HARDY-RAMANUJAN JOURNAL

(A Journal devoted to primes, diophantine equations, transcendental numbers and other questions on 1,2,3,4,5,...)

VOLUME 28
2005
Date of issue: 22.12.2005
(To be put on the internet around this time)

Hardy Ramanujan Journal
Vol. 28(2005) 1-9

Further Variations on the Six Exponentials Theorem

Michel Waldschmidt

Abstract. Let $\widetilde{\mathcal{L}}$ denote the set of linear combinations, with algebraic coefficients, of 1 and logarithms of algebraic numbers. The Strong Six Exponentials Theorem of D. Roy gives sufficient conditions for a 2×3 matrix

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}$$

whose entries are in $\widetilde{\mathcal{L}}$ to have rank 2.

Here we give sufficient conditions so that one at least of the three 2×2 determinants

$$\begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix}, \quad \begin{vmatrix} \Lambda_{12} & \Lambda_{13} \\ \Lambda_{22} & \Lambda_{23} \end{vmatrix}, \quad \begin{vmatrix} \Lambda_{13} & \Lambda_{11} \\ \Lambda_{23} & \Lambda_{21} \end{vmatrix}$$

is not in $\widetilde{\mathcal{L}}$

1. Main result

We denote by $\mathbb{Q}$ the field of rational numbers, by $\overline{\mathbb{Q}}$ the field of algebraic numbers (algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$), by $\mathcal{L}$ the $\mathbb{Q}$-vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \{\lambda \in \mathbb{C}\,;\ e^\lambda \in \overline{\mathbb{Q}}^\times\} = \{\log \alpha\,;\ \alpha \in \overline{\mathbb{Q}}^\times\} = \exp^{-1}(\overline{\mathbb{Q}}^\times)$$

and by $\widetilde{\mathcal{L}}$ the $\overline{\mathbb{Q}}$-vector subspace of $\mathbb{C}$ spanned by $\{1\} \cup \mathcal{L}$. Hence $\widetilde{\mathcal{L}}$ is the set of linear combinations of 1 and logarithms of algebraic numbers with algebraic coefficients:

$$\widetilde{\mathcal{L}} = \Big\{\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n;\\ n \geq 0,\, (\alpha_1, \ldots, \alpha_n) \in (\overline{\mathbb{Q}}^\times)^n,\ (\beta_0, \beta_1, \ldots, \beta_n) \in \overline{\mathbb{Q}}^{n+1}\Big\}.$$

Here is the so-called strong six exponentials Theorem of D. Roy (([**5**] Corollary 2 §4 p. 38; see also [**7**] Corollary 11.16):

Key words and phrases. Transcendental numbers, logarithms of algebraic numbers, four exponentials Conjecture, six exponentials Theorem, algebraic independence.

Acknowledgements: A suggestion by D. Roy in Banff in november 2004 turned out to be a key point in the proof of the main result. Thanks also to him and to Guy Diaz for their comments on previous versions of this text.

THEOREM 1.1. *Let M be a 2×3 matrix with entries in $\widetilde{\mathcal{L}}$:*

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}.$$

Assume that the two rows of M are linearly independent over $\overline{\mathbb{Q}}$ and also that the three columns are linearly independent over $\overline{\mathbb{Q}}$. Then M has rank 2.

Consider the three 2×2 determinants

$$\Delta_1 = \Lambda_{12}\Lambda_{23} - \Lambda_{13}\Lambda_{22}, \ \Delta_2 = \Lambda_{13}\Lambda_{21} - \Lambda_{11}\Lambda_{23}, \ \Delta_3 = \Lambda_{11}\Lambda_{22} - \Lambda_{12}\Lambda_{21}.$$

From the relation

$$\Delta_1 \begin{pmatrix} \Lambda_{11} \\ \Lambda_{21} \end{pmatrix} + \Delta_2 \begin{pmatrix} \Lambda_{12} \\ \Lambda_{22} \end{pmatrix} + \Delta_3 \begin{pmatrix} \Lambda_{13} \\ \Lambda_{23} \end{pmatrix} = 0,$$

it follows from the assumptions of Theorem 1.1 that one at least of the three numbers $\Delta_1, \Delta_2, \Delta_3$ is transcendental. We want to prove that one at least of these three numbers is not in $\widetilde{\mathcal{L}}$.

