College of Science, October 9, 2008
Salahaddin University, Hawler (Erbil)

On the so-called Pell–Fermat Equation $x^2 - dy^2 = \pm 1$

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Institut de Mathématiques de Jussieu & CIMPA
http://www.math.jussieu.fr/∼miw/

The so-called Pell–Fermat equation

The equation $x^2 - dy^2 = \pm 1$, where the unknowns x and y are positive integers while d is a fixed positive integer which is not a square, has been mistakenly called with the name of Pell by Euler. It was investigated by Indian mathematicians since Brahmagupta (628) who solved the case $d = 92$, next by Bhaskara II (1150) for $d = 61$ and Narayana (during the 14-th Century) for $d = 103$. The smallest solution for these values of d are respectively

$$1\,151^2 - 92 \cdot 120^2 = 1, \qquad 29\,718^2 - 61 \cdot 3\,805^2 = -1$$

and

$$227\,528^2 - 103 \cdot 22\,419^2 = 1,$$

hence they have not been found by a brute force search !
After a short introduction to this long history we explain the connection with Diophantine approximation and continued fractions, next we say a few words on more recent development of the subject.

Archimedes cattle problem

The sun god had a herd of cattle consisting of bulls and cows, one part of which was white, a second black, a third spotted, and a fourth brown.

The Bovinum Problema

Among the bulls, the number of white ones was one half plus one third the number of the black greater than the brown.

The number of the black, one quarter plus one fifth the number of the spotted greater than the brown.

The number of the spotted, one sixth and one seventh the number of the white greater than the brown.

First system of equations

$B =$ white bulls, $N =$ black bulls,
$T =$ brown bulls , $X =$ spotted bulls

$$B - \left(\frac{1}{2} + \frac{1}{3}\right) N = N - \left(\frac{1}{4} + \frac{1}{5}\right) X$$
$$= X - \left(\frac{1}{6} + \frac{1}{7}\right) B = T.$$

Up to a multiplicative factor, the solution is

$$B_0 = 2226,\ N_0 = 1602,\ X_0 = 1580,\ T_0 = 891.$$

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Among the cows, the number of white ones was one third plus one quarter of the total black cattle.

The number of the black, one quarter plus one fifth the total of the spotted cattle ;

The number of spotted, one fifth plus one sixth the total of the brown cattle ;

The number of the brown, one sixth plus one seventh the total of the white cattle.

What was the composition of the herd ?

Second system of equations

$b =$ white cows, $n =$ black cows,
$t =$ brown cows, $x =$ spotted cows

$$b = \left(\frac{1}{3} + \frac{1}{4}\right)(N + n), \quad n = \left(\frac{1}{4} + \frac{1}{5}\right)(X + x),$$
$$t = \left(\frac{1}{6} + \frac{1}{7}\right)(B + b), \quad x = \left(\frac{1}{5} + \frac{1}{6}\right)(T + t).$$

Since the solutions b, n, x, t are requested to be integers, one deduces

$$(B, N, X, T) = k \times 4657 \times (B_0, N_0, X_0, T_0).$$

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Archimedes Cattle Problem

If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise.

The Bovinum Problema

But come, understand also all these conditions regarding the cattle of the Sun.

When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude.

Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking.

Arithmetic constraints

$$B+N = \text{a square},$$
$$T+X = \text{a triangular number}.$$

As a function of the integer k, we have $B+N=4Ak$ with $A=3\cdot 11\cdot 29\cdot 4657$ squarefree. Hence $k=AU^2$ with U an integer. On the other side if $T+X$ is a triangular number $(=m(m+1)/2)$, then $8(T+X)+1$ is a square $(2m+1)^2=V^2$. Writing $T+X=Wk$ with $W=7\cdot 353\cdot 4657$, we get

$$V^2-DU^2=1$$

with $D=8AW=(2\cdot 4657)^2\cdot 2\cdot 3\cdot 7\cdot 11\cdot 29\cdot 353$.

