`Updated:` *May 16, 2010*

Diophantine approximation, irrationality and transcendence

Michel Waldschmidt

Course N°8, *May 12, 2010*

These are informal notes of my course given in April – June 2010 at IMPA (*Instituto Nacional de Matematica Pura e Aplicada*), Rio de Janeiro, Brazil.

Addition to Lemma 81.
In [1], § 4, there is a variant of the matrix formula (64) for the simple continued fraction of a real number.

Given integers $a_0, a_1, \dots$ with $a_i > 0$ for $i \geq 1$ and writing, for $n \geq 0$, as usual, $p_n/q_n = [a_0, a_1, \dots, a_n]$, one checks, by induction on n, the two formulae

$$\left.\begin{array}{ll}\begin{pmatrix}1 & a_0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ a_1 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & a_n\\ 0 & 1\end{pmatrix} = \begin{pmatrix}p_{n-1} & p_n\\ q_{n-1} & q_n\end{pmatrix} & \text{if } n \text{ is even}\\ \begin{pmatrix}1 & a_0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ a_1 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & 0\\ a_n & 1\end{pmatrix} = \begin{pmatrix}p_n & p_{n-1}\\ q_n & q_{n-1}\end{pmatrix} & \text{if } n \text{ is odd}\end{array}\right\} \qquad (95)$$

Define two matrices U (up) and L (low) in $\mathrm{GL}_2(\mathbf{R})$ of determinant $+1$ by

$$U = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

For p and q in $\mathbf{Z}$, we have

$$U^p = \begin{pmatrix}1 & p\\ 0 & 1\end{pmatrix} \quad \text{and} \quad L^q = \begin{pmatrix}1 & 0\\ q & 1\end{pmatrix},$$

so that these formulae (95) are

$$U^{a_0}L^{a_1}\cdots U^{a_n} = \begin{pmatrix}p_{n-1} & p_n\\ q_{n-1} & q_n\end{pmatrix} \quad \text{if } n \text{ is even}$$

and

$$U^{a_0}L^{a_1}\cdots L^{a_n} = \begin{pmatrix}p_n & p_{n-1}\\ q_n & q_{n-1}\end{pmatrix} \quad \text{if } n \text{ is odd.}$$

The connexion with Euclid's algorithm is

$$U^{-p}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-pc & b-pd \\ c & d \end{pmatrix} \quad \text{and} \quad L^{-q}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c-qa & d-qb \end{pmatrix}.$$

The corresponding variant of Lemma 81 is also given in [1], § 4: *If a, b, c, d are rational integers satisfying $b > a > 0$, $d > c \geq 0$ and $ad - bc = 1$, then there exist rational integers $a_0, \ldots, a_n$ with n even and $a_1, \ldots, a_n$ positive, such that*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix}$$

These integers are uniquely determined by $b/d = [a_0, \ldots, a_n]$ with n even.

6.3.12 Periodic continued fractions

An infinite sequence $(a_n)_{n\geq 0}$ is said to be *ultimately periodic* if there exists $n_0 \geq 0$ and $s \geq 1$ such that

$$a_{n+s} = a_n \quad \text{for all } n \geq n_0. \tag{96}$$

The set of s satisfying this property (6.3.12) is the set of positive multiples of an integer s_0, and $(a_{n_0}, a_{n_0+1}, \ldots, a_{n_0+s_0-1})$ is called *the fundamental period*.

A continued fraction with a sequence of partial quotients satisfying (96) will be written

$$[a_0, a_1, \ldots, a_{n_0-1}, \overline{a_{n_0}, \ldots, a_{n_0+s-1}}].$$

Example. For D a positive integer which is not a square, setting $a_0 = \lfloor \sqrt{D} \rfloor$, we have by Theorem 87

$$a_0 + \sqrt{D} = [\overline{2a_0, a_1, \ldots, a_{s-1}}] \quad \text{and} \quad \frac{1}{\sqrt{D} - a_0} = [\overline{a_1, \ldots, a_{s-1}, 2a_0}].$$

Lemma 97 (Euler 1737)**.** *If an infinite continued fraction*

$$x = [a_0, a_1, \ldots, a_n, \ldots]$$

is ultimately periodic, then x is a quadratic irrational number.

