Early history of irrational and transcendental numbers

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Abstract

The transcendence proofs for constants of analysis are essentially all based on the seminal work by Ch. Hermite : his proof of the transcendence of the number e in 1873 is the prototype of the methods which have been subsequently developed. We first show how the founding paper by Hermite was influenced by earlier authors (Lambert, Euler, Fourier, Liouville), next we explain how his arguments have been expanded in several directions : Padé approximants, interpolation series, auxiliary functions.

Numbers : rational, irrational

Numbers = real or complex numbers $\mathbf{R}$, $\mathbf{C}$.

Natural integers : $\mathbf{N} = \{0, 1, 2, \ldots\}$.

Rational integers : $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Rational numbers :
a/b with a and b rational integers, $b > 0$.

Irreducible representation :
p/q with p and q in $\mathbf{Z}$, $q > 0$ and $\gcd(p, q) = 1$.

Irrational number : a real (or complex) number which is not rational.

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Sums and products of rational numbers are rational numbers :

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

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The sum of an rational number and an irrational number is irrational. This is a consequence of the fact that the sum of two rational numbers is rational.

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Infinite series

Main question : Is the sum of a convergent series of rational numbers a rational or an irrational number ?

Answer : It may be rational or irrational !

Example of a rational sum (geometric series) :

$$2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

Example of an irrational sum :

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

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Algebraic number : a complex number which is root of a non-zero polynomial with rational coefficients.

Examples :
rational numbers : a/b, root of $bX - a$.
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For instance a polynomial with rational coefficients vanishing at $\sqrt{2}+\sqrt{3}$ is

$$(X-\sqrt{2}-\sqrt{3})(X-\sqrt{2}+\sqrt{3})(X+\sqrt{2}-\sqrt{3})(X+\sqrt{2}+\sqrt{3}).$$

In general if

$$\prod_{i=1}^{m}(X-\alpha_i) \quad \text{and} \quad \prod_{j=1}^{n}(X-\beta_j)$$

have rational coefficients then

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where j is root of X^2+X+1, so that

$$X^3-5=(X-\sqrt[3]{5})(X-j\sqrt[3]{5})(X-j^2\sqrt[3]{5}).$$

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Another proof of this fact is the following : *a complex number α is algebraic if and only if the vector space spanned by $1, \alpha, \alpha^2, \alpha^3 \ldots$ over the rational has finite dimension.*

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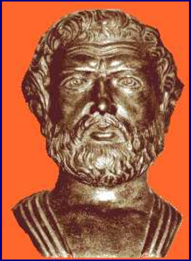
Pythagoreas school



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Sulba Sutras, Vedic civilization in India, ∼800-500 BC.

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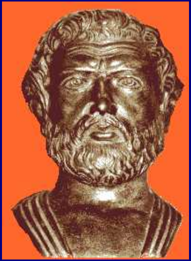

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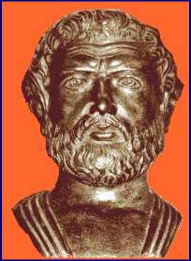

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Assume $\sqrt{2} = p/q$ with p and q without common factor.
Hence one at least of p, q is odd.
By definition of $\sqrt{2}$ we have $p^2/q^2 = 2$, which means

$$p^2 = 2q^2.$$

Hence p is even : write $p = 2a$.
Now $p^2 = 4a^2$ and $4a^2 = 2q^2$.
This yields

$$2a^2 = q^2$$

and q is even !

Contradiction.

