Royal University of Phnom Penh RUPP
Université Royale de Phnom Penh URPP

Michel Waldschmidt

Master of Science in Mathematics

Master Training Program

Complex Analysis MMA 106

Assignment, May 9, 2008

1. Let $n \geq 1$ be a positive integer. Solve the equation $z^{n+1} = z$.

2. Which are the values of $t \in \mathbf{C}$ such that the set $\{z \in \mathbf{C}\ ;\ 2z^3 - 3z^2 = t\}$ has exactly two elements? For each of these values of t find the two roots of the polynomial $2z^3 - 3z^2 - t$.

3. What is the multiplicity at $z = 0$ of the function $(z-2)e^z + 2 + z$?

4. Let $f(z)$ be the function $1/z$ on $\mathbf{C} \setminus \{0\}$. Let $z_0 \in \mathbf{C}$, $z_0 \neq 0$. Write the expansion of $f(z)$ as a power series in $(z - z_0)$. What is the radius of convergence?

5. For which values of the positive real number t is the open set

$$\bigl\{z \in \mathbf{C}\ ;\ 1 < |z| < 2\ ;\ 0 < \Re e(z) < t\bigr\}$$

connected?

6. Give an example of an open set U in $\mathbf{C}$, an analytic function f in U and a closed path γ in U such that

$$\int_\gamma f(z)dz \neq 0.$$

Can you give such an example where U is a disc?

7. Let f be an analytic function in the unit disc $\{z \in \mathbf{C}\ ;\ |z| < 1\}$. Assume $f(1/n) = 1/n^2$ for all integers $n \geq 2$. Compute $f(i/2)$.

8. Which are the functions f analytic in $\mathbf{C}$ satisfying $|f(z)| = |z|^3$ for all $z \in \mathbf{C}$?

9. Let $n \in \mathbf{Z}$. Compute

$$\int_{|z|=1} \frac{e^{iz}}{z^n} dz.$$

10. Find the Laurent expansions of the function $\dfrac{1}{z(z-1)}$ for $0 < |z| < 1$ and for $|z| > 1$.

miw@math.jussieu.fr Tél. : [33] (0)6 83 17 29 06 http://www.math.jussieu.fr/∼miw/

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Analyse complexe MMA 106

Contrôle, vendredi 9 Mai 2008

1. *Soit $n \geq 1$ un entier positif. Résoudre l'équation $z^{n+1} = z$.*

2. *Quels sont les nombres $t \in \mathbf{C}$ tels que l'ensemble $\{z \in \mathbf{C}\ ;\ 2z^3 - 3z^2 = t\}$ ait exactement deux éléments ? Pour chacune de ces valeurs de t, trouver les deux racines du polynôme $2z^3 - 3z^2 - t$.*

3. *Quelle est la multiplicité à l'orgine de la fonction $(z-2)e^z + 2 + z$?*

4. *On désigne par $f(z)$ la fonction $1/z$ sur $\mathbf{C} \setminus \{0\}$. Soit $z_0 \in \mathbf{C}$, $z_0 \neq 0$. Écrire le développement de $f(z)$ en série entière en $(z - z_0)$. Quel est le rayon de convergence ?*

5. *Pour quelles valeurs du paramètre réel $t > 0$ l'ensemble ouvert*

$$\bigl\{z \in \mathbf{C}\ ;\ 1 < |z| < 2\ ;\ 0 < \Re e(z) < t\bigr\}$$

est-il connexe ?

6. *Donner un exemple d'un ouvert U de $\mathbf{C}$, d'une fonction f analytique dans U et d'un chemin fermé γ dans U tel que*

$$\int_\gamma f(z)dz \neq 0.$$

Pouvez-vous donner un tel exemple en prenant pour U un disque ?

7. *Soit f une fonction analytique dans le disque unité $\{z \in \mathbf{C}\ ;\ |z| < 1\}$. On suppose $f(1/n) = 1/n^2$ pour tout entier $n \geq 2$. Calculer $f(i/2)$.*

8. *Quelles sont les fonctions f analytiques dans $\mathbf{C}$ qui vérifient $|f(z)| = |z|^3$ pour tout $z \in \mathbf{C}$?*

9. *Soit $n \in \mathbf{Z}$. Calculer*

$$\int_{|z|=1} \frac{e^{iz}}{z^n} dz.$$

10. *Quel est le développement de Laurent de la fonction $\dfrac{1}{z(z-1)}$ dans $0 < |z| < 1$? Et dans $|z| > 1$.*

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Assignment May 9, 2008 : Correction

Solution exercise (1). The polynomial $z^{n+1} - z^n$ has $n+1$ distinct roots which are 0 and $e^{2i\pi k/n}$ for $k = 1, \ldots, n$. Also $e^{2i\pi k/n} = \cos(2\pi k/n) + i \sin(2\pi k/n)$.