$$2\cdot 3\cdot 7\cdot 11\cdot 29\cdot 353=4\,729\,494.$$

$$D=(2\cdot 4657)^2\cdot 4\,729\,494=410\,286\,423\,278\,424.$$

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Cattle problem

If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

History

Archimedes : 287–212 AC – lettre to Eratosthenes of Cyrene

Odyssey d'Homer - the Sun God Herd

Gotthold Ephraim Lessing : 1729–1781 – Library Herzog August, Wolfenbüttel, 1773

C.F. Meyer, 1867

A. Amthor, 1880 : the smallest solution has $206\,545$ digits, starting with 776.

B. Krumbiegel and A. Amthor, Das Problema Bovinum des Archimedes, Historisch-literarische Abteilung der Zeitschrift für Mathematik und Physik, **25** *(1880), 121–136, 153–171.*

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History (continued)

A.H. Bell, The “Cattle Problem” by Archimedies 251 BC, Amer. Math. Monthly **2** (1895), 140–141.
Computation of the first 31 and last 12 decimal digits.

“Since it has been calculated that it would take the work of a thousand men for a thousand years to determine the complete number [of cattle], it is obvious that the world will never have a complete solution”
Pre-computer-age thinking from a letter to The New York Times, January 18, 1931

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I. Vardi, Archimedes' Cattle Problem, Amer. Math. Monthly **105** (1998), 305-319.

H.W. Lenstra Jr, Solving the Pell Equation, Notices of the A.M.S. **49** (2) (2002) 182–192.

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The solution

Equation $x^2 - 410\,286\,423\,278\,424 y^2 = 1$.

Print out of the smallest solution with $206\,545$ decimal digits : 47 pages (H.G. Nelson, 1980).

$$77602714 \ldots\ldots 237983357 \ldots\ldots 55081800$$

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Large numbers

A number written with only 3 digits, but having nearly 370 millions decimal digits

The number of decimal digits of 9^{9^9} *is*

$$\left\lceil 9^9 \frac{\log 9}{\log 10} \right\rceil = 369\,693\,100.$$

$10^{10^{10}}$ has $1 + 10^{10}$ decimal digits.

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Antti Nygrén, "A simple solution to Archimedes' cattle problem", University of Oulu Linnanmaa, Oulu, Finland Acta Universitatis Ouluensis Scientiae Rerum Naturalium, 2001.

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Solution of Pell's equation

H.W. Lenstra Jr,
Solving the Pell Equation,
Notices of the A.M.S.
49 (2) (2002) 182–192.

Solution of Archimedes Problem

All solutions to the cattle problem of Archimedes

$$w = 300\,426607\,914281\,713365 \cdot \sqrt{609} + 84\,129507\,677858\,393258 \cdot \sqrt{7766}$$

$$k_j = (w^{4658 \cdot j} - w^{-4658 \cdot j})^2 / 368\,238304 \qquad (j = 1,\ 2,\ 3,\ \ldots)$$

jth solution	*bulls*	*cows*	*all cattle*
white	$10\,366482 \cdot k_j$	$7\,206360 \cdot k_j$	$17\,572842 \cdot k_j$
black	$7\,460514 \cdot k_j$	$4\,893246 \cdot k_j$	$12\,353760 \cdot k_j$
dappled	$7\,358060 \cdot k_j$	$3\,515820 \cdot k_j$	$10\,873880 \cdot k_j$
brown	$4\,149387 \cdot k_j$	$5\,439213 \cdot k_j$	$9\,588600 \cdot k_j$
all colors	$29\,334443 \cdot k_j$	$21\,054639 \cdot k_j$	$50\,389082 \cdot k_j$

Figure 4.

H.W. Lenstra Jr,
Solving the Pell Equation,
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Brahmagupta (628)

Brahmasphutasiddhanta : Solve in integers the equation

$$x^2 - 92y^2 = 1$$

The smallest solution is

$$x = 1151, \qquad y = 120.$$

Composition method : *samasa*.

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Bhaskara II (12-th Century)

Lilavati Ujjain (India)

(*Bijaganita*, 1150)

$$x^2 - 61y^2 = 1$$

$$x = 1\,766\,319\,049, \qquad y = 226\,153\,980.$$

Cyclic method (Chakravala) due to Brahmagupta.