If the five rows of the matrix $\begin{pmatrix} M \\ I_3 \end{pmatrix}$ (where I_3 is the 3×3 identity matrix) are linearly dependent over $\overline{\mathbb{Q}}$, which means that there exists $(\gamma_1, \gamma_2) \in \overline{\mathbb{Q}}^2 \setminus \{0\}$ such that the three numbers

$$\delta_j = \gamma_1 \Lambda_{1j} + \gamma_2 \Lambda_{2j} \quad (j = 1, 2, 3)$$

are algebraic, then the three numbers $\Delta_1, \Delta_2, \Delta_3$ are in $\widetilde{\mathcal{L}}$. Indeed, if (j, h, k) denotes any of the triples $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, then

$$\gamma_1 \Delta_j = \delta_h \Lambda_{2k} - \delta_k \Lambda_{2h} \quad \text{and} \quad \gamma_2 \Delta_j = \delta_k \Lambda_{1h} - \delta_h \Lambda_{1k}.$$

Here is the main result of this paper.

THEOREM 1.2. *Let M be a 2×3 matrix with entries in $\widetilde{\mathcal{L}}$:*

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}.$$

Assume that the five rows of the matrix

$$\begin{pmatrix} M \\ I_3 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are linearly independent over $\overline{\mathbb{Q}}$ *and that the five columns of the matrix*

$$(I_2, M) = \begin{pmatrix} 1 & 0 & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ 0 & 1 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}$$

are linearly independent over $\overline{\mathbb{Q}}$*. Then one at least of the three numbers*

$$\Delta_1 = \begin{vmatrix} \Lambda_{12} & \Lambda_{13} \\ \Lambda_{22} & \Lambda_{23} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \Lambda_{13} & \Lambda_{11} \\ \Lambda_{23} & \Lambda_{21} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix}$$

is not in $\widetilde{\mathcal{L}}$*.*

If M is $d \times \ell$ matrix of rank 1, with $d \geq 2$ and $\ell \geq 2$, whose columns are $\overline{\mathbb{Q}}$-linearly independent, then the $d + \ell$ columns of the matrix $(I_d \ M)$ are also $\overline{\mathbb{Q}}$-linearly independent. Hence on the one hand Theorem 1.2 generalizes Theorem 1.1. On the other hand, as noticed by G. Diaz, when one of the six numbers Λ_{ij} is algebraic, Theorem 1.2 reduces to the next consequence of Theorem 1.1 (further related results are given in [**1**] and [**8**]).

COROLLARY 1.3. *Let* Λ_1, Λ_2, Λ_3 *be three elements of* $\widetilde{\mathcal{L}}$ *Assume that* Λ_1 *is transcendental and that the three numbers* 1, Λ_2, Λ_3 *are* $\overline{\mathbb{Q}}$*-linearly independent. Then one at least of the two numbers* $\Lambda_1\Lambda_2$, $\Lambda_1\Lambda_3$ *is not in* $\widetilde{\mathcal{L}}$*.*

The simple example

$$M = \begin{pmatrix} 0 & \Lambda_2 & \Lambda_3 \\ \Lambda_1 & 0 & 0 \end{pmatrix}.$$

shows that the assumptions of Theorem 1.2 are not sufficient to ensure that none of the three determinants is in $\widetilde{\mathcal{L}}$.

