RUPP Master in Mathematics
Number Theory

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1. Let p and q be distinct primes. Show that pq divides $p^{q-1} + q^{p-1} - 1$.

2. Let $a \geq 2$ and $m \geq 2$ be integers. Check that

$$\gcd\left(\frac{a^m - 1}{a-1}\ ,\ a-1\right) = \gcd(m, a-1).$$

3. Let a be a positive integer, P the product of its divisors, D the number of its divisors. Check

$$P^2 = a^D.$$

4. Let x, y,z be positive integers satisfying $x^2 + y^2 = z^2$. Check that the product xyz is divisible by 60.

5. Which are the prime numbers p such that p divides $2^p + 1$?

6. Let n be a positive integer. What is the class of $1 + 2 + \cdots + (n-1)$ modulo n?

7. Let R be a non commutative ring and x, y two elements in R. Assume $1 - xy$ is a unit in R. Prove that $1 - yx$ is a unit in R.

8. Prove that in a finite group of even order, there is an element of order 2.

9. Show that a finite integral domain is a field.

10. Which are the irreducible polynomials in $\mathbf{R}[X]$?

11. Prove that in a finite field, any element is a sum of two squares.

12. Let K be a field, a and b two elements in K. Check that the quotient ring of $K[X]$ by the ideal generated by $(X-a)(X-b)$ is isomorphic to $K \times K$.