INTERNATIONAL CONFERENCE

on ALGEBRA and NUMBER THEORY

Hyderabad, December 11–16, 2003

Dependence of logarithms on commutative algebraic groups

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Transcendence of periods of $K3$ surfaces

Conjecture. *Let $\wp$ and $\wp^*$ be two non isogeneous Weierstraß elliptic functions with algebraic invariants g_2, g_3 and g_2^*, g_3^* respectively. Let $\{\omega_1, \omega_2\}$ and $\{\omega_1^*, \omega_2^*\}$ be a pair of fundamental periods of $\wp$ and $\wp^*$ respectively. Then the number*

$$\omega_1\omega_2^* - \omega_1^*\omega_2$$

is transcendental.

Multiplicative analog

Conjecture. *Let* α_{11}, α_{12}, α_{21}, α_{22} *be four non zero algebraic numbers. For* $i = 1, 2$ *and* $j = 1, 2$, *let* $\lambda_{ij} \in \mathbf{C}$ *satisfy* $e^{\lambda_{ij}} = \alpha_{ij}$. *Then the number*

$$\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}$$

is either 0 *or else transcendental.*

Multiplicative analog

Conjecture. *Let* α_{11}, α_{12}, α_{21}, α_{22} *be four non zero algebraic numbers. For* $i=1,2$ *and* $j=1,2$, *let* $\lambda_{ij}\in\mathbf{C}$ *satisfy* $e^{\lambda_{ij}}=\alpha_{ij}$. *Then the number*

$$\lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21}$$

is either 0 *or else transcendental.*

Notice:

$$\lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21}=\det\begin{vmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{vmatrix}.$$

Vanishing of $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}$

Denote by $\mathcal{L}$ the $\mathbf{Q}$ vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \{\lambda \in \mathbf{C}\ ;\ e^\lambda \in \overline{\mathbf{Q}}\} = \{\log \alpha\ ;\ \alpha \in \overline{\mathbf{Q}}^\times\} = \exp^{-1}(\overline{\mathbf{Q}}^\times).$$

For $r \in \mathbf{Q}$, $\lambda \in \mathcal{L}$ and $\lambda' \in \mathcal{L}$,

$$\det \begin{vmatrix} \lambda & \lambda' \\ r\lambda & r\lambda' \end{vmatrix} = 0 \qquad \text{and} \qquad \det \begin{vmatrix} \lambda & r\lambda \\ \lambda' & r\lambda' \end{vmatrix} = 0.$$

Four Exponentials Conjecture. *For* $i = 1, 2$ *and* $j = 1, 2$, *let* α_{ij} *be a non zero algebraic number and* λ_{ij} *a complex number satisfying* $e^{\lambda_{ij}} = \alpha_{ij}$. *Assume* $\lambda_{11}, \lambda_{12}$ *are linearly independent over* $\mathbf{Q}$ *and also* $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$. *Then*

$$\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}.$$

Four Exponentials Conjecture. *For* $i=1,2$ *and* $j=1,2$, *let* α_{ij} *be a non zero algebraic number and* λ_{ij} *a complex number satisfying* $e^{\lambda_{ij}}=\alpha_{ij}$. *Assume* $\lambda_{11},\lambda_{12}$ *are linearly independent over* $\mathbf{Q}$ *and also* $\lambda_{11},\lambda_{21}$ *are linearly independent over* $\mathbf{Q}$. *Then*

$$\lambda_{11}\lambda_{22}\neq\lambda_{12}\lambda_{21}.$$

A. Selberg, C.L. Siegel, Th Schneider, S. Lang, K. Ramachandra.

Algebraic independence of logarithms of algebraic numbers

Conjecture. *Let* $\alpha_1, \ldots, \alpha_n$ *be non zero algebraic numbers. For* $1 \le j \le n$ *let* $\lambda_j \in \mathbf{C}$ *satisfy* $e^{\lambda_j} = \alpha_j$. *Assume* $\lambda_1, \ldots, \lambda_n$ *are linearly independent over* $\mathbf{Q}$. *Then* $\lambda_1, \ldots, \lambda_n$ *are algebraically independent.*

Write $\lambda_j = \log \alpha_j$.

