November 2, 2009 Khon Kaen University.

Number Theory Days in KKU

http ://202.28.94.202/math/thai/

History of irrational and transcendental numbers

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Abstract

The transcendence proofs for constants of analysis are essentially all based on the seminal work by Ch. Hermite : his proof of the transcendence of the number e in 1873 is the prototype of the methods which have been subsequently developed. We first show how the founding paper by Hermite was influenced by earlier authors (Lambert, Euler, Fourier, Liouville), next we explain how his arguments have been expanded in several directions : Padé approximants, interpolation series, auxiliary functions.

Numbers : rational, irrational

Numbers = real or complex numbers $\mathbf{R}$, $\mathbf{C}$.

Natural integers : $\mathbf{N} = \{0, 1, 2, \ldots\}$.

Rational integers : $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Rational numbers :
a/b with a and b rational integers, $b > 0$.

Irreducible representation :
p/q with p and q in $\mathbf{Z}$, $q > 0$ and $\gcd(p, q) = 1$.

Irrational number : a real (or complex) number which is not rational.

Numbers : algebraic, transcendental

Algebraic number : a complex number which is root of a non-zero polynomial with rational coefficients.

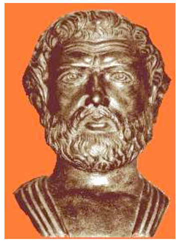
Examples :
rational numbers : a/b, root of $bX - a$.
$\sqrt{2}$, root of $X^2 - 2$.
i, root of $X^2 + 1$.

The sum and the product of algebraic numbers are algebraic numbers. The set of complex algebraic numbers is a field, the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$.

A transcendental number is a complex number which is not algebraic.

Irrationality of $\sqrt{2}$

Pythagoreas school



Hippasus of Metapontum (around 500 BC).

Sulba Sutras, Vedic civilization in India, ~800-500 BC.

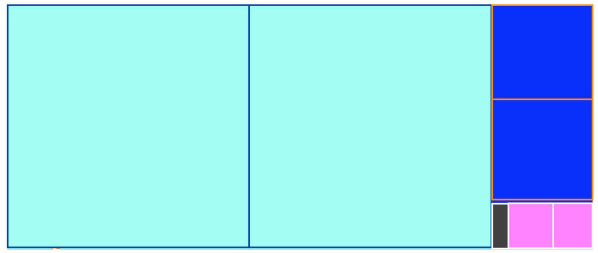
Irrationality of $\sqrt{2}$: geometric proof

- Start with a rectangle have side length 1 and $1+\sqrt{2}$.
- Decompose it into two squares with sides 1 and a smaller rectangle of sides $1+\sqrt{2}-2=\sqrt{2}-1$ and 1.
- This second small rectangle has side lenghts in the proportion

$$\frac{1}{\sqrt{2}-1}=1+\sqrt{2},$$

which is the same as for the large one.
- Hence the second small rectangle can be split into two squares and a third smaller rectangle, the sides of which are again in the same proportion.
- This process does not end.

Rectangles with proportion $1+\sqrt{2}$



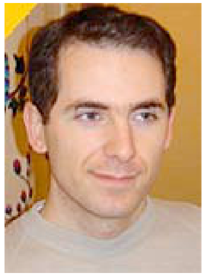
Irrationality of $\sqrt{2}$: geometric proof

If we start with a rectangle having integer side lengths, then this process stops after finitely may steps (the side lengths are positive decreasing integers).

Also for a rectangle with side lengths in a rational proportion, this process stops after finitely may steps (reduce to a common denominator and scale).

Hence $1+\sqrt{2}$ is an irrational number, and $\sqrt{2}$ also.

The fabulous destiny of $\sqrt{2}$



Éditions Le Pommier

• Benoît Rittaud, Éditions *Le Pommier* (2006).

http://www.math.univ-paris13.fr/~rittaud/RacineDeDeux

Continued fraction

The number

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\ \dots$$

satisfies

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1}.$$

Hence

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2}+1}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

We write the continued fraction expansion of $\sqrt{2}$ using the shorter notation

$$\sqrt{2} = [1;\ 2,\ 2,\ 2,\ 2,\ 2,\ \dots] = [1;\ \overline{2}].$$

Continued fractions

• H.W. Lenstra Jr,
Solving the Pell Equation,
Notices of the A.M.S.
49 (2) (2002) 182–192.

