9th Canadian Number Theory conference
(CNTA 9)
University of British Columbia, Vancouver
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Report on recent progress in Diophantine approximation

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http://www.math.jussieu.fr/∼miw/transparents.html

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Abstract

After the works by J. Liouville, A. Thue, C.L. Siegel, F.J. Dyson, A.O. Gel'fond, Th. Schneider and K.F. Roth, the question of approximation of a single real algebraic irrational number by rational numbers is better understood; however many questions are not yet answered.

We first introduce some of them.

Next we consider simultaneous Diophantine approximation.

The pioneer work of W.M. Schmidt and his subspace theorem provide a lot of information on this subject when one considers algebraic numbers.

Several recent progress have been made by D. Roy, M. Laurent, Y. Bugeaud, T. Rivoal and S. Fischler, among others. We review some of these works.

Hilbert's seventh problem and some of its developments

- Hilbert's seventh problem (1900): *transcendence of* α^β *for algebraic* α *and* β *with* $\alpha \neq 0$, $\alpha \neq 1$, $\beta \notin \mathbf{Q}$.
- Solution by A.O. Gel'fond and Th. Schneider in 1934.
- Consequences: *transcendence of*

$$2^{\sqrt{2}}, \quad e^\pi = (e^{i\pi})^{-i}, \quad e^{\pi\sqrt{163}} = a - 10^{-12}b$$

where

$$a = 262\ 537\ 412\ 640\ 768\ 744 \in \mathbf{Z}, \quad b = 0.7499274\cdots$$

and of

$$\frac{\log 2}{\log 3}.$$

Hint: $3^{\log 2/\log 3} = 2.$

Arithmetic tool: Liouville's inequality

- Let n be a non-zero rational integer. Then $|n| \geq 1$.
- Let x be a non-zero rational number and let $q \in \mathbf{Z}_{>0}$ be such that $qx \in \mathbf{Z}$. Then $|x| \geq 1/q$.
- Let γ be a non-zero algebraic integer with conjugates $\gamma_1, \ldots, \gamma_d$. Then $|\gamma_1| \geq (|\gamma_2 \cdots \gamma_d|)^{-1}$.
- Liouville's inequality. *Let α be an algebraic number of degree d; there exists $c = c(\alpha) > 0$ such that, for any rational number $p/q \neq \alpha$,*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^d}.$$

- More generally, Liouville's argument yields a lower bound for $|P(\alpha)|$ when α is an algebraic number and $P \in \mathbf{Z}[X]$ a polynomial such that $P(\alpha) \neq 0$.

Problem of Gel'fond and Schneider

- Algebraic independence of numbers of the form α^β: raised by A.O. Gel'fond in 1948 and Th. Schneider in 1952.
- **Conjecture**: *if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an irrational algebraic number of degree d, then the $d-1$ numbers*

$$\alpha^\beta,\ \alpha^{\beta^2},\ \ldots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Schanuel's Conjecture

- *Let* $x_1, \ldots, x_n$ *be* $\mathbf{Q}$*-linearly independent complex numbers. Then* n *at least of the* $2n$ *numbers* $x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n}$ *are algebraically independent.*
- The conclusion can be phrased in terms of the transcendence degree over $\mathbf{Q}$:

$$\mathrm{trdeg}_{\mathbf{Q}}\mathbf{Q}(x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n}) \geq n.$$

- Special case: take $x_i = \beta^{i-1}\log\alpha$. The conclusion is, for β algebraic number of degree d,

$$\mathrm{trdeg}_{\mathbf{Q}}\mathbf{Q}(\log\alpha, \alpha^{\beta},\ \alpha^{\beta^2},\ \ldots, \alpha^{\beta^{d-1}}) = d.$$

Algebraic independence method of A.O. Gel'fond (1948)

- A.O. Gel'fond (1948): *algebraic independence of* $2^{\sqrt[3]{2}}$ *and* $2^{\sqrt[3]{4}}$.
- More generally (Gel'fond again) *if* β *has degree* $d \geq 3$, *then at least* 2 *among the* $d-1$ *numbers*

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots, \alpha^{\beta^{d-1}}$$

are algebraically independent.

- Best known result to date (after work by A.O. Gel'fond, A.A. Smelev, W.D. Brownawell, G.V. Chudnovskii, P. Philippon, Yu.V. Nesterenko, G. Diaz and others): *for* β *algebraic number of degree* $d \geq 2$, *among the numbers*

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots, \alpha^{\beta^{d-1}},$$

at least $[(d+1)/2]$ *are algebraically independent.*

Tools of Gel'fond's algebraic independence method

- Gel'fond–Schneider transcendence method (based on Hermite–Lindemann-Siegel earlier work)
- A *zero estimate* for exponential polynomials (earlier results were due to G. Pòlya, K. Mahler. . .)
- A so-called *transcendence criterion*.
  - One needs a replacement for Liouville's inequality, namely a lower bound for $|P(x)|$ where now x is transcendental.
  - But there is no such lower bound for a non-zero transcendental number!
  - One of Gel'fond's main ideas here is that the transcendence proof yields not only one non-zero number $|P(x)|$, but a sequence of $|P_n(x)|$, $n \geq 0$.
  - **There is no non-trivial uniform sequence of polynomials taking small values at a given transcendental number.**

§ 2. Rational approximation to a real number

- Since $\mathbf{Q}$ is dense in $\mathbf{R}$, *for any* $\xi \in \mathbf{R}$ *and any* $\epsilon > 0$ *there exists* $b/a \in \mathbf{Q}$ *such that*

$$\left|\xi - \frac{b}{a}\right| < \epsilon.$$

- Write the conclusion

$$|a\xi - b| < \epsilon a.$$

- It is easy to improve this estimate: *let* $a \in \mathbf{Z}_{>0}$ *and let* b *be the nearest integer to* $a\xi$. *Then*

$$|a\xi - b| \leq 1/2.$$

Existence of good rational approximations

Let ξ be a real number.