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Irrationality of $\sqrt{2}$: geometric proof

- Start with a rectangle have side length 1 and $1+\sqrt{2}$.
- Decompose it into two squares with sides 1 and a smaller rectangle of sides $1+\sqrt{2}-2=\sqrt{2}-1$ and 1.
- This second small rectangle has side lenghts in the proportion
$$\frac{1}{\sqrt{2}-1}=1+\sqrt{2},$$
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- Hence the second small rectangle can be split into two squares and a third smaller rectangle, the sides of which are again in the same proportion.
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- Start with a rectangle have side length 1 and $1+\sqrt{2}$.
- Decompose it into two squares with sides 1 and a smaller rectangle of sides $1+\sqrt{2}-2=\sqrt{2}-1$ and 1.
- This second small rectangle has side lenghts in the proportion
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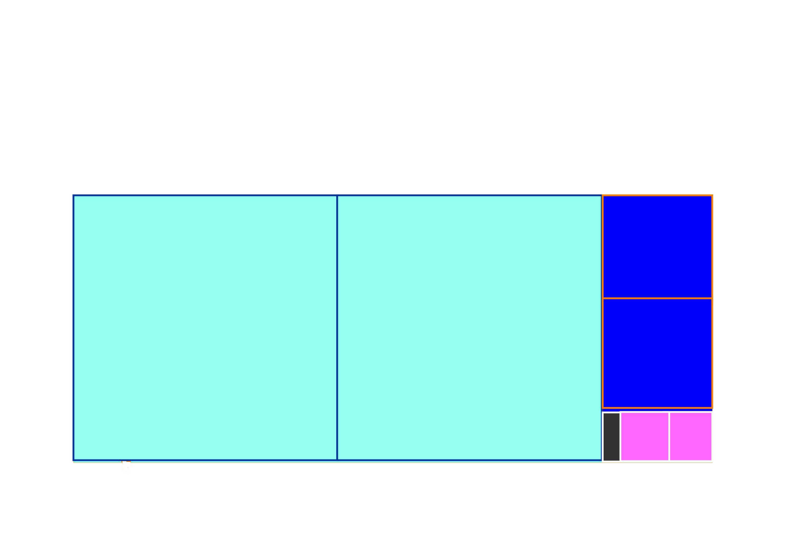
Rectangles with proportion $1 + \sqrt{2}$



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If we start with a rectangle having integer side lengths, then this process stops after finitely may steps (the side lengths are positive decreasing integers).

Also for a rectangle with side lengths in a rational proportion, this process stops after finitely may steps (reduce to a common denominator and scale).

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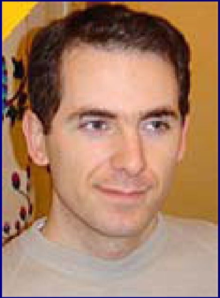

- Benoît Rittaud, Éditions *Le Pommier* (2006).

http://www.math.univ-paris13.fr/~rittaud/RacineDeDeux

Continued fraction

The number

$$\sqrt{2} = 1,414\,213\,562\,373\,095\,048\,801\,688\,724\,209 \ \ldots$$

satisfies

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1}.$$

Hence

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2}+1}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

We write the continued fraction expansion of $\sqrt{2}$ using the shorter notation

$$\sqrt{2} = [1;\ 2,\ 2,\ 2,\ 2,\ 2,\ \ldots] = [1;\ \overline{2}].$$

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- H.W. Lenstra Jr,
Solving the Pell Equation,
Notices of the A.M.S.
49 (2) (2002) 182–192.

Irrationality criteria

A real number is rational if and only if its continued fraction expansion is finite.

A real number is rational if and only if its binary (or decimal, or in any basis $b \geq 2$) expansion is *ultimately periodic*.

Consequence : it should not be so difficult to decide whether a given number is rational or not.

To prove that certain numbers (occurring as constants in analysis) are irrational is most often an impossible challenge. However to construct irrational (even transcendental) numbers is easy.

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Euler–Mascheroni constant

Euler's Constant is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$
$$= 0,577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$

Is–it a rational number ?

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The function
$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$
was studied by Euler (1707– 1783)
for integer values of s
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Euler : for any even integer value of $s \geq 2$, the number $\zeta(s)$ is a rational multiple of π^s.

Examples : $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450 \cdots$

Denominators : Bernoulli numbers.

