof` of (10) and (11). We have

$$|x| = |\beta_{i_0} + \gamma_{i_0} y| \leq |\beta_{i_0}| + |\gamma_{i_0}| y. \tag{12}$$

From $|\beta_{i_0}| \leq |\beta_i|$ for $i = 1, 2, \ldots, d$ and $\beta_1 \cdots \beta_d = m$, we deduce $|\beta_{i_0}| \leq |m|^{1/d}$, hence

$$|x| \leq |m|^{1/d} + |\gamma_{i_0}| y \leq |m|^{1/d} + \lambda_0 \lambda^a y.$$

Using the assumption (9), we check $|m|^{1/d} \leq \lambda_0 \lambda^a y$, whereupon the inequality (10) is secured.

We also have

$$|\beta_{i_0}|^{d-1} \max_{1 \leq i \leq d} |\beta_i| \leq |m|. \tag{13}$$

For $i = 1, 2, \ldots, d$, we write

$$\beta_i = \beta_{i_0} + y(\gamma_{i_0} - \gamma_i). \tag{14}$$

We have

$$|\alpha_1 \alpha_2 \cdots \alpha_d| \geq 1$$

(recall $a_0 = 1$), hence

$$\frac{1}{\lambda_0} \leq |\alpha_i| \leq \lambda_0 \quad \text{for } i = 1, 2, \ldots, d.$$

We choose an index $j_0 \neq i_0$ as follows:

• If $|\upsilon_{i_0}| \leq \lambda^{1/(2(d-1))}$, we take $j_0 = d$ so that, with the help of (7), we have $|\upsilon_{j_0}| \geq \lambda^{1/(d-1)}$, whereupon with the help of (9) we obtain

$$\left|\frac{\gamma_{i_0}}{\gamma_{j_0}}\right| < \frac{1}{2}.$$

• If $|\upsilon_{i_0}| > \lambda^{1/(2(d-1))}$, we take $j_0 = 1$ so that, again with the help of (7), we have $|\upsilon_{j_0}| \leq \lambda^{-1/(d-1)}$, whereupon with the help of (9) we obtain

$$\left|\frac{\gamma_{j_0}}{\gamma_{i_0}}\right| < \frac{1}{2}.$$

In both cases, we deduce

$$|\gamma_{j_0} - \gamma_{i_0}| \geq \frac{1}{2} \max\{|\gamma_{j_0}|, |\gamma_{i_0}|\} \geq \frac{\lambda^{a/(2(d-1))}}{2\lambda_0}$$

and therefore, using (9) again together with (13) and (14), we obtain

$$|\beta_{j_0}| \geq |\gamma_{j_0} - \gamma_{i_0}|y - |\beta_{i_0}| \geq \frac{\lambda^{a/(2(d-1))}y}{2\lambda_0} - |m|^{1/d} \geq \lambda^{a/(2d)}y.$$

Since $\max_{1\leq i\leq d} |\beta_i| \geq \lambda^{a/(2d)}y$, from (13) we deduce

$$|\beta_{i_0}| \leq \left(\frac{|m|}{y\lambda^{a/(2d)}}\right)^{1/(d-1)}. \tag{15}$$

In particular, thanks to (9), we have

$$|\beta_{i_0}| \leq \frac{1}{2}. \tag{16}$$

Using the assumption $|x| \geq 1$ together with (12), we deduce

$$\frac{|x|}{2} \leq |\gamma_{i_0}|y \leq |x| + |\beta_{i_0}| \leq \frac{3|x|}{2}. \tag{17}$$

Let $i \neq i_0$. The upper bound

$$|\gamma_i - \gamma_{i_0}| \leq 2\max\{|\gamma_{i_0}|, |\gamma_i|\}$$

is trivial, while the lower bound

$$|\gamma_i - \gamma_{i_0}| \geq \max\{|\gamma_{i_0}|, |\gamma_i|\}e^{-\kappa_{23}\chi} \tag{18}$$

follows from (8) and from Corollary 7. We first use the lower bound

$$|\gamma_i - \gamma_{i_0}| \geq |\gamma_{i_0}|e^{-\kappa_{23}\chi}.$$

Using (17), we obtain

$$|\gamma_i - \gamma_{i_0}| \geq \frac{1}{2y}e^{-\kappa_{23}\chi} \geq \frac{2}{y}e^{-\kappa_{24}\chi} \tag{19}$$