Irrationality criteria

A real number is rational if and only if its continued fraction expansion is finite.

A real number is rational if and only if its binary (or decimal, or in any basis $b \geq 2$) expansion is *ultimately periodic*.

Consequence : it should not be so difficult to decide whether a given number is rational or not.

To prove that certain numbers (occurring as constants in analysis) are irrational is most often an impossible challenge. However to construct irrational (even transcendental) numbers is easy.

Euler–Mascheroni constant

Euler's Constant is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$
$$= 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$

Is–it a rational number ?

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right) = \int_1^{\infty}\left(\frac{1}{[x]}-\frac{1}{x}\right)dx$$
$$= -\int_0^1\int_0^1 \frac{(1-x)dxdy}{(1-xy)\log(xy)}.$$

Euler's constant

Recent work by *J. Sondow* inspired by the work of F. Beukers on Apéry's proof.

F. Beukers

Jonathan Sondow

http://home.earthlink.net/~jsondow/

Jonathan Sondow http://home.earthlink.net/~jsondow/

$$\gamma = \int_0^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^2\binom{t+k}{k}}dt$$

$$\gamma = \lim_{s\to 1+}\sum_{n=1}^{\infty}\left(\frac{1}{n^s}-\frac{1}{s^n}\right)$$

$$\gamma = \int_1^{\infty}\frac{1}{2t(t+1)}F\begin{pmatrix}1, & 2, & 2\\ 3, & & t+2\end{pmatrix}dt.$$

Riemann zeta function

The function
$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s}$
was studied by Euler (1707– 1783)
for integer values of s
and by Riemann (1859) for complex values of s.

Euler : for any even integer value of $s \geq 2$, the number $\zeta(s)$ is a rational multiple of π^s.

Examples : $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450\cdots$

Coefficients : Bernoulli numbers.

Introductio in analysin infinitorum

Leonhard Euler

(1707 – 1783)

Introductio in analysin infinitorum

Divergent series

Euler :

$$1-1+1-1+1-1+\cdots=\frac{1}{2}$$

$$1+1+1+1+1+\cdots=-\frac{1}{2}$$

$$1+2+3+4+5+\cdots=-\frac{1}{12}$$

$$1^2+2^2+3^2+4^2+5^2+\cdots=0.$$

Srinivasa Ramanujan (1887 – 1920)

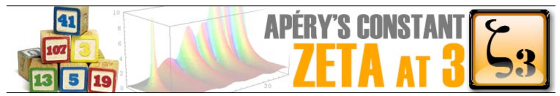
Letter of Ramanujan
to M.J.M. Hill in 1912

$$1+2+3+\cdots+\infty=-\frac{1}{12}$$

$$1^2+2^2+3^2+\cdots+\infty^2=0$$

$$1^3+2^3+3^3+\cdots+\infty^3=\frac{1}{120}$$

Riemann zeta function



The number

$$\zeta(3)=\sum_{n\geq 1}\frac{1}{n^3}=1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511\ldots$$

is irrational *(Apéry 1978)*.

Recall that $\zeta(s)/\pi^s$ is rational for any even value of $s\geq 2$.

Open question : Is the number $\zeta(3)/\pi^3$ irrational ?

Riemann zeta function

Is the number

$$\zeta(5) = \sum_{n\geq 1} \frac{1}{n^5} = 1.036\,927\,755\,143\,369\,926\,331\,365\,486\,457\ldots$$

irrational ?

T. Rivoal (2000) : infinitely many $\zeta(2n+1)$ are irrational.

Infinitely many odd zeta are irrational

Tanguy Rivoal (2000)

Let $\epsilon > 0$. For any sufficiently large odd integer a, the dimension of the $\mathbf{Q}$–vector space spanned by the numbers $1, \zeta(3), \zeta(5), \cdots, \zeta(a)$ is at least

$$\frac{1-\epsilon}{1+\log 2} \log a.$$

Open problems (irrationality)

- Is the number

$$e + \pi = 5.859\,874\,482\,048\,838\,473\,822\,930\,854\,632\ldots$$

irrational ?

- Is the number

$$e\pi = 8.539\,734\,222\,673\,567\,065\,463\,550\,869\,546\ldots$$

irrational ?