Proof. Since the continued fraction of x is infinite, x is irrational. Assume first that the continued fraction is periodic, namely that (96) holds with $n_0 = 0$:

$$x = [\overline{a_0, \ldots, a_{s-1}}].$$

This can be written

$$x = [a_0, \dots, a_{s-1}, x].$$

Hence

$$x = \frac{p_{s-1}x + p_{s-2}}{q_{s-1}x + q_{s-2}}\cdot$$

It follows that

$$q_{s-1}X^2 + (q_{s-2} - p_{s-1})X - p_{s-2}$$

is a non–zero quadratic polynomial with integer coefficients having x as a root. Since x is irrational, this polynomial is irreducible and x is quadratic.

In the general case where (96) holds with $n_0 > 0$, we write

$$x = [a_0, a_1, \dots, a_{n_0-1}, \overline{a_{n_0}, \dots, a_{n_0+s-1}}] = [a_0, a_1, \dots, a_{n_0-1}, y],$$

where $y = [\overline{a_{n_0}, \dots, a_{n_0+s-1}}]$ is a periodic continued fraction, hence is quadratic. But

$$x = \frac{p_{n_0-1}y + p_{n_0-2}}{q_{n_0-1}y + q_{n_0-2}},$$

hence $x \in \mathbf{Q}(y)$ is also quadratic irrational.

$\square$

Lemma 98 (Lagrange, 1770)**.** *If x is a quadratic irrational number, then its continued fraction*

$$x = [a_0, a_1, \dots, a_n, \dots]$$

is ultimately periodic.

Proof. For $n \geq 0$, define $d_n = q_n x - p_n$. According to Corollary 72, we have $|d_n| < 1/q_{n+1}$.

Let $AX^2 + BX + C$ with $A > 0$ be an irreducible quadratic polynomial having x as a root. For each $n \geq 2$, we deduce from (70) that the convergent x_n is a root of a quadratic polynomial $A_n X^2 + B_n X + C_n$, with

$$\begin{aligned}
A_n &= Ap_{n-1}^2 + Bp_{n-1}q_{n-1} + Cq_{n-1}^2,\\
B_n &= 2Ap_{n-1}p_{n-2} + B(p_{n-1}q_{n-2} + p_{n-2}q_{n-1}) + 2Cq_{n-1}q_{n-2},\\
C_n &= A_{n-1}.
\end{aligned}$$

Using $Ax^2 + Bx + C = 0$, we deduce

$$\begin{aligned}
A_n &= (2Ax + B)d_{n-1}q_{n-1} + Ad_{n-1}^2,\\
B_n &= (2Ax + B)(d_{n-1}q_{n-2} + d_{n-2}q_{n-1}) + 2Ad_{n-1}d_{n-2}.
\end{aligned}$$

There are similar formulae expressing A, B, C as homogeneous linear combinations of A_n, B_n, C_n, and since $(A, B, C) \neq (0, 0, 0)$, it follows that $(A_n, B_n, C_n) \neq (0, 0, 0)$. Since x_n is irrational, one deduces $A_n \neq 0$.

From the inequalities

$$q_{n-1}|d_{n-2}| < 1, \quad q_{n-2}|d_{n-1}| < 1, \quad q_{n-1} < q_n, \quad |d_{n-1}d_{n-2}| < 1,$$

one deduces

$$\max\{|A_n|,\ |B_n|/2, |C_n|\} < A + |2Ax + B|.$$

This shows that $|A_n|$, $|B_n|$ and $|C_n|$ are bounded independently of n. Therefore there exists $n_0 \geq 0$ and $s > 0$ such that $x_{n_0} = x_{n_0+s}$. From this we deduce that the continued fraction of x_{n_0} is purely periodic, hence the continued fraction of x is ultimately periodic. □

A *reduced quadratic irrational number* is an irrational number $x > 1$ which is a root of a degree 2 polynomial $ax^2 + bx + c$ with rational integer coefficients, such that the other root x' of this polynomial, which is the *Galois conjugate of* x, satisfies $-1 < x' < 0$. If x is reduced, then so is $-1/x'$.