with $\kappa_{24} > 0$. Using the contrapositive of Lemma 5 with

$$U = a\frac{\log^\star \lambda}{\log^\star \lambda_0}, \quad V = \frac{1}{\kappa_{25}} \log^\star \lambda,$$

we deduce from (9) that

$$\chi \leq \kappa_{25} a \log^\star \lambda.$$

Recall that κ_{21} is sufficiently large, hence κ_{25} is sufficiently small. Now from (15), the inequality $|m| \le e^{|a|/\kappa_{21}}$ and (19) we deduce

$$|\beta_{i_0}| \le |m|^{1/(d-1)}\lambda^{-a/(2d(d-1))} \le \lambda^{-\kappa_{26}a} \le e^{-\kappa_{24}\chi} \le \frac{1}{2}y|\gamma_i - \gamma_{i_0}|.$$

Therefore, for $i \ne i_0$, using (14), we deduce

$$\frac{1}{2}y|\gamma_i - \gamma_{i_0}| \le |\beta_i| \le \frac{3}{2}y|\gamma_i - \gamma_{i_0}|.$$

Using once more (18), we obtain (11) for $i \ne i_0$. We also deduce

$$|\beta_i| > \lambda^{-a/(2d)} \quad \text{for} \quad i \ne i_0. \tag{20}$$

Recall

$$\mathrm{N}(\gamma) = \gamma_1\gamma_2\cdots\gamma_d = \mathrm{N}(\alpha)\mathrm{N}(\upsilon)^a = \pm\mathrm{N}(\alpha) \quad \text{and} \quad \mathrm{N}(\beta) = \beta_1\beta_2\cdots\beta_d = m.$$

The estimate (11) for $i = i_0$ follows from the relations

$$\Psi_1\Psi_2\cdots\Psi_d\mathrm{N}(\gamma) = 1,$$

$$\frac{m}{\beta_{i_0}} = \prod_{i\ne i_0}\beta_i = y^{d-1}\gamma_{i_0}^{i_0-1}\gamma_{i_0+1}\cdots\gamma_d\prod_{i\ne i_0}\Psi_i$$

and

$$\frac{\mathrm{N}(\gamma)}{\gamma_{i_0}^{i_0-1}\gamma_{i_0+1}\cdots\gamma_d} = \frac{\gamma_1\cdots\gamma_{i_0-1}}{\gamma_{i_0}^{i_0-2}}\cdot$$

From (9) and (11), we deduce

$$|\beta_{i_0}| < \frac{|m|}{y^{d-1}}|\gamma_1|e^{\kappa_{22}\chi} < \lambda^{-a/(2d)},$$

hence from (20) we infer $|\beta_{i_0}| < |\beta_i|$ for $i \ne i_0$. It follows that β_{i_0} is real, and therefore γ_{i_0}, α_{i_0} and υ_{i_0} also. □

`Step 2`. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ be a basis of the group of units of K given by Proposition 10. From Lemma 11, it follows that there exists $\tilde{\beta} \in \mathbb{Z}_K$ and $b_1, b_2, \ldots, b_r$ in $\mathbb{Z}$ with

$$\beta = \tilde{\beta}\epsilon_1^{b_1}\epsilon_2^{b_2}\cdots\epsilon_r^{b_r}$$

and

$$|m|^{1/d}e^{-\kappa_{17}R} \le |\tilde{\beta}_i| \le |m|^{1/d}e^{\kappa_{17}R} \quad \text{for} \quad i = 1, 2, \ldots, d.$$

We set

$$B = \kappa_{27}(R + a\log\lambda + \log y) \tag{21}$$

with a sufficiently large constant κ_{27}. We want to prove that

$$\max_{1\le i\le r}|b_i| \le B.$$

`Proof.` We consider the system of d linear forms in r variables with real coefficients

$$L_i(X_1, X_2, \ldots, X_r) = \sum_{j=1}^{r} X_j \log|\sigma_i(\epsilon_j)|, \quad (i = 1, 2, \ldots, d).$$