If $\log \alpha_1, \ldots, \log \alpha_n$ *are* $\mathbf{Q}$*-linearly independent then they are algebraically independent.*

Recall that $\mathcal{L}$ denotes the $\mathbf{Q}$ vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \{\lambda \in \mathbf{C}\,;\; e^\lambda \in \overline{\mathbf{Q}}\} = \{\log \alpha\,;\; \alpha \in \overline{\mathbf{Q}}^\times\} = \exp^{-1}(\overline{\mathbf{Q}}^\times).$$

The conjecture on algebraic independence of logarithms of algebraic numbers can be stated:

The injection of $\mathcal{L}$ into $\mathbf{C}$ extends into an injection of the symmetric algebra $\mathrm{Sym}_{\mathbf{Q}}(\mathcal{L})$ on $\mathcal{L}$ into $\mathbf{C}$.

Consequence of the conjecture on algebraic independence of logarithms of algebraic numbers

1 **Four Exponentials Conjecture.** *For* $i = 1, 2$ *and* $j = 1, 2$, *let* α_{ij} *be a non zero algebraic number and* λ_{ij} *a complex number satisfying* $e^{\lambda_{ij}} = \alpha_{ij}$. *Assume* $\lambda_{11}, \lambda_{12}$ *are linearly independent over* $\mathbf{Q}$ *and also* $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$. *Then*

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1 **Four Exponentials Conjecture.** *For* $i = 1, 2$ *and* $j = 1, 2$, *let* α_{ij} *be a non zero algebraic number and* λ_{ij} *a complex number satisfying* $e^{\lambda_{ij}} = \alpha_{ij}$. *Assume* $\lambda_{11}, \lambda_{12}$ *are linearly independent over* $\mathbf{Q}$ *and also* $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$. *Then*

$$\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}.$$

$$\lambda_{11} - \frac{\lambda_{12}\lambda_{21}}{\lambda_{22}} \neq 0, \qquad \frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} \neq 0,$$

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} \neq 0, \qquad \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \neq 0.$$

Consequence of the conjecture on algebraic independence of logarithms of algebraic numbers

Let λ_{ij} $(i = 1, 2,\ j = 1, 2)$ be four non zero logarithms of algebraic numbers.

2 Assume

$$\lambda_{11} - \frac{\lambda_{12}\lambda_{21}}{\lambda_{22}} \in \overline{\mathbf{Q}}.$$

Then

$$\lambda_{11}\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Consequence of the conjecture on algebraic independence of logarithms of algebraic numbers

Let λ_{ij} $(i = 1, 2,\ j = 1, 2)$ be four non zero logarithms of algebraic numbers.

3 Assume

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} \in \overline{\mathbf{Q}}.$$

Then

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} \in \mathbf{Q}.$$

Consequence of the conjecture on algebraic independence of logarithms of algebraic numbers

Let λ_{ij} $(i = 1, 2,\ j = 1, 2)$ be four non zero logarithms of algebraic numbers.

4 Assume

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} \in \overline{\mathbf{Q}}.$$

Then

- either $\lambda_{11}/\lambda_{12} \in \mathbf{Q}$ and $\lambda_{21}/\lambda_{22} \in \mathbf{Q}$
- or $\lambda_{12}/\lambda_{22} \in \mathbf{Q}$ and

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} \in \mathbf{Q}.$$

Remark:

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{b\lambda_{11} - a\lambda_{12}}{b\lambda_{12}} = \frac{a}{b}.$$

Consequence of the conjecture on algebraic independence of logarithms of algebraic numbers

Let λ_{ij} $(i = 1, 2,\ j = 1, 2)$ be four non zero logarithms of algebraic numbers.

5 Assume

$$\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \in \overline{\mathbf{Q}}.$$

Then

$$\lambda_{11}\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Algebraic independence of elliptic logarithms

Conjecture. *Let* $\mathcal{E}_1,\ldots,\mathcal{E}_n$ *be elliptic curves which are pairwise non isogeneous. For* $1\le h\le n$, *let* $\wp_h$ *be the Weierstraß elliptic function associated to* $\mathcal{E}_h$; *assume its invariants* g_{2h}, g_{3h} *are algebraic. Let* $k_h=\mathrm{End}(\mathcal{E}_h)\otimes_{\mathbf{Z}}\mathbf{Q}$ *be the field of endomorphisms of* $\mathcal{E}_h$ *and* $u_{1h},\ldots,u_{r_hh}$ *be complex numbers which are linearly independent over* k_h *such that* $\wp_h(u_{ih})\in\overline{\mathbf{Q}}$ *for* $1\le i\le r_h$. *Then the* $r_1+\cdots+r_h$ *numbers*

$$u_{ih},\qquad (1\le i\le r_h,\quad 1\le h\le n)$$

are algebraically independent.