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Narayana (14-th Century)

Narayana cows (*Tom Johnson*)

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References to Indian mathematics

André Weil
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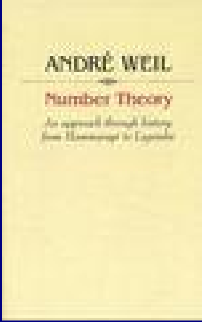
An approach through history. From Hammurapi to Legendre.
Birkhäuser Boston, Inc., Boston, Mass., (1984) 375 pp.
MR 85c :01004



History

John Pell : 1610–1685

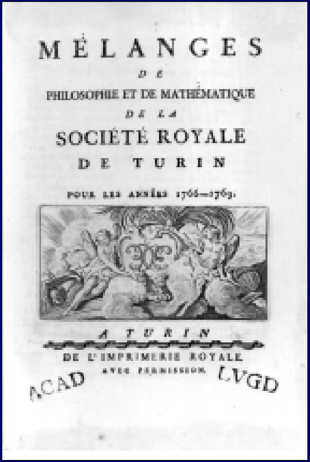
Pierre de Fermat : 1601–1665
Letter to Frenicle in 1657

Lord William Brounckner : 1620–1684

Leonard Euler : 1707–1783
Book of algebra in1770, + continued fractions

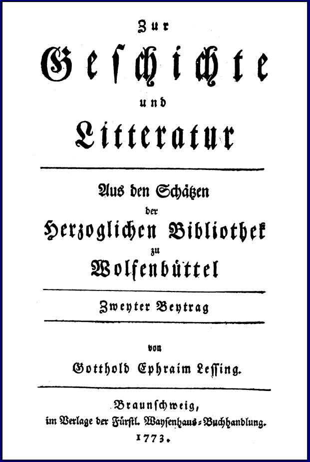
Joseph–Louis Lagrange : 1736–1813

1773 : Lagrange and Lessing



MÉLANGES
DE
PHILOSOPHIE ET DE MATHÉMATIQUE
DE LA
SOCIÉTÉ ROYALE
DE TURIN
POUR LES ANNÉES 1766–1769.

A TURIN
DE L'IMPRIMERIE ROYALE
AVEC PERMISSION.
ACAD LVGD



Zur
Geschichte
und
Litteratur

Aus den Schätzen
der
Herzoglichen Bibliothek
zu
Wolfenbüttel

Zweyter Beytrag

von
Gotthold Ephraim Lessing.

Braunschweig,
im Verlage der Fürstl. Waysenhaus-Buchhandlung.
1773.

Figures 1 and 2. Title pages of two publications from 1773. The first (far left) contains Lagrange's proof of the solvability of Pell's equation, already written and submitted in 1768. The second contains Lessing's discovery of the cattle problem of Archimedes.

The trivial solution $(x, y) = (1, 0)$

Let d be a nonzero integer. Consider the equation $x^2 - dy^2 = \pm 1$ in positive integers x and y.

The *trivial* solution is $x = 1$, $y = 0$. We are interested with nontrivial solutions.

In case $d \leq -2$, there is no nontrivial solution.

For $d = -1$ there is only $x = 0$, $y = 1$.

Assume now d is positive.

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If d is the square of an integer e, there is no nontrivial solution :

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A multiplicative group

Given two solutions (x_1, y_1) and (x_2, y_2) in rational integers, one deduces a third one (x_3, y_3) by writing

$$(x_1 + y_1\sqrt{d})(x_2 + y_2\sqrt{d}) = x_3 + y_3\sqrt{d}.$$

Also, given one solution (x, y), one deduces another one (x', y') by writing

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If there is a nontrivial solution (x_1, y_1) in positive integers, there are infinitely many of them, which are obtained by writing

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for $n = 1, 2, \ldots$.

We list the solutions by increasing values of $x + y\sqrt{d}$ (it amounts to the same to take the ordering given by x, or the one given by y).