The rank is r. By Proposition 10(ii),

$$\log|\sigma_i(\epsilon_j)| \leq \kappa_{28} R.$$

For $i = 1, 2, \ldots, d$, define $e_i = L_i(b_1, b_2, \ldots, b_r)$. We have

$$e_i = \log|\sigma_i(\beta/\tilde{\beta})| = \log|\beta_i/\tilde{\beta}_i|,$$

hence, using the inequality $|m| \leq e^{|a|/\kappa_{21}}$ and (11), we deduce

$$|e_i| \leq \kappa_{29}(R + a\log\lambda + \log y).$$

Computing $b_1, b_2, \ldots, b_r$ by means of the system of linear equations

$$L_i(b_1, b_2, \ldots, b_r) = e_i \quad (i = 1, 2, \ldots, d)$$

and using Proposition 10(iii), we deduce

$$\max_{1\leq j\leq r} |b_j| \leq \kappa_{30} \max_{1\leq i\leq d} |e_i| \leq B.$$

□

`Step 3`. From the inequality (3.2) in [1, Theorem 3] (see also [2, Th. 9.6.2]), thanks to (9), we deduce the following upper bound for $|x|$ and $|y|$ in terms of a, λ, λ_0, m and R: there exists a positive effectively computable constant κ_{31} depending only on d such that

$$\log\max\{|x|, y\} \leq \kappa_{31} R(\log^\star R)\big(R + a\log\lambda\big). \tag{22}$$

`Step 4`. Assume $c\gamma_i\beta_j \neq \gamma_k\beta_\ell$ for some indices i, j, k, ℓ in $\{1, \ldots, d\}$ and some $c \in \{1, -1\}$. Then there exists $\kappa_{32} > 0$ such that

$$\left| c\frac{\gamma_i\beta_j}{\gamma_k\beta_\ell} - 1 \right| \geq \exp\Bigg\{ -\kappa_{32}(\log\lambda)(R + \log|m| + \log\lambda_0 + \log\lambda)R \times \log\left(\frac{Ra\log\lambda}{R + \log|m| + \log\lambda_0 + \log\lambda}\right)\Bigg\}.$$

`Proof.` This lower bound follows from Proposition 6(ii) with

$$\frac{c\gamma_i\beta_j}{\gamma_k\beta_\ell} = \eta_1^{c_1}\eta_2^{c_2}\cdots\eta_s^{c_s},$$

where $s = r+2$ and

$$\eta_t = \frac{\sigma_j(\epsilon_t)}{\sigma_\ell(\epsilon_t)}\ (1 \le t \le r), \quad \eta_{r+1} = \frac{\sigma_i(\upsilon)}{\sigma_k(\upsilon)}, \quad \eta_{r+2} = \frac{c\sigma_j(\tilde{\beta})\sigma_i(\alpha)}{\sigma_\ell(\tilde{\beta})\sigma_k(\alpha)},$$

$$c_t = b_t\ (1 \le t \le r), \quad c_{r+1} = a, \quad c_{r+2} = 1,$$

$$H_t = \max\{1, 2\mathrm{h}(\epsilon_t)\}\ (1 \le t \le r),$$

$$H_{r+1} = \max\{1, 2\log\lambda\}, \quad H_{r+2} = \kappa_{33}(R + \log|m| + \log\lambda_0 + \log\lambda),$$

$$C = 2 + \frac{2a\log\lambda + RB}{H_{r+2}}\cdot$$

Using Proposition 10(i) together with part (b) of Lemma 8, we deduce

$$H_1 H_2 \cdots H_r \le \kappa_{34} R.$$

Finally we deduce from the steps 2 and 3 that

$$\log C \le \kappa_{35} \log\left(\frac{Ra\log\lambda}{R + \log|m| + \log\lambda_0 + \log\lambda}\right),$$

and this secures the lower bound for $\left|c\frac{\gamma_i\beta_j}{\gamma_k\beta_l} - 1\right|$ announced above. □