- From Dirichlet's box principle (or from Minkowski's Theorem in the geometry of numbers) one deduces that *for each real number $H > 1$, there exists $q \in \mathbf{Z}$ and $p \in \mathbf{Z}$ with $1 \le q < H$ such that*

$$|q\xi - p| < \frac{1}{H}. \tag{1}$$

- As a consequence, *if ξ is irrational, then there exist infinitely many $p/q \in \mathbf{Q}$ such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}. \tag{2}$$

- Is it possible to improve (1) (uniform)?
Same question for (2) (asymptotic)?

Irrationality measures

Let ξ be a real number.

- (Asymptotic) irrationality measure of ξ :

$$\omega(\xi) = \sup\Big\{w;\ \text{there exist infinitely many } (p,q) \in \mathbf{Z}^2,\ q \ge 1,\quad |q\xi - p| \le q^{-w}\Big\}.$$

  - An upper bound for $\omega(\xi)$ is an *irrationality measure* for ξ, namely a lower bound for $|\xi - p/q|$ when $p/q \in \mathbf{Q}$.

- Irrationality exponent of ξ :

$$\mu(\xi) = \omega(\xi) + 1 = \sup\Big\{\mu;\ \text{there exist infinitely many } (p,q) \in \mathbf{Z}^2,\quad q \ge 1,\quad \left|\xi - \frac{p}{q}\right| \le q^{-\mu}\Big\}.$$

- Hence for any $\xi \in \mathbf{R} \setminus \mathbf{Q}$,

$$1 \le \omega(\xi) \le +\infty \quad \text{and} \quad 2 \le \mu(\xi) \le +\infty.$$

- Capelli: for almost all ξ, $\omega(\xi) = 1$ and $\mu(\xi) = 2$.

Uniform irrationality measures

Let ξ be a real number.

- Uniform irrationality measure of ξ :

$$\widehat{\omega}(\xi) = \sup\Big\{w;\ \text{for any } H \geq 1, \text{there exists } (p,q) \in \mathbf{Z}^2, \\ 1 \leq q \leq H \text{ and } |q\xi - p| \leq H^{-w}\Big\}.$$

- Hence for any $\xi \in \mathbf{R} \setminus \mathbf{Q}$,

$$\omega(\xi) \geq \widehat{\omega}(\xi) \geq 1.$$

- Khintchine and Roth: *for almost all* $\xi \in \mathbf{R}$ *and for all algebraic* $\xi \in \mathbf{R} \setminus \mathbf{Q}$,

$$\omega(\xi) = \widehat{\omega}(\xi) = 1.$$

Hurwitz, Markoff et al.

- Continued fractions or Farey series yield: *for any irrational real number* ξ *there exist infinitely many* $p/q \in \mathbf{Q}$ *such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

  and *this is best possible*
  – for instance for the Golden Number $(1+\sqrt{5})/2$.
- Markoff Spectrum: $\sqrt{5}$, $\sqrt{8}$, $\sqrt{221}/5$, $\sqrt{1517}/13, \ldots$
- Liouville (1844): *there exist real numbers* ξ *such that, for any* $m \geq 1$, *there is a rational approximation* p/q *with*

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^m}.$$

Spectrum of $\big(\omega(\xi), \widehat{\omega}(\xi)\big)$

- The exponent $\omega(\xi)$ for (2) and asymptotic rational approximation can take any value in the range $[1, +\infty]$.
- Khintchine: *for any* $\xi \in \mathbf{R} \setminus \mathbf{Q}$, *the exponent* $\widehat{\omega}(\xi)$ *for (1) and uniform rational approximation satisfies*

$$\widehat{\omega}(\xi) = 1.$$

- Hence

$$\big\{\big(\omega(\xi), \widehat{w}(\xi)\big)\,;\ \xi \in \mathbf{R} \setminus \mathbf{Q}\big\} = [1, +\infty] \times \{1\}.$$

$\widehat{\omega}(\xi) = 1$

- Recall (1):
  Let $\xi \in \mathbf{R}$. *For each real number* $H > 1$, *there exists* $a \in \mathbf{Z}$ *and* $b \in \mathbf{Z}$ *with* $1 \le a < H$ *such that*

$$|a\xi - b| < \frac{1}{H}.$$

  Hence $\widehat{\omega}(\xi) \ge 1$ for any $\xi \in \mathbf{R} \setminus \mathbf{Q}$.
- *Let* ξ *be a real number. Assume that for each sufficiently large integer* H, *there exists* $a \in \mathbf{Z}$ *and* $b \in \mathbf{Z}$ *with* $1 \le a < H$ *and*

$$|a\xi - b| < \frac{1}{2H}.$$

  Then ξ *is rational and* $a\xi = b$ *for each* $H \ge H_0$.
  Hence $\widehat{\omega}(\xi) \le 1$ for any $\xi \in \mathbf{R} \setminus \mathbf{Q}$.