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Introductio in analysin infinitorum

Leonhard Euler

(**15 Avril 1707** – 1783)

Introductio in analysin infinitorum

Divergent series

Euler :

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$$

$$1 + 1 + 1 + 1 + 1 + \cdots = -\frac{1}{2}$$

$$1 + 2 + 3 + 4 + 5 + \cdots = -\frac{1}{12}$$

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Geometric series

- Let x be a real number and n an integer. Consider the sum of n terms

$$S_n(x) = x + x^2 + x^3 + \cdots + x^n.$$

- For instance $S_n(1) = n$ for all n.
- **Formula :** pour $x \neq 1$,

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Proof

$$S_n(x) = x + x^2 + x^3 + \cdots + x^n$$

Multiply by x :

$$xS_n(x) = x^2 + x^3 + x^4 + \cdots + x^{n+1}$$

Substract

$$(1 - x)S_n(x) = x - x^{n+1}$$

Since $x \neq 1$ we may divide by $1 - x$:

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Example : $S_n(1/2) = 1/2 + 1/4 + 1/8 + \cdots + 1/2^n$

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$$S_n(1/2) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n}$$

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For $-1 < x < 1$ we have

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The right hand side at $x = -1$ is $1/2$. Hence the value $1/2$ given by Euler to the infinite sum

$$S = 1 - 1 + 1 - 1 + \cdots$$

We obtain the same value $S = 1/2$ by writing

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Since

$$4A = 4 + 8 + 12 + \cdots$$

we obtain by substraction

$$A - 4A = -3A = B$$

hence

$$A = -\frac{1}{12}.$$

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Since

$$4A = 4 + 8 + 12 + \cdots$$

we obtain by substraction

$$A - 4A = -3A = B$$

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Letter of Ramanujan
to M.J.M. Hill in 1912

$$1 + 2 + 3 + \cdots + \infty = -\frac{1}{12}$$

$$1^2 + 2^2 + 3^2 + \cdots + \infty^2 = 0$$

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Answer of M.J.M. Hill in 1912

$$1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n+1)$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

First letter from Ramanujan to Hardy (January 16, 1913)

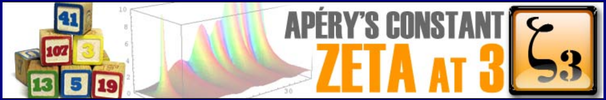
$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

$$1 - 1! + 2! - 3! + \cdots = .596\cdots$$

Answer from Hardy (February 8, 1913)

I was exceedingly interested by your letter and by the theorems which you state. You will however understand that, before I can judge properly of the value of what you have done, it is essential that I should see proofs of some of your assertions. Your results seem to me to fall into roughly three classes :
(1) there are a number of results that are already known, or easily deducible from known theorems ;
(2) there are results which, so far as I know, are new and interesting, but interesting rather from their curiosity and apparent difficulty than their importance ;
(3) there are results which appear to be new and important...

Riemann zeta function



The number

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511\ \ldots$$

is irrational *(Apéry 1978).*

Recall that $\zeta(s)/\pi^s$ is rational for any even value of $s \geq 2$.

Open question : Is the number $\zeta(3)/\pi^3$ irrational ?

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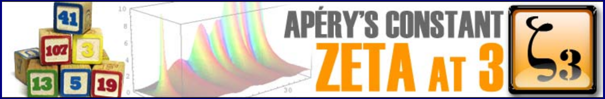

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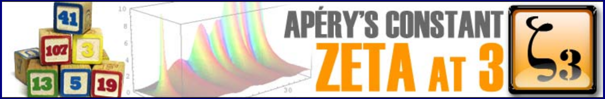

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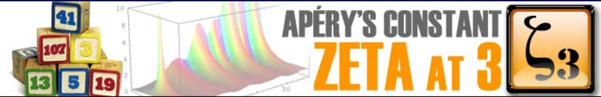

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T. Rivoal (2000) : infinitely many $\zeta(2n+1)$ are irrational.