`Step 5.` We will prove Theorem 3 by assuming

$$\max_{1\le i\le d} \max\{|\Psi_i|, |\Psi_i|^{-1}\} > \mu^{a/4}.$$

Using (11), we deduce from our assumption

$$\frac{a}{4}\log\mu < \kappa_{22}\chi,$$

hence

$$a \le \frac{4\kappa_{22}}{\log\mu}(\log^\star\lambda_0)(\log^\star\lambda)\log^\star\left(a\frac{\log^\star\lambda}{\log^\star\lambda_0}\right).$$

With

$$U = \frac{a\log^\star\lambda}{\log^\star\lambda_0} \quad\text{and}\quad V = \frac{4\kappa_{22}(\log^\star\lambda)^2}{\log\mu},$$

we have $U \le V\log^\star U$, and we conclude that we can use Lemma 5 to deduce

$$a \le \frac{8\kappa_{22}(\log^\star\lambda_0)(\log^\star\lambda)}{\log\mu}\log\left(\frac{4\kappa_{22}(\log^\star\lambda)^2}{\log\mu}\right),$$

and the conclusion of Theorem 3 follows.

In the rest of the paper, we assume

$$\max_{1\le i\le d} \max\{|\Psi_i|, |\Psi_i|^{-1}\} \le \mu^{a/4}. \tag{23}$$

`Step 6`. Our next goal is to prove the following results.

(a) Assume $1 \le i_0 \le d-2$ and

$$\frac{|\upsilon_{d-1}|}{|\upsilon_{i_0}|} \ge \sqrt{\mu}.$$

Then

$$0 < \left|\frac{\gamma_{d-1}\beta_d}{\gamma_d\beta_{d-1}} - 1\right| \le 4\lambda_0^2\mu^{-a/4}.$$

(b) Assume $3 \le i_0 \le d$ and

$$\frac{|\upsilon_{i_0}|}{|\upsilon_2|} \ge \sqrt{\mu}.$$

Then

$$0 < \left|\frac{\beta_1}{\beta_2} - 1\right| \le 2\lambda_0^2\mu^{-a/4}.$$

(c) Assume $2 \le i_0 \le d-1$ and

$$\min\left\{\frac{|\upsilon_{i_0}|}{|\upsilon_1|}, \frac{|\upsilon_d|}{|\upsilon_{i_0}|}\right\} \ge \mu.$$

Then

$$\left|\frac{\gamma_{i_0}\beta_d}{\gamma_d\beta_1} + 1\right| \le 4|m|\lambda_0^4\mu^{-a/2}.$$

`Proof` (a) We approximate β_d by $-\gamma_d y$, β_{d-1} by $-\gamma_{d-1}y$ and we eliminate y. Since γ has degree d, we have

$$\beta_d\gamma_{d-1} - \beta_{d-1}\gamma_d = x(\gamma_{d-1} - \gamma_d) \ne 0.$$

From (17) we deduce $|x| \le 2|\gamma_{i_0}y|$ and

$$|\beta_d\gamma_{d-1} - \beta_{d-1}\gamma_d| \le 2|\gamma_{i_0}|(|\gamma_d| + |\gamma_{d-1}|)y.$$

Using $\beta_{d-1} = \gamma_{d-1}y\Psi_{d-1}$ together with the assumption

$$|\upsilon_d| \ge |\upsilon_{d-1}| \ge \sqrt{\mu}|\upsilon_{i_0}|,$$

we deduce

$$\left|\frac{\gamma_{d-1}\beta_d}{\gamma_d\beta_{d-1}} - 1\right| \le \frac{2|\gamma_{i_0}|(|\gamma_{d-1}| + |\gamma_d|)}{|\gamma_{d-1}\gamma_d|}|\Psi_{d-1}|^{-1} \le 4\lambda_0^2\mu^{-a/2}|\Psi_{d-1}|^{-1}.$$

The conclusion of (a) follows from (23).

(b) We approximate β_1 and β_2 by x and we eliminate x. Since $\gamma_1 \ne \gamma_2$, we have

$$|\beta_1 - \beta_2| = |(\gamma_2 - \gamma_1)y| \ne 0.$$

From $\beta_2 = \gamma_{i_0} y \Psi_2$ and the assumption

$$|\upsilon_1| \leq |\upsilon_2| \leq \mu^{-1/2} |\upsilon_{i_0}|,$$

we deduce

$$\left| \frac{\beta_1}{\beta_2} - 1 \right| \leq \frac{|\gamma_2| + |\gamma_1|}{|\gamma_{i_0}|} |\Psi_2|^{-1} \leq 2\lambda_0^2 \mu^{-a/2} |\Psi_2|^{-1}.$$

Again, the conclusion of (b) follows from (23).