- Is the number

$$\log \pi = 1.144\,729\,885\,849\,400\,174\,143\,427\,351\,353\ldots$$

irrational ?

Catalan's constant

Is Catalan's constant

$$\sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^2} = 0.915\,965\,594\,177\,219\,015\,0\ldots$$

an irrational number ?

This is the value at $s = 2$ of the Dirichlet L–function $L(s, \chi_{-4})$ associated with the Kronecker character

$$\chi_{-4}(n) = \left(\frac{n}{4}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv -1 \pmod 4. \end{cases}$$

which is the quotient of the Dedekind zeta function of $\mathbf{Q}(i)$ and the Riemann zeta function.

Euler Gamma function

Is the number

$\Gamma(1/5) = 4.590\,843\,711\,998\,803\,053\,204\,758\,275\,929\,152\,\ldots$

irrational?

$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = \int_0^{\infty} e^{-t} t^z \cdot \frac{dt}{t}$$

Here is the set of rational values for z for which the answer is known (and, for these arguments, the Gamma value is a transcendental number) :

$$r \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}\right\} \pmod 1.$$

Known results

Irrationality of the number π :

Āryabhaṭa, b. 476 AD : $\pi \sim 3.1416$.

Nīlakaṇṭha Somayājī, b. 1444 AD : *Why then has an approximate value been mentioned here leaving behind the actual value? Because it (exact value) cannot be expressed.*

K. Ramasubramanian, *The Notion of Proof in Indian Science*, 13th World Sanskrit Conference, 2006.

Irrationality of π

Johann Heinrich Lambert (1728 - 1777)
Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,
Mémoires de l'Académie des Sciences de Berlin, **17** (1761), p. 265-322 ;
read in 1767 ; Math. Werke, t. II.

$\tan(v)$ is irrational for any rational value of $v \neq 0$ and $\tan(\pi/4) = 1$.

Lambert and Frederick II, King of Prussia

— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez–vous ?
— De moi-même !

Continued fraction expansion of tan(x)

$$\tan(x) = \frac{1}{i}\tanh(ix), \qquad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cfrac{x^2}{9 - \cfrac{x^2}{\ddots}}}}}}.$$

S.A. Shirali – *Continued fraction for e*, Resonance, vol. **5** N°1, Jan. 2000, 14–28.
http ://www.ias.ac.in/resonance/

Leonard Euler (1707 – 1783)

Leonhard Euler (1707 – 1783)
De fractionibus continuis dissertatio,
Commentarii Acad. Sci. Petropolitanae,
9 (1737), 1744, p. 98–137 ;
Opera Omnia Ser. **I** vol. **14**,
Commentationes Analyticae, p. 187–215.

$$e = \lim_{n\to\infty}(1+1/n)^n$$
$$= 2.718\,281\,828\,459\,045\,235\,360\,287\,471\,352\ldots$$
$$= 1 + 1 + \frac{1}{2}\cdot\Big(1 + \frac{1}{3}\cdot\Big(1 + \frac{1}{4}\cdot\Big(1 + \frac{1}{5}\cdot(1+\cdots)\Big)\Big)\Big).$$

Continued fraction expansion for e (Euler)

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}}}$$
$$= [2\,;\,1,\,2,\,1,\,1,\,4,\,1,\,1,\,6,\ldots]$$
$$= [2; \overline{1,\,2m,\,1}]_{m\ge 1}.$$

Continued fraction expansion for e (Euler)

The continued fraction expansion for e is infinite not periodic.

Leonhard Euler
(1707– 1783)

Johann Heinrich Lambert
(1728 - 1777)

Joseph-Louis Lagrange
(1736 - 1813)

e is neither rational (J-H. Lambert, 1767) nor quadratic irrational (J-L. Lagrange, 1770).

Continued fraction expansion for $e^{1/a}$

Starting point : $y = \tanh(x/a)$ satisfies the differential equation $ay' + y^2 = 1$.
This leads Euler to

$$e^{1/a} = [1\,;\ a-1,\ 1,\ 1,\ 3a-1,\ 1,\ 1,\ 5a-1, \dots] = [1, \overline{(2m+1)a-1,\ 1}]_{m\geq 0}.$$

Geometric proof of the irrationality of e

Jonathan Sondow
http://home.earthlink.net/~jsondow/
A geometric proof that e is irrational and a new measure of its irrationality,
Amer. Math. Monthly **113** (2006) 637-641.