Here is a simple result which follows from Theorem 1.2: *Let* $\Lambda_1, \Lambda_2, \Lambda_3$ *be three elements in* $\widetilde{\mathcal{L}}$ *such that* $1, \Lambda_1, \Lambda_2, \Lambda_3$ *are linearly independent over* $\overline{\mathbb{Q}}$*. Then one at least of the three numbers*

$$\Lambda_1^2 - \Lambda_2\Lambda_3, \quad \Lambda_2^2 - \Lambda_3\Lambda_1, \quad \Lambda_3^2 - \Lambda_1\Lambda_2$$

is not in $\widetilde{\mathcal{L}}$*.*

In §3 we shall deduce from Theorem 1.2 the following corollary.

COROLLARY 1.4. *Let* M *be a* 2×3 *matrix with entries in* $\mathcal{L}$*:*

$$M = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \end{pmatrix}.$$

Assume that the two rows of M *are linearly independent over* $\mathbb{Q}$ *and also that the three columns of* M *are linearly independent over* $\mathbb{Q}$*. Then one at* *least of the three numbers*

$$\lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21},\quad \lambda_{12}\lambda_{23}-\lambda_{13}\lambda_{22},\quad \lambda_{13}\lambda_{21}-\lambda_{11}\lambda_{23} \tag{1.5}$$

is not in $\widetilde{\mathcal{L}}$.

The six exponentials Theorem of S. Lang ([**3**], Chap. II § 1) and K. Ramachandra ([**4**] II § 4) states that, under the assumptions of Corollary 1.4, one at least of the three numbers (1.5) is not zero.

It is expected that a result similar to Theorem 1.2 holds when M is replaced by a 2×2 matrix:

CONJECTURE 1.6. *Let* M *be a* 2×2 *matrix with entries in* $\widetilde{\mathcal{L}}$*:*

$$M=\begin{pmatrix}\Lambda_{11} & \Lambda_{12}\\ \Lambda_{21} & \Lambda_{22}\end{pmatrix}.$$

Assume that the four rows of the matrix

$$\begin{pmatrix}M\\ I_2\end{pmatrix}=\begin{pmatrix}\Lambda_{11} & \Lambda_{12}\\ \Lambda_{21} & \Lambda_{22}\\ 1 & 0\\ 0 & 1\end{pmatrix}$$

are linearly independent over $\overline{\mathbb{Q}}$ *and that the four columns of the matrix*

$$\bigl(I_2, M\bigr)=\begin{pmatrix}1 & 0 & \Lambda_{11} & \Lambda_{12}\\ 0 & 1 & \Lambda_{21} & \Lambda_{22}\end{pmatrix}$$

are linearly independent over $\overline{\mathbb{Q}}$*. Then the number*

$$\Delta=\begin{vmatrix}\Lambda_{11} & \Lambda_{12}\\ \Lambda_{21} & \Lambda_{22}\end{vmatrix}$$

is not in $\widetilde{\mathcal{L}}$.

Conjecture 1.6 follows from the conjecture (see for instance [**3**], Historical Note of Chapter III, [**2**], Chap. 6 p. 259 and [**7**], Conjecture 1.15 and [**8**] Conjecture 1.1) that $\mathbb{Q}$-linearly independent logarithms of algebraic numbers are algebraically independent.

2. A consequence of the Linear Subgroup Theorem

Let n be a positive integer and Y a $\overline{\mathbb{Q}}$-vector subspace of $\mathbb{C}^n$. We define

$$\mu(Y,\mathbb{C}^n)=\min_{V\subset\mathbb{C}^n}\frac{\dim_{\overline{\mathbb{Q}}}\bigl(Y/Y\cap V\bigr)}{\dim_{\mathbb{C}}\bigl(\mathbb{C}^n/V\bigr)},$$

where V runs over the set of $\mathbb{C}$-vector subspaces of $\mathbb{C}^n$ with $V\neq\mathbb{C}^n$.

For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ we denote by $x \cdot y$ the scalar product

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n.$$

For X and Y two subsets of $\mathbb{C}^n$, we denote by $X \cdot Y$ the set of scalar products $x \cdot y$ where x ranges over the set X and y over Y.