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If one is interested to get all solutions $(x, y) \in \mathbf{Z} \times \mathbf{Z}$ of $x^2 - dy^2 = \pm 1$, one let n run over $\mathbf{Z}$ and one considers also $(x_1 - y_1\sqrt{d})^n$.

Hence the multiplicative group associated with all solutions in $\mathbf{Z} \times \mathbf{Z}$ has rank ≤ 1.

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Units of a real quadratic number field

The Dirichlet *unit theorem* for a real quadratic number field states that the group of units of $\mathbf{Q}(\sqrt{d})$ has rank one, which means that there is always a nontrivial solution (hence infinitely many of them).

The classical proof relies on Minkowski's *geometry of numbers*.

The group of units of a real quadratic number field is isomorphic to $\{\pm 1\} \times \mathbf{Z}$ *: there is a fundamental unit* $\epsilon > 1$ *such that all units can be written* $\pm\epsilon^n$ *with* $n \in \mathbf{Z}$*.*

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- If the fundamental solution $x_1^2 - dy_1^2 = \pm 1$ produces the $+$ sign, then the equation $x_1^2 - dy_1^2 = -1$ has no solution. This is the case where *the fundamental unit of the ring* $\mathbf{Z}[\sqrt{d}]$ *has norm* $+1$.

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$$x_2 = x_1^2 + dy_1^2, \qquad y_2 = 2x_1y_1.$$

The solutions of $x_1^2 - dy_1^2 = 1$ are the (x_n, y_n) with n even, the solutions of $x_1^2 - dy_1^2 = -1$ are obtained with n odd. This is the case where *the fundamental unit of the ring* $\mathbf{Z}[\sqrt{d}]$ *has norm* -1.

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Algorithm for the fundamental solution

All the problem now is to find the fundamental solution.

Here is the idea. If x, y is a solution, then the equation $x^2 - dy^2 = \pm 1$, written as

$$\frac{x}{y} - \sqrt{d} = \pm \frac{1}{y(x + y\sqrt{d})},$$

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The algorithm of continued fractions

Let $x \in \mathbf{R}$.

• Perform the Euclidean division of x by 1 :

$$x = [x] + \{x\} \quad \text{with } [x] \in \mathbf{Z} \text{ and } 0 \le \{x\} < 1.$$

• In case x is an integer, this is the end of the algorithm. If x is not an integer, then $\{x\} \neq 0$ and we set $x_1 = 1/\{x\}$, so that

$$x = [x] + \frac{1}{x_1} \quad \text{with } [x] \in \mathbf{Z} \text{ and } x_1 > 1.$$

• In the case where x_1 is an integer, this is the end of the algorithm. If x_1 is not an integer, then we set $x_2 = 1/\{x_1\}$:

$$x = [x] + \cfrac{1}{[x_1] + \cfrac{1}{x_2}} \quad \text{with } x_2 > 1.$$

Continued fraction expansion

Set $a_0 = [x]$ and $a_i = [x_i]$ for $i \geq 1$.

- Then :

$$x = [x] + \cfrac{1}{[x_1] + \cfrac{1}{[x_2] + \cfrac{1}{\ddots}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

The algorithm stops after finitely many steps if and only if x is rational.

- We shall use the notation

$$x = [a_0,\ a_1,\ a_2,\ a_3 \dots]$$

- Remark : if $a_k \geq 2$, then

$[a_0,\ a_1,\ a_2,\ a_3, \dots, a_k] = [a_0,\ a_1,\ a_2,\ a_3, \dots, a_k - 1, 1]$.

Continued fractions and rational Diophantine approximation

For

$$x = [a_0,\ a_1,\ a_2, \dots, a_k, \dots],$$

the sequence of rational numbers

$$p_k/q_k = [a_0,\ a_1,\ a_2, \dots, a_k] \qquad (k = 1, 2, \dots)$$

produces rational approximations to x, and a classical result is that there are *the best possible ones* in terms of the quality of the approximation compared with the *size of the denominator.*

Continued fractions of a positive rational integer d

Receipt : let d be a positive integer which is not a square. Then the continued fraction of the number $\sqrt{d}$ is periodic.