Lemma 99. *A continued fraction*

$$x = [a_0, a_1, \ldots, a_n \ldots]$$

is purely periodic if and only if x is a reduced quadratic irrational number. In this case, if $x = [\overline{a_0, a_1, \ldots, a_{s-1}}]$ and if x' is the Galois conjugate of x, then

$$-1/x' = [\overline{a_{s-1}, \ldots, a_1, a_0}]$$

Proof. Assume first that the continued fraction of x is purely periodic:

$$x = [\overline{a_0, a_1, \ldots, a_{s-1}}].$$

From $a_s = a_0$ we deduce $a_0 > 0$, hence $x > 1$. From $x = [a_0, a_1, \ldots, a_{s-1}, x]$ and the unicity of the continued fraction expansion, we deduce

$$x = \frac{p_{s-1}x + p_{s-2}}{q_{s-1}x + q_{s-2}} \quad \text{and} \quad x = x_s.$$

Therefore x is a root of the quadratic polynomial

$$P_s(X) = q_{s-1}X^2 + (q_{s-2} - p_{s-1})X - p_{s-2}.$$

This polynomial P_s has a positive root, namely $x > 1$, and a negative root x', with the product $xx' = -p_{s-2}/q_{s-1}$. We transpose the relation

$$\begin{pmatrix} p_{s-1} & p_{s-2} \\ q_{s-1} & q_{s-2} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{s-1} & 1 \\ 1 & 0 \end{pmatrix}$$

and obtain

$$\begin{pmatrix} p_{s-1} & q_{s-1} \\ p_{s-2} & q_{s-2} \end{pmatrix} = \begin{pmatrix} a_{s-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Define

$$y = [\overline{a_{s-1}, \dots, a_1, a_0}],$$

so that $y > 1$,

$$y = [a_{s-1}, \dots, a_1, a_0, y] = \frac{p_{s-1}y + q_{s-1}}{p_{s-2}y + q_{s-2}}$$

and y is the positive root of the polynomial

$$Q_s(X) = p_{s-2}X^2 + (q_{s-2} - p_{s-1})X - q_{s-1}.$$

The polynomials P_s and Q_s are related by $Q_s(X) = -X^2 P_s(-1/X)$. Hence $y = -1/x'$.

For the converse, assume $x > 1$ and $-1 < x' < 0$. Let $(x_n)_{n\geq 1}$ be the sequence of complete quotients of x. For $n \geq 1$, define x'_n as the Galois conjugate of x_n. One deduces by induction that $x'_n = a_n + 1/x'_{n+1}$, that $-1 < x'_n < 0$ (hence x_n is reduced), and that a_n is the integral part of $-1/x'_{n+1}$.

If the continued fraction expansion of x were not purely periodic, we would have

$$x = [a_0, \dots, a_{h-1}, \overline{a_h, \dots, a_{h+s-1}}]$$

with $a_{h-1} \neq a_{h+s-1}$. By periodicity we have $x_h = [a_h, \dots, a_{h+s-1}, x_h]$, hence $x_h = x_{h+s}$, $x'_h = x'_{h+s}$. From $x'_h = x'_{h+s}$, taking integral parts, we deduce $a_{h-1} = a_{h+s-1}$, a contradiction. □

Corollary 100. *If $r > 1$ is a rational number which is not a square, then the continued fraction expansion of $\sqrt{r}$ is of the form*

$$\sqrt{r} = [a_0, \overline{a_1, \dots, a_{s-1}, 2a_0}]$$

with $a_1, \dots, a_{s-1}$ a palindrome and $a_0 = [\sqrt{r}]$.
Conversely, if the continued fraction expansion of an irrational number $t > 1$ is of the form

$$t = [a_0, \overline{a_1, \dots, a_{s-1}, 2a_0}]$$

with $a_1, \dots, a_{s-1}$ a palindrome, then t^2 is a rational number.