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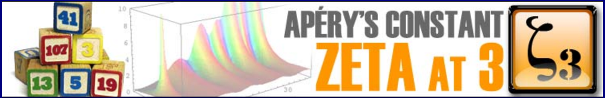

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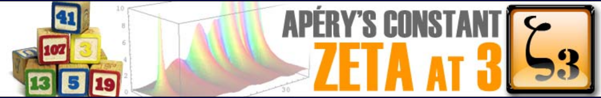

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Open problems (irrationality)

- Is the number

$$e + \pi = 5,859\,874\,482\,048\,838\,473\,822\,930\,854\,632\ldots$$

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Catalan's constant

Is Catalan's constant

$$\sum_{n\geq 1}\frac{(-1)^n}{(2n+1)^2}$$

$= 0,915\,965\,594\,177\,219\,015\,0\ldots$

an irrational number ?

This is the value at $s=2$ of the Dirichlet L–function $L(s,\chi_{-4})$ associated with the Kronecker character

$$\chi_{-4}(n)=\left(\frac{n}{4}\right),$$

which is the quotient of the Dedekind zeta function of $\mathbf{Q}(i)$ and the Riemann zeta function.

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Is the number

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$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = \int_0^{\infty} e^{-t} t^z \cdot \frac{dt}{t}$$

Here is the set of rational values for z for which the answer is known (and, for these arguments, the Gamma value is a transcendental number) :

$$r \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}\right\} \pmod 1.$$

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Known results

Irrationality of the number π :

Āryabhaṭa, b. 476 AD : $\pi \sim 3.1416$.

Nīlakaṇṭha Somayājī, b. 1444 AD : *Why then has an approximate value been mentioned here leaving behind the actual value ? Because it (exact value) cannot be expressed.*

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Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,
Mémoires de l'Académie des Sciences de Berlin, **17** (1761), p. 265-322 ; read in 1767 ; Math. Werke, t. II.

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Continued fraction expansion of $\tan(x)$

$$\tan(x) = \frac{1}{i}\tanh(ix), \qquad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\cdot$$

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cfrac{x^2}{9 - \cfrac{x^2}{\ddots}}}}}}\cdot$$

S.A. Shirali – *Continued fraction for e*, Resonance, vol. **5** N°1, Jan. 2000, 14–28.
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Leonard Euler (**April 15, 1707** – 1783)

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De fractionibus continuis dissertatio,
Commentarii Acad. Sci. Petropolitanae,
9 (1737), 1744, p. 98–137 ;
Opera Omnia Ser. I vol. **14**,
Commentationes Analyticae, p. 187–215.

$$e = \lim_{n\to\infty} (1+1/n)^n$$
$$= 2,718\,281\,828\,459\,045\,235\,360\,287\,471\,352\ldots$$
$$= 1 + 1 + \frac{1}{2}\cdot\left(1 + \frac{1}{3}\cdot\left(1 + \frac{1}{4}\cdot\left(1 + \frac{1}{5}\cdot(1+\cdots)\right)\right)\right).$$

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= [2\,;\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\dots]\\
= [2; \overline{1,\ 2m,\ 1}]_{m\ge 1}.
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e is neither rational (J-H. Lambert, 1766) nor quadratic irrational (J-L. Lagrange, 1770).

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Continued fraction expansion for $e^{1/a}$

Starting point : $y = \tanh(x/a)$ satisfies the differential equation $ay' + y^2 = 1$.

This leads Euler to

$$e^{1/a} = [1\,;\ a-1,\ 1,\ 1,\ 3a-1,\ 1,\ 1,\ 5a-1, \dots]$$
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Geometric proof of the irrationality of e

Jonathan Sondow
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A geometric proof that e is irrational and a new measure of its irrationality,
Amer. Math. Monthly **113** (2006) 637-641.

Start with an interval I_1 with length 1. The interval I_n will be obtained by splitting the interval I_{n-1} into n intervals of the same length, so that the length of I_n will be $1/n!$.