Proof of $\widehat{\omega}(\xi) \le 1$

- Goal: *Let ξ be a real number. Assume that there exists H_0 such that, for each integer $H \ge H_0$, there exists $a_H \in \mathbf{Z}$ and $b_H \in \mathbf{Z}$ with $1 \le a_H < H$ and*

$$|a_H\xi - b_H| < \frac{1}{2H}.$$

Then ξ is rational and $a_H\xi = b_H$ for each $H \ge H_0$.

- **Proof.** Let $H \ge H_0$. Write (a,b) for (a_H, b_H) and (a',b') for (a_{H+1}, b_{H+1}):

$$|a\xi - b| < \frac{1}{2H}, \quad |a'\xi - b'| < \frac{1}{2H+2},$$

$$1 \le a \le H-1, \quad 1 \le a' \le H$$

and

$$ab' - a'b = a(b' - a'\xi) + a'(a\xi - b).$$

Proof of $\widehat{\omega}(\xi) \le 1$ (continued)

- From $1 \le a \le H, \quad 1 \le a' \le H$,

$$|a\xi - b| < \frac{1}{2H}, \quad |a'\xi - b'| < \frac{1}{2H},$$

and

$$ab' - a'b = a(b' - a'\xi) + a'(a\xi - b)$$

one deduces $|ab' - a'b| < 1$, hence $ab' = a'b$.

- Therefore $a_H/b_H = a_{H+1}/b_{H+1}$ does not depend on $H \ge H_0$. Since

$$\lim_{H\to\infty} \frac{a_H}{b_H} = \xi,$$

it follows that ξ is rational and $\xi = a_H/b_H$ for all $H \ge H_0$.

- Consequence: for any irrational number ξ , $\widehat{\omega}(\xi) = 1$.
- Alternative argument for $\widehat{\omega}(\xi) = 1$ (M. Laurent): use continued fraction expansions.

The exponent $\nu(\xi)$ of S. Fischler and T. Rivoal

Let $\xi \in \mathbf{R} \setminus \mathbf{Q}$.

- When $\underline{u} = (u_n)_{n\geq 1}$ is an increasing sequence of positive integers, define $\underline{v} = (v_n)_{n\geq 1}$ by $|u_n\xi - v_n| < 1/2$ and set

$$\alpha_\xi(\underline{u}) = \limsup_{n\to\infty} \frac{|u_{n+1}\xi - v_{n+1}|}{|u_n\xi - v_n|},$$

$$\beta(\underline{u}) = \limsup_{n\to\infty} \frac{u_{n+1}}{u_n}\cdot$$

- Set
$$\nu(\xi) = \inf \log \sqrt{\alpha_\xi(\underline{u})\beta(\underline{u})},$$
where $\underline{u}$ ranges over the sequences which satisfy $\alpha_\xi(\underline{u}) < 1$ and $\beta(\underline{u}) < +\infty$.
With $\inf_\emptyset = +\infty$.

Properties of $\nu(\xi)$

- Connection with the irrationality exponent:
$$\mu(\xi) \leq 1 - \frac{\log \beta(\underline{u})}{\log \alpha_\xi(\underline{u})}\cdot$$
- **Corollary**: *if* $\nu(\xi) < +\infty$, *then* $\mu(\xi) < +\infty$.
  Apéry like – proofs of irrationality + measures.
- If ξ is quadratic, Fischler and Rivoal produce a sequence $\underline{u}$ with $\alpha_\xi(\underline{u})\beta(\underline{u}) = 1$, hence $\nu(\xi) = 0$.
- Works of R. Apéry, A. Baker, F. Beukers, M. Hata, ... and S. Fischler– T. Rivoal:

$$\nu(2^{1/3}) \leq (3/2)\log 2,\ \nu(\zeta(3)) \leq 3,\ \nu(\pi^2) \leq 2,\ \nu(\log 2) \leq 1.$$

Also $\nu(\pi) \leq 21$.

Spectrum of $\nu(\xi)$

- For any $\xi \in \mathbf{R} \setminus \mathbf{Q}$, the inequalities $0 \le \nu(\xi) \le +\infty$ hold.
- Fischler & Rivoal: *for almost all $\xi \in \mathbf{R}$, $\nu(\xi) = 0$.*
- B. Adamczewski , S. Fischler and T. Rivoal: *any irrational algebraic real number ξ has $\nu(\xi) < +\infty$.*
- There are examples of $\xi \in \mathbf{R} \setminus \mathbf{Q}$ for which $\nu(\xi) = +\infty$. All known examples so far have $\mu(\xi) = +\infty$.
- *Is it true that $\nu(\xi) < +\infty$ implies $\mu(\xi) = 2$? Are there numbers ξ with $0 < \nu(\xi) < +\infty$?*

§ 3. Polynomial approximations

Let ξ be a complex number.

- From Dirichlet's box principle one deduces that *there exists a constant $c(\xi)$ such that, for each positive integer n and each real number $H > 1$, there exists a non zero polynomial $P \in \mathbf{Z}[X]$ of degree $\le n$ and usual height $\le H$ such that*
$$|P(\xi)| \le H^{-c(\xi)n}.$$
- Two main special cases: either n fixed (bounded) as above, or require the same upper bound N for n and $\log H$:
For each positive integer $N \ge 1$ there exists a non zero polynomial $P \in \mathbf{Z}[X]$ of degree $\le N$ and height $\le e^N$ with
$$|P(\xi)| \le e^{-c(\xi)N^2}.$$

Approximation of a complex number by an algebraic number

Let ξ be a complex number.
Connection between

- polynomial approximation and study of $|P(\xi)|$ for $P \in \mathbf{Z}[X]$ (of degree $\le n$)

and

- approximation by algebraic numbers and study of $|\xi - \alpha|$ for α algebraic number (of degree $\le n$).