If k is the smallest period (that means that any period is a positive integer multiple of k), this continued fraction can be written

$$\sqrt{d} = [a_0\ ; \overline{a_1, a_2, \ldots, a_k}],$$

with $a_k = 2a_0$ and $a_0 = [\sqrt{d}]$.

Further, $(a_1, a_2, \ldots, a_{k-1})$ is a palindrom :

$$a_j = a_{k-j} \quad \text{for} \quad 1 \le j < k-1.$$

Fact : the rational number given by the continued fraction $[a_0; a_1, \ldots, a_{k-1}]$ is a good rational approximation to $\sqrt{d}$.

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Parity of the length of the palindrom

If k is even, the fundamental solution of the equation $x^2 - dy^2 = 1$ is given by the fraction

$$[a_0; a_1, a_2, \dots, a_{k-1}] = \frac{x_1}{y_1}.$$

In this case the equation $x^2 - dy^2 = -1$ has no solution.

Parity of the length of the palindrom

If k is odd, the fundamental solution (x_1, y_1) of the equation $x^2 - dy^2 = -1$ is given by the fraction

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Remark. *In both cases where k is either even or odd, we obtain all the sequence $(x_n, y_n)_{n\geq 1}$ of all solutions by repeating $n-1$ times $a_1, a_2, \dots, a_k$ followed by $a_1, a_2, \dots, a_{k-1}$.*

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The simplest Pell equation $x^2 - 2y^2 = \pm 1$

Euclides, Elements, II § 10, 300 BC. :

$$17^2 - 2 \cdot 12^2 = 289 - 2 \cdot 144 = 1.$$

$$99^2 - 2 \cdot 70^2 = 9\,801 - 2 \cdot 4900 = 1.$$

$$577^2 - 2 \cdot 408^2 = 332\,929 - 2 \cdot 166\,464 = 1.$$

Pythagorean triples

Which are the rectangle triangles with integer sides such that the two sides of the right angle are consecutive integers ?

$$x^2 + y^2 = z^2, \qquad y = x + 1.$$

$$2x^2 + 2x + 1 = z^2$$

$$(2x + 1)^2 - 2z^2 = -1$$

$$X^2 - 2Y^2 = -1$$

$1^2 - 2 \cdot 1^2 = -1$
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$x^2 - 2y^2 = \pm 1$

$$\sqrt{2} = 1,4142135623730950488016887242 \ \ldots$$

satisfies

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1}.$$

Hence the continued fraction expansion is periodic with period length 1 :

$$\sqrt{2} = [1,\ 2,\ 2,\ 2,\ 2,\ 2,\ \ldots] = [1;\ \overline{2}],$$

The fundamental solution of $x^2 - 2y^2 = -1$ is $x_1 = 1$, $y_1 = 1$

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The fundamental unit of the field $\mathbb{Q}(\sqrt{2})$ is $1+\sqrt{2}$, with norm -1.

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Pell's equation $x^2 - 2y^2 = 1$

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is $x = 3$, $y = 2$, given by

$$[1; 2] = 1 + \frac{1}{2} = \frac{3}{2}.$$

The number $3 + 2\sqrt{2} = (1 + \sqrt{2})^2$ is a unit of norm 1 in $\mathbf{Q}(\sqrt{2})$.

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The number $3 + 2\sqrt{2} = (1 + \sqrt{2})^2$ is a unit of norm 1 in $\mathbf{Q}(\sqrt{2})$.

$x^2 - 3y^2 = 1$

The continued fraction expansion of the number

$$\sqrt{3} = 1,7320508075688772935274463415\dots$$

is

$$\sqrt{3} = [1,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ \dots] = [1;\ \overline{1,\ 2}],$$

because

$$\sqrt{3} + 1 = 2 + \cfrac{1}{1 + \cfrac{1}{\sqrt{3}+1}}.$$

The fundamental solution of $x^2 - 3y^2 = 1$ is $x = 2$, $y = 1$, corresponding to

$$[1;1] = 1 + \frac{1}{1} = \frac{2}{1}.$$

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$$\sqrt{3} = 1,7320508075688772935274463415\ldots$$

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There is no unit of norm -1 in $\mathbf{Q}(\sqrt{3})$.