(c) We approximate β_1 by x, β_d by $-y\gamma_d$ and x by $y\gamma_{i_0}$, then we eliminate x and y. More precisely we have

$$\beta_1 \gamma_d + \beta_d \gamma_{i_0} = (\gamma_d + \gamma_{i_0}) \beta_{i_0} + \gamma_{i_0}^2 y - \gamma_1 \gamma_d y.$$

Hence

$$\frac{\gamma_{i_0} \beta_d}{\gamma_d \beta_1} + 1 = \frac{(\gamma_d + \gamma_{i_0}) \beta_{i_0}}{\gamma_d \beta_1} + \frac{\gamma_{i_0}^2 y}{\gamma_d \beta_1} - \frac{\gamma_1 y}{\beta_1}.$$

We have $\beta_1 = \gamma_{i_0} y \Psi_1$. Therefore we have

$$\frac{|\gamma_{i_0}|^2 y}{|\gamma_d \beta_1|} = \frac{|\gamma_{i_0}|}{|\gamma_d|} |\Psi_1|^{-1} \leq \lambda_0^2 \left| \frac{\upsilon_{i_0}}{\upsilon_d} \right|^a |\Psi_1|^{-1}$$

and

$$\frac{|\gamma_1| y}{|\beta_1|} = \frac{|\gamma_1|}{|\gamma_{i_0}|} |\Psi_1|^{-1} \leq \lambda_0^2 \left| \frac{\upsilon_1}{\upsilon_{i_0}} \right|^a |\Psi_1|^{-1}.$$

Finally, from

$$|\beta_{i_0}| \leq \frac{|m|}{y^{d-1}} |\gamma_1 \Psi_{i_0}|$$

we deduce

$$\begin{aligned} \frac{(|\gamma_d| + |\gamma_{i_0}|)|\beta_{i_0}|}{|\gamma_d \beta_1|} &\leq (1 + \lambda_0^2) \left| \frac{\beta_{i_0}}{\beta_1} \right| \leq (1 + \lambda_0^2) \frac{|m|}{y^d} \frac{|\gamma_1 \Psi_{i_0}|}{|\gamma_{i_0} \Psi_1|} \\ &\leq \lambda_0^2 (1 + \lambda_0^2) \frac{|m|}{y^d} \left| \frac{\upsilon_1}{\upsilon_{i_0}} \right|^a \frac{|\Psi_{i_0}|}{|\Psi_1|}. \end{aligned}$$

Hence from the assumptions

$$|\upsilon_1| \leq \mu^{-1} |\upsilon_{i_0}| \quad \text{and} \quad |\upsilon_{i_0}| \leq \mu^{-1} |\upsilon_d|,$$

we deduce

$$\left| \frac{\gamma_{i_0} \beta_d}{\gamma_d \beta_1} + 1 \right| \leq 4|m| \lambda_0^4 \mu^{-a} \frac{|\Psi_{i_0}|}{|\Psi_1|}.$$

The conclusion of (c) follows from (23). $\square$

`Step 7.` (a) Assume $|\upsilon_{i_0}| = |\upsilon_1|$. Since $\upsilon_{i_0} \in \mathbb{R}$, we deduce from Lemma 12 that

$$|\upsilon_1| < |\upsilon_{d-1}|.$$

If $|\upsilon_2| < |\upsilon_{d-1}|$, then

$$\frac{|\upsilon_{d-1}|}{|\upsilon_{i_0}|} \geq \frac{|\upsilon_{d-1}|}{|\upsilon_2|} = \mu$$

and we are in the case (a) of the step 6.

If $|\upsilon_2| = |\upsilon_{d-1}|$, then $i_0 = 1$, we have

$$\frac{|\upsilon_{d-1}|}{|\upsilon_1|} \geq \mu$$

and again we are in the case (a) of the step 6.