Roughly speaking,

- If $|P(\xi)|$ is small then ξ is close to a root α of P,

Conversely,

- if $|\xi - \alpha|$ is small then the minimal polynomial of α assumes a small value at ξ.

Existence of algebraic approximations

Let ξ be a real number and n a positive integer. Assume ξ is not algebraic of degree $\le n$.

- E. Wirsing (1960): *There exist infinitely many algebraic numbers α of degree $\le n$ with*

$$|\xi - \alpha| \le c(\xi, n) H(\alpha)^{-(n+3)/2}.$$

- H. Davenport and W.M. Schmidt (1967): for $n = 2$ replace $(n+3)/2 = 5/2$ by 3.
  This is optimal for the approximation of a real number by quadratic algebraic numbers.

Wirsing's Conjecture

- Wirsing's Conjecture: *For any $\epsilon > 0$ there exists $c(\xi, n, \epsilon) > 0$ for which there are infinitely many algebraic numbers α of degree $\leq n$ with*

$$|\xi - \alpha| \leq c(\xi, n, \epsilon) H(\alpha)^{-n-1+\epsilon}.$$

- Recent work by V. Bernik, K. Tishchenko,...

Tijdeman's version of Gel'fond's criterion

- *Let $\xi \in \mathbf{C}$. Assume there is a sequence P_N of non-zero polynomials in $\mathbf{Z}[X]$, where P_N has degree $\leq N$ and (usual) height $\leq e^N$, such that*

$$|P_N(\xi)| \leq e^{-7N^2}.$$

Then ξ is algebraic and $P_N(\xi) = 0$ for all $N \geq N_0$.

- **Sketch of proof.** Fix $H \geq H_0$. Since $|P_H(\xi)|$ is small, ξ is close to a root α_H of P_H, hence P_H is divisible by a power Q_H of the irreducible polynomial of α_H and $|Q_H(\xi)|$ is small. The resultant of the two polynomials Q_H and Q_{H+1} has absolute value < 1, hence it vanishes, and therefore α_H does not depend on H.

Gel'fond's transcendence criterion with fixed degree

Let ξ be a complex number and n a positive integer.

- *Assume that for each positive integer $H \geq H_0$ there exists a non zero polynomial $P_H \in \mathbf{Z}[X]$ of degree $\leq n$ and usual height $\leq H$ such that*

$$|P_H(\xi)| \leq H^{-7n}.$$

Then ξ is algebraic and $P_H(\xi) = 0$ for each $H \geq H_0$.

- **Sketch of proof**: the same!
- Refinement by H. Davenport and W.M. Schmidt (1969): *exponent $2n-1$ in place of $7n$.*
- **Sketch of proof**: refined elimination argument.

Transcendence measures: $w_n(\xi)$ and $\widehat{w}_n(\xi)$

Let ξ be a real number and n a positive integer. Assume ξ is not algebraic of degree $\leq n$.

- Denote by $w_n(\xi)$ the supremum of $w \in \mathbf{R}$ such that there exist infinitely many positive integers H for which the system

$$|x_0 + x_1\xi + \cdots + x_n\xi^n| \leq H^{-w}, \quad 0 < \max_{0 \leq i \leq n} |x_i| \leq H$$

  has a solution in rational integers $x_0, x_1, \ldots, x_n$.
- Upper bound for $w_n(\xi)$ = transcendence measure for ξ.
- Denote by $\widehat{w}_n(\xi)$ the supremum of $w \in \mathbf{R}$ such that, for any sufficiently large integer H, the same system has a solution.
- An upper bound for $\widehat{w}_n(\xi)$ is a uniform transcendence measure for ξ.

Spectrum of exponents $w_n(\xi)$ and $\widehat{w}_n(\xi)$

- For $n=1$, $w_1(\xi)=\omega(\xi)$ and $\widehat{w}_1(\xi)=\widehat{\omega}(\xi)$.
- From the definition: $w_n(\xi)\geq \widehat{w}_n(\xi)$.
- From Dirichlet's box principle one deduces $\widehat{w}_n(\xi)\geq n$.
- The above mentioned result by Davenport and Schmidt can be read:
$$\widehat{w}_n(\xi)\leq 2n-1.$$
- Liouville numbers have $w_n(\xi)=+\infty$ for $n=1$, hence for all $n\geq 1$ (since $w_n\leq w_{n+1}$).
- Sprindzuck: *For almost all numbers $\xi\in\mathbf{R}$,*
$$w_n(\xi)=\widehat{w}_n(\xi)=n \text{ for all } n\geq 1.$$
- Consequence of W.M. Schmidt's subspace Theorem: *For all $n\geq 1$ and all irrational algebraic numbers of degree $>n$,*
$$w_n(\xi)=\widehat{w}_n(\xi)=n.$$

Simultaneous approximation measure: $w'_n(\xi)$

Let ξ be a real number and n a positive integer. Assume ξ is not algebraic of degree $\leq n$.