Proof. If $t^2 = r$ is rational > 1, then for and $a_0 = [\sqrt{t}]$ the number $x = t + a_0$ is reduced. Since $t' + t = 0$, we have

$$-\frac{1}{x'} = \frac{1}{x - 2a_0}\cdot$$

Hence

$$x = [\overline{2a_0, a_1, \ldots, a_{s-1}}], \qquad -\frac{1}{x'} = [\overline{a_{s-1}, \ldots, a_1, 2a_0}]$$

and $a_1, \ldots, a_{s-1}$ a palindrome.

Conversely, if $t = [a_0, \overline{a_1, \ldots, a_{s-1}, 2a_0}]$ with $a_1, \ldots, a_{s-1}$ a palindrome, then $x = t + a_0$ is periodic, hence reduced, and its Galois conjugate x' satisfies

$$-\frac{1}{x'} = [\overline{a_1, \ldots, a_{s-1}, 2a_0}] = \frac{1}{x - 2a_0},$$

which means $t + t' = 0$, hence $t^2 \in \mathbf{Q}$. □

Lemma 101 (Serret, 1878)**.** *Let x and y be two irrational numbers with continued fractions*

$$x = [a_0, a_1, \ldots, a_n \ldots] \quad \text{and} \quad y = [b_0, b_1, \ldots, b_m \ldots]$$

respectively. Then the two following properties are equivalent.

(i) *There exists a matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with rational integer coefficients and determinant* ± 1 *such that*

$$y = \frac{ax + b}{cx + d}\cdot$$

(ii) *There exists $n_0 \geq 0$ and $m_0 \geq 0$ such that $a_{n_0+k} = b_{m_0+k}$ for all $k \geq 0$.*

Condition (i) means that x *and* y *are equivalent modulo the action of* $\mathrm{GL}_2(\mathbf{Z})$ *by homographies.*

Condition (ii) means that there exists integers n_0, m_0 and a real number $t > 1$ such that

$$x = [a_0, a_1, \ldots, a_{n_0-1}, t] \quad \text{and} \quad y = [b_0, b_1, \ldots, b_{m_0-1}, t].$$

Example.

$$\text{If } x = [a_0, a_1, x_2], \text{ then} \quad -x = \begin{cases} [-a_0 - 1, 1, a_1 - 1, x_2] & \text{if } a_1 \geq 2, \\ [-a_0 - 1, 1 + x_2] & \text{if } a_1 = 1. \end{cases} \tag{102}$$

Proof. We already know by (70) that if x_n is a complete quotient of x, then x and x_n are equivalent modulo $\mathrm{GL}_2(\mathbf{Z})$. Condition (ii) means that there is a partial quotient of x and a partial quotient of y which are equal. By transitivity of the $\mathrm{GL}_2(\mathbf{Z})$ equivalence, (ii) implies (i).

Conversely, assume (i):

$$y = \frac{ax+b}{cx+d}\cdot$$

Let n be a sufficiently large number. From

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} u_n & u_{n-1} \\ v_n & v_{n-1} \end{pmatrix}$$

with

$$\begin{aligned} u_n &= ap_n + bq_n, \quad u_{n-1} = ap_{n-1} + bq_{n-1}, \\ v_n &= cp_n + dq_n, \quad v_{n-1} = cp_{n-1} + dq_{n-1}, \end{aligned}$$

we deduce

$$y = \frac{u_n x_{n+1} + u_{n-1}}{v_n x_{n+1} + v_{n-1}}\cdot$$

We have $v_n = (cx+d)q_n + c\delta_n$ with $\delta_n = p_n - q_n x$. We have $q_n \to \infty$, $q_n \geq q_{n-1} + 1$ and $\delta_n \to 0$ as $n \to \infty$. Hence, for sufficiently large n, we have $v_n > v_{n-1} > 0$. From part 1 of Corollary 84, we deduce