(b) Assume $|\upsilon_{i_0}| = |\upsilon_d|$. Using Lemma 12, we deduce

$$|\upsilon_d| > |\upsilon_2|.$$

If $|\upsilon_2| < |\upsilon_{d-1}|$, then

$$\frac{|\upsilon_{i_0}|}{|\upsilon_1|} \geq \frac{|\upsilon_{d-1}|}{|\upsilon_2|} = \mu$$

and we are in the case (b) of the step 6.

If $|\upsilon_2| = |\upsilon_{d-1}|$, then $i_0 = d$, we have

$$\frac{|\upsilon_d|}{|\upsilon_2|} \geq \mu$$

and again we are in the case (b) of the step 6.

(c) Assume finally $|\upsilon_1| < |\upsilon_{i_0}| < |\upsilon_d|$. In particular we have $2 \leq i_0 \leq d-1$. Assume that we are neither in the case (a) nor in the case (b) of the step 6. From

$$\frac{|\upsilon_{d-1}|}{|\upsilon_{i_0}|} < \sqrt{\mu} \quad \text{and} \quad \frac{|\upsilon_{i_0}|}{|\upsilon_2|} < \sqrt{\mu}$$

we deduce

$$\frac{|\upsilon_{d-1}|}{|\upsilon_2|} < \mu.$$

Given the definition of μ, it follows that we have $|\upsilon_2| = |\upsilon_{d-1}|$. Since υ_{i_0} is real, Lemma 12 implies $d = 3$ and therefore $i_0 = 2$, $|\upsilon_1| < |\upsilon_2| < |\upsilon_3|$ and

$$\mu = \min\left\{\frac{|\upsilon_3|}{|\upsilon_2|}, \frac{|\upsilon_2|}{|\upsilon_1|}\right\}.$$

From

$$|\gamma_1| \leq \lambda_0 |\upsilon_1|^a \leq \lambda_0 \lambda^{-a/2} < 1, \qquad |\beta_2| = |\beta_{i_0}| < 1$$

and

$$|\gamma_2\beta_3| = |\gamma_2\gamma_3\Psi_3| y \geq \frac{y|\Psi_3|}{|\gamma_1|} \geq \lambda_0^{-1}\lambda^{a/2}|\Psi_3| > 1,$$

we deduce $|\gamma_1\beta_2| < 1 < |\gamma_2\beta_3|$, hence

$$\gamma_1\beta_2 + \gamma_2\beta_3 \neq 0.$$

There is an element of the Galois group of the Galois closure of the cubic field $\mathbb{Q}(\upsilon)$ which maps υ_1 to υ_2, υ_2 to υ_3, υ_3 to υ_1. Therefore,

$$\gamma_2\beta_3 + \gamma_3\beta_1 \neq 0.$$

From part (c) of the step 6 we deduce

$$0 < \left|\frac{\gamma_2\beta_3}{\gamma_3\beta_1} + 1\right| \leq 4m\lambda_0^4\mu^{-a/2}.$$

`Step 8`. Combining the steps 6 and 7 with the step 4 where we choose

$$\begin{cases} i = \ell = d-1,\ j = k = d,\ c = 1 & \text{in the case (a)}, \\ i = k = i_0,\ j = 1,\ \ell = d,\ c = 1 & \text{in the case (b)}, \\ i = i_0,\ j = k = d,\ \ell = 1,\ c = -1 & \text{in the case (c)}. \end{cases}$$

we deduce

$$\begin{aligned} a\log\mu \leq \kappa_{36} R(R + \log|m| + \log\lambda_0 + \log\lambda)(\log\lambda) \\ \times \log\left(\frac{Ra\log\lambda}{R + \log|m| + \log\lambda_0 + \log\lambda}\right)\cdot \end{aligned}$$

For

$$U = \frac{Ra\log\lambda}{R + \log|m| + \log\lambda_0 + \log\lambda} \quad \text{and} \quad V = \kappa_{37}\frac{R^2(\log\lambda)^2}{\log\mu},$$

we have $U \leq V\log^\star U$. Therefore we use Lemma 5 to obtain the conclusion of Theorem 3.