Start with an interval I_1 with length 1. The interval I_n will be obtained by splitting the interval I_{n-1} into n intervals of the same length, so that the length of I_n will be $1/n!$.

Geometric proof of the irrationality of e

The origin of I_n will be

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Hence we start from the interval $I_1 = [2,3]$. For $n \geq 2$, we construct I_n inductively as follows : split I_{n-1} into n intervals of the same length, and call the second one I_n :

$$I_1 = \left[1 + \frac{1}{1!}\,,\, 1 + \frac{2}{1!}\right] = [2,3],$$

$$I_2 = \left[1 + \frac{1}{1!} + \frac{1}{2!}\,,\, 1 + \frac{1}{1!} + \frac{2}{2!}\right] = \left[\frac{5}{2!}\,,\,\frac{6}{2!}\right],$$

$$I_3 = \left[1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\,,\, 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{2}{3!}\right] = \left[\frac{16}{3!}\,,\,\frac{17}{3!}\right].$$

Irrationality of e, following J. Sondow

The origin of I_n is

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \frac{a_n}{n!},$$

the length is $1/n!$, hence $I_n = [a_n/n!, (a_n+1)/n!]$.

The number e is the intersection point of all these intervals, hence it is inside each I_n, therefore it cannot be written $a/n!$ with a an integer.
Since

$$\frac{p}{q} = \frac{(q-1)!\,p}{q!},$$

we deduce that the number e is irrational.

Joseph Fourier (1768 - 1830)

Course of analysis at the École Polytechnique Paris, 1815.

Irrationality of e, following J. Fourier

$$e = \sum_{n=0}^{N} \frac{1}{n!} + \sum_{m \geq N+1} \frac{1}{m!}.$$

Multiply by $N!$ and set

$$B_N = N!, \qquad A_N = \sum_{n=0}^{N} \frac{N!}{n!}, \quad R_N = \sum_{m \geq N+1} \frac{N!}{m!},$$

so that $B_N e = A_N + R_N$. Then A_N and B_N are in $\mathbf{Z}$, $R_N > 0$ and

$$R_N = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{e}{N+1}.$$

Irrationality of e, following J. Fourier

In the formula

$$B_N e - A_N = R_N,$$

the numbers A_N and $B_N = N!$ are integers, while the right hand side is > 0 and tends to 0 when N tends to infinity. Hence $N!\ e$ is not an integer, therefore e is irrational.

Irrationality of e^{-1}, following C.L. Siegel

C.L. Siegel (1949) : even simpler by considering e^{-1} (alternating series).

The sequence $(1/n!)_{n \geq 0}$ is decreasing and tends to 0, hence for odd N,

$$1 - \frac{1}{1!} + \frac{1}{2!} - \cdots - \frac{1}{N!} < e^{-1} < 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(N+1)!}.$$

Set

$$a_N = N! - \frac{N!}{1!} + \frac{N!}{2!} - \cdots + \frac{(N-1)!}{N!} - 1 \in \mathbf{Z}$$

Then $0 < N!e^{-1} - a_N < 1$, and therefore $N!e^{-1}$ is not an integer.

The number e is not quadratic

Since e is irrational, the same is true for $e^{1/b}$ when b is a positive integer. That e^2 is irrational is a stronger statement.

Recall (Euler, 1737) : $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$ which is not a periodic expansion. J.L. Lagrange (1770) : it follows that e is not a quadratic number.

Assume $ae^2 + be + c = 0$. Replacing e and e^2 by the series and truncating does not work : the denominator is too large and the *remainder* does not tend to zero.

Liouville (1840) : Write the quadratic equation as $ae + b + ce^{-1} = 0$.

Joseph Liouville

J. Liouville (1809 - 1882) proved also that e^2 is not a quadratic irrational number in 1840.

Sur l'irrationalité du nombre $e = 2,718\dots$,
J. Math. Pures Appl.
(1) **5** (1840), p. 192 and p. 193-194.

1844 : J. Liouville proved the existence of transcendental numbers by giving explicit examples (continued fractions, series).