- Denote by $w'_n(\xi)$ the supremum of $w\in\mathbf{R}$ such that there exist infinitely many positive integers H for which the system
$$\max_{0\leq i\leq n}|x_i-x_0\xi^i|\leq H^{-w},\quad 0<\max_{0\leq i\leq n}|x_i|\leq H$$
has a solution in rational integers $x_0,x_1,\dots,x_n$.
- An upper bound for $w'_n(\xi)$ is a simultaneous approximation measure for $\xi,\xi^2,\dots,\xi^n$.

Uniform simultaneous approximation measure: $\widehat{w}'_n(\xi)$

Let ξ be a real number and n a positive integer. Assume ξ is not algebraic of degree $\leq n$.

- Denote by $\widehat{w}'_n(\xi)$ the supremum of $w \in \mathbf{R}$ such that, for any sufficiently large integer H, the same system

$$\max_{0\leq i\leq n} |x_i - x_0\xi^i| \leq H^{-w}, \quad 0 < \max_{0\leq i\leq n} |x_i| \leq H$$

has a solution in rational integers $x_0, x_1, \ldots, x_n$.
- For $n=1$, $w'_1(\xi) = \omega(\xi)$ and $\widehat{w}'_1(\xi) = \widehat{\omega}(\xi)$.

Spectrum of $w'_n(\xi)$ and $\widehat{w}'_n(\xi)$

- Dirichlet's box principle: *for all* ξ *and* n, $w'_n(\xi) \geq \widehat{w}'_n(\xi) \geq 1/n$.
- Sprindzuck: *for almost all real numbers* ξ, $w'_n(\xi) = \widehat{w}'_n(\xi) = 1/n$ *for all* n.
- Consequence of W.M. Schmidt's subspace Theorem: *for all* n *and for all algebraic real numbers* ξ *of degree* $> n$, $w'_n(\xi) = \widehat{w}'_n(\xi) = 1/n$.

Transfer principle: polar convex bodies

Let ξ be a real number and n a positive integer. Assume ξ is not algebraic of degree $\leq n$.
H. Davenport and W.M. Schmidt (1969), with a refinement by Y. Bugeaud and O. Teulié (2000):
Let $\xi \in \mathbf{R} \setminus \mathbf{Q}$ and $n \geq 1$.
Assume $\widehat{w}'_n(\xi) \leq \lambda$. Then there exists $c(n,\xi) > 0$ such that there are infinitely many

- *algebraic numbers α of degree n*
- *algebraic integers α of degree $n+1$*
- *algebraic units α of degree $n+2$*
- ...

such that

$$|\xi - \alpha| \leq c(n,\xi) H(\alpha)^{-\kappa} \quad \textit{with} \quad \kappa = \frac{1}{\lambda} + 1.$$

Upper bounds for $\widehat{w}'_n(\xi)$

Let ξ be a real number and n a positive integer. Assume ξ is not algebraic of degree $\leq n$.

- H. Davenport and W.M. Schmidt (1969).
  Let $\xi \in \mathbf{R} \setminus \mathbf{Q}$ and $n \geq 2$.
  Assume ξ is not algebraic of degree $\leq [n/2]$. Then

$$\widehat{w}'_n(\xi) \leq [n/2]^{-1} = \begin{cases} 2/n & \textit{if } n \textit{ is even,} \\ 2/(n-1) & \textit{if } n \textit{ is odd.} \end{cases}$$

- Refinement by M. Laurent (2003): replaces $[n/2]$ (twice) by $\lceil n/2 \rceil$: for not algebraic of degree $\leq \lceil n/2 \rceil$,

$$\widehat{w}'_n(\xi) \leq \lceil n/2 \rceil^{-1} = \begin{cases} 2/n & \text{if } n \text{ is even,} \\ 2/(n+1) & \text{if } n \text{ is odd.} \end{cases}$$

Quadratic approximation: estimates for $\widehat{w}_2'$

Consider the special case $n = 2$. Let $\xi \in \mathbf{R}$ which is neither rational nor quadratic.

- Dirichlet's box principle: *for all such* ξ, $\widehat{w}_2'(\xi) \geq 1/2$.
- Khintchine: *for almost all* ξ, $\widehat{w}_2'(\xi) = 1/2$.
- Consequence of W.M. Schmidt's subspace Theorem: *for all algebraic* ξ *(of degree* ≥ 3*)*, $\widehat{w}_2'(\xi) = 1/2$.
- Davenport and Schmidt (1969): *for all* ξ,

$$\widehat{w}_2'(\xi) \leq 1/\gamma = 0.618\dots.$$

- Comment by H. Davenport and W.M. Schmidt in their 1969 paper:
  We have no reason to think that the exponents in these theorems are best possible.

Simultaneous approximation of a number and its square

- D. Roy (2003): Examples of transcendental numbers ξ for which

$$\widehat{w}_2'(\xi) = 1/\gamma = 0.618\dots$$

- Start with $f_1 = b$ and $f_2 = a$ and define (concatenation): $f_n = f_{n-1}f_{n-2}$.
- Hence $f_3 = ab$ $\quad f_4 = aba$ $\quad f_5 = abaab$
  $f_6 = abaababa$ $\quad f_7 = abaababaabaab$
  $f_8 = abaababaabaababaababa$ ...
- *The Fibonacci word*

$$w = abaababaabaababaababaabaababaabaab\dots$$

is the fixed point of the morphism $a \mapsto ab$, $b \mapsto a$.