$$\begin{pmatrix} u_n & u_{n-1} \\ v_n & v_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_s & 1 \\ 1 & 0 \end{pmatrix}$$

with $a_0, \dots, a_s$ in $\mathbf{Z}$ and $a_1, \dots, a_s$ positive. Hence

$$y = [a_0, a_1, \dots, a_s, x_{n+1}].$$

□

A computational proof of (i) ⇒ (ii). Another proof is given by Bombieri [2] (Theorem A.1 p. 209). He uses the fact that $\mathrm{GL}_2(\mathbf{Z})$ is generated by the two matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The associated fractional linear transformations are K and J defined by

$$K(x) = x + 1 \quad \text{and} \quad J(x) = 1/x.$$

We have $J^2 = 1$ and

$$K([a_0, t]) = [a_0 + 1, t], \quad K^{-1}([a_0, t]) = [a_0 - 1, t].$$

Also $J([a_0, t]) = [0, a_0, t])$ if $a_0 > 0$ and $J([0, t]) = [t])$. According to (102), the continued fractions of x and $-x$ differ only by the first terms. This completes the proof. [13]

$\square$

6.4 Diophantine approximation and simple continued fractions

Lemma 103 (Lagrange, 1770). *The sequence $(|q_n x - p_n|)_{n\geq 0}$ is strictly decreasing: for $n \geq 1$ we have*

$$|q_n x - p_n| < |q_{n-1} x - p_{n-1}|.$$

Proof. We use Lemma 71 twice: on the one hand

$$|q_n x - p_n| = \frac{1}{x_{n+1} q_n + q_{n-1}} < \frac{1}{q_n + q_{n-1}}$$

because $x_{n+1} > 1$, on the other hand

$$|q_{n-1} x - p_{n-1}| = \frac{1}{x_n q_{n-1} + q_{n-2}} > \frac{1}{(a_n + 1) q_{n-1} + q_{n-2}} = \frac{1}{q_n + q_{n-1}}$$

because $x_n < a_n + 1$. $\square$

Corollary 104. *The sequence $(|x - p_n/q_n|)_{n\geq 0}$ is strictly decreasing: for $n \geq 1$ we have*

$$\left| x - \frac{p_n}{q_n} \right| < \left| x - \frac{p_{n-1}}{q_{n-1}} \right|.$$

Proof. For $n \geq 1$, since $q_{n-1} < q_n$, we have

$$\left| x - \frac{p_n}{q_n} \right| = \frac{1}{q_n} |q_n x - p_n| < \frac{1}{q_n} |q_{n-1} x - p_{n-1}| = \frac{q_{n-1}}{q_n} \left| x - \frac{p_{n-1}}{q_{n-1}} \right| < \left| x - \frac{p_{n-1}}{q_{n-1}} \right|.$$

$\square$

Here is the *law of best approximation* of the simple continued fraction.

[13] Bombieri in [2] gives formulae for $J([a_0, t])$ when $a_0 \leq -1$. He distinguishes eight cases, namely four cases when $a_0 = -1$ ($a_1 > 2$, $a_1 = 2$, $a_1 = 1$ and $a_3 > 1$, $a_1 = a_3 = 1$), two cases when $a_0 = -2$ ($a_1 > 1$, $a_1 = 1$) and two cases when $a_0 \leq -3$ ($a_1 > 1$, $a_1 = 1$). Here, (102) enables us to simplify his proof by reducing to the case $a_0 \geq 0$.

Lemma 105. *Let $n \geq 0$ and $(p, q) \in \mathbf{Z} \times \mathbf{Z}$ with $q > 0$ satisfy*

$$|qx - p| < |q_n x - p_n|.$$

Then $q \geq q_{n+1}$.

Proof. The system of two linear equations in two unknowns u, v

$$\begin{cases} p_n u + p_{n+1} v & = p \\ q_n u + q_{n+1} v & = q \end{cases} \tag{106}$$

has determinant ± 1, hence there is a solution $(u, v) \in \mathbf{Z} \times \mathbf{Z}$.

Since $p/q \neq p_n/q_n$, we have $v \neq 0$.