Existence of transcendental numbers (1844)

J. Liouville (1809 - 1882)

gave the first examples of transcendental numbers.
For instance

$$\sum_{n\geq 1} \frac{1}{10^{n!}} = 0.110\,001\,000\,000\,0\dots$$

is a transcendental number.

The number e^2 is not quadratic

The irrationality of e^4, hence of $e^{4/b}$ for b a positive integer, follows.

It does not seem that this kind of argument will suffice to prove the irrationality of e^3, even less to prove that the number e is not a cubic irrational.

Fourier's argument rests on truncating the exponential series, it amounts to approximate e by $a/N!$ where $a \in \mathbf{Z}$. Better rational approximations exist, involving other denominators than $N!$.

The denominator $N!$ arises when one approximates the exponential series of e^z by polynomials $\sum_{n=1}^{N} z^n/n!$.

Idea of Ch. Hermite

Ch. Hermite (1822 - 1901).
approximate the exponential function e^z by rational fractions $A(z)/B(z)$.

For proving the irrationality of e^a, (a an integer ≥ 2), approximate e^a par $A(a)/B(a)$.

If the function $B(z)e^z - A(z)$ has a zero of high multiplicity at the origin, then this function has a small modulus near 0, hence at $z = a$. Therefore $|B(a)e^a - A(a)|$ is small.

Charles Hermite

A rational function $A(z)/B(z)$ is *close* to a complex analytic function f if $B(z)f(z) - A(z)$ has a zero of high multiplicity at the origin.

Goal : find $B \in \mathbf{C}[z]$ such that the Taylor expansion at the origin of $B(z)f(z)$ has a big gap : $A(z)$ will be the part of the expansion before the gap, $R(z) = B(z)f(z) - A(z)$ the remainder.

Irrationality of e^r and π *(Lambert, 1767)*

Charles Hermite (1873)

Carl Ludwig Siegel (1929, 1949)

Yuri Nesterenko (2005)



Irrationality of e^r and π *(Lambert, 1767)*

We wish to prove the irrationality of e^a for a a positive integer.

Goal : write $B_n(z)e^z = A_n(z) + R_n(z)$ with A_n and B_n in $\mathbf{Z}[z]$ and $R_n(a) \neq 0$, $\lim_{n\to\infty} R_n(a) = 0$.

Substitute $z = a$, set $q = B_n(a)$, $p = A_n(a)$ and get

$$0 < |qe^a - p| < \epsilon.$$

Rational approximation to exp

Given $n_0 \geq 0$, $n_1 \geq 0$, find A and B in $\mathbf{R}[z]$ of degrees $\leq n_0$ and $\leq n_1$ such that $R(z) = B(z)e^z - A(z)$ has a zero at the origin of multiplicity $\geq N+1$ with $N = n_0 + n_1$.

Theorem *There is a non-trivial solution, it is unique with B monic. Further, B is in $\mathbf{Z}[z]$ and $(n_0!/n_1!)A$ is in $\mathbf{Z}[z]$. Furthermore A has degree n_0, B has degree n_1 and R has multiplicity exactly $N+1$ at the origin.*

$B(z)e^z = A(z) + R(z)$

Proof. Unicity of R, hence of A and B.
Let $D = d/dz$. Since A has degree $\leq n_0$,

$$D^{n_0+1}R = D^{n_0+1}(B(z)e^z)$$

is the product of e^z with a polynomial of the same degree as the degree of B and same leading coefficient.
Since $D^{n_0+1}R(z)$ has a zero of multiplicity $\geq n_1$ at the origin, $D^{n_0+1}R = z^{n_1}e^z$. Hence R is the unique function satisfying $D^{n_0+1}R = z^{n_1}e^z$ with a zero of multiplicity $\geq n_0$ at 0 and B has degree n_1.

Siegel's algebraic point of view

C.L. Siegel, 1949.
Solve $D^{n_0+1}R(z) = z^{n_1}e^z$.
The operator $J\varphi = \int_0^z \varphi(t)dt$, inverse of D, satisfies

$$J^{n+1}\varphi = \int_0^z \frac{1}{n!}(z-t)^n\varphi(t)dt.$$

Hence

$$R(z) = \frac{1}{n_0!}\int_0^z (z-t)^{n_0}t^{n_1}e^t dt.$$

Also $A(z) = -(-1+D)^{-n_1-1}z^{n_0}$ and $B(z) = (1+D)^{-n_0-1}z^{n_1}$.