The period of the continued fraction

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is $[1,\ 2]$ of even length 2.

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is $[1,\ 2]$ of even length 2.

Small values of d

$$x^2 - 2y^2 = \pm 1,\ \sqrt{2} = [1;\overline{2}],\ k = 1,\ (x_1, y_1) = (1, 1),$$
$$1^2 - 2 \cdot 1^2 = -1.$$

$$x^2 - 3y^2 = \pm 1,\ \sqrt{3} = [1;\overline{1, 2}],\ k = 2,\ (x_1, y_1) = (2, 1),$$
$$2^2 - 3 \cdot 1^2 = 1.$$

$$x^2 - 5y^2 = \pm 1,\ \sqrt{5} = [2;\overline{4}],\ k = 1,\ (x_1, y_1) = (2, 1),$$
$$2^2 - 5 \cdot 1^2 = -1.$$

$$x^2 - 6y^2 = \pm 1,\ \sqrt{6} = [2;\overline{2, 4}],\ k = 2,\ (x_1, y_1) = (5, 4),$$
$$5^2 - 6 \cdot 2^2 = 1.$$

$$x^2 - 7y^2 = \pm 1,\ \sqrt{7} = [2;\overline{1, 1, 1, 4}],\ k = 4,\ (x_1, y_1) = (8, 3),$$
$$8^2 - 7 \cdot 3^2 = 1.$$

$$x^2 - 8y^2 = \pm 1,\ \sqrt{8} = [2;\overline{1, 4}],\ k = 2,\ (x_1, y_1) = (3, 1),$$
$$3^2 - 8 \cdot 1^2 = 1.$$

Brahmagupta's Problem (628)

The continued fraction expansion of $\sqrt{92}$ is

$$\sqrt{92} = [9; \overline{1,\ 1,\ 2,\ 4,\ 2,\ 1,\ 1,\ 18}].$$

The fundamental solution of the equation $x^2 - 92y^2 = 1$ is given by

$$[9;\ 1,\ 1,\ 2,\ 4,\ 2,\ 1,\ 1] = \frac{1151}{120}.$$

Indeed, $1151^2 - 92 \cdot 120^2 = 1\,324\,801 - 1\,324\,800 = 1.$

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Narayana's equation $x^2 - 103y^2 = 1$

$$\sqrt{103} = [10; \overline{6,\ 1,\ 2,\ 1,\ 1,\ 9,\ 1,\ 1,\ 2,\ 1,\ 6,\ 20}]$$

$$[10; 6,\ 1,\ 2,\ 1,\ 1,\ 9,\ 1,\ 1,\ 2,\ 1,\ 6] = \frac{227\,528}{22\,419}$$

Fundamental solution : $x = 227\,528$, $y = 22\,419$.

$$227\,528^2 - 103 \cdot 22\,419^2 = 51\,768\,990\,784 - 51\,768\,990\,783 = 1.$$

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Equation of Bhaskhara II $x^2 - 61y^2 = \pm 1$

$$\sqrt{61} = [7; \overline{1,\ 4,\ 3,\ 1,\ 2,\ 2,\ 1,\ 3,\ 4,\ 1,\ 14}]$$

$$[7; 1, 4, 3, 1, 2, 2, 1, 3, 4, 1] = \frac{29\ 718}{3\ 805}$$

$29\ 718^2 = 883\ 159\ 524, \qquad 61 \cdot 3805^2 = 883\ 159\ 525$
is the fundamental solution of $x^2 - 61y^2 = -1$.

The fundamental solution of $x^2 - 61y^2 = 1$ is

$$[7; 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1] = \frac{1\,766\,319\,049}{226\,153\,980}$$

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Correspondence from Fermat to Brounckner

" *pour ne vous donner pas trop de peine*" (Fermat)

$$X^2 - DY^2 = 1, \text{ with } D = 61 \text{ and } D = 109.$$

Solutions respectively :

$$(1\,766\,319\,049\ ,\ 226\,153\,980)$$
$$(158\,070\,671\,986\,249,\ 15\,140\,424\,455\,100)$$

$$158\,070\,671\,986\,249 + 15\,140\,424\,455\,100\sqrt{109} = \left(\frac{261 + 25\sqrt{109}}{2}\right)^6.$$

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Around 2008

For $d = 2007$ the smallest solution is

$$224^2 - 2007 \cdot 5^2 = 1$$

.