Mixed conjecture of algebraic independence

Conjecture. *Let $\mathcal{E}_h$, $\wp_h$, u_{ih} be as in the conjecture of algebraic independence of elliptic logarithms. Let $u_{10}, \ldots, u_{r_0 0}$ be $\mathbf{Q}$-linearly independent of $\mathcal{L}$. Then the $r_0 + r_1 + \cdots + r_h$ numbers*

$$u_{ih}, \qquad (1 \le i \le r_h, \quad 0 \le h \le n)$$

are algebraically independent.

General Conjectures

- **Generalized Grothendieck Conjecture of periods**
  (Yves André)

- **Special case of 1-motives**

$$[\mathbf{Z}^r \to \mathbf{G}_m^s \prod_{h=1}^n \mathcal{E}_h]$$

Elliptico-toric Conjecture *(Cristiana Bertolin)*

- **For $n=0$:**

$$[\mathbf{Z}^r \to \mathbf{G}_m^s]$$

Schanuel's Conjecture.

Schanuel's Conjecture. *Let* $x_1, \ldots, x_n$ *be* $\mathbf{Q}$*-linearly independent complex numbers. Then the transcendence degree of the field*

$$\mathbf{Q}\bigl(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}\bigr)$$

is at least n.

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is at least n*.*

Special case: Taking $x_i \in \mathcal{L}$ for $1 \leq i \leq n$ one recovers the conjecture of algebraic independence of logarithms of algebraic numbers.

Elliptic conjecture of C. Bertolin. Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be pairwise non isogeneous elliptic curves with modular invariants $j(\mathcal{E}_h)$. Let ω_{1h}, ω_{2h} be a pair of fundamental periods of $\wp_h$ and η_{1h}, η_{2h} the associated quasi-periods, P_{ih} points on $\mathcal{E}_h(\mathbf{C})$, p_{ih} and d_{ih} elliptic integrals of the first (resp. second) kind associated to P_{ih}. Define $\kappa_h = [k_h : \mathbf{Q}]$ and let d_h be the dimension of the k_h-subspace of $\mathbf{C}/(k_h\omega_{1h} + k_h\omega_{2h})$ spanned by $p_{1h}, \ldots, p_{r_h h}$. Then the transcendence degree of the field

$$\mathbf{Q}\Big(\big\{j(\mathcal{E}_h), \omega_{1h}, \omega_{2h}, \eta_{1h}, \eta_{2h}, P_{ih}, p_{ih}, d_{ih}\big\}_{\substack{1\le i\le r_h\\ 1\le h\le n}}\Big)$$

is at least

$$2\sum_{h=1}^{n} d_h + 4\sum_{h=1}^{n} \kappa_h^{-1} - n + 1.$$

Example of a mixed situation

Conjecture. *Let* $\wp$ *be a Weierstraß elliptic function with algebraic invariants* g_2, g_3 *and let* ω_1, ω_2 *be a pair of fundamental periods of* $\wp$*. Let* λ_1, λ_2 *be two non zero elements of* $\mathcal{L}$*. Then the determinant*

$$\begin{vmatrix} \lambda_1 & \lambda_2 \\ \omega_1 & \omega_2 \end{vmatrix}$$

does not vanish.

$$\begin{vmatrix} \lambda_1 & \lambda_2 \\ \omega_1 & \omega_2 \end{vmatrix} \neq 0.$$

Special case: $\lambda_1 = 2i\pi$:

Define $\tau = \omega_2/\omega_1$, $q = e^{2i\pi\tau}$, $J(q) = j(\tau)$.

Theorem *(K. Barré-Sirieix, F. Gramain, G. Philibert).* If $q \in \mathbf{C}$ satisfies $0 < |q| < 1$, then one at least of the two numbers

$$q, \; J(q)$$

is transcendental.

Transcendence results

Six Exponentials Theorem. *Let* x_1, x_2 *be two complex numbers which are* $\mathbf{Q}$*-linearly independent. Let* y_1, y_2, y_3 *be three complex numbers which are* $\mathbf{Q}$*-linearly independent. Then one at least of the six numbers*

$$e^{x_i y_j} \quad (i = 1, 2,\ j = 1, 2, 3)$$

is transcendental.