Irrationality of logarithms including π

The irrationality of e^r for $r \in \mathbf{Q}^\times$, is equivalent to the irrationality of $\log s$ for $s \in \mathbf{Q}_{>0}$.

The same argument gives the irrationality of $\log(-1)$, meaning $\log(-1) = i\pi \notin \mathbf{Q}(i)$.

Hence $\pi \notin \mathbf{Q}$.

Simultaneous approximation and transcendence

Irrationality proofs involve rational approximation to a single real number ϑ.

We wish to prove transcendence results.

A complex number ϑ is transcendental if and only if the numbers

$$1, \vartheta, \vartheta^2, \ldots, \vartheta^m, \ldots$$

are $\mathbf{Q}$–linearly independent.

Hence our goal is to prove linear independence, over the rational number field, of complex numbers.

$L = a_0 + a_1\vartheta_1 + \cdots + a_m\vartheta_m$

Let $\vartheta_1, \ldots, \vartheta_m$ be real numbers and $a_0, a_1, \ldots, a_m$ rational integers, not all of which are zero. We wish to prove that the number

$$L = a_0 + a_1\vartheta_1 + \cdots + a_m\vartheta_m$$

is not zero. Approximate simultaneously $\vartheta_1, \ldots, \vartheta_m$ by rational numbers $b_1/b_0, \ldots, b_m/b_0$.
Let $b_0, b_1, \ldots, b_m$ be rational integers. For $1 \le k \le m$ set

$$\epsilon_k = b_0\vartheta_k - b_k.$$

Then $b_0L = A + R$ with

$$A = a_0b_0 + \cdots + a_mb_m \in \mathbf{Z} \quad \text{and} \quad R = a_1\epsilon_1 + \cdots + a_m\epsilon_m \in \mathbf{R}.$$

If $0 < |R| < 1$, then $L \neq 0$.

Simultaneous approximation to the exponential function

Irrationality results follow from rational approximations $A/B \in \mathbf{Q}(x)$ to the exponential function e^x.

One of Hermite's ideas is to consider *simultaneous rational approximations to the exponential function*, in analogy with Diophantine approximation.

Let $B_0, B_1, \ldots, B_m$ be polynomials in $\mathbf{Z}[x]$. For $1 \le k \le m$ define

$$R_k(x) = B_0(x)e^{kx} - B_k(x).$$

Set $b_j = B_j(1)$, $0 \le j \le m$ and

$$R = a_0 + a_1R_1(1) + \cdots + a_mR_m(1).$$

If $0 < |R| < 1$, then $a_0 + a_1e + \cdots + a_me^m \neq 0$.

Charles Hermite and Ferdinand Lindemann

Hermite (1873) :
Transcendence of e
$e = 2.718\,281\,828\,4\ldots$

Lindemann (1882) :
Transcendence of π
$\pi = 3.141\,592\,653\,5\ldots$

Hermite–Lindemann Theorem

For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.

Corollaries : Transcendence of $\log\alpha$ and of e^β for α and β non-zero algebraic complex numbers, provided $\log\alpha \neq 0$.

Hermite : approximation to the functions $1, e^{\alpha_1 x}, \dots, e^{\alpha_m x}$

Let $\alpha_1, \dots, \alpha_m$ be pairwise distinct complex numbers and $n_0, \dots, n_m$ be rational integers, all ≥ 0. Set $N = n_0 + \cdots + n_m$.

Hermite constructs explicitly polynomials $B_0, B_1, \dots, B_m$ with B_j of degree $N - n_j$ such that each of the functions

$$B_0(z)e^{\alpha_k z} - B_k(z), \quad (1 \leq k \leq m)$$

has a zero at the origin of multiplicity at least N.

Padé approximants

Henri Eugène Padé (1863 - 1953)
Approximation of complex analytic functions by rational functions.

Transcendental functions

A complex function is called transcendental if it is transcendental over the field $\mathbf{C}(z)$, which means that the functions z and $f(z)$ are algebraically independent : if $P \in \mathbf{C}[X, Y]$ is a non-zero polynomial, then the function $P(z, f(z))$ is not 0.

