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Diophantine approximation, irrationality and transcendence

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These are informal notes of my course given in April – June 2010 at IMPA (*Instituto Nacional de Matematica Pura e Aplicada*), Rio de Janeiro, Brazil.

Content of the course:

1. Algebraic independence of the two functions $\wp(z)$ and e^z.
Legendre's relation $\eta_2\omega_1 - \eta_1\omega_2 = 2i\pi$. Proof: integrate $\zeta(z)dz$ on a fundamental parallelogram.
Application: algebraic independence of the two functions $az + b\zeta(z)$ and $\wp(z)$.

2. Section § 10.7.2: Morphisms between elliptic curves. The modular invariant.

3. Section § 10.7.3: Endomorphisms of an elliptic curve; complex multiplications.
Algebraic independence of $\wp$ and $\wp^*$.
Schneider's Theorem on the transcendence of $j(\tau)$ (corollary 174).

11 Algebraic independence

11.1 Chudnovskii's results

References: [1], [3], Lecture 8. [5] § 5.2.

$\boxed{\text{The text below is taken from [5] § 5.2.}}$

In the 1970's G.V. Chudnovsky proved strong results of algebraic independence (small transcendence degree) related with elliptic functions. One of his most spectacular contributions was obtained in 1976:

Theorem 178 (G.V. Chudnovsky, 1976). *Let $\wp$ be a Weierstraß elliptic function with invariants g_2, g_3. Let (ω_1, ω_2) be a basis of the lattice period of $\wp$ and $\eta_1 = \eta(\omega_1)$, $\eta_2 = \eta(\omega_2)$ the associated quasi-periods of the associated Weierstraß zeta function. Then at least two of the numbers $g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2$ are algebraically independent.*

A more precise result is that, for any non-zero period ω, at least two of the four numbers $g_2,\ g_3,\ \omega/\pi,\ \eta/\omega$ (with $\eta = \eta(\omega)$) are algebraically independent.

In the case where g_2 and g_3 are algebraic one deduces from Theorem 178 that two among the four numbers $\omega_1,\ \omega_2,\ \eta_1,\ \eta_2$ are algebraically independent; this statement is also a consequence of the next result:

Theorem 179 (G.V. Chudnovsky, 1981). *Assume that g_2 and g_3 are algebraic. Let ω be a non-zero period of $\wp$, set $\eta = \eta(\omega)$ and let u be a complex number which is not a period such that u and ω are $\mathbf{Q}$-linearly independent: $u \notin \mathbf{Q}\omega \cup \Omega$. Assume $\wp(u) \in \overline{\mathbf{Q}}$. Then the two numbers*

$$\zeta(u) - \frac{\eta}{\omega}u, \quad \frac{\eta}{\omega}$$

are algebraically independent.

From Theorem 178 or Theorem 179 one deduces:

Corollary 180. *Let ω be a non-zero period of $\wp$ and $\eta = \eta(\omega)$. If g_2 and g_3 are algebraic, then the two numbers π/ω and η/ω are algebraically independent.*

The following consequence of Corollary 180 shows that in the CM case, Chudnovsky's results are sharp:

Corollary 181. *Assume that g_2 and g_3 are algebraic and the elliptic curve has complex multiplications. Let ω be a non-zero period of $\wp$. Then the two numbers ω and π are algebraically independent.*

As a consequence of formulae (162) and (163), one deduces:

Corollary 182. *The numbers π and $\Gamma(1/4)$ are algebraically independent. Also the numbers π and $\Gamma(1/3)$ are algebraically independent.*

References

[1] G. V. Chudnovsky –"Algebraic independence of values of exponential and elliptic functions", in *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)* (Helsinki), Acad. Sci. Fennica, 1980, p. 339–350.

[2] M. Waldschmidt, *Les travaux de G. V. Čudnovskiĭ sur les nombres transcendants*, in Séminaire Bourbaki, Vol. 1975/76, 28e année, Exp. No. 488, Springer, Berlin, 1977, pp. 274–292. Lecture Notes in Math., Vol. 567.
`http://archive.numdam.org/article/SB_1975-1976__18__274_0.pdf`

[3] — , *Transcendence methods*, vol. 52 of Queen's Papers in Pure and Applied Mathematics, Queen's University, Kingston, Ont., 1979.
`http://www.math.jussieu.fr/ miw/articles/pdf/QueensPaper52.pdf`

[4] — , *Elliptic curves and complex multiplication* English translation by Franz Lemmermeyer of notes by A. Faisant, R. Lardon and G. Philibert, Sém Arithm. Univ. St Etienne, 1981-82, N°4, 23 p.
`http://www.math.jussieu.fr/ miw/articles/ps/eccm.ps`

[5] — , *Elliptic functions and transcendence*, in Surveys in number theory, vol. 17 of Dev. Math., Springer, New York, 2008, pp. 143–188.