THEOREM 2.1. *Let X and Y be two $\overline{\mathbb{Q}}$-vector subspaces of $\mathbb{C}^n$. Assume X has dimension d with $d > n$. Assume further*

$$\mu(Y, \mathbb{C}^n) > \frac{d}{d-n}\cdot$$

Then the set $X \cdot Y$ is not contained in $\widetilde{\mathcal{L}}$.

PROOF. This is essentially Proposition 6.1 of [**6**], where $\mathbb{Q}$ is replaced by $\overline{\mathbb{Q}}$ and the $\mathbb{Q}$-vector space $\mathcal{L}$ by the $\overline{\mathbb{Q}}$-vector space $\widetilde{\mathcal{L}}$. Henceforth the proof runs as follows.

Like in Lemme 5.2 of [**6**], one checks that if X and Y are two vector subspaces of $\mathbb{C}^n$ over $\overline{\mathbb{Q}}$, of dimensions d and ℓ respectively, then there exist a positive integer $n' \leq n$ and two vector subspaces X' and Y' of $\mathbb{C}^{n'}$, of dimensions d' and ℓ' respectively, such that

$$\mu(X', \mathbb{C}^{n'}) = \frac{d'}{n'} \geq \frac{d}{n}, \quad \mu(Y', \mathbb{C}^{n'}) = \frac{\ell'}{n'} \geq \mu(Y, \mathbb{C}^n)$$

and

$$X' \cdot Y' \subset X \cdot Y. \tag{2.2}$$

This shows that for the proof of Theorem 2.1, there is no loss of generality to assume $\mu(X, \mathbb{C}^n) = d/n$ and $\mu(Y, \mathbb{C}^n) = \ell/n$. The assumption $\mu(Y, \mathbb{C}^n) > d/(d-n)$ reduces to $\ell d > n(\ell + d)$.

Following the argument of Lemme 5.4 in [**6**], one proves that if X and Y are two vector subspaces of $\mathbb{C}^n$ over $\overline{\mathbb{Q}}$, of dimensions d and ℓ respectively, X_1 a subspace of X of dimension d_1 and Y_1 a subspace of Y of dimension ℓ_1 such that $X_1 \cdot Y_1 = \{0\}$, then

$$(d - d_1)\mu(Y, \mathbb{C}^n) + (\ell - \ell_1)\mu(X, \mathbb{C}^n) \geq n\mu(X, \mathbb{C}^n)\mu(Y, \mathbb{C}^n). \tag{2.3}$$

In Lemme 5.4 in [**6**] an extra assumption is required, namely

$$\mu(X, \mathbb{C}^n)\mu(Y, \mathbb{C}^n) \geq \mu(X, \mathbb{C}^n) + \mu(Y, \mathbb{C}^n),$$

but we do not need it here, since our assumption $X_1 \cdot Y_1 = \{0\}$ is stronger than the assumption in Lemme 5.4 of [**6**] that $X_1 \cdot Y_1$ has rank ≤ 1.

Next we introduce the coefficient $\theta(M)$ attached to a $d \times \ell$ matrix M with entries in $\mathbb{C}$. It is defined as follows:

$$\theta(M) = \min \frac{\ell'}{d'},$$

where (d', ℓ') ranges over the set of pairs of integers satisfying $0 \le \ell' \le \ell$, $1 \le d' \le d$, such that there exist a $d \times d$ regular matrix P and a regular $\ell \times \ell$ regular matrix Q, both with entries in $\overline{\mathbb{Q}}$, with

$$PMQ = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{matrix} \}d^* \\ \}d' \end{matrix}$$
$$\qquad\qquad \underbrace{}_{\ell'} \; \underbrace{}_{\ell^*}$$