Exercise. *An entire function (analytic in $\mathbf{C}$) is transcendental if and only if it is not a polynomial.*

Example. *The transcendental entire function e^z takes an algebraic value at an algebraic argument z only for $z = 0$.*

Weierstrass question

Is–it true that a transcendental entire function f takes usually transcendental values at algebraic arguments ?

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain...
If S is a countable subset of $\mathbf{C}$ and T is a dense subset of $\mathbf{C}$, there exist transcendental entire functions f mapping S into T, as well as all its derivatives.
Also there are transcendental entire functions f such that $D^k f(\alpha) \in \mathbf{Q}(\alpha)$ for all $k \geq 0$ and all algebraic α.

Integer valued entire functions

An integer valued entire function is a function f, which is analytic in $\mathbf{C}$, and maps $\mathbf{N}$ into $\mathbf{Z}$.

Example : 2^z *is an integer valued entire function, not a polynomial.*

Question : *Are-there integer valued entire function growing slower than 2^z without being a polynomial ?*

Let f be a transcendental entire function in $\mathbf{C}$. For $R > 0$ set

$$|f|_R = \sup_{|z|=R} |f(z)|.$$

Integer valued entire functions

G. Pólya (1914) :
if f is not a polynomial and $f(n) \in \mathbf{Z}$ for $n \in \mathbf{Z}_{\geq 0}$, then

$$\limsup_{R\to\infty} 2^{-R}|f|_R \geq 1.$$

Further works on this topic by G.H. Hardy, G. Pólya, D. Sato, E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross,...

Arithmetic functions

Pólya's proof starts by expanding the function f into a *Newton interpolation series* at the points $0, 1, 2, \dots$:

$$f(z) = a_0 + a_1 z + a_2 z(z-1) + a_3 z(z-1)(z-2) + \cdots$$

Since $f(n)$ is an integer for all $n \geq 0$, the coefficients a_n are rational and one can bound the denominators. If f does not grow fast, one deduces that these coefficients vanish for sufficiently large n.

Newton interpolation series

Sir Isaac Newton (1643 - 1727)

From

$$f(z) = f(\alpha_1) + (z-\alpha_1) f_1(z),$$
$$f_1(z) = f_1(\alpha_2) + (z-\alpha_2) f_2(z) + \ldots$$

we deduce

$$f(z) = a_0 + a_1(z-\alpha_1) + a_2(z-\alpha_1)(z-\alpha_2) + \cdots$$

with

$$a_0 = f(\alpha_1), \quad a_1 = f_1(\alpha_2), \ldots, \quad a_n = f_n(\alpha_{n+1}).$$

An identity due to Ch. Hermite

$$\frac{1}{x-z} = \frac{1}{x-\alpha} + \frac{z-\alpha}{x-\alpha} \cdot \frac{1}{x-z}.$$

Repeat :

$$\frac{1}{x-z} = \frac{1}{x-\alpha_1} + \frac{z-\alpha_1}{x-\alpha_1} \cdot \left(\frac{1}{x-\alpha_2} + \frac{z-\alpha_2}{x-\alpha_2} \cdot \frac{1}{x-z} \right).$$

An identity due to Ch. Hermite

Inductively we deduce the next formula due to Hermite :

$$\frac{1}{x-z} = \sum_{j=0}^{n-1} \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_j)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} + \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)} \cdot \frac{1}{x-z}.$$

Newton interpolation expansion

Application. Multiply by $(1/2i\pi) f(z)$ and integrate :

$$f(z) = \sum_{j=0}^{n-1} a_j (z-\alpha_1)\cdots(z-\alpha_j) + R_n(z)$$

with

$$a_j = \frac{1}{2i\pi} \int_C \frac{F(x)dx}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} \quad (0 \le j \le n-1)$$

and

$$R_n(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n) \cdot \frac{1}{2i\pi} \int_C \frac{F(x)dx}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)(x-z)}.$$

Integer valued entire function on $\mathbf{Z}[i]$

A.O. Gel'fond (1929) : growth of entire functions mapping the Gaussian integers into themselves.
Newton interpolation series at the points in $\mathbf{Z}[i]$.

An entire function f which is not a polynomial and satisfies $f(a+ib) \in \mathbf{Z}[i]$ for all $a+ib \in \mathbf{Z}[i]$ satisfies

$$\limsup_{R\to\infty} \frac{1}{R^2} \log |f|_R \geq \gamma.$$

F. Gramain (1981) : $\gamma = \pi/(2e)$.
This is best possible : *D.W. Masser (1980).*

Transcendence of e^π

A.O. Gel'fond (1929).