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The origin of I_n will be

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Hence we start from the interval $I_1 = [2, 3]$. For $n \geq 2$, we construct I_n inductively as follows : split I_{n-1} into n intervals of the same length, and call the second one I_n :

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Hence we start from the interval $I_1=[2,3]$. For $n\geq 2$, we construct I_n inductively as follows : split I_{n-1} into n intervals of the same length, and call the second one I_n :

$$I_1=\left[1+\frac{1}{1!}\,,\,1+\frac{2}{1!}\right]=[2,3],$$

$$I_2=\left[1+\frac{1}{1!}+\frac{1}{2!}\,,\,1+\frac{1}{1!}+\frac{2}{2!}\right]=\left[\frac{5}{2!}\,,\,\frac{6}{2!}\right],$$

$$I_3=\left[1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\,,\,1+\frac{1}{1!}+\frac{1}{2!}+\frac{2}{3!}\right]=\left[\frac{16}{3!}\,,\,\frac{17}{3!}\right].$$

Irrationality of e, following J. Sondow

The origin of I_n is

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \frac{a_n}{n!},$$

the length is $1/n!$, hence $I_n = [a_n/n!, (a_n + 1)/n!]$.

The number e is the intersection point of all these intervals, hence it is inside each I_n, therefore it cannot be written $a/n!$ with a an integer.
Since

$$\frac{p}{q} = \frac{(q-1)!\,p}{q!},$$

we deduce that the number e is irrational.

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Irrationality measure for e, following J. Sondow

For any integer $n > 1$,

$$\frac{1}{(n+1)!} < \min_{m \in \mathbf{Z}} \left| e - \frac{m}{n!} \right| < \frac{1}{n!} \cdot$$

Smarandache function : $S(q)$ is the least positive integer such that $S(q)!$ is a multiple of q :

$S(1) = 1,\; S(2) = 2,\; S(3) = 3,\; S(4) = 4,\; S(5) = 5,\; S(6) = 3 \ldots$

$S(p) = p$ for p prime. Also $S(n!) = n$.
Irrationality measure for e : *for* $q > 1$,

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Joseph Fourier

Course of analysis at the École Polytechnique Paris, 1815.

Irrationality of e, following J. Fourier

$$e = \sum_{n=0}^{N} \frac{1}{n!} + \sum_{m \geq N+1} \frac{1}{m!}.$$

Multiply by $N!$ and set

$$B_N = N!, \qquad A_N = \sum_{n=0}^{N} \frac{N!}{n!}, \quad R_N = \sum_{m \geq N+1} \frac{N!}{m!},$$

so that $B_N e = A_N + R_N$. Then A_N and B_N are in $\mathbf{Z}$, $R_N > 0$ and

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In the formula

$$B_N e - A_N = R_N,$$

the numbers A_N and $B_N = N!$ are integers, while the right hand side is > 0 and tends to 0 when N tends to infinity.

Hence $N!\, e$ is not an integer, therefore e is irrational.

Since e is irrational, the same is true for $e^{1/b}$ when b is a positive integer. That e^2 is irrational is a stronger statement.

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The number e is not quadratic

Recall (Euler, 1737) : $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ which is not a periodic expansion. J.L. Lagrange (1770) : it follows that e is not a quadratic number.

Assume $ae^2 + be + c = 0$. Replacing e and e^2 by the series and truncating does not work : the denominator is too large and the *remainder* does not tend to zero.

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Sur l'irrationalité du nombre $e = 2,718\ldots$, J. Math. Pures Appl. (1) **5** (1840), p. 192 and p. 193-194.

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The irrationality of e^4, hence of $e^{4/b}$ for b a positive integer, follows.

It does not seem that this kind of argument will suffice to prove the irrationality of e^3, even less to prove that the number e is not a cubic irrational (A. Hurwitz, 1896).

Fourier's argument rests on truncating the exponential series, it amounts to approximate e by $a/N!$ where $a \in \mathbf{Z}$. Better rational approximations exist, involving other denominators than $N!$.

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approximate the exponential function e^z by rational fractions $A(z)/B(z)$.

For proving the irrationality of e^a, (a an integer ≥ 2), approximate e^a par $A(a)/B(a)$.

If the function $B(z)e^z - A(z)$ has a zero of high multiplicity at the origin, then this function has a small modulus near 0, hence at $z = a$. Therefore $|B(a)e^a - A(a)|$ is small.