For $d = 2005$, 2006, 2008 and 2009 the solutions are huge.

After that, for 2010, they become reasonable :

$$269^2 - 2010 \cdot 6^2 = 1$$

.

wims : interactive server

reference : `http ://wims.unice.fr/wims/`

Accueil WIMS | Références | Aide | A propos | Aides WIMS

Contfrac

Développement en fraction continue de n = sqrt(2008) :

$44.8107130048161585821765948748832308407094151495360665937 61 = 44 + {}^1/_{1}+ {}^1/_{4}+ {}^1/_{3}+ {}^1/_{1}+ {}^1/_{1}+ {}^1/_{6}+ {}^1/_{1}+ {}^1/_{9}+ {}^1/_{11}+ {}^1/_{9}+ {}^1/_{1}+ {}^1/_{6}+ {}^1/_{1}+ {}^1/_{1}+ {}^1/_{3}+ {}^1/_{4}+ {}^1/_{1}+ {}^1/_{88}+ {}^1/_{1}+ {}^1/_{4}+ {}^1/_{3}+ {}^1/_{1}+ {}^1/_{1}+ {}^1/_{6}+ {}^1/_{1}+ {}^1/_{9}+ {}^1/_{11}+ {}^1/_{9}+ {}^1/_{1}+ {}^1/_{6}+ {}^1/_{1}+ {}^1/_{1}+ {}^1/_{3}+ {}^1/_{4}+ {}^1/_{1}+ {}^1/_{88}+ {}^1/_{1}+ {}^1/_{4}+ {}^1/_{3}+ {}^1/_{1}+ {}^1/_{1}+ {}^1/_{6}+ {}^1/_{1}+ {}^1/_{9}+ {}^1/_{11}+ {}^1/_{9}+ {}^1/_{1}+ {}^1/_{6}+ {}^1/_{1}+ \cdots$

Avec javascript, placer la souris sur un dénominateur fera afficher le convergent du terme correspondant (précision limitée) :

The continued fraction expansion is computed with PARI version 2.2.1.

Pell's equation $x^2 - 2008y^2 = \pm 1$

Développement en fraction continue de n = 3832352837/85523139 :

3832352837/85523139 = 44 + 1/1+ 1/4+ 1/3+ 1/1+ 1/1+ 1/6+ 1/1+ 1/9+ 1/11+ 1/9+ 1/1+ 1/6+ 1/1+ 1/1+ 1/3+ 1/5

$$x = 3\,832\,352\,837, \qquad y = 85\,523\,139.$$

$$3\,832\,352\,837^2 - 2008 \cdot 85\,523\,139^2 = 1.$$

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Back to Archimedes

$$x^2 - 410\,286\,423\,278\,424y^2 = 1$$

Computation of the continued fraction of $\sqrt{410\,286\,423\,278\,424}$.

In 1867, C.F. Meyer performed the first 240 steps of the algorithm and then gave up.

The *length of the period* has now be computed : it is $203\,254$.

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Solution by Amthor – Lenstra

$d = (2 \cdot 4657)^2 \cdot d' \qquad d' = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 29 \cdot 353.$

Length of the period for $\sqrt{d'}$: 92.

Fundamental unit : $u = x' + y'\sqrt{d'}$

$$u = \Big(300\,426\,607\,914\,281\,713\,365 \cdot \sqrt{609} + 84\,129\,507\,677\,858\,393\,258\sqrt{7766}\Big)^2$$

Fundamental solution of the Archimedes equation :

$$x_1 + y_1\sqrt{d} = u^{2329}.$$

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Length of the period and regulator

Estimating the length L_d of the period in terms of d :

$$\frac{\log 2}{2} L_d \le R_d \le \frac{\log(4d)}{2} L_d, \qquad R_d = \log(x_1 + y_1\sqrt{d})$$

with

$$\log(2\sqrt{d}) < R_d < \sqrt{d}\bigl(\log(4d) + 2\bigr).$$

Any method for solving the Pell–Fermat equation which requires to produce the digits of the fundamental solution has an exponential complexity.