Result of D. Roy (2003)

- *Let A and B be two distinct positive integers. Let $\xi \in (0,1)$ be the real number whose* continued fraction expansion *is obtained from the Fibonacci word w by replacing the letters a and b by A and B:*

  $$[0; A, B, A, A, B, A, B, A, A, B, A, A, B, A, B, A, A, \dots]$$

  Then $\widehat{w}_2'(\xi) = 1/\gamma$.
- Further more recent results on simultaneous approximation of a number and its square (hence on approximation of real numbers by quadratic integers) and on quadratic approximation of numbers associated with *Sturmian words* by M. Laurent, Y. Bugeaud, S. Fischler, D. Roy...

Quadratic approximation: the four exponents $(w_2(\xi), \widehat{w}_2(\xi), w_2'(\xi), \widehat{w}_2'(\xi))$

Problem: describe the spectrum of the 4-tuples

$$(w_2(\xi), \widehat{w}_2(\xi), w_2'(\xi), \widehat{w}_2'(\xi))$$

when ξ ranges over the set of non quadratic irrational real numbers.

- Jarnik's formula:

  $$\widehat{w}_2'(\xi) = 1 - \frac{1}{\widehat{w}_2(\xi)}\cdot$$

- Hence for any irrational number ξ which is not quadratic,

  $$2 \le \widehat{w}_2(\xi) \le \frac{3+\sqrt{5}}{2}\cdot$$

Transcendence criterion for quadratic polynomials

Recall $\widehat{w}_2(\xi) \leq \gamma + 1$. An explicit transcendence criterion due to B. Arbour and D. Roy (2004) is the following:

Let ξ be a real number. Assume that for any sufficiently large H there exist a polynomial $P \in \mathbf{Z}[X]$ of degree ≤ 2 and height $\leq H$ such that

$$|P(\xi)| \leq \frac{1}{4} H^{-\gamma-1},$$

where γ denotes the Golden Number $(1+\sqrt{5})/2$. Then ξ is algebraic and all these values $P(\xi)$ are zero.

Cubic approximation (following D. Roy)

Laurent's refined inequality (for ξ not algebraic of degree $\leq \lceil n/2 \rceil$)

$$\frac{1}{n} \leq \widehat{w}'_n(\xi) \leq \lceil n/2 \rceil^{-1} = \begin{cases} 2/n & \text{if } n \text{ is even,} \\ 2/(n+1) & \text{if } n \text{ is odd,} \end{cases}$$

valid for all $n \geq 2$ was already known by Davenport and Schmidt in the special case $n = 3$ and yields

$$\frac{1}{3} \leq \widehat{w}'_3(\xi) \leq \frac{1}{2}.$$

The lower bound is optimal. The upper bound has very recently been improved by D. Roy (cf. his lecture at 11:25 this morning).

§ 4. Simultaneous rational approximation of several numbers

- Polynomial approximation to a complex number ξ is the study of $|P(\xi)|$ for $P \in \mathbf{Z}[X]$. Negative results on the existence of polynomial approximations lead to transcendence measures.
- This is a special case of the study of linear combinations in $\xi_1, \ldots, \xi_n$ where $\xi_i = \xi^{i-1}$ $(1 \le i \le n)$.
- Another (less) special case is the simultaneous approximation to
$$\xi_1^{a_1} \cdots \xi_m^{a_m}$$
which is related to measures of algebraic independence of $\xi_1, \ldots, \xi_m$.
- Further special cases: simultaneous approximation of dependent quantities, approximation on a manifold.

Connection with algebraic independence

- Simultaneous approximation and algebraic independence
- Criteria of algebraic independence: "Cassels' counterexample" (due to Khintchine, 1926), criteria of G.V. Chudnovskii, P. Philippon, Yu.V. Neterenko, M. Ably, C. Jadot...
- May include multiplicities
- More recent work by M. Laurent, D. Roy (1999): approximation by algebraic sets
- New approach to Schanuel's conjecture by D. Roy (2000).

Four exponents

Given $\theta = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$, there are (at least) two points of view for studying simultaneous approximation to $\xi_1, \ldots, \xi_n$.

- On the one hand one may consider linear forms

$$|x_0 + x_1\xi_1 + \cdots + x_n\xi_n|$$

- On the other hand one may investigate the existence of simultaneous approximation by rational numbers

$$\max_{0\le i\le n} \left|\xi_i - \frac{x_i}{x_0}\right|.$$

- Since each of these question has two versions, an asymptotic one (with w) and a uniform one (with $\widehat{w}$), that makes 4 exponents

$$\omega(\theta), \quad \widehat{\omega}(\theta), \quad \omega({}^t\theta), \quad \widehat{\omega}({}^t\theta).$$

- Special case $\xi_i = \xi^i$, $1 \le i \le n$: for $\theta = (\xi, \xi^2, \ldots, \xi^n)$,
  $\omega(\theta) = w_n(\xi)$, $\widehat{\omega}(\theta) = \widehat{w}_n(\xi)$, $\omega({}^t(\theta)) = w'_n(\xi)$, $\widehat{\omega}({}^t(\theta)) = \widehat{w}'_n(\xi)$.

Simultaneous approximation by linear forms

Let $\xi_1, \ldots, \xi_n$ be real numbers. Set

$$\theta = (\xi_1, \ldots, \xi_n)$$

- Denote by $\omega(\theta)$ the supremum of $w \in \mathbf{R}$ such that there exist infinitely many positive integers H for which the system

$$|x_0 + x_1\xi_1 + \cdots + x_n\xi_n| \le H^{-w}, \quad 0 < \max_{0\le i\le n} |x_i| \le H$$

  has a solution in rational integers $x_0, x_1, \ldots, x_n$.
- An upper bound for $\omega(\theta)$ is a linear independence measure for $1, \xi_1, \ldots, \xi_n$.
- Denote by $\widehat{\omega}(\theta)$ the supremum of $w \in \mathbf{R}$ such that, for any sufficiently large integer H, the same system has a solution.
- Hence $\widehat{\omega}(\theta) \ge \omega(\theta)$.