If $u = 0$, then $v = q/q_{n+1} > 0$, hence $v \geq 1$ and $q_n \geq q_{n+1}$.

We now assume $uv \neq 0$.

Since q, q_n and q_{n+1} are > 0, it is not possible for u and v to be both negative. In case u and v are positive, the desired result follows from the second relation of (106). Hence one may suppose u and v of opposite signs. Since $q_n x - p_n$ and $q_{n+1} x - p_{n+1}$ also have opposite signs, the numbers $u(q_n x - p_n)$ and $v(q_{n+1} x - p_{n+1})$ have same sign, and therefore

$$|q_n x - p_n| = |u(q_n x - p_n)| + |v(q_{n+1} x - p_{n+1})| = |qx - p| < |q_n x - p_n|,$$

which is a contradiction.

□

A consequence of Lemma 105 is that the sequence of p_n/q_n produces the best rational approximations to x in the following sense: any rational number p/q with denominator $q < q_n$ has $|qx - p| > |q_n x - p_n|$. This is sometimes referred to as *best rational approximations of type* 0.

Corollary 107. *The sequence $(q_n)_{n \geq 0}$ of denominators of the convergents of a real irrational number x is the increasing sequence of positive integers for which*

$$\|q_n x\| < \|qx\| \quad \textit{for} \quad 1 \leq q < q_n.$$

As a consequence,

$$\|q_n x\| = \min_{1 \leq q \leq q_n} \|qx\|.$$

The theory of continued fractions is developed starting from Corollary 107 as a definition of the sequence $(q_n)_{n \geq 0}$ in Cassels's book [5].

Corollary 108. *Let $n \geq 0$ and $p/q \in \mathbf{Q}$ with $q > 0$ satisfy*

$$\left| x - \frac{p}{q} \right| < \left| x - \frac{p_n}{q_n} \right|.$$

Then $q > q_n$.

Proof. For $q \leq q_n$ we have

$$\left| x - \frac{p}{q} \right| = \frac{1}{q} |qx - p| > \frac{1}{q} |q_n x - p_n| \frac{q_n}{q} \left| x - \frac{p_n}{q_n} \right| \geq \left| x - \frac{p_n}{q_n} \right|.$$

□

Corollary 108 shows that the denominators q_n of the convergents are also among the *best rational approximations of type* 1 in the sense that

$$\left| x - \frac{p}{q} \right| > \left| x - \frac{p_n}{q_n} \right| \quad \text{for} \quad 1 \leq q < q_n,$$

but they do not produce the full list of them: to get the complete set, one needs to consider also some of the rational fractions of the form

$$\frac{p_{n-1} + a p_n}{q_{n-1} + a q_n}$$

with $0 \leq a \leq a_{n+1}$ (*semi–convergents*) – see for instance [7], Chap. II, § 16.

Lemma 109 (Vahlen, 1895)**.** *Among two consecutive convergents p_n/q_n and p_{n+1}/q_{n+1}, one at least satisfies $|x - p/q| < 1/2q^2$.*

Proof. Since $x - p_n/q_n$ and $x - p_{n-1}/q_{n-1}$ have opposite signs,

$$\left| x - \frac{p_n}{q_n} \right| + \left| x - \frac{p_{n-1}}{q_{n-1}} \right| = \left| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right| = \frac{1}{q_n q_{n-1}} < \frac{1}{2q_n^2} + \frac{1}{2q_{n-1}^2}.$$

The last inequality is $ab < (a^2 + b^2)/2$ for $a \neq b$ with $a = 1/q_n$ and $b = 1/q_{n-1}$. Therefore,

$$\text{either} \quad \left| x - \frac{p_n}{q_n} \right| < \frac{1}{2q_n^2} \quad \text{or} \quad \left| x - \frac{p_{n-1}}{q_{n-1}} \right| < \frac{1}{2q_{n-1}^2}.$$

□

Lemma 110 (É. Borel, 1903). *Among three consecutive convergents* p_{n-1}/q_{n-1}*,* p_n/q_n *and* p_{n+1}/q_{n+1}*, one at least satisfies* $|x - p/q| < 1/\sqrt{5}q^2$*.*

This completes the proof of the irrationality criterion Proposition 4 including (i) ⇒ (vi) in § 2.1.