From (2.3) with $d_1 = d'$ and $\ell_1 = \ell^*$ it follows that if

$$X = \overline{\mathbb{Q}}x_1 + \cdots + \overline{\mathbb{Q}}x_d \quad \text{and} \quad Y = \overline{\mathbb{Q}}y_1 + \cdots + \overline{\mathbb{Q}}y_\ell$$

are again two vector subspaces of $\mathbb{C}^n$ over $\overline{\mathbb{Q}}$, of dimensions d and ℓ respectively, satisfying $\mu(X, \mathbb{C}^n) = d/n$, then the matrix

$$M = \bigl(x_i \cdot y_j\bigr)_{1 \le i \le d,\, 1 \le j \le \ell} \tag{2.4}$$

has

$$\theta(M) \ge \frac{n}{d} \cdot \mu(Y, \mathbb{C}^n).$$

In particular if $\mu(X, \mathbb{C}^n) = d/n$ and $\mu(Y, \mathbb{C}^n) = \ell/n$, then $\theta(M) = \ell/d$. Finally Theorem 4 in [**5**] (which is Proposition 11.19 or Theorem 12.19 in [**7**]) shows that the rank r of a $d \times \ell$ matrix M with entries in $\widetilde{\mathcal{L}}$ satisfies

$$r \ge \frac{d\theta}{1 + \theta},$$

where $\theta = \theta(M)$. Using this result for the matrix M given by (2.4) whose rank r is $\le n$, one concludes that if $\mu(X, \mathbb{C}^n) = d/n$ and $\mu(Y, \mathbb{C}^n) = \ell/n$ with $X \cdot Y \subset \widetilde{\mathcal{L}}$, then

$$n \ge \frac{\ell d}{\ell + d}.$$

Theorem 2.1 follows. □

Remark. Theorem 1.1 is equivalent with the case $n = 1$ of Theorem 2.1.

3. Proof of the main results

In this section we prove Theorem 1.2 and Corollary 1.4.

PROOF OF THEOREM 1.2. Assume that the hypotheses of Theorem 1.2 are satisfied. Define elements $v_1, \ldots, v_5$ in $\widetilde{\mathcal{L}}^2$ by

$$v_1 = e_1, \quad v_2 = e_2, \qquad v_{2+j} = (\Lambda_{1j}, \Lambda_{2j}), \quad (j = 1, 2, 3),$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For $v = (x, y) \in \mathbb{C}^2$, set $v' = (-y, x)$, so that $v' \cdot v = 0$. Consider the 5×5 matrix

$$A = \bigl(v'_i \cdot v_j\bigr)_{1 \le i,j \le 5}.$$

From its very definition, it is plain that A has rank 2. Explicitly one has

$$A = \begin{pmatrix} 0 & 1 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ -1 & 0 & -\Lambda_{11} & -\Lambda_{12} & -\Lambda_{13} \\ -\Lambda_{21} & \Lambda_{11} & 0 & \Delta_3 & -\Delta_2 \\ -\Lambda_{22} & \Lambda_{12} & -\Delta_3 & 0 & \Delta_1 \\ -\Lambda_{23} & \Lambda_{13} & \Delta_2 & -\Delta_1 & 0 \end{pmatrix}.$$

Let X be the $\overline{\mathbb{Q}}$-vector space spanned by $v_1, \ldots, v_5$ in $\mathbb{C}^2$ and similarly let Y be the subspace of $\mathbb{C}^2$ spanned by $v'_1, \ldots, v'_5$ over $\overline{\mathbb{Q}}$. We claim

$$\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) \ge 2. \tag{3.1}$$

The equality $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2)$ follows from the fact that the map $(x, y) \mapsto (-y, x)$ is an automorphism of $\mathbb{C}^2$.

Since the five columns of $\bigl(I_2 \;\; M\bigr)$ are linearly independent over $\overline{\mathbb{Q}}$, $\dim_{\overline{\mathbb{Q}}} X = 5$.