Solution exercise (2). The polynomial $2z^3 - 3z^2 - t$ has degree 3, it has three complex roots counting multiplicities, hence the number of distinct roots is less than 3 if and only if it has a multiple root, that is a root which is also root of the derivative $6z^2 - 6z = 6z(z-1)$. So if it has a multiple root z it is with $z = 0$ or $z = 1$, hence $t = 0$ or $t = -1$.

For $t = 0$ there are two roots, $z = 0$ which is a double root (zero of multiplicity 2) and $z = 3/2$ which is a simple root.

For $t = -1$ there is a double root $z = 1$ and a single root $z = -1/2$.

Solution exercise (3). The expansion at the origin of $(z-2)e^z$ starts with

$$-2 - z + \frac{z^3}{6} + \frac{z^4}{6} + \cdots$$

hence the function $(z-2)e^z + 2 + z$ has multiplicity 3 at the origin.

Solution exercise (4). The expansion of $f(z) = 1/z$ as a power series in $(z - z_0)$ for $z_0 \neq 0$ has been considered many times during the course :

$$\frac{1}{z} = \sum_{n \geq 0} (-1)^n \frac{(z - z_0)^n}{z_0^{n+1}}.$$

It follows for instance from the geometric series expansion, writing

$$\frac{1}{z} = \frac{1}{z_0} \cdot \frac{1}{1 + \dfrac{z - z_0}{z_0}}.$$

Another solution is to compute the Taylor coefficients

$$\frac{1}{n!}\left(\frac{d}{dz}\right)^n\left(\frac{1}{z}\right)\Big|_{z=z_0}=\frac{(-1)^n}{z_0^{n+1}}\cdot$$

The radius of convergence is $|z_0|$: the series converges in the disc $|z-z_0|<|z_0|$, which is the largest disc in $\mathbf{C}$ centered at z_0 not containing 0.

Solution exercise (5)**.** For $0<t<1$ the set

$$\{z\in\mathbf{C}\;;\;1<|z|<2\;;\;0<\Re e(z)<t\}$$

is a union of two disjoint components, one in the upper half plane $\Im m(z)>0$ and the other in the half plane $\Im m(z)<0$, while in case $t>1$ this set is connected.

Solution exercise (6)**.** An example with $U=\mathbf{C}\setminus\{0\}$ is $f(z)=1/z$ and the circumference $|z|=1$, since the integral

$$\int_{|z|=1}f(z)dz$$

has the value $2\pi i$.

If U is a disc, then any analytic function in U has a primitive, hence the integral over a closed path is 0.

Solution exercise (7)**.** Let f be an analytic function in the unit disc such that $f(1/n)=1/n^2$ for all integers $n\geq 2$. The zeros of the function $f(z)-z^2$ in the unit disc are not isolated,. Since a disc is connected, this function is 0, hence $f(z)=z^2$ for all $|z|<1$. In particular $f(i/2)=(i/2)^2=-1/4$.

Solution exercise (8)**.** By Cauchy's inequalities, f is a polynomial (this is a refinement of Liouville's Theorem that bounded entire functions are constant). The only zero of f is 0, hence $f(z)=cz^d$ for some $c\in\mathbf{C}$ and $d\geq 0$. From the assumption $|f(z)|=|z|^3$ we deduce $d=3$ and $|c|=1$. Conversely, for any $c\in\mathbf{C}$ with $|c|=1$ the function $f(z)=cz^3$ satisfies $|f(z)|=|z|^3$.

Solution exercise (9)**.** Let $n\in\mathbf{Z}$. We have

$$e^{iz}=\sum_{k\geq 0}\frac{(iz)^k}{k!}\quad\text{and}\quad\frac{e^{iz}}{z^n}=\sum_{k\geq 0}\frac{i^kz^{k-n}}{k!}\cdot$$

The integral

$$\int_{|z|=1} z^{k-n} dz$$

is 0 for $k-n \neq -1$ and $2i\pi$ for $k-n=-1$, that is for $k=n-1$. Hence the integral

$$\int_{|z|=1} \frac{e^{iz}}{z^n} dz$$

is 0 for $n \leq 0$ and

$$2i\pi \frac{i^{n-1}}{(n-1)!}$$

for $n \geq 1$. Recall that i^m is 1 if m is a multiple of 4, it is -1 if m is even but not a multiple of 4, it is i if $m = 4k+1$ and $-i$ if $m = 4k-1$ for some integer k.

Solution exercise (10)**.** Write the rational fraction $f(z) = \dfrac{1}{z(z-1)}$ as sum a simple fractions

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}\cdot$$

First assume $0 < |z| < 1$. The Taylor expansion of $1/(z-1)$ at the origin is

$$\frac{1}{z-1} = -1 - z - z^2 - z^3 - \cdots$$

with convergence radius 1. Hence for $0 < |z| < 1$ the Laurent expansion of the function f is

$$-\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots$$

Next assume $|z| > 1$. The Laurent expansion of $1/(z-1)$ is given by

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

hence the Laurent expansion of $f(z)$ in $|z| > 1$ is

$$\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$