The fact that the constant $\sqrt{5}$ cannot be replaced by a larger one was proved in Lemma 41. This is true for any number with a continued fraction expansion having all but finitely many partial quotients equal to 1 (which means the Golden number Φ and all rational numbers which are equivalent to Φ modulo $\mathrm{GL}_2(\mathbf{Z})$).

Proof. Recall Lemma 71: for $n \geq 0$,

$$q_n x - p_n = \frac{(-1)^n}{x_{n+1}q_n + q_{n-1}}\cdot$$

Therefore $|q_n x - p_n| < 1/\sqrt{5}q_n$ if and only if $|x_{n+1}q_n + q_{n-1}| > \sqrt{5}q_n$. Define $r_n = q_{n-1}/q_n$. Then this condition is equivalent to $|x_{n+1} + r_n| > \sqrt{5}$.

Recall the inductive definition of the convergents:

$$x_{n+1} = a_{n+1} + \frac{1}{x_{n+2}}\cdot$$

Also, using the definitions of r_n, r_{n+1}, and the inductive relation $q_{n+1} = a_{n+1}q_n + q_{n-1}$, we can write

$$\frac{1}{r_{n+1}} = a_{n+1} + r_n.$$

Eliminate a_{n+1}:

$$\frac{1}{x_{n+2}} + \frac{1}{r_{n+1}} = x_{n+1} + r_n.$$

Assume now

$$|x_{n+1} + r_n| \leq \sqrt{5} \quad \text{and} \quad |x_{n+2} + r_{n+1}| \leq \sqrt{5}.$$

We deduce

$$\frac{1}{\sqrt{5} - r_{n+1}} + \frac{1}{r_{n+1}} \leq \frac{1}{x_{n+2}} + \frac{1}{r_{n+1}} = x_{n+1} + r_n \leq \sqrt{5},$$

which yields

$$r_{n+1}^2 - \sqrt{5}r_{n+1} + 1 \leq 0.$$

The roots of the polynomial $X^2 - \sqrt{5}X + 1$ are $\Phi = (1+\sqrt{5})/2$ and $\Phi^{-1} = (\sqrt{5}-1)/2$. Hence $r_{n+1} > \Phi^{-1}$ (the strict inequality is a consequence of the irrationality of the Golden ratio). .

This estimate follows from the hypotheses $|q_n x - p_n| < 1/\sqrt{5}q_n$ and $|q_{n+1}x - p_{n+1}| < 1/\sqrt{5}q_{n+1}$. If we also had $|q_{n+2}x - p_{n+2}| < 1/\sqrt{5}q_{n+2}$, we would deduce in the same way $r_{n+2} > \Phi^{-1}$. This would give

$$1 = (a_{n+2} + r_{n+1})r_{n+2} > (1+\Phi^{-1})\Phi^{-1} = 1,$$

which is impossible.

□

Lemma 111 (Legendre, 1798)**.** *If $p/q \in \mathbf{Q}$ satisfies $|x - p/q| \leq 1/2q^2$, then p/q is a convergent of x.*

Proof. Let r and s in $\mathbf{Z}$ satisfy $1 \leq s < q$. From

$$1 \leq |qr - ps| = |s(qx-p) - q(sx-r)| \leq s|qx-p| + q|sx-r| \leq \frac{s}{2q} + q|sx-r|$$

one deduces

$$q|sx - r| \geq 1 - \frac{s}{2q} > \frac{1}{2} \geq q|qx - p|.$$

Hence $|sx - r| > |qx - p|$ and therefore Lemma 105 implies that p/q is a convergent of x.

□

References

[1] P. Flajolet, B. Vallée, I. Vardi, *Continued fractions from Euclid to the present day*, 44p.
http://www.lix.polytechnique.fr/Labo/Ilan.Vardi/continued_fractions.ps