R_d is the regulator of the kernel of the norm

$$(\mathbf{Z}[\sqrt{d}])^\times \to \mathbf{Z}^\times = \{\pm 1\}$$

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Riemannian varieties with negative curvature

Arithmetic varieties
Nicolas Bergeron (Paris VI) : "Topologies arising from arithmetic constructions"

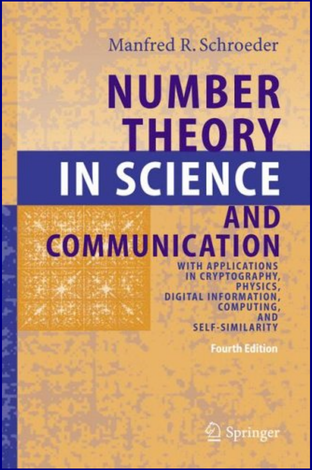
Substitutions in Christoffel's word

J. Riss, 1974

J-P. Borel et F. Laubie, Quelques mots sur la droite projective réelle ; Journal de Théorie des Nombres de Bordeaux, **5** 1 (1993), 23–51

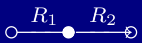
Number Theory in Science and communication

M.R. Schroeder.
Number theory in science and communication :
with applications in cryptography, physics, digital information, computing and self similarity
Springer series in information sciences **7** 1986.
4th ed. (2006) 367 p.



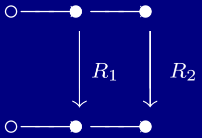
Electric networks

- The resistance of a network in series



is the sum $R_1 + R_2$.

- The resistance R of a network in parallel

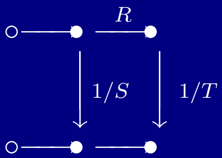


satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Electric networks and continued fractions

The resistance U of the circuit



is given by

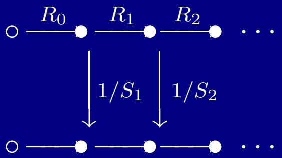
$$U = \cfrac{1}{S + \cfrac{1}{R + \cfrac{1}{T}}}$$

Decomposition of a square in squares

• The resistance of the network belo is given by a continued fraction expansion

$$[R_0; S_1, R_1, S_2, R_2 \dots]$$

pour le circuit



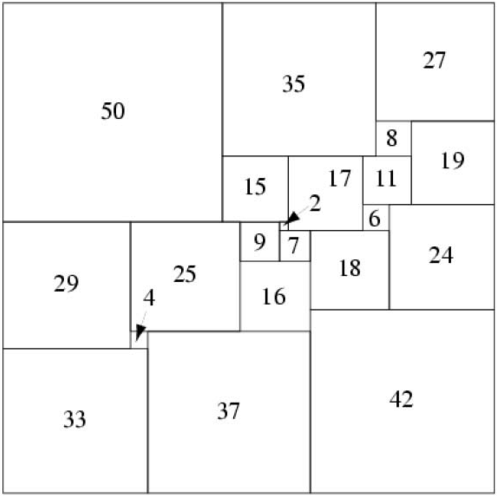
R_i : resistances in series

$1/S_j$: resistances in parallel

• For instance, for $R_i = S_j = 1$, we obtain the quotients of consecutive Fibonacci numbers.

• Electric networks and continued fraction have been used to find the first solution to the problem of decomposing an integer square into a disjoint union of integer squares, all of which are distinct.

Squaring the square



21-*square perfect square*

There is a unique simple perfect square of order 21 (the lowest possible order), discovered in 1978 by A. J. W. Duijvestijn (Bouwkamp and Duijvestijn 1992). It is composed of 21 squares with total side length 112, and is illustrated above.