5 Proofs of Theorem 1 and of Corollary 2

Proof of Theorem 1. Since $d \geq 3$, under the assumptions of Lemma 13 we have

$$\log\frac{|\upsilon''|}{|\upsilon'|} \geq \frac{\kappa_{18}(\log\lambda)^2}{\lambda^{d^2(d+2)/2}}\cdot$$

From Lemma 12, we deduce that under the assumptions of Theorem 1 and with the notations of Theorem 3, we have

$$\log\mu \geq \frac{\kappa_{38}(\log\lambda)^2}{\lambda^{d^2(d+2)/2}}\cdot$$

Hence Theorem 3 implies Theorem 1. $\square$

Proof of Corollary 2. The conclusion of Corollary 2 follows from Theorem 1 thanks to the upper bound (2). □

ACKNOWLEDGEMENTS: This research project was initiated in June 2016 at the research institute AIMS M'BOUR (Africa) and the authors are grateful to the director, professor Aissa Wade, for her invitation to give lectures at the center. The authors are also thankful to Paul Voutier for his remarks. The first author benefitted from a grant from NSERC.

References

[1] Y. Bugeaud & K. Győry, *Bounds for the solutions of Thue-Mahler equations and norm form equations*, "Acta Arith." **74** (1996), 273–292.
`http://www-irma.u-strasbg.fr/~bugeaud/travaux/BG2debut.ps`

[2] J-H. Evertse & K. Győry, *Unit equations in Diophantine number theory*, "Cambridge studies in advanced mathematics" **146**, Cambridge Univ. Press (2015).

[3] X. Gourdon & B. Salvy, *Effective asymptotics of linear recurrences with rational coefficients*, "Discrete Mathematics" **153** (1996), 145–163.

[4] C. Levesque and M. Waldschmidt, *Familles d'équations de Thue–Mahler n'ayant que des solutions triviales*, "Acta Arith." **155** (2012), 117–138.
`arXiv:1312.7202 [math.NT]`

[5] — , *Families of cubic Thue equations with effective bounds for the solutions*, "Springer Proceedings in Mathematics & Statistics" **43** (2013), 229–243.
`arXiv:1312.7204 [math.NT]`

[6] — , *Solving effectively some families of Thue Diophantine equations*, "Moscow J. of Combinatorics and Number Theory" **3**, 3–4 (2013), 118–144.
`arXiv:1312.7205 [math.NT]`

[7] — , *Familles d'équations de Thue associées à un sous-groupe de rang 1 d'unités totalement réelles d'un corps de nombres*, in SCHOLAR – a Scientific Celebration Highlighting Open Lines of Arithmetic Research (volume dedicated to Ram Murty), CRM collection "Contemporary Mathematics", AMS, **655** (2015), 117–134.
`http://www.ams.org/books/conm/655/`
`arXiv: 1505.06656 [math.NT]`

[8] — , *A family of Thue equations involving powers of units of the simplest cubic fields*, "J. Théor. Nombres Bordx." **27**, No. 2 (2015), 537–563.
`arXiv:1505.06708 [math.NT]`

[9] — , *Solving simultaneously Thue Diophantine equations: almost totally imaginary case*, "Ramanujan Mathematical Society, Lecture Notes Series" **23** (2016), 137–156.
`arXiv: 1505.06653 [math.NT]`

[10] T.N. Shorey and R. Tijdeman, *Exponential Diophantine equations*, "Cambridge Tracts in Mathematics" **87**, Cambridge Univ. Press (1986).

[11] C.L. Siegel, *Abschätzung von Einheiten,* "Nachr. Akad. Wiss. Göttingen Math.-Phys." Kl. II (1969), 71–86. Gesammelte Abhandlungen IV, Springer Collected Works in Mathematics, §88 (1979), 66–81.

[12] M. Waldschmidt, *Diophantine approximation on linear algebraic groups*, "Grundlehren der Mathematischen Wissenschaften" **326**, Springer-Verlag (2000).

Claude Levesque
Département de mathématiques et de statistique
Université Laval
Québec (Québec)
Canada G1V 0A6
Claude.Levesque@mat.ulaval.ca

Michel Waldschmidt
Sorbonne Universités
UPMC Univ Paris 06
UMR 7586 IMJ-PRG
F – 75005 Paris, France
michel.waldschmidt@imj-prg.fr