Transcendence results

Six Exponentials Theorem *(again).* *For* $i = 1, 2$ *and* $j = 1, 2, 3$, *let* α_{ij} *be a non zero algebraic number and* λ_{ij} *a complex number satisfying* $e^{\lambda_{ij}} = \alpha_{ij}$. *Assume* $\lambda_{11}, \lambda_{12}, \lambda_{13}$ *are linearly independent over* $\mathbf{Q}$ *and also* $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$. *Then the matrix*

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \end{pmatrix}$$

has rank 2.

Transcendence results

1 *(Six exponentials)*

$$(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21},\ \lambda_{11}\lambda_{23} - \lambda_{13}\lambda_{21}) \neq (0,\ 0).$$

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$$(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21},\ \lambda_{11}\lambda_{23} - \lambda_{13}\lambda_{21}) \neq (0,\ 0).$$

2

$$\left(\lambda_{12} - \frac{\lambda_{11}\lambda_{22}}{\lambda_{21}}, \quad \lambda_{13} - \frac{\lambda_{11}\lambda_{23}}{\lambda_{21}}\right) \notin \overline{\mathbf{Q}} \times \overline{\mathbf{Q}}.$$

Transcendence results

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$$\left(\lambda_{12} - \frac{\lambda_{11}\lambda_{22}}{\lambda_{21}}, \quad \lambda_{13} - \frac{\lambda_{11}\lambda_{23}}{\lambda_{21}}\right) \notin \overline{\mathbf{Q}} \times \overline{\mathbf{Q}}.$$

3

$$\left(\frac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}}, \quad \frac{\lambda_{13}\lambda_{21}}{\lambda_{11}\lambda_{23}}\right) \notin \overline{\mathbf{Q}} \times \overline{\mathbf{Q}}.$$

Transcendence results

1 *(Six exponentials)*

$$(\lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21},\ \lambda_{11}\lambda_{23}-\lambda_{13}\lambda_{21}) \neq (0,\ 0).$$

2

$$\left(\lambda_{12}-\frac{\lambda_{11}\lambda_{22}}{\lambda_{21}},\quad \lambda_{13}-\frac{\lambda_{11}\lambda_{23}}{\lambda_{21}}\right) \notin \overline{\mathbf{Q}}\times\overline{\mathbf{Q}}.$$

3

$$\left(\frac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}},\quad \frac{\lambda_{13}\lambda_{21}}{\lambda_{11}\lambda_{23}}\right) \notin \overline{\mathbf{Q}}\times\overline{\mathbf{Q}}.$$

4

$$\left(\frac{\lambda_{12}}{\lambda_{11}}-\frac{\lambda_{22}}{\lambda_{21}},\quad \frac{\lambda_{13}}{\lambda_{11}}-\frac{\lambda_{23}}{\lambda_{21}}\right) \notin \overline{\mathbf{Q}}\times\overline{\mathbf{Q}}.$$

Transcendence results

1 *(Six exponentials)*

$$(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21},\ \lambda_{11}\lambda_{23} - \lambda_{13}\lambda_{21}) \neq (0,\ 0).$$

2 $$\left(\lambda_{12} - \frac{\lambda_{11}\lambda_{22}}{\lambda_{21}}, \quad \lambda_{13} - \frac{\lambda_{11}\lambda_{23}}{\lambda_{21}}\right) \notin \overline{\mathbf{Q}} \times \overline{\mathbf{Q}}.$$

3 $$\left(\frac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}}, \quad \frac{\lambda_{13}\lambda_{21}}{\lambda_{11}\lambda_{23}}\right) \notin \overline{\mathbf{Q}} \times \overline{\mathbf{Q}}.$$

4 $$\left(\frac{\lambda_{12}}{\lambda_{11}} - \frac{\lambda_{22}}{\lambda_{21}}, \quad \frac{\lambda_{13}}{\lambda_{11}} - \frac{\lambda_{23}}{\lambda_{21}}\right) \notin \overline{\mathbf{Q}} \times \overline{\mathbf{Q}}.$$

5 $$\left(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}, \quad \lambda_{11}\lambda_{23} - \lambda_{13}\lambda_{21}\right) \notin \overline{\mathbf{Q}} \times \overline{\mathbf{Q}}\ ?$$

Theorem 1. *Let* λ_{ij} *(*$i = 1, 2,\ j = 1, 2, 3, 4, 5$*) be ten non zero logarithms of algebraic numbers. Assume*

- $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$

and

- $\lambda_{11}, \ldots, \lambda_{15}$ *are linearly independent over* $\mathbf{Q}$.