If

$$e^\pi = 23,140\,692\,632\,779\,269\,005\,729\,086\,367\ldots$$

is rational, then the function $e^{\pi z}$ takes values in $\mathbf{Q}(i)$ when the argument z is in $\mathbf{Z}[i]$.

Expand $e^{\pi z}$ into an interpolation series at the Gaussian integers.

Hilbert's seventh problem

A.O. Gel'fond and Th. Schneider (1934).
Solution of Hilbert's seventh problem :
transcendence of α^β
and of $(\log\alpha_1)/(\log\alpha_2)$
for algebraic α, β, α_2 and α_2.


Dirichlet's box principle

Gel'fond and Schneider
use an *auxiliary function*,
the existence of which follows
from Dirichlet's box principle
(pigeonhole principle,
Thue-Siegel Lemma).

Johann Peter Gustav Lejeune Dirichlet
(1805 – 1859)

Auxiliary functions

C.L. Siegel (1929) :
Hermite's explicit formulae can be replaced by Dirichlet's box principle (Thue–Siegel Lemma) which shows the existence of suitable *auxiliary functions*.

M. Laurent (1991) : instead of using the pigeonhole principle for proving the existence of solutions to homogeneous linear systems of equations, consider the matrices of such systems and take determinants.

Slope inequalities in Arakelov theory

J–B. Bost (1994) :
matrices and determinants require choices of bases.
Arakelov's Theory produces *slope inequalities* which avoid the need of bases.

Périodes et isogénies des variétés abéliennes sur les corps de nombres, (d'après D. Masser et G. Wüstholz).
Séminaire Nicolas Bourbaki, Vol. 1994/95.

Rational interpolation

René Lagrange (1935).

$$\frac{1}{x-z} = \frac{\alpha-\beta}{(x-\alpha)(x-\beta)} + \frac{x-\beta}{x-\alpha}\cdot\frac{z-\alpha}{z-\beta}\cdot\frac{1}{x-z}.$$

Iterating and integrating yield

$$f(z) = \sum_{n=0}^{N-1} B_n \frac{(z-\alpha_1)\cdots(z-\alpha_n)}{(z-\beta_1)\cdots(z-\beta_n)} + \tilde{R}_N(z).$$

Hurwitz zeta function

T. Rivoal (2006) : consider Hurwitz zeta function

$$\zeta(s,z) = \sum_{k=1}^{\infty} \frac{1}{(k+z)^s}.$$

Expand $\zeta(2,z)$ as a series in

$$\frac{z^2(z-1)^2\cdots(z-n+1)^2}{(z+1)^2\cdots(z+n)^2}.$$

The coefficients of the expansion belong to $\mathbf{Q}+\mathbf{Q}\zeta(3)$. This produces a new proof of Apéry's Theorem on the irrationality of $\zeta(3)$.
In the same way : new proof of the irrationality of $\log 2$ by expanding

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+z}.$$

Mixing C. Hermite and R. Lagrange

T. Rivoal (2006) : new proof of the irrationality of $\zeta(2)$ by expanding

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+z}\right)$$

as a Hermite–Lagrange series in

$$\frac{\bigl(z(z-1)\cdots(z-n+1)\bigr)^2}{(z+1)\cdots(z+n)}.$$

Taylor series and interpolation series

Taylor series are the special case of Hermite's formula with a single point and multiplicities — they give rise to Padé approximants.

Multiplicities can also be introduced in René Lagrange interpolation.

There is another duality between the methods of Gel'fond and Schneider : Fourier-Borel transform.

Further develoments

Transcendence and algebraic independence of values of modular functions (*méthode stéphanoise* and work of Yu.V. Nesterenko).

Measures : transcendence, linear independence, algebraic independence…

Finite characteristic :

Federico Pellarin - *Aspects de l'indépendance algébrique en caractéristique non nulle [d'après Anderson, Brownawell, Denis, Papanikolas, Thakur, Yu,…]*
Séminaire Nicolas Bourbaki, Dimanche 18 mars 2007.
`http://www.bourbaki.ens.fr/seminaires/2007/Prog_mars.07.html`