Simultaneous approximation by rational numbers

Let $\xi_1, \dots, \xi_n$ be real numbers. Set ${}^t\theta = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$

- Denote by $\omega({}^t\theta)$ the supremum of $w \in \mathbf{R}$ such that there exist infinitely many positive integers H for which the system

$$\max_{0\le i\le n} |x_i - x_0\xi_i| \le H^{-w}, \quad 0 < \max_{0\le i\le n} |x_i| \le H$$

has a solution in rational integers $x_0, x_1, \dots, x_n$.
- An upper bound for $\omega({}^t\theta)$ is a simultaneous approximation measure for $1, \xi_1, \dots, \xi_n$.
- Denote by $\widehat{\omega}({}^t\theta)$ the supremum of $w \in \mathbf{R}$ such that, for any sufficiently large integer H, the same system has a solution.
- Again $\widehat{\omega}({}^t\theta) \ge \omega({}^t\theta)$.

Further exponents (M. Laurent)

For each d in the range $0 \le d \le n-1$, M. Laurent introduces two exponents, one for asymptotic approximation $\omega_d(\theta)$ and one for uniform approximation $\widehat{\omega}_d(\theta)$, so that

$$\omega_0(\theta) = \omega(\theta), \qquad \omega_{n-1}(\theta) = \omega({}^t\theta)$$

and

$$\widehat{\omega}_0(\theta) = \widehat{\omega}(\theta), \qquad \widehat{\omega}_{n-1}(\theta) = \widehat{\omega}({}^t\theta).$$

Definitions for higher dimensional approximation

$$\begin{array}{ccc} \mathbf{R}^n & \subset & \mathbf{P}^n(\mathbf{R}) \\ \theta = (\xi_1, \ldots, \xi_n) & \mapsto & (\xi_1 : \cdots : \xi_n : 1) \end{array}$$

$$0 \le d \le n-1.$$

- $\omega_d(\theta) = \sup \Big\{ \omega;$ there exist infinitely many vectors $X = x_0 \wedge \cdots \wedge x_d \in \Lambda^{d+1}(\mathbf{Z}^{n+1})$ such that
$$|X \wedge \theta| \le |X|^{-\omega} \Big\}.$$

- $\widehat{\omega}_d(\theta) = \sup \Big\{ \omega;$ for any sufficiently large H, there exists $X = x_0 \wedge \cdots \wedge x_d \in \Lambda^{d+1}(\mathbf{Z}^{n+1})$ such that
$$0 < |X| \le H \quad \text{and} \quad |X \wedge \theta| \le H^{-\omega} \Big\}.$$

Hence $\omega_d(\theta) \ge \widehat{\omega}_d(\theta)$.

Distances

The multivector $X = x_0 \wedge \cdots \wedge x_d$ is a system of Plücker coordinates of the linear projective subvariety
$$L = \langle x_0, \ldots, x_d \rangle \subset \mathbf{P}^n(\mathbf{R}).$$
Then
$$\frac{|X \wedge \theta|}{|X||\theta|} \sim d(\theta, L) = \min_{x \in L} d(\theta, x).$$

Equivalent definitions

- $\omega_d(\theta) = \sup \Big\{ \omega;$ there exist infinitely many L rational over $\mathbf{Q}$, $\dim L = d$ and $d(\theta, L) \le H(L)^{-\omega-1} \Big\}$

- $\widehat{\omega}_d(\theta) = \sup \Big\{ \omega;$ for any sufficiently large H, there exists L rational, $\dim L = d$, $H(L) \le H$ and $d(\theta, L) \le H(L)^{-1} H^{-\omega} \Big\}$

Extremal cases $d = 0$ and $d = n - 1$

- $$\begin{aligned} \omega_0(\theta) &= \omega({}^t\theta), & \widehat{\omega}_0(\theta) &= \widehat{\omega}({}^t\theta), \\ \omega_{n-1}(\theta) &= \omega(\theta), & \widehat{\omega}_{n-1}(\theta) &= \widehat{\omega}(\theta). \end{aligned}$$
- Minkowski's convex body Theorem yields *lower bounds* (valid for all θ):
$$\widehat{\omega}_d(\theta) \geq \frac{d+1}{n-d} \quad \text{for all } d = 0, \ldots, n-1.$$
- In particular for $d = n - 1$ and $d = 0$ respectively, one recovers
$$\omega(\theta) \geq n \quad \text{and } \omega({}^t\theta) \geq 1/n.$$
- Khintchine (1926): $\omega(\theta) = n$ if and only if $\omega({}^t(\theta)) = 1/n$.
- Generic: *for almost all* $\theta \in \mathbf{R}^n$, *for* $0 \leq d \leq n-1$,
$$\omega_d(\theta) = \widehat{\omega}_d(\theta) = \frac{d+1}{n-d}.$$