Then one at least of the four numbers

$$\lambda_{1j}\lambda_{21} - \lambda_{2j}\lambda_{11}, \quad (j = 2, 3, 4, 5)$$

is transcendental.

Elliptic Analogue

Theorem 2. *Let* $\wp$ *and* $\wp^*$ *be two non isogeneous Weierstraß elliptic functions with algebraic invariants* g_2, g_3 *and* g_2^*, g_3^* *respectively. For* $1 \le j \le 9$ *let* u_j *(resp.* u_j^**) be a non zero logarithm of an algebraic point of* $\wp$ *(resp.* $\wp^*$*). Assume* $u_1, \ldots, u_9$ *are* $\mathbf{Q}$*-linearly independent. Then one at least of the eight numbers*

$$u_j u_1^* - u_j^* u_1 \quad (j = 2, \ldots, 9)$$

is transcendental

Sketch of Proofs

Theorem 3. *Let* $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2$ *be a commutative algebraic group over* $\overline{\mathbf{Q}}$ *of dimension* $d = d_0 + d_1 + d_2$, V *a hyperplane of the tangent space* $T_e(G)$, Y *a finitely generated subgroup of* V *of rank* ℓ_1 *such that* $\exp_G(Y) \subset G(\overline{\mathbf{Q}})$ *with*

$$\ell_1 > (d-1)(d_1 + 2d_2).$$

Then V *contains a non zero algebraic Lie subalgebra of* $T_e(G)$ *defined over* $\overline{\mathbf{Q}}$.

Sketch of Proof of Theorem 1

Assume

$$\lambda_{1j}\lambda_{21} - \lambda_{2j}\lambda_{11} = \gamma_j \quad \text{for} \quad 1 \leq j \leq \ell_1.$$

Take $G = \mathbf{G}_a \times \mathbf{G}_m^2$, $d_0 = 1$, $d_1 = 2$, $G_2 = \{0\}$, V is the hyperplane

$$z_0 = \lambda_{21}z_1 - \lambda_{11}z_2$$

and $Y = \mathbf{Z}y_1 + \cdots + \mathbf{Z}y_{\ell_1}$ with

$$y_j = (\gamma_j, \lambda_{1j}, \lambda_{2j}), \quad (1 \leq j \leq \ell_1).$$

Since $(d-1)(d_1 + 2d_2) = 4$, we need $\ell_1 \geq 5$.

Sketch of Proof of Theorem 2

Assume

$$u_j u_1^* - u_j^* u_1 = \gamma_j \quad \text{for} \quad 1 \le j \le \ell_1.$$

Take $G = \mathbf{G}_a \times \mathcal{E} \times \mathcal{E}^*$, $d_0 = 1$, $d_1 = 0$, $d_2 = 2$, V is the hyperplane

$$z_0 = u_1^* z_1 - u_1 z_2$$

and $Y = \mathbf{Z} y_1 + \cdots + \mathbf{Z} y_{\ell_1}$ with

$$y_j = (\gamma_j, u_j, u_j^*), \quad (1 \le j \le \ell_1).$$

Since $(d-1)(d_1 + 2d_2) = 8$, we need $\ell_1 \ge 9$.

In both cases we need to check that V does not contain a non zero Lie subalgebra of $T_e(G)$. For Theorem 1 this follows from the assumption

$$\lambda_{11},\ \lambda_{21} \quad \text{are } \mathbf{Q}\text{-linearly independent},$$

while for Theorem 2 this follows from the assumption

$$\mathcal{E},\ \mathcal{E}^* \quad \text{are not isogeneous.}$$

Remarks.