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A rational function $A(z)/B(z)$ is *close* to a complex analytic function f if $B(z)f(z) - A(z)$ has a zero of high multiplicity at the origin.

Goal : find $B \in \mathbf{C}[z]$ such that the Taylor expansion at the origin of $B(z)f(z)$ has a big gap : $A(z)$ will be the part of the expansion before the gap, $R(z) = B(z)f(z) - A(z)$ the remainder.

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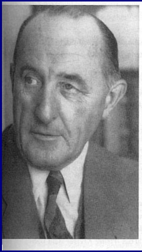
Irrationality of e^r and π

Charles Hermite (1873)

Carl Ludwig Siegel (1929, 1949)

Yuri Nesterenko (2005)



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We wish to prove the irrationality of e^a for a a positive integer.

Goal : write $B_n(z)e^z = A_n(z) + R_n(z)$ with A_n and B_n in $\mathbf{Z}[z]$ and $R_n(a) \neq 0$, $\lim_{n\to\infty} R_n(a) = 0$.

Substitute $z = a$, set $q = B_n(a)$, $p = A_n(a)$ and get

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Rational approximation to exp

Given $n_0 \geq 0$, $n_1 \geq 0$, *find* A *and* B *in* $\mathbf{R}[z]$ *of degrees* $\leq n_0$ *and* $\leq n_1$ *such that* $R(z) = B(z)e^z - A(z)$ *has a zero at the origin of multiplicity* $\geq N+1$ *with* $N = n_0 + n_1$.

Theorem *There is a non-trivial solution, it is unique with* B *monic. Further,* B *is in* $\mathbf{Z}[z]$ *and* $(n_0!/n_1!)A$ *is in* $\mathbf{Z}[z]$. *Furthermore* A *has degree* n_0, B *has degree* n_1 *and* R *has multiplicity exactly* $N+1$ *at the origin.*

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$B(z)e^z = A(z) + R(z)$

Proof. Unicity of R, hence of A and B.
Let $D = d/dz$. Since A has degree $\leq n_0$,

$$D^{n_0+1}R = D^{n_0+1}(B(z)e^z)$$

is the product of e^z with a polynomial of the same degree as the degree of B and same leading coefficient.
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Irrationality of logarithms including π

The irrationality of e^r for $r \in \mathbf{Q}^\times$, is equivalent to the irrationality of $\log s$ for $s \in \mathbf{Q}_{>0}$.

The same argument gives the irrationality of $\log(-1)$, meaning $\log(-1) = i\pi \notin \mathbf{Q}(i)$.

Hence $\pi \notin \mathbf{Q}$.

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C.L. Siegel, 1949.

Solve $D^{n_0+1}R(z) = z^{n_1}e^z$.

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inverse of D, satisfies

$$J^{n+1}\varphi = \int_0^z \frac{1}{n!}(z-t)^n\varphi(t)dt.$$

Hence

$$R(z) = \frac{1}{n_0!}\int_0^z (z-t)^{n_0}t^{n_1}e^t dt.$$

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Irrationality proofs involve rational approximation to a single real number θ.

We wish to prove transcendence results.

A complex number θ is transcendental if and only if the numbers

$$1, \theta, \theta^2, \ldots, \theta^m, \ldots$$

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$L = a_0 + a_1 x_1 + \cdots + a_m x_m$

Let $x_1, \ldots, x_m$ be real numbers and a_0, $a_1, \ldots,$ a_m rational integers, not all of which are zero. We wish to prove that the number

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is not zero. Approximate simultaneously $x_1, \ldots, x_m$ by rational numbers $b_1/b_0, \ldots, b_m/b_0$.
Let $b_0, b_1, \ldots, b_m$ be rational integers. For $1 \le k \le m$ set

$$\epsilon_k = b_0 x_k - b_k.$$

Then $b_0 L = A + R$ with

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Simultaneous approximation to the exponential function

Irrationality results follow from rational approximations $A/B \in \mathbf{Q}(x)$ to the exponential function e^x.