Devissage of Khintchine transfer principles

- **Theorem (*M. Laurent*).** *Set*
$$\omega_d = \omega_d(\xi_1, \ldots, \xi_n), \quad 0 \leq d \leq n-1.$$
(i) Going up transfer principle.
$$\omega_{d+1} \geq \frac{(n-d)\omega_d + 1}{n-d-1}, \quad 0 \leq d \leq n-2.$$
(ii) Going down transfer principle.
$$\omega_{d-1} \geq \frac{d\omega_d}{\omega_d + d + 1}, \quad 1 \leq d \leq n-1.$$

- **Corollary (*Khintchine transfer principle*).**
$$\begin{aligned} \omega_{n-1} &\geq n\omega_0 + n - 1 \\ \omega_0 &\geq \frac{\omega_{n-1}}{(n-1)\omega_{n-1} + n} \end{aligned}$$
M. Laurent (2006): *these estimates are optimal.*

§ 5. Dimension 2

Let ξ and η be two real numbers with $1, \xi, \eta$ linearly independent over $\mathbf{Q}$.

- Khintchine's transfert Theorem

$$\frac{\omega(\xi,\eta)}{\omega(\xi,\eta)+2} \le \omega\binom{\xi}{\eta} \le \frac{\omega(\xi,\eta)-1}{2}.$$

- Optimal: Jarnik's formula:

$$\widehat{\omega}\binom{\xi}{\eta} = 1 - \frac{1}{\widehat{\omega}(\xi,\eta)}.$$

Spectrum of each exponent in dimension 2

- $\omega(\xi,\eta)$ takes any value in the range $[2,+\infty]$.
- $\omega\binom{\xi}{\eta}$ takes any value in the range $[1/2,1]$.
- $\widehat{\omega}(\xi,\eta)$ takes any value in the range $[2,+\infty]$.
- $\widehat{\omega}\binom{\xi}{\eta}$ takes any value in the range $[1/2,1]$.
- Generic: *for almost all* $(\xi,\eta)\in\mathbf{R}^2$,

$$\omega(\xi,\eta)=\widehat{\omega}(\xi,\eta)=2, \quad \omega\binom{\xi}{\eta}=\widehat{\omega}\binom{\xi}{\eta}=\frac{1}{2}.$$

Spectrum of the four exponents in dimension 2

Théorème (M. Laurent) *Assume $1, \xi, \eta$ are linearly independent over $\mathbf{Q}$. The four exponents*

$$w = \omega(\xi,\eta), \quad w' = \omega\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \quad \widehat{w} = \widehat{\omega}(\xi,\eta), \quad \widehat{w}' = \widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

are related by

$$2 \le \widehat{w} \le +\infty, \quad \widehat{w}' = \frac{\widehat{w}-1}{\widehat{w}}, \quad \frac{w(\widehat{w}-1)}{w+\widehat{w}} \le w' \le \frac{w-\widehat{w}+1}{\widehat{w}}.$$

For $w = +\infty$ this means

$$\widehat{w} - 1 \le w' \le +\infty.$$

Conversely, for any $(w, w', \widehat{w}, \widehat{w}')$ in $(\mathbf{R}_{>0} \cup \{+\infty\})^4$ satisfying the previous inequalities there exists $(\xi,\eta) \in \mathbf{R}^2$, with $1, \xi, \eta$ linearly independent over $\mathbf{Q}$, such that

$$w = \omega(\xi,\eta), \quad w' = \omega\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \quad \widehat{w} = \widehat{\omega}(\xi,\eta), \quad \widehat{w}' = \widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

Consequences

- **Corollary 1**. *The exponents $w = \omega(\xi,\eta)$, $\widehat{w} = \widehat{\omega}(\xi,\eta)$ are related by*

  $$w \ge \widehat{w}(\widehat{w}-1) \quad \text{and} \quad \widehat{w} \ge 2.$$

  Conversely, for any $(w, \widehat{w})$ satisfying these conditions, there exists (ξ,η) such that

  $$(\omega(\xi,\eta), \widehat{\omega}(\xi,\eta)) = (w, \widehat{w}).$$

- **Corollary 2**. *The exponents $w' = \omega\begin{pmatrix}\xi\\ \eta\end{pmatrix}$, $\widehat{w}' = \widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$ are related by*

  $$w' \ge \frac{\widehat{w}'^2}{1-\widehat{w}'} \quad \text{and} \quad \frac{1}{2} \le \widehat{w}' \le 1.$$

  Conversely, for any $(w', \widehat{w}')$ satisfying these conditions, there exists (ξ,η) with

  $$(\omega\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix}) = (w', \widehat{w}').$$

Open problems

- Describe for each $n \geq 2$ the spectrum of the set

$$(w_n(\xi),\ \widehat{w}_n(\xi),\ w'_n(\xi),\ \widehat{w}'_n(\xi))$$

where ξ ranges over the set of real numbers which are not algebraic of degree $\leq n$.

- Is there an extension of Jarnik's *equality*

$$\widehat{\omega}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = 1 - \frac{1}{\widehat{\omega}(\xi,\eta)}$$

in higher dimension relating $\widehat{\omega}(\theta)$ and $\widehat{\omega}'(\theta)$ for $\theta \in \mathbf{R}^n$?

Report on recent progress in Diophantine approximation

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/transparents.html

9th Canadian Number Theory conference (CNTA 9)
University of British Columbia in Vancouver, July 9-14, 2006
http://www.pims.math.ca/science/2006/06cnta/

1. Algebraic independence
2. Rational approximation to a real number
3. Polynomial approximation to a complex number
4. Simultaneous rational approximation
5. Dimension 2