1. The proofs of results 1 (*six exponentials theorem*), 2, 3, 4 above are similar: they also rest on Theorem **3**.

2. Theorem **3** also contains Baker's Theorem on linear independence of logarithms.

3. A refinement of Theorem **3** is available (and useful). If V contains a subspace W of dimension ℓ_0 which is rational over $\overline{\mathbf{Q}}$, then the condition

$$\ell_1 > (d-1)(d_1+2d_2)$$

can be replaced by

$$\ell_1 > (d-\ell_0-1)(d_1+2d_2).$$

Theorem 4. *Let* $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2$ *be a commutative algebraic group over* $\overline{\mathbf{Q}}$ *of dimension* $d = d_0 + d_1 + d_2$*,* V *a hyperplane of the tangent space* $T_e(G)$*,* Y *a finitely generated subgroup of* V *of rank* ℓ_1 *and* W *a subspace of* $T_e(G)$ *of dimension* ℓ_0 *which is rational over* $\overline{\mathbf{Q}}$ *and contained in* V*. Assume* $\exp_G(Y) \subset G(\overline{\mathbf{Q}})$ *and*

$$\ell_1 > (d - \ell_0 - 1)(d_1 + 2d_2).$$

Then V *contains a non zero algebraic Lie subalgebra of* $T_e(G)$ *defined over* $\overline{\mathbf{Q}}$*.*

A further extension to subspaces of $T_e(G)$ (in place of hyperplanes) yields the *Algebraic subgroup theorem*. The case $V = W$ is due to G. Wüstholz.

Algebraic independence of logarithms *(again)*

Conjecture. Let n be a positive integer, X an affine algebraic variety of $\mathbf{C}^n$ defined over $\overline{\mathbf{Q}}$, P a point on X with coordinates on $\mathcal{L}$ and V the smallest vector subspace of $\mathbf{C}^n$ defined over $\mathbf{Q}$ which contains P. Then V is contained in X.

Algebraic independence of logarithms *(again)*

Conjecture. Let n be a positive integer, X an affine algebraic variety of $\mathbf{C}^n$ defined over $\overline{\mathbf{Q}}$, P a point on X with coordinates on $\mathcal{L}$ and V the smallest vector subspace of $\mathbf{C}^n$ defined over $\mathbf{Q}$ which contains P. Then V is contained in X.

Conjecture. Let M be a 4×4 skew symmetric matrix with entries in $\mathcal{L}$ and with $\mathbf{Q}$-linearly independent rows. Assume that the $\mathbf{Q}$-vector space spanned by the columns of M in $\mathbf{C}^4$ does not contain a non zero element of $\mathbf{Q}^4$. Then the rank of M is ≥ 3.

Special case: the four exponentials conjecture.

$$\begin{pmatrix} 0 & \lambda_{11} & \lambda_{12} & 0 \\ -\lambda_{11} & 0 & 0 & -\lambda_{21} \\ -\lambda_{12} & 0 & 0 & -\lambda_{22} \\ 0 & \lambda_{21} & \lambda_{22} & 0 \end{pmatrix}$$

Rank of matrices whose entries are logarithms of algebraic numbers

Let M be a matrix whose entries are in $\mathcal{L}$. Let $\lambda_1, \ldots, \lambda_r$ be a basis of the $\mathbf{Q}$-vector space spanned by the entries of M. Write

$$\mathsf{M} = M_1\lambda_1 + \cdots + M_r\lambda_r$$

where $M_1, \ldots, M_r$ have rational entries. Denote by $r_{\mathrm{s}tr}(\mathsf{M})$ the rank of the matrix

$$\mathsf{M} = M_1X_1 + \cdots + M_rX_r$$

in the field $\mathbf{Q}(X_1, \ldots, X_r)$.

Conjecture. The rank of M is $r_{\mathrm{s}tr}(\mathsf{M})$.

Proposition *(D. Roy).* *This conjecture is* **equivalent** *to the conjecture of* homogeneous *algebraic independence of logarithms of algebraic numbers:*
if $\lambda_1, \ldots, \lambda_n$ *are* $\mathbf{Q}$*-linearly independent elements of* $\mathcal{L}$ *and if* P *is a non zero* homogeneous *polynomial with rational coefficients, then*

$$P(\lambda_1, \ldots, \lambda_n) \neq 0.$$

Lemma *(D. Roy).* *Any polynomial* $A \in \mathbf{Z}[X_1, \ldots, X_n]$ *is the determinant of a matrix with entries in the* $\mathbf{Z}$*-module* $\mathbf{Z} + \mathbf{Z}X_1 + \cdots + \mathbf{Z}X_n$.

Further developments

- Quadratic relations among logarithms of algebraic numbers.
- Abelian varieties in place of product of elliptic curves.
- Semi abelian varieties. Commutative algebraic groups.
- Taking periods into account.