Let V is a vector subspace of $\mathbb{C}^2$ of dimension 1 and let $t_1 z_1 + t_2 z_2 = 0$ be an equation of V in $\mathbb{C}^2$, with $(t_1, t_2) \in \mathbb{C}^2 \setminus \{0\}$. Consider the linear map

$$\begin{array}{rccc} p: & \mathbb{C}^2 & \to & \mathbb{C} \\ & (z_1, z_2) & \mapsto & t_1 z_1 + t_2 z_2 \end{array}$$

whose kernel is V. Since the five rows of $\begin{pmatrix} M \\ I_3 \end{pmatrix}$ are $\overline{\mathbb{Q}}$-linearly independent,

$$\dim_{\overline{\mathbb{Q}}}\bigl((X \cap V)/V\bigr) = \dim_{\overline{\mathbb{Q}}} p(X) \ge 2.$$

This completes the proof of (3.1).

From (3.1) we deduce that the hypothesis $\mu(Y, \mathbb{C}^2) > d/(d - n)$ of Theorem 2.1 is satisfied with $d = 5$ and $n = 2$, hence the set $X \cdot Y$ is not contained in $\widetilde{\mathcal{L}}$. Consequently one at least of the three numbers $\Delta_1, \Delta_2, \Delta_3$ is not in $\widetilde{\mathcal{L}}$.

This completes the proof of the Main Theorem 1.2. □

Remark. In (3.1) we may have equality: for instance if $\Lambda_{22} = \Lambda_{23} = 0$ then $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) = 2$.

However the proof of Theorem 2.1 shows that in the case $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) < 5/2$, Theorem 1.2 should follow from Theorem 1.1. Indeed after a change of variables rational over $\overline{\mathbb{Q}}$ one needs only to consider a matrix

$$M = \begin{pmatrix} 0 & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & 0 & 0 \end{pmatrix},$$

which is the situation of Corollary 1.3. If X is the $\overline{\mathbb{Q}}$-subspace of $\mathbb{C}^2$ spanned by

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (0, \Lambda_{21}), \quad v_4 = (\Lambda_{12}, 0), \quad v_4 = (\Lambda_{13}, 0)$$

and Y the subspace spanned by

$$v_1' = (0, 1),\ v_2' = (-1, 0),\ v_3' = (-\Lambda_{21}, 0),\ v_4' = (0, \Lambda_{12}),\ v_4' = (0, \Lambda_{13}),$$

then

$$X' = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\Lambda_{12} + \overline{\mathbb{Q}}\Lambda_{13} \quad \text{and} \quad Y' = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\Lambda_{21}$$

are $\overline{\mathbb{Q}}$-subspaces of $\mathbb{C}$ satisfying (2.2). Here $\mu(X', \mathbb{C}) = 3 > d/n = 5/2$ and $\mu(Y', \mathbb{C}) = 2 = \mu(Y, \mathbb{C}^2)$.

Proof of Corollary 1.4. From Baker's Theorem it follows that if Y_0 is a $\mathbb{Q}$-vector subspace of $\mathcal{L}^n$ of dimension ℓ, then the $\overline{\mathbb{Q}}$-vector subspace of $\widetilde{\mathcal{L}}^n$ spanned by $\overline{\mathbb{Q}}^n \cup Y_0$ has dimension $\ell + n$ (see Exercise 1.5 (iii) of [**7**]). Taking firstly $n = 2$, $\ell = 3$, and secondly $n = 3$, $\ell = 2$, we deduce that the matrix M of corollary 1.4 satisfies the assumptions of Theorem 1.2. Corollary 1.4 follows. □

4. Erratum to [8]

We take the opportunity of this paper to point out a mistake in the statement of Corollary 2.12 p. 347 of [8]: the assumption that Λ_{21} *is not zero and* $\Lambda_{11}/\Lambda_{21}$ *is transcendental* should be replaced by the assumption that *the three numbers* 1, Λ_{11} *and* Λ_{21} *are linearly independent over the field of algebraic numbers.* Otherwise a counterexample is obtained for instance with $\Lambda_{21} = 1$ and $\Lambda_{2j} = 0$ for $2 \leq j \leq 5$.

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2000 Mathematics Subject Classification. 11J81 11J86 11J89.
Email address: miw@math.jussieu.fr
URL:http://www.math.jussieu.fr/~miw/