One of Hermite's ideas is to consider *simultaneous rational approximations to the exponential function*, in analogy with Diophantine approximation.

Let $B_0, B_1, \ldots, B_m$ be polynomials in $\mathbf{Z}[x]$. For $1 \leq k \leq m$ define

$$R_k(x) = B_0(x)e^{kx} - B_k(x).$$

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Hermite–Lindemann Theorem

For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.

Hermite (1873) : transcendence of e.

Lindemann (1882) : transcendence of π.

Corollaries : transcendence of $\log \alpha$ and of e^β for α and β non-zero algebraic complex numbers, with $\log \alpha \neq 0$.

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Hermite : approximation to the functions $1, e^{\alpha_1 x}, \ldots, e^{\alpha_m x}$

Let $\alpha_1, \ldots, \alpha_m$ be pairwise distinct complex numbers and $n_0, \ldots, n_m$ be rational integers, all ≥ 0. Set $N = n_0 + \cdots + n_m$.

Hermite constructs explicitly polynomials B_0, $B_1, \ldots$, B_m with B_j of degree $N - n_j$ such that each of the functions

$$B_0(z)e^{\alpha_k z} - B_k(z), \quad (1 \leq k \leq m)$$

has a zero at the origin of multiplicity at least N.

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Approximation of complex analytic functions by rational functions.

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Transcendental functions

A complex function is called transcendental if it is transcendental over the field $\mathbf{C}(z)$, which means that the functions z and $f(z)$ are algebraically independent : if $P \in \mathbf{C}[X, Y]$ is a non-zero polynomial, then the function $P\big(z, f(z)\big)$ is not 0.

Exercise. *An entire function (analytic in $\mathbf{C}$) is transcendental if and only if it is not a polynomial.*

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Weierstrass question

Is–it true that a transcendental entire function f takes usually *transcendental values at algebraic arguments ?*

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain...

If S is a countable subset of $\mathbf{C}$ and T is a dense subset of $\mathbf{C}$, there exist transcendental entire functions f mapping S into T, as well as all its derivatives.

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An integer valued entire function is a function f, which is analytic in $\mathbf{C}$, and maps $\mathbf{N}$ into $\mathbf{Z}$.

Example : 2^z *is an integer valued entire function, not a polynomial.*

Question : *Are-there integer valued entire function growing slower than* 2^z *without being a polynomial ?*

Let f be a transcendental entire function in $\mathbf{C}$. For $R > 0$ set

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if f is not a polynomial
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Further works on this topic by G.H. Hardy, G. Pólya, D. Sato, E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross,. . .

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Since $f(n)$ is an integer for all $n \geq 0$, the coefficients a_n are rational and one can bound the denominators. If f does not grow fast, one deduces that these coefficients vanish for sufficiently large n.

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$$f(z) = f(\alpha_1) + (z - \alpha_1) f_1(z), \quad f_1(z) = f_1(\alpha_2) + (z - \alpha_2) f_2(z), \ldots$$

we deduce

$$f(z) = a_0 + a_1(z - \alpha_1) + a_2(z - \alpha_1)(z - \alpha_2) + \cdots$$

with

$$a_0 = f(\alpha_1), \quad a_1 = f_1(\alpha_2), \ldots, \quad a_n = f_n(\alpha_{n+1}).$$

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An identity due to Ch. Hermite

$$\frac{1}{x-z} = \frac{1}{x-\alpha} + \frac{z-\alpha}{x-\alpha} \cdot \frac{1}{x-z}.$$

Repeat :

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Inductively we deduce the next formula due to Hermite :

$$\frac{1}{x-z} = \sum_{j=0}^{n-1} \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_j)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} + \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)} \cdot \frac{1}{x-z}.$$

Newton interpolation expansion

Application. Multiply by $(1/2i\pi)f(z)$ and integrate :

$$f(z) = \sum_{j=0}^{n-1} a_j(z-\alpha_1)\cdots(z-\alpha_j) + R_n(z)$$

with

$$a_j = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{F(x)dx}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} \qquad (0 \le j \le n-1)$$

and

$$R_n(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)\cdot \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{F(x)dx}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)(x-z)}.$$

Integer valued entire function on $\mathbf{Z}[i]$

A.O. Gel'fond (1929) : growth of entire functions mapping the Gaussian integers into themselves.
Newton interpolation series at the points in $\mathbf{Z}[i]$.

An entire function f which is not a polynomial and satisfies $f(a+ib) \in \mathbf{Z}[i]$ for all $a+ib \in \mathbf{Z}[i]$ satisfies

$$\limsup_{R\to\infty} \frac{1}{R^2} \log |f|_R \geq \gamma.$$

F. Gramain (1981) : $\gamma = \pi/(2e)$.
This is best possible : *D.W. Masser (1980).*

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Transcendence of e^π

A.O. Gel'fond (1929).

If

$$e^\pi = 23,140\,692\,632\,779\,269\,005\,729\,086\,367\ \ldots$$

is rational, then the function $e^{\pi z}$ takes values in $\mathbf{Q}(i)$ when the argument z is in $\mathbf{Z}[i]$.

Expand $e^{\pi z}$ into an interpolation series at the Gaussian integers.

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Hilbert's seventh problem

A.O. Gel'fond and Th. Schneider (1934).

Solution of Hilbert's seventh problem :

transcendence of α^β

and of $(\log \alpha_1)/(\log \alpha_2)$

for algebraic α, β, α_1 *and* α_2.

Dirichlet's box principle

Gel'fond and Schneider
use an *auxiliary function*,
the existence of which follows
from Dirichlet's box principle
(pigeonhole principle,
Thue-Siegel Lemma).

Auxiliary functions

C.L. Siegel (1929) :
Hermite's explicit formulae
can be replaced by
Dirichlet's box principle
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Slope inequalities in Arakelov theory

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matrices and determinants require choices of bases.
Arakelov's Theory produces *slope inequalities* which avoid the need of bases.

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Rational interpolation

René Lagrange (1935).

$$\frac{1}{x-z} = \frac{\alpha-\beta}{(x-\alpha)(x-\beta)} + \frac{x-\beta}{x-\alpha}\cdot\frac{z-\alpha}{z-\beta}\cdot\frac{1}{x-z}.$$

Iterating and integrating yield

$$f(z) = \sum_{n=0}^{N-1} B_n \frac{(z-\alpha_1)\cdots(z-\alpha_n)}{(z-\beta_1)\cdots(z-\beta_n)} + \tilde{R}_N(z).$$

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Hurwitz zeta function

T. Rivoal (2006) : consider Hurwitz zeta function

$$\zeta(s,z) = \sum_{k=1}^{\infty} \frac{1}{(k+z)^s}.$$

Expand $\zeta(2,z)$ as a series in

$$\frac{z^2(z-1)^2\cdots(z-n+1)^2}{(z+1)^2\cdots(z+n)^2}.$$

The coefficients of the expansion belong to $\mathbf{Q}+\mathbf{Q}\zeta(3)$. This produces a new proof of Apéry's Theorem on the irrationality of $\zeta(3)$.

In the same way : new proof of the irrationality of $\log 2$ by expanding

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Mixing C. Hermite and R. Lagrange

T. Rivoal (2006) : new proof of the irrationality of $\zeta(2)$ by expanding

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+z}\right)$$

as a Hermite–Lagrange series in

$$\frac{\bigl(z(z-1)\cdots(z-n+1)\bigr)^2}{(z+1)\cdots(z+n)}.$$

Taylor series and interpolation series

Taylor series are the special case of Hermite's formula with a single point and multiplicities — they give rise to Padé approximants.

Multiplicities can also be introduced in René Lagrange interpolation.

There is another duality between the methods of Gel'fond and Schneider : Fourier-Borel transform.

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Further develoments

Transcendence and algebraic independence of values of modular functions (*méthode stéphanoise* and work of Yu.V. Nesterenko).

Measures : transcendence, linear independence, algebraic independence. . .

Finite characteristic :

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Early history of irrational and transcendental numbers

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/