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Complex Analysis MMA 106

Exercises *02/05/2008*

In these exercises, log *is always the function defined in the open disc of center* 1 *and radius* 1 *by the power series*

$$\log z = -\sum_{n\geq 1} \frac{(1-z)^n}{n}\cdot$$

For $t\in\mathbf{C}$ *and* $|z-1|<1$ *the notation* z^t *stands for* $e^{t\log z}$.

For z_0 *a complex numbers and* r *a positive real number, we denote by* $C(z_0,r)$ *the map* $t\mapsto z_0+re^{2i\pi t}$ *from* $[0,1]$ *onto the circle of center* z_0 *and radius* r. *We also write*

$$\int_{|z-z_0|=r} f(z)dz \quad \textit{in place of} \quad \int_{C(z_0,r)} f(z)dz.$$

1. Solve the following equations (i.e. find all $z\in\mathbf{C}$ satisfying the given equation)

$$z^2+(1+i)z-i=0,\quad e^z=\log 2,\quad e^z=(1+i)/2,\quad e^z=0,\quad \sin z=0,\quad \cos z=0.$$

2. How many $z\in\mathbf{C}$ are there such that $z^4-z^3=1$?
More generally, for $n\geq 1$ a positive integer and $t\in\mathbf{C}$, how many $z\in\mathbf{C}$ are there such that $z^{n+1}-z^n=t$?

3. Show that the set of 2×2 matrices

$$\left\{\begin{pmatrix} a & b\\ -b & a\end{pmatrix}\;;\;(a,b)\in\mathbf{R}\times\mathbf{R}\right\}$$

is a field for the usual rules on matrices. Which field is-it ?

4. Check that the composition $f\circ g$ of two functions which are complex differentiable is also complex differentiable. Write the result carefully.

5. Let α and z be complex numbers with $z\overline{\alpha}\neq 1$. Set

$$u=\frac{z-\alpha}{1-\overline{\alpha}z}\cdot$$

Depending on $|z|$ and $|\alpha|$, decide whether $|u|$ is >1 or $=1$ or else <1.

6. What is $\log((1+i)/2)$?

7. What is $(1/2)^{i\pi/\log 2}$?

8. Let $t \in \mathbf{C}$ and let f be the function defined in the unit disc by $f(z) = (1-z)^t$. Show that f is complex differentiable and satisfies the differential equation

$$-(1-z)f'(z) = tf(z).$$

9. Which are the power series $\sum_{n\geq 1} a_n z^n$ solutions of $f'(z) = e^{f(z)}$? Compute the radius of convergence.

10. Find the unique solution y of the differential equation $y'' + y = 0$ for each of the following initial conditions
a) $y(0) = 1, \quad y'(0) = 0.$
b) $y(0) = 0, \quad y'(0) = 1.$
c) $y(0) = 1, \quad y'(0) = i.$
d) $y(0) = 1, \quad y'(0) = -i.$

11. For each of the open subsets U below, decide whether U is connected or not.
a) For z_1, z_2 two complex numbers and r_1, r_2 two positive real numbers,

$$U = D(z_1, r_1) \cup D(z_2, r_2).$$

b) For a_1, a_2, b_1, b_2 real numbers,

$$U = \left(\left\{z \,;\, \Re e(z) > a_1\right\} \cap \left\{z \,;\, \Im m(z) > b_1\right\}\right) \cup \left(\left\{z \,;\, \Re e(z) < a_2\right\} \cap \left\{z \,;\, \Im m(z) < b_2\right\}\right).$$

12. What is the order of multiplicity at $z = 0$ of the following functions ?
a) $\sin z - z\cos z$.
b) $e^z + 1 + \log(1+z)$
c) $e^z - 1 - \log(1+z)$

13. Show that there is a unique polynomial P of degree 2 such that the function

$$(z^2 - 4z + 6)e^z - P(z)$$

has a zero of multiplicity 4 at the origin. Find it.

14. Write the polygonal path going from 0 to 1 to $(1+i\sqrt{3})/2$ and back to 0 (equilateral triangle).
Write the polygonal path which is the regular polygon with n sides centered at 0 starting at 1.

15. Let $u : [0,1] \to \mathbf{C}$ be a C^1 function. Check

$$\left| \int_0^1 u(t)dt \right| \le \int_0^1 |u(t)|dt.$$

16. Compute, for $n \in \mathbf{Z}$,

$$\int_{|z|=1} \frac{e^z}{z^n} dz, \qquad \int_{|z|=1} \frac{\cos(z)}{z^n} dz, \qquad \int_{|z|=1} \frac{\sin(z)}{z^n} dz.$$

17. Let $n \in \mathbf{Z}$ be any integer. Let δ be a closed regular polygonal line centered at 0. Check the equality between the two integrals

$$\int_{|z|=1} z^n dz = \int_\delta z^n dz.$$

`Hint.` *The case $n \neq -1$ is easy and does not require any computation. For the case $n = -1$, use the fact that the integral along the closed path, which consists of a side of the polygon and the corresponding arc on the circle, is* 0.

18. Let z_0 and z_1 be complex numbers and r a positive real number. Assume $|z_1 - z_0| \neq r$. Check

$$\int_{|z-z_0|=r} \frac{dz}{z - z_1} = \begin{cases} 0 & \text{if } |z_1 - z_0| > r, \\ 2i\pi & \text{if } |z_1 - z_0| < r. \end{cases}$$

`Hint.` *The case $|z_1 - z_0| > r$ is easy and does not require any computation. The case $z_1 = z_0$ is easy. For the case $0 < |z_1 - z_0| < r$, write $1/(z - z_1)$ by means of the geometric series in $(z_1 - z_0)/(